Class 20 Principal bundle (Chap 10)

Before introducing principal bundle, let's review materials in previous lectures:

Part I: manifolds

topological manifold: M para, Haus topo space s.t. each pt has nbhd homeo to $\mathbb{R}^n$

$$pt \in U \subset M \quad \phi_U : U \xrightarrow{\cong} \mathbb{R}^n \quad \text{(Global view)}$$

Construct by charts and transition functions (Local view)

Atlas $\mathcal{U}$, for $U, V \in \mathcal{U}$

$$h_{VU} = \phi_V \circ \phi_U^{-1} : \underset{\mathbb{R}^n}{\phi_U(U \cap V)} \xrightarrow{\cong} \underset{\mathbb{R}^n}{\phi_V(U \cap V)}$$

(They satisfy the cocycle condition $h_{VU} \circ h_{UV} = Id$

$h_{VU} \circ h_{UW} \circ h_{WV} = Id$)

$$M = \coprod_{U \in \mathcal{U}} \mathbb{R}^n_U \Big/ x \in \mathbb{R}^n_U \sim h_{VU}(x) \in \mathbb{R}^n_V$$

Smooth mfd: h_{VU} are smooth

(The smoothness only makes sense for maps btw $\mathbb{R}^n$)

Inverse/Implicit function thm: $\psi : \mathbb{R}^m \to \mathbb{R}^n$

If for any pt in $\psi^{-1}(a)$, the Jacobian $\{\frac{\partial \psi^i}{\partial x^j}\}$ is surjective, then $\psi^{-1}(a)$ is a smooth mfd

Lie group: smooth manifold with group structure.

s.t. $(a,b) \mapsto ab$, $a \mapsto a^{-1}$ are smooth

Ex: $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n)$, $SO(n)$

$GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $U(n)$, $SU(n)$

($O(n)$, $SO(n)$, $U(n)$, $SU(n)$: compact)

Part II. vector bundle

E smooth mfd : $\pi : E \to M$

① $\forall p \in M$ $\exists U \subset M$ containing p

s.t. $\varphi_U : E|_U = \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ diffeo

② zero section $\hat{0} : M \to E$ $\pi \circ \hat{0} = Id$

③ $\exists$ smooth map $\mu : \mathbb{R} \times E \to E$

or $\mathbb{C} \times E \to E$

satisfying some conditions.

(Global view)

Construct by (locally finite) charts and bundle transition functions

$g_{VU} : U \cap V \longrightarrow GL(n,\mathbb{R})$ with cocycle cond

$$E = \coprod_{U \in \mathcal{U}} U \times \mathbb{R}^n \Big/ (p,u) \in U \times \mathbb{R}^n \sim (p, g_{VU} u) \in V \times \mathbb{R}^n$$

(Local view)

v.b. are generalization of product manifold $M \times \mathbb{R}^n$

it is also a mfd. but with a fiber str

tangent bundle : $g_{vu} = (h_{vu})_*$ TM

cotangent bundle : $g_{vu} = ((h_{vu})_*^{-1})^T$ T^*M

dim M = dim fiber

Sections : $s : M \to E$

$C^\infty(M;E)$ or $C^\infty(E)$ space of sections

operation : E^*, $E_1 \oplus E_2$, $E_1 \otimes E_2$, $Hom(E_1, E_2)$

$Sym^k E$ $\Lambda^k E$

$End(E) = Hom(E, E) = E^* \otimes E$

Def Principal bundle P over M (↑ mfd.) with fiber G (↑ Lie gp)

has the following structures

1) a smooth map $m : G \times P \to P$ with
   ① $m(e, p) = p$ e is identity in G
   ② $m(h, m(g, p)) = m(hg, p)$
   usually write $m(g, p)$ by pg^{-1}

2) a smooth map $\pi : P \to M$ surjective with
   $\pi(pg^{-1}) = \pi(p)$. π is called projection

3) Any pt in M has nbhd U with

$$\varphi_U : P|_U = \pi^{-1}(U) \longrightarrow U \times G \text{ diffeo}$$

s.t. if $\varphi_U(p) = (\pi(p), h(p))$

then $\varphi_U(pg^{-1}) = (\pi(p), h(p)g^{-1})$

(Global view)

Def 2 Given a locally finite open cover $\mathcal{U}$ of M

and $g_{VU} : U \cap V \longrightarrow G$ satisfying

$$g_{VU} = g_{UV}^{-1} \quad g_{VU}\, g_{UW}\, g_{WV} = Id$$

$$P = \coprod_{U \in \mathcal{U}} U \times G \Big/ (p, g) \in U \times G \sim (p, g_{VU} g) \in V \times G$$

The G action $m(h; (p, g)) = (p, gh^{-1})$

Rem generalization of product pricipal bundle $M \times G$

Def A bundle homomorphism $f : P \longrightarrow P'$ over M

Satisfies $\pi(f(p)) = \pi'(p)$

$$f(pg^{-1}) = f(p)g^{-1}$$

P and P' are iso if f has an inverse

Def: A fiber bundle E is a smooth mfd with

① $\pi : E \to B$. B called base manifold

② $\forall p \in B$. $\exists U \subset B$ containing p

s.t. $\exists \varphi_U : \pi^{-1}(U) \to U \times F$ diffeo

F mfd. called fiber (Global view)

Rem Can also be defined by bundle transition function

$g_{VU} : U \cap V \to \mathrm{Diff}(F)$

↑ diffeo group

(Local version)

A vector bundle is a fiber bundle with fiber $\mathbb{R}^n, \mathbb{C}^n$

+ scalar multiplication

A principal bundle is a fiber bundle with fiber G

+ group multiplication.

Note that $GL(n,\mathbb{R})$ contains linear transformation of $\mathbb{R}^n$

vector bundle E $\xrightarrow{\text{framed bundle}}$ principal bundle
$\xleftarrow[\text{associated vector bundle}]{}$

fiber $\mathbb{R}^n$ fiber $GL(n,\mathbb{R})$

Framed bundle of E: recall the transition function of E is $g_{VU}: U \cap V \to GL(n,\mathbb{R})$

$$E = \coprod_{U \in \mathcal{U}} U \times \mathbb{R}^n \Big/ (p,u) \in U \times \mathbb{R}^n \sim (p, g_{VU} u) \in V \times \mathbb{R}^n$$

$$P = \coprod_{U \in \mathcal{U}} U \times G \Big/ (p,g) \in U \times G \sim (p', g_{VU} g) \in V \times G$$

Let $G = GL(n,\mathbb{R})$ $\quad P = P_{GL(E)}$

Associated vector bundle: a <u>representation</u> of G is a homomorphism $\rho: G \longrightarrow GL(n,\mathbb{R})$

i.e. $\rho(e) = Id$ $\quad \rho(gh) = \rho(g)\rho(h)$ $\quad \rho(g) = \rho(g)^{-1}$

Given $g_{VU}: U \cap V \to G$ for P

Construct vector bundle E by

$$g_{VU}: U \cap V \to G \xrightarrow{\rho} GL(n,\mathbb{R})$$

$$E = P \times_\rho \mathbb{R}^n$$

Part III metrics

A metric g on E is a section $M \to E^* \otimes E^*$ s.t.

1) $g|_p(v,v) > 0$ for $v \neq 0 \in E|_p$ (positive definite)

2) $g|_p(v,w) = g|_p(w,v)$ $v, w \in E|_p$ (Symmetric)

Fact. 1) metric always exists
(construct locally and use partion of unity to glue globally)

2) g_1, g_2 are metrics $\Rightarrow sg_1, \ tg_1 + (1-t)g_2$ are metrics
$s, t > 0$

$\Rightarrow$ the space of metrics is convex. Contractible

3) The existence of metric can help us choose
an orthonormal basis $e_1, \cdots, e_n$ locally (only on U)
which is different from the coordinate basis
$\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ of $TM|_U$ $dx^1, \cdots, dx^n$ of $T^*M|_U$

Hence we can choose transition function
$g_{VU} : U \cap V \to O(n)$

If furthermore E is orientable, i.e. $\Lambda^{\dim E} E \cong M \times \mathbb{R}$
then we have $\det g_{VU} > 0$. $g_{VU} : U \cap V \to SO(n)$

Then we can construct principal bundle $P_{O(n)}$ or $P_{SO(n)}$ just by gvu.

Conversely, if we construct an associate bundle from some principal bundle with fiber $SO(n)$, then the vector bundle is orientable

Class 21 Review

Last time, we review Part I manifold and

Part II vector bundle, and construct principal bundle

	mfd M	v.b. E	p.b. P
chart (Global)	$\phi_U : U \to \mathbb{R}^m$	$\varphi_U : E\vert_U \to U \times \mathbb{R}^n$	$\varphi_U : P\vert_U \to U \times G$
transition (Local)	$h_{VU} : \phi_U(U \cap V) \subset \mathbb{R}^m \to \phi_V(U \cap V) \subset \mathbb{R}^m$	$g_{VU} : U \cap V \to GL(n, \mathbb{R})$	$g_{VU} : U \cap V \to G$
action	/	scalar action $\mathbb{R} \times E \to E$, $\mathbb{C} \times E \to E$	group action $G \times E \to E$

fiber bundle: $F \to E$: $\varphi_U : E\vert_U \to U \times F$

$E \downarrow B$ no action

vector bundle $\xrightarrow{\text{framed bundle}}$ principal bundle

vector bundle $\xleftarrow{\text{associate v.b.}}$ principal bundle

need a rep: $\rho : G \to GL(n, \mathbb{R}^n)$

Today, we review more constructions in

Part III metrics, IV derivatives

V metric compatible derivatives.

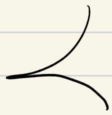
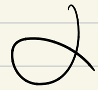
In Part III, we also study Riemannian metric (on TM) and introduce the length l_γ of a smooth path

$$\gamma: I=[0,1] \to M \qquad l_\gamma = \int_I g\left(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}\right)^{\frac{1}{2}} dt$$

$\gamma_*: TI \to TM$ is the tangent map (push forward)

For $p, q \in M$ $\quad d(p,q) = \inf_{\substack{\gamma(0)=p \\ \gamma(1)=q}} l_\gamma$

Geodesic thm says if M is compact, then the path minimizing l_γ always exists, called geodesic.

It is embedded (no  or )

and we can set $g\left(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}\right) = 1$

Locally, it is obtained by solving the geodesic equation $\ddot{\gamma}^i + \Gamma^i_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0 \quad \dot{\gamma} = \frac{d\gamma}{dt} \quad \ddot{\gamma} = \frac{d^2\gamma}{dt^2}$

$$\Gamma^i_{jk} = \frac{1}{2} \sum_{l=1}^{n} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

called Christoffel symbol

This equation can be regarded as Euler-Lagrange equation from the functional

$$S: \{\gamma : I \to \mathbb{R}^n \mid \gamma(0)=p,\ \gamma(1)=q\} \longrightarrow \mathbb{R}$$

$$\gamma \longmapsto L_\gamma$$

i.e. For any $\eta : I \to \mathbb{R}^n$ $\eta(0)=\eta(1)=0$

$$\gamma_s = \gamma + s\eta \qquad \left.\frac{dS(\gamma_s)}{ds}\right|_{s=0} = 0$$

Later, when we introduce LC conn, this equation can be written as $\nabla^{LC}_{\dot\gamma}\dot\gamma = 0$

i.e. $\dot\gamma$ itself is parallel section along γ about ∇^{LC}

Part IV derivatives.

$\Omega^k(M) = C^\infty(M; \Lambda^k T^*M)$ the space of k-forms

We construct exterior derivative $d : \Omega^k \to \Omega^{k+1}$

by (locally) $d(f\,dx^1\wedge\cdots\wedge dx^k) = df\wedge dx^1\cdots\wedge dx^k$

$$= \frac{\partial f}{\partial x^m} dx^m \wedge dx^1\wedge\cdots\wedge dx^m$$

Since $\dfrac{\partial^2 f}{\partial x^i \partial x^j} = \dfrac{\partial^2 f}{\partial x^j \partial x^i}$, we have $d^2=0$

$H^k_{dR}(M) = \ker(d : \Omega^k \to \Omega^{k+1}) / \mathrm{Im}(d : \Omega^{k-1} \to \Omega^k)$

de Rham cohomology. it is an $\mathbb{R}$-vector space

only depends on the diffeo type of M

(Indeed also doesn't depend on smooth str

because $H^k_{dR}(M) = H^k_{sing}(M;\mathbb{R})$)

$f : M \to N$ induces

① push forward $f_* : T_pM \to T_{f(p)}N$

② pull-back of bundle $\pi : E \to N$ f^*E

also pull-back of principal bundle f^*P $f^*P_{GL(E)} = P_{GL(f^*E)}$

③ pull-back of forms $f^* : \Omega^k(N) \to \Omega^k(M)$

$f^*\omega(v_1, \dots, v_k) = \omega(f_* v_1, \dots, v_k)$

④ pull-back of de Rham cohomology class

because $d f^*\omega = f^* d\omega$ $\Rightarrow f^* : H^k_{dR}(N) \to H^k_{dR}(M)$

homotopic maps f, g induce the same map.

($\Phi : I \times M \to N$. $\Phi(0,-) = f$ $\Phi(1,-) = g$)

Lie derivative : $\mathcal{L}_V : \Omega^k \to \Omega^k$ for V. vector field

Cartan formula $\mathcal{L}_V \omega = (\iota_V d + d\iota_V)\omega$

$\iota_V \alpha(v_1, \dots, v_{k-1}) = \alpha(V, v_1, \dots, v_{k-1})$

Covariant derivative : $\nabla : C^\infty(E) \to C^\infty(E \otimes T^*M)$

satisfying $\nabla(fs) = s \otimes df + f\nabla s$

If $v \in TM$, we can define $\nabla_v : C^\infty(E) \to C^\infty(E)$

similar to the partial derivative along v in $\mathbb{R}^n$

Fact. ① Cov. der exists (construct locally and

use partition of unity, locally. $s = (s_1, \cdots, s_n)$

$s_i : U \to \mathbb{R}$ $\quad (s_i)_* : T_pU \to T_{s_i(p)}\mathbb{R} \cong \mathbb{R}$

$((s_1)_*, \cdots, (s_n)_*)$ is a section of $(E \otimes T^*M)|_U$

written as ds. In particular, for $M \times \mathbb{R}$

the exterior derivative $d : \Omega^0 \to \Omega^1$ is a cov der.

② Any ∇, ∇', we have $\nabla' - \nabla = \alpha$

$\alpha \in C^\infty(\mathrm{End}(E) \otimes T^*M)$ matrix-valued 1-form.

③ Combining d, we have

$d_\nabla : C^\infty(E \otimes \Lambda^k T^*M) \to C^\infty(E \otimes \Lambda^{k+1} T^*M)$

$d_\nabla^2 s = F_\nabla s$ the curvature

Locally $F_\nabla = d\alpha + \alpha \wedge \alpha$

(on TM. $\nabla_i = \nabla_{\frac{\partial}{\partial x^i}}$ $\quad \nabla_i \nabla_j - \nabla_j \nabla_i = F_\nabla$)

④ Given g on TM, $\exists$ unique ∇ on TM satisfies

$$\begin{cases} \nabla g = 0 & \text{(metric compatible)} \\ T_\nabla = d_\nabla - d = 0 & \text{(torsion free)} \end{cases}$$, called

we also study curvature of LC conn

⑤ Bianchi indentity $d_\nabla F_\nabla = 0$

$$C_k(E) = \left[\frac{1}{(2\pi\sqrt{-1})^k} tr(\underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{k}) \right] \in H^{2k}_{dR}(M)$$

indep of choice of ∇.

Property of Chern class

$f: M \to N$ $\quad \pi: E \to N$ $\quad C_k(f^*E) = f^* C_k(E)$

$C_k(E^*) = (-1)^k C_k(E)$

$C(E) = 1 + C_1(E) + C_2(E) + \cdots$

$C(E_1 \oplus E_2) = C(E_1) \wedge C(E_2)$

For real vector bundle E, we can define pontryagin class $p_k(E) = c_{2k}(\underbrace{E \otimes_{\mathbb{R}} \mathbb{C}}_{\text{roughly } E \oplus \bar{E}}) \in H^{4k}_{dR}(M;\mathbb{R})$

The odd Chern class of complexification is determined by Stiefel-Whitney class of the original bundle

For more discussion of characteristic classes, see [Hatcher vector bundles and K-theory. Chap 3.]

Roughly speaking. Char classes are obstructions of the triviality of the bundle: Chern class $c_k \in H^{2k}(M;\mathbb{Z})$, pontryagin class $p_k \in H^{4k}(M;\mathbb{Z})$

Stiefel-Whitney $w_k \in H^k(M;\mathbb{Z}/2\mathbb{Z})$. Euler class $e \in H^2(M;\mathbb{Z})$

Class 22 Connection on principal bundle (Chap 11.4)

Suppose $\pi : P \to M$ is a principal bundle with fiber G.

First, we study the tangent space of G at e, denoted by

$\mathfrak{g} = T_eG$, called Lie algebra of G (lie(G) in Cliff's book)

Since G is a Lie group, for any fixed $g \in G$, define

$L_g : G \to G$, $R_g : G \to G$ they are diffeomorphism.

$h \mapsto gh$ $\qquad$ $h \mapsto hg$

$(L_g)_* : T_eG \to T_gG$ is a vector space isomorphism.

$TG \cong G \times T_eG$ by the following map.

For $(g, v) \in G \times T_eG$, define $f(g,v) = (L_g)_* v \in TG$

Def A vector field $v : G \to TG$ is <u>left-invariant</u> if

$$v(gh) = (L_g)_* v(h)$$

Note that left-invariant v.f. is 1-1 cor to vector in T_eG

Let $\mathfrak{g} = \text{Lie}(G)$ be the set of all left-inv v.f.
\mathfrak{g}

Recall that there is a 1-1 correspondence btw

vector field v and derivative $\mathcal{L}_v : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$

Given v, w v.f. we can define $[v,w]$ by

$$\mathcal{L}_{[v,w]}(f) = \mathcal{L}_v \mathcal{L}_w(f) - \mathcal{L}_w \mathcal{L}_v(f)$$

Then ① $[av+bw, u] = a[v,u] + b[w,u]$

$[v, aw+bu] = a[v,w] + b[v,u]$

$a, b \in \mathbb{R}$ v, w, u v.f.

② $[v,v] = 0 \Rightarrow [v,w] = -[w,v]$

③ $[v,[w,u]] + [u,[v,w]] + [w,[u,v]] = 0$

(Jacobi identity)

$[-,-]$ is called <u>Lie bracket</u> (of vector fields)

In the case of $\mathfrak{g}$, we can check for left inv v, w,

$[v,w]$ is still left inv, so we have a map

$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$,

$\mathfrak{g}$ is called Lie algebra associated to G.

as a vector space, it is just T_eG

Ex.

G	$\mathfrak{g}$
$GL(n,\mathbb{R})$ ($\det \neq 0$) $GL(n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{R}) = M(n,\mathbb{R})$ $\mathfrak{gl}(n,\mathbb{C}) = M(n,\mathbb{C})$ $[X,Y] = XY - YX$
$SL(n,\mathbb{R})$ ($\det = 1$) $SL(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{R}) \subset M(n,\mathbb{R})$ $\mathrm{tr}\,X = 0$ (traceless)
$O(n)$ ($AA^T = Id$) $U(n)$ ($AA^* = Id$)	$X + X^T = 0$ (skew-symmetric) $X + X^* = 0$
$SO(n)$ $\begin{pmatrix} AA^T = Id \\ \det A = 1 \end{pmatrix}$	$\mathfrak{so}(n)$ $\mathrm{tr}\,X = 0$ $X + X^T = 0$ $\begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&-1&0 \end{pmatrix}$ $\mathfrak{su}(2) = \mathfrak{so}(3)$ $\begin{pmatrix} 0&1&0 \\ -1&0&0 \\ 0&0&0 \end{pmatrix}$ $\begin{pmatrix} 0&0&1 \\ 0&0&0 \\ -1&0&0 \end{pmatrix}$
$SU(n)$ $\begin{pmatrix} AA^* = Id \\ \det A = 1 \end{pmatrix}$	$\mathfrak{su}(n)$ $\mathrm{tr}\,X = 0$ $X + X^* = 0$ Pauli matrices $\begin{pmatrix} 0&-1 \\ 1&0 \end{pmatrix}$ $\begin{pmatrix} 0&i \\ i&0 \end{pmatrix}$ $\begin{pmatrix} i&0 \\ 0&-i \end{pmatrix}$

$\pi : P \to M$ is a smooth (surjective map), we can

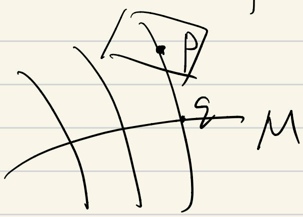
consider the tangent map $\pi_* : T_pP \to T_{\pi(p)}M$

$\ker \pi_*$ is a sub-(vector)-bundle of TP

s.t. sections in $\ker \pi_*$ are sent to zero sections of TM

So $(\ker \pi_*)$ over $P|_q = \pi^{-1}(q) \cong G$ for $q \in M$

is isomorphic to $TP|_q \cong T_eG = \mathfrak{g}$



also, we consider $\pi^* TM$ as a bundle over P.

We have a sequence of vector bundles

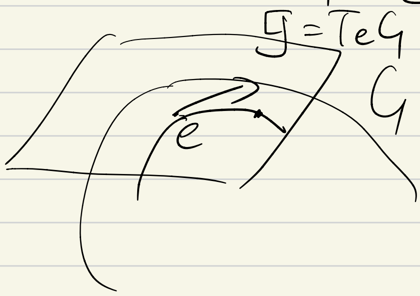
$$\ker \pi_* \to TP \to \pi^* TM$$

s.t. at each fiber, it is exact ($\ker(TP \to \pi^*TM) = \mathrm{Im}(\ker\pi_* \to TP)$)

Indeed $\ker \pi_* \cong P \times \mathfrak{g}$

For $(p, x) \in P \times \mathfrak{g}$, consider the map $t \mapsto p\exp(tx)$

where $\exp : \mathfrak{g} \to G$ is the exponential map



define $f(p,x) \in \ker \pi_*$

to be the tangent vector at $t = 0$

Any $g \in G$ define $\psi_g : P \xrightarrow{\cong} P$ by $p \mapsto pg^{-1}$

It induces $(\psi_g)_* \; T_pP \to T_{pg^{-1}}P$

Then we have $f(pg^{-1}, (L_g)_*(R_{g^{-1}})_* x) = (\Psi_g)_*(f(p,x))$

$$gxg^{-1}$$

Def a connection A on P is one of the following:

1) A section $P \longrightarrow T^*P \otimes \ker \pi_*$.

(or $\mathrm{Hom}(TP, \mathfrak{g})$ where $\mathfrak{g}$ is the product v.b. $P\times\mathfrak{g}$)

satisfies ① $\langle A|_p, f(p,x)\rangle = x \in \mathfrak{g}$

② $\langle \Psi_g^*(A|_{pg^{-1}}), v\rangle = g\langle A|_p, v\rangle g^{-1}$ $\forall g\in G, v\in TP|_p$

2) Horizontal subspaces $H \subset TP$ tangent to fibers of π i.e

$TP = H \oplus \ker \pi_*$

(A choice of splitting in $\ker\pi_* \longrightarrow TP \longrightarrow \pi^*TM$)

H is isomorphic to π^*TM, but not canonically.

Also. H need to be preserved under $(\Psi_g)_*$

1) $\Rightarrow$ 2). Let H_A be kernel of $A: TP \to \ker\pi_*$

2) $\Rightarrow$ 1) Use H to define the A by projection

If A and A' are two connections, then $A-A$ is

a map from π^*TM to $\ker\pi_* = P\times\mathfrak{g}$

which is compatible with the action of g

Class 23 connection and covariant derivative

Last time, we introduced Lie algebra of a Lie gp as left-invariant vector field. the space is identified with $T_e G = \mathfrak{g}$

$$L_g : G \xrightarrow{\cong} G \qquad R_g : G \xrightarrow{\cong} G$$
$$h \mapsto gh \qquad\qquad h \mapsto hg$$

Left inv: $(L_g)_* v = v$ (usually write v for v.f. but x for element in $\mathfrak{g}$)

Define ad: $G \longrightarrow GL(\mathfrak{g})$

$$G \times \mathfrak{g} \longrightarrow \mathfrak{g} \qquad g \cdot x = (L_g)_*(R_{g^{-1}})_* x = gxg^{-1}$$

This is called adjoint representation of G

Recall a rep of G is a homomorphism $\rho : G \longrightarrow GL(n, \mathbb{R})$. now $\mathfrak{g}$ itself is a vector space (with a Lie bracket)

(Consider tangent map, we have ad: $\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, where $\mathfrak{gl}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g})$, $ad(x)(y) = [x, y]$

$$= M(\dim \mathfrak{g}, \mathbb{R}) \qquad = xy - yx$$

This is called the adjoint representation of Lie algebra $\mathfrak{g}$)

Recall for G-principal bundle P and a rep $\rho: G \to GL(n,\mathbb{R})$, we can define an associated vector bundle $P \times_\rho \mathbb{R}^n$

by transition functions

$$g_{UV} : U \cap V \to G \xrightarrow{\rho} GL(n,\mathbb{R})$$

We can use adjoint rep to obtain $P \times_{Ad} \mathfrak{g}$

It is a vector bundle over M

we want to argue for two connections A, A'

the difference $A' - A$ is a section of $P \times_{Ad} \mathfrak{g}$

Note $\mathfrak{g}$ can be regarded as a subspace of matrices,

this is related to the matrix-valued 1-form in

$$\nabla' - \nabla = \alpha \in C^\infty(M; End(E) \otimes \Lambda^1 T^*M)$$

Recall the definition of connections:

we have an exact sequence of v.b. over P

$$\ker \pi_* \to TP \to \pi^* TM$$

$\ker \pi_* \cong P \times \mathfrak{g}$ (product $\mathfrak{g}$ bundle)

a connection is either ① a bundle map $TP \to \ker \pi_*$

s.t. $\ker \pi_* \hookrightarrow TP \to \ker \pi_*$ is identity.

or ② a subbundle H s.t. $H \oplus \ker \pi_* = TP$

Both satisfy some G-equivariant conditions.

H is called horizontal space, because it is tangent to the fiber G

From ①. A is a $\mathfrak{g}$-valued 1-form on P

$$\Psi_g : P \xrightarrow{\simeq} P, \quad p \mapsto pg^{-1}$$

G-equivariant means.

$$\langle \Psi_g^*(A|_{pg^{-1}}), v\rangle = g\langle A|_p, v\rangle g^{-1} \quad \forall g \in G, v \in TP|_p$$

② G-equivariant of H means $(\Psi_g)_* H_p = H_{pg^{-1}}$

$$H_A = \ker A \qquad H \cong \pi^* TM$$

The difference $A' - A : TP \longrightarrow \ker \pi_*$

s.t. $\ker \pi_* \hookrightarrow TP \xrightarrow{A'-A} \ker \pi_*$ is zero

So it induces a map $\pi^* TM = TP/_{\ker \pi_*} \longrightarrow \ker \pi_*$

that is G-equivariant, so this gives a section

in $P \times_{ad} \mathfrak{g} = \{(p,x) \in P \times \mathfrak{g}\} / (p,x) \sim (pg^{-1}, gxg^{-1})$

↑ from G-equivariant condition

We show a general argument about associated bundle

Let $V = \mathbb{R}^n$ or $\mathbb{C}^n$ $\mathrm{End}(V) = GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$

Let $\rho : G \to \mathrm{End}(V)$ be an representation

$$E = P \times_\rho V = \{(p,v) \in P \times V\} / (p,g) \sim (pg^{-1}, \rho(g)v)$$

$$= P \times V / \text{action of } G$$

Then a section $s : M \to E$ is the same as

G-equ map $\$: P \to V$

Given a connection A (or H_A) on P, we can define

cov der ∇ on E as follows.

First, take $d\$: TP \to V$, regarded as

a section $\underline{V} \otimes T^*P$, where $\underline{V}$ denote the bundle

$P \times V$. We have $\langle (L_g)^*(d\$|_{pg^{-1}}), w \rangle = \rho(g) \langle d\$|_p, w \rangle$

for $w \in TP|_p$

For $v|_p \in \pi^*TM|_p$ (or TM), we pick the unique horizontal

lift $v_A|_p \in H_p \subset T_pP$ s.t. $\pi_* v_A|_p = v|_p$

we have $(L_g)_* v_A|_p = v_A|_{pg^{-1}}$

Let $\nabla \tilde{s}$ be the G-equ map from $\pi^* TM$ to V

by $\langle \nabla \tilde{s}, v \rangle = \langle d\tilde{s}, v_A \rangle$

we have

$$\langle \nabla((\pi^* f)\tilde{s}), v \rangle = \langle d((\pi^* f)\tilde{s}), v_A \rangle$$

$$= \tilde{s} \langle \pi^* df, v_A \rangle + \pi^* f \langle d\tilde{s}, v_A \rangle$$

$$= \tilde{s} \langle df, v \rangle + \pi^* f \langle \nabla \tilde{s}, v \rangle$$

Hence, $\nabla \tilde{s}$ induces a covariant derivative ∇s

Locally, we have $\varphi_u : P|_u \to U \times G$

a connection A on P can be written explicit

$$A = \varphi_u^* (g^{-1} dg + g^{-1} \alpha_u g)$$

($\mathfrak{g}$-valued 1-form on $U \times G$, not just U)

α_u is a $\mathfrak{g}$-valued 1-form on U

$g^{-1} dg$ comes from $T^* G$ part g^{-1} identity $T_g G$ with $T_e G$

$g^{-1} \alpha_u g$ comes from G-equivariant.

Class 24 Horizontal lift and Yang-Mills equation

$\pi: P \to M$ principal bundle with fiber G

we have $\ker \pi_* \hookrightarrow TP \twoheadrightarrow \pi^* TM$ exact

a connection A is a G-equivariant map

$TP \to \ker \pi_*$ s.t. $\ker \pi_* \hookrightarrow TP \xrightarrow{A} \ker \pi_*$

is identity. take horizontal space H_A as $\ker A$

we have $\pi_*: H_A \subset TP \longrightarrow TM$

is isomorphism.

Given $v \in TM$, we get $v \in \pi^* TM$,

pick $v_A \in H_A$ s.t. $\pi_* v_A = v$

v_A is called horizontal lift of v.

On associated bundle $E = P \times_\rho V$, we can define

∇s by lift $s: M \to E$ to

G-equ $\tilde{s}: P \to V$. and $d\tilde{s}: TP \to V$

$\langle \nabla s, v \rangle = \langle d\tilde{s}, v_A \rangle$

Given a path $\gamma: [0,1] \to M$, we want to

lift γ to $\gamma_A: [0,1] \to P$ horizontally.

Prop. Given $t_0 \in [0,1]$, $p \in P|_{\gamma(s)}$, $\exists$ unique

$\gamma_A : [0,1] \to P$ satisfying

1) $\gamma_A(t_0) = p$

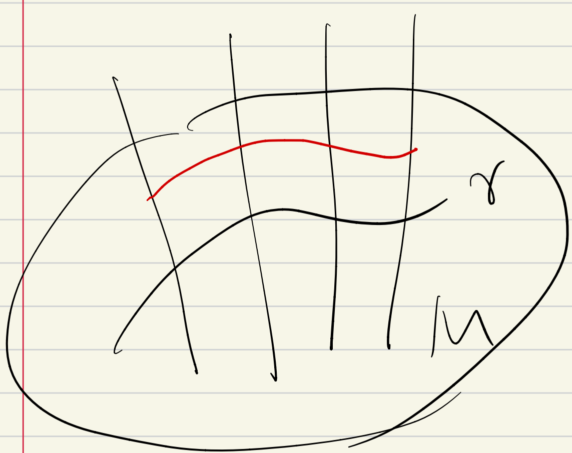
2) $\pi(\gamma_A) = \gamma$

3) $\dot{\gamma}_A \subset H_A$

Moreover, this lift is G-equ. i.e.

the path $\gamma_A' : t \mapsto \gamma_A(t) g^{-1}$ is the horizontal lift

of γ with $\gamma_A'(t_0) = p g^{-1}$



Recall in the case of E, ∇.

a section S is parallel along γ

if $\nabla_{\dot{\gamma}} S = 0$

If $E = P \times_\rho V$ is the associated bundle

a section $s : M \to E$ corresponds to G-equ. $\$: P \to V$

$S(\gamma(t_0)) \in E|_{\gamma(t_0)}$ $\langle \nabla s, \dot{\gamma} \rangle = 0$

$\Rightarrow \langle d\$, \dot{\gamma}_A \rangle = 0$

$\Rightarrow$ partial derivative along $\dot{\gamma}_A$ vanishes

To prove the existence of horizontal lift.
We need local model of the connection

Locally, we have $\varphi_u : P|_U \to U \times G$

a connection A on P can be written explicit

$$A = \varphi_u^*(g^{-1}dg + g^{-1}\alpha_u g)$$

($\mathfrak{g}$-valued 1-form on $U \times G$, not just U)

α_u is a $\mathfrak{g}$-valued 1-form on U

This α_u is the same one as in $\nabla s_u = ds_u + \alpha_u s_u$

In another chart $V \times G$, we have

$$(x, g_v) = (x, g_{vu}(x) g_u)$$

$$g_v^{-1} dg_v + g_v^{-1} \alpha_v g_v$$

$$= g_u^{-1} dg_u + g_u^{-1}(g_{vu}^{-1} \alpha_v g_{vu} + g_{vu}^{-1} dg_{vu}) g_u$$

$$\Rightarrow \alpha_u = g_{vu}^{-1} \alpha_v g_{vu} + g_{vu}^{-1} dg_{vu}$$

In chart $U \times G$, we have

$\gamma_A(t) = (\gamma(t), g(t))$

$\dot{\gamma}_A(t) = (\dot{\gamma}(t), g^{-1}(t)\dot{g}(t))$ ← identify T_gG with T_eG by g^{-1}

$\dot{\gamma}_A \subset H_A \Rightarrow$ the action

$g^{-1}(t)dg(t) + g^{-1}(t)\alpha_U(\gamma(t))g(t)$ vanishes

$\Rightarrow g^{-1}(t)\dot{g}(t) + g^{-1}(t)\langle \alpha_U(\gamma(t)), \dot{\gamma}(t)\rangle g(t) = 0$

$\Rightarrow \dot{g}(t) + \langle \alpha_U(\gamma(t)), \dot{\gamma}(t)\rangle g(t) = 0$

This is an ODE, given $g(t_0)$, $\exists$ unique solution

Yang-Mills equation

$\pi : P \to M$ principal bundle

$$0 \to \ker \pi_* \to TP \to \pi^* TM \to 0 \quad \text{exact}$$

$$A : TP \to \ker \pi_*, \qquad H_A \hookrightarrow TP, \qquad H_A \xrightarrow{0} \ker \pi_*$$

A connection $A : TP \to \ker \pi_*$ G-equ

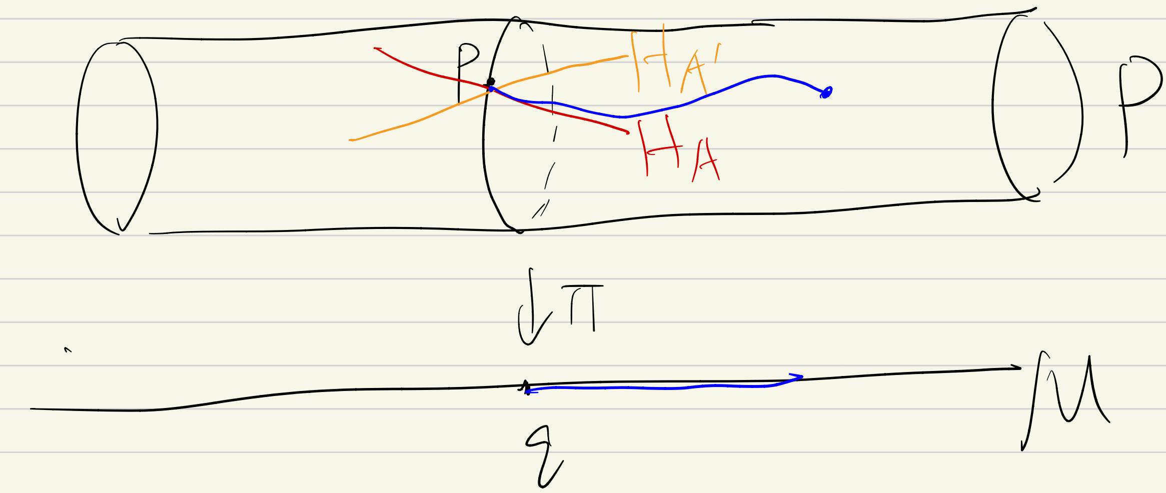
$\operatorname{Ker} A = H_A$ horizontal space

$H_A \cong \pi^* TM \qquad \ker \pi_* \oplus H_A = TP$

$v \in TM \qquad v \in \pi^* TM$

$v_A \in H_A$ s.t. $\pi_* v_A = v$ horizontal lift of v

$G = S^1$



$$\nabla_A : C^\infty(\mathrm{ad}P) \longrightarrow C^\infty(\mathrm{ad}P \otimes T^*M)$$

$$\mathrm{ad}P = P \times_{\mathrm{ad}} \mathfrak{g} = P \times \mathfrak{g} / (p,x) \sim (pg^{-1}, gxg^{-1})$$

$$\langle \nabla_A s, v \rangle = \langle d\tilde{s}, v_A \rangle$$

For a path $\gamma : [0,1] \longrightarrow M$ and $p \in P|_{\gamma(0)}$,

we can find unique horizontal lift $\gamma_{A,p} : [0,1] \longrightarrow P$

s.t. $\pi \circ \gamma_{A,p} = \gamma$ $\quad \gamma_{A,p}(0) = p$ $\quad \dot{\gamma}_{A,p} \subset H_A$

$$\gamma_{A,pg^{-1}} = \gamma_{A,p}\, g^{-1}$$

Locally (on $P|_U \cong U \times G$), $A = g^{-1}dg + g^{-1}\alpha_U g$

with $\alpha_U = g_{VU}^{-1}\, \alpha_V\, g_{VU} + g_{VU}^{-1}\, dg_{VU}$

$\alpha_U \in C^\infty(\mathfrak{g} \otimes T^*U)$

matrix valued 1-form

We can consider the exterior covariant derivative

$$d_A : C^\infty(\mathrm{ad}P \otimes \Lambda^k T^*M) \longrightarrow C^\infty(\mathrm{ad}P \otimes \Lambda^{k+1} T^*M)$$

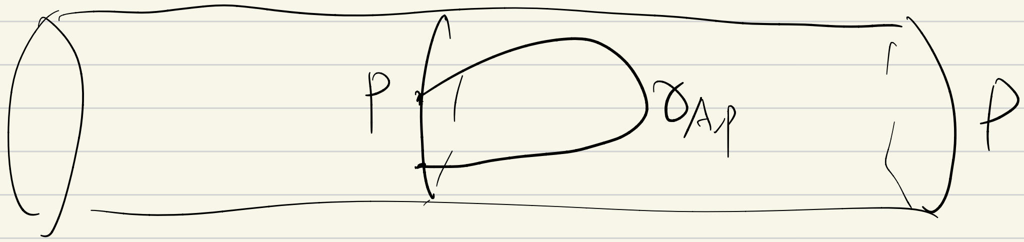
and $F_A s = d_A^2 s$ for $s \in C^\infty(\mathrm{ad}P)$

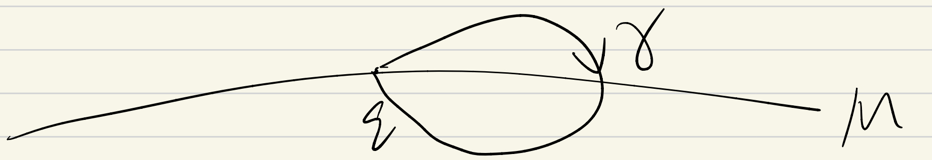
Locally, $F_A = d\alpha_U + \alpha_U \wedge \alpha_U \in C^\infty(\mathrm{ad}P \otimes \Lambda^2 T^*M)$

matrix valued 2-form.

If $F_A = 0$ A is called a <u>flat connection</u>

In such case, $\gamma_{A,p}(1)$ only depends on homotopy class of γ





Define $hol_{A,p} : \pi_1(M,q) \longrightarrow G$ holonomy map

$$[\gamma] \longmapsto \gamma_{A,p}(1)$$

where $\pi_1(M,q) = \{\gamma : [0,1] \to M \quad \gamma(0) = \gamma(1) = q\} / \text{homotopy}$

we have $hol_{A,pg^{-1}}([\gamma]) = g \, hol_{A,p}([\gamma]) g^{-1}$

$$hol_{A,p}([\gamma_1]\cdot[\gamma_2]) = hol_{A,p}([\gamma_1]) \cdot hol_{A,p}([\gamma_2])$$

$$[\gamma_1]\cdot[\gamma_2] = [\gamma]$$

$$\text{s.t. } \gamma(t) = \begin{cases} \gamma_1(2t) & t \in [0, \tfrac{1}{2}] \\ \gamma_2(2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases}$$

So flat connection determines a representation of $\pi_1(M)$ up to conjugation

Conversely, the universal cover $\widetilde{M}$ ($\pi_1(\widetilde{M}) = \{e\}$)

is a principal $\pi_1(M)$ bundle over M

a representation $\rho: \pi_1(M) \longrightarrow G$

induces a principal G-bundle

$$M \times_\rho G = \widetilde{M} \times G \,/\, (q, g) \sim (\gamma \cdot q \,,\, \rho(\gamma)\, g\, \rho(\gamma)^{-1})$$

Let $(\tilde{\gamma}, e)$ be the horizontal lift of $\gamma = \pi(\tilde{\gamma})$

This defines a flat connection.

$$\{\text{flat conn}\}/_{\text{iso}} \longleftrightarrow \{\pi_1 \text{ rep}\}/_{\text{conj}}$$

Given g on TM, we can define the volume form $dvol$ by picking orthonormal basis $e^1, \dots, e^n$ of T^*M and $dvol = e^1 \wedge \dots \wedge e^n$.

There exists an operator $* : \Omega^k \longrightarrow \Omega^{n-k}$, $n = \dim M$

s.t. $v \wedge * v = |v|^2 dvol$,

where $|\cdot|$ is induced by g.

Explicitly, $*(e^1 \wedge \dots \wedge e^k) = e^{k+1} \wedge \dots \wedge e^n$

$*(e^{i_1} \wedge \dots \wedge e^{i_k}) = \pm e^{i_{k+1}} \wedge \dots \wedge e^{i_n}$

where $\{i_1, \dots, i_k, \dots, i_n\} = \{1, \dots, n\}$

we have $*^2 = (-1)^{k(n-k)} : \Omega^k \to \Omega^k$

In particular, let $n=4$ $k=2$. we have $*^2=1$

$\Omega^2 = \Omega^+ \oplus \Omega^-$ $\quad * = \pm 1$ on $\Omega^\pm$

$\Omega^\pm$ is generated by $e_1 \wedge e_2 \pm e_3 \wedge e_4$

$e_1 \wedge e_3 \pm e_4 \wedge e_2$

$e_1 \wedge e_4 \pm e_2 \wedge e_3$

F_A is a matrix valued 2-form.

we also have $*F_A$.

Let $F_A^\pm = \frac{1}{2}(F_A \pm * F_A)$ $\quad * F_A^\pm = \pm F_A^\pm$

A is anti-self-dual if $F_A^+ = 0$

$\Longleftrightarrow F_A = - * F_A$

This is the Yang-Mills equation

recall the space of ∇ is affine over

$C^\infty(\mathrm{End}(E) \otimes T^*M)$.

$F_A^+ = 0$ is an equation on the space of conn

the space of conn is also affine over

$C^\infty(\mathfrak{g} \otimes T^*M)$ $\mathfrak{g}$-valued 1-form

(Note we should start with cmfd M with g on TM)

In particular $G = U(1)$ $\mathfrak{g} = \{ c \in \mathbb{C} \mid c + c^* = 0 \}$

$= i\mathbb{R}$

$\mathcal{A}$ is affine over $C^\infty(i\mathbb{R} \otimes T^*M)$

$F_A^+ = 0$ is indeed the Maxwell's equations

for electromagnetism in physics

$G = SU(2)$ $\mathfrak{g} = \{ X \in M(2, \mathbb{C}) \mid \operatorname{tr} X = 0 \;\; X + X^* = 0 \}$

nonabelian

$= \operatorname{span} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

Pauli matrices

$F_A^+ = 0$ is the original one studied in physics

Class 25/26 Topological invs from Yang-Mills equation

Last time, we focus on $\dim M = 4$ with g on TM and introduce the Yang-Mills equation $F_A^+ = 0$.

Let's review the set-up (replace M by X)

Suppose X is closed (compact + no boundary) and oriented (orientable + an orientation)

$\pi : P \to X$ is a principal $G = SU(2)$ bundle.

Given g on TX, define dvol by orthonormal basis

$$dvol = e^1 \wedge \cdots \wedge e^n.$$

Define $* : \Omega^k \to \Omega^{4-k}$ s.t. $v \wedge * v = |v|^2 dvol$

$*^2 = 1$ on Ω^2 split it into $\Omega^+ \oplus \Omega^-$

Given $w \in \Omega^2$, we have $w \pm * w \in \Omega^{\pm}$

For a conn A on P, we can define ∇_A, d_A, F_A

$$F_A^+ = \frac{1}{2}(F_A + * F_A) \text{ is the ASD part of } F_A.$$

Classification of principal G-bundle:

Thm: If $\psi, \phi : M \to N$ are homotopic,

i.e. $\exists\ \Phi : [0,1] \times M \to N$ $\Phi(0,-) = \psi$ $\Phi(1,-) = \phi$

For $\pi : P \to N$, $\psi^* P$ is isomorphic to $\phi^* P$

Pf: Fix a connection A on $\Phi^* P \to [0,1] \times M$

Given $p \in \psi^* P$, let $\gamma_{A,p}(t)$ be the

horizontal lift of $\gamma(t) = \Phi(t, \pi(p))$

(This path has the same image on M)

Then $\gamma_{A,p}(1) \in \phi^* P$

The map $p \mapsto \gamma_{A,p}(1)$ is an isomorphism

from $\psi^* P$ to $\phi^* P$. it is G-equ

because $\gamma_{A,pg^{-1}}(t) = \gamma_{A,p}(t)\, g^{-1}$

Cor. If M is homotopic to $\mathbb{R}^n$ (or contractible), then any principal bundle over M is product G bundle

Fact For any G, $\exists$ a universal classifying space BG (unique up to homotopy), s.t.

$$\{\text{principal } G\text{-bundle over } M\}/_{\text{iso}}$$

$$\Longleftrightarrow \{\text{maps from } X \text{ to } BG\}/_{\text{homotopy}} =: [X, BG]$$

Ex: $G = S^1 = U(1)$ $\quad BG = \mathbb{C}P^\infty$ $(\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1})$

$G = O(n)$ $\quad BG = \text{Grass}(n, \mathbb{R}^\infty)$ ($\mathbb{R}^n$ in $\mathbb{R}^\infty$)

Another understanding of charateristic classes (for p.b. or v.b. by framed bundle)

$H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$ $\quad \deg x = 2$

first Chern class of P is the pull-back of the generator $x \in H^2(\mathbb{C}P^\infty)$ by

$$M \longrightarrow BU(1) = \mathbb{C}P^\infty$$

Fact complex line bundle (or principal $U(1)$ bundle) is classified by c_1 because $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$

Def $K(\mathbb{Z}, n)$ Eilenberg–MacLane space

A topological space (unique up to weak homotop equ) s.t. $\pi_i(K(\mathbb{Z}, n)) = \begin{cases} \mathbb{Z} & i = n \\ \{e\} & i \neq n \end{cases}$

Prop $[X, K(\mathbb{Z}, n)] \cong H^n(X; \mathbb{Z})$

$n = 2$. principal $U(1)$ bundle $\Longleftrightarrow$ $c_1 \in H^2(X; \mathbb{Z})$

Fact. Principal $SU(2)$ bundle on 4d manifold X has $c_1(P) = c_1(P \times_{ad} \mathfrak{g}) = 0$, and is classified by

$$C_2(P) = \frac{1}{(2\pi\sqrt{-1})^2}[tr(F_A \wedge F_A)] \in H^4(X; \mathbb{Z}) \cong \mathbb{Z}$$

This is because $SU(2) \cong S^3$

$$\pi_i(BSU(2)) \cong \pi_{i-1}(SU(2)) = \pi_{i-1}(S^3)$$

$$= \begin{cases} ?? & i-1 > 3 \\ \mathbb{Z} & i-1 = 3 \\ \{e\} & i-1 < 3 \end{cases}$$

$BSU(2) \longrightarrow K(\mathbb{Z}, 4)$ inclusion, induces

$$[X, BSU(2)] \cong [X, K(\mathbb{Z}, 4)] \cong H^4(X; \mathbb{Z})$$

The orientation on X means picking a generator in $\mathbb{Z}$

Then we can write $C_2(P)$ as an integer $k \in \mathbb{Z}$

Let $\mathcal{A}_K$ be the space of all conns on P

with $c_2(P) = K$. it is affine over $C^\infty(\mathrm{ad}P \otimes T^*X)$

Define $\mathrm{Ad}P = P \times_{\mathrm{Ad}} G = P \times G / (p,h) \sim (pg^{-1}, ghg^{-1}) \ \forall g$

It is a bundle of Lie group $(\mathrm{ad}\,P = P \times_{\mathrm{ad}} \mathfrak{g})$

Let $\mathcal{G}_K = C^\infty(\mathrm{Ad}P)$ called gauge group

There is an action $\mathcal{G}_K \times \mathcal{A}_K \to \mathcal{A}_K$

$$(g, A) \mapsto g^*A = A + g^{-1} d_A g$$

This induces $F_{g^*A} = g \cdot F_A g^{-1}$

So A is ASD. $\Rightarrow$ g^*A is also ASD.

Let $M_K = \{\text{ASD conn in } \mathcal{A}_K\} / \mathcal{G}_K$

be the moduli space. the set of ASD conn

and $\mathcal{G}_K$ are both infinite dimensional,

but $\dim M_K$ is finite

Note that M_K also depends on g because $*$ is.
For any g, M_K is not always manifold
To prove M_K is a smooth manifold for generic g.
(generic : roughly means dense set in the space of metrics)
we have to do completion of $\mathcal{A}_K$ and $\mathcal{G}_K$
under some norms, s.t. they become Banach manifold
(Locally Banach space = infinite dim space Complete)
C^∞ not Banach $C^r(X)$ or $L^r_k(X)$ sobolev space
Then we can apply infinite dim version of
implicit function thm and Sard thm.

For generic g, M_K is a smooth manifold with (possibly containing singular part)

$$\dim M_K = 8k - 3(b_2^+ - b^1 + 1)$$

$\uparrow$ $SU(2) \cong \mathbb{R}^3$

b^1: $\dim H^1_{dR}(X)$

$H^2_{dR}(X) = H^+_{dR}(X) \oplus H^-_{dR}(X)$

$b_2^+ = \dim H^+_{dR}(X)$

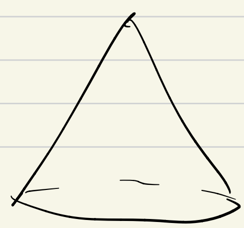
Ex. $X = S^4$ round metric $k=1$ $M_K = \mathring{B}^5$

$X = \overline{\mathbb{CP}}^2$ ($\mathbb{CP}^2$ with opposite orientation)

g = Fubini-Study metric induced from $\mathbb{C}^n$

$k=1$ M_K = Cone of $\overline{\mathbb{CP}}^2$

$[0,1) \times \overline{\mathbb{CP}}^2 / \{0\} \times \overline{\mathbb{CP}}^2$



The cone pt is the reducible solution
(comes from $S^1 = U(1) \subset SU(2)$ connection)

when $\dim M_K < 8$, after adding boundary,
M_K is compact (> 8, some bubble phenomenon
of codim 8)

In particular, if $\dim M_K = 0$, this is just
finitely many points. We can introduce orientations
on M_K, and in the case of $\dim M_K = 0$,
we count points with signs.

Fact: under some assumptions (like $b_2^+(X) > 1$),
the number of pts in M_K with $\dim M_K = 0$
is independent of g. Hence this is an invariant
that only depends on the diffeomorphism type of X

Sometimes, we also consider $SO(3)$ principal bundle
and count solutions of $F_A^+ = 0$ (note $SO(3) = SU(2)$)

$\exists$ smooth manifold X_1, X_2 s.t. $X_1 \cong_{homeo} X_2$,
but the number of solutions are different. $X_1 \not\cong_{diff} X_2$

These are called <u>exotic pair</u>

Ex For K3 surface, the signed counting of solution is 1, but $\exists X_k \cong_{homeo} K3$, with the counting $(2k+1)$ for any $k \in \mathbb{Z}$.

Donaldson diagonal thm: Suppose X is a closed, oriented, connected, <u>simply-connected</u> <u>smooth</u> 4-manifold,
$(\pi_1(X) = \{e\})$

Consider the cup product (wedge product on $\mathbb{Z}$ coeff)

$$H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \longrightarrow H^4(X;\mathbb{Z})$$

as a bilinear form, Q_X.

If Q_X is negative definite, then $\exists A \in GL(n;\mathbb{Z})$

$$A^T Q_X A = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

(over $\mathbb{R}$, $A \in GL(n;\mathbb{R})$ also exists by linear algebra)

(over $\mathbb{Z}$, a necessary condition of diagnonalizable is that all entries on the diagnol are -1

simplest counterexample : $E8$ $=$

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}$$

i.e. If Q_X is not diagonalizable. (over $\mathbb{Z}$)

there is no smooth structure on X.

The proof uses the moduli space M_1

$\dim M_1 = 8 - 3 = 5$. $\partial M_1 = X$

M_1 is orientable, and M_1 has cones on $\overline{\mathbb{CP}}^2$

instanton Floer homology

Y closed oriented 3-manifold

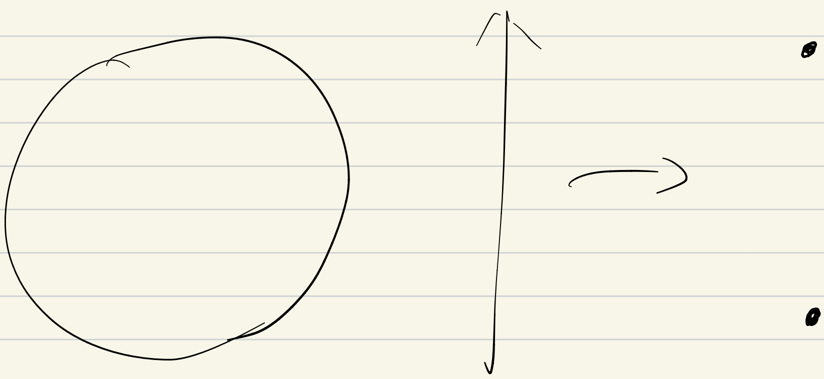
$H_i(Y;\mathbb{Z}) = H_i(S^3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0,3 \\ 0 & \text{other} \end{cases}$

$I(Y) = H(CI(Y), d)$

$CI(Y)$ generated by irreducible (Im ρ is nonabelian) flat conns on Y

$\{\text{flat conn}\}/_{\text{iso}} \overset{1-1}{\longleftrightarrow} \{\rho: \pi_1(Y) \to SU(2)\}/_{\text{conj}}$

If the space is not 0-d. choose some perturbation



Given two flat conns α, β. Consider moduli space

$M_1(\alpha,\beta)$ on $Y \times \mathbb{R}$

(α, β become boundary condition. 1 means 1d)

$n_{\alpha\beta} = \# M_1(\alpha,\beta)/\mathbb{R}$ ⟵ translation

$d\alpha = \sum_\beta n_{\alpha\beta}\beta$

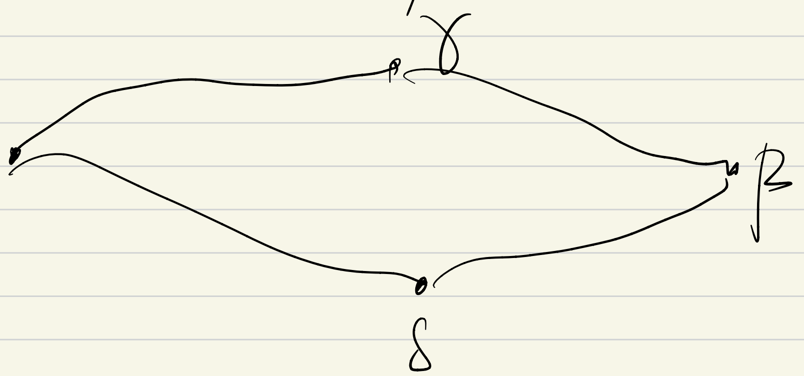
To prove $d^2 = 0$.

Consider $M_2(\alpha, \beta)/\mathbb{R}$.

It can be compactified and oriented

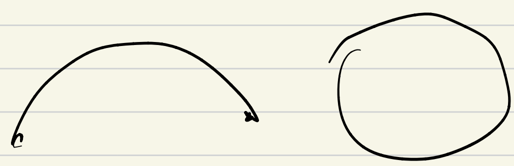
$\partial \overline{M_2}(\alpha,\beta)/\mathbb{R}$ 0 d mfd contains

"broken trajectory" contribution to d^2



$\# \partial \overline{M_2}(\alpha,\beta)/\mathbb{R} = 0$ because it is

boundary of 1-mfd



Fact if $I(Y) \neq 0$, $\exists \rho : \pi_1(Y) \to SU(2)$

s.t. $\operatorname{Im}\rho$ is nonabelian

$\Rightarrow \pi_1(Y) \neq \{e\}$