Class 4 Vector bundle (Chap 3)

Manifold + group = Lie group

Manifold + vector space = vector bundle

Def 1: Let M be a smooth mfd of dim m.

A smooth mfd E is called a (real) vector bundle over M with fiber dimension n if we have the following

1) There is a smooth map $\pi: E \to M$

for each $p \in M$, $\exists\, U_p \subset M$ $\lambda_U : \pi^{-1}(U_p) \to \mathbb{R}^n$

s.t. for each $x \in U_p$. $\lambda_U : \pi^{-1}(x) \to \mathbb{R}^n$

is an diffeomorphism

2) There is a smooth map $\hat{0}: M \to E$ s.t. $\pi \circ \hat{0} = Id$

3) There is a smooth map $\mu : \mathbb{R} \times E \to E$ s.t.

a) $\pi(\mu(r, v)) = \pi(v)$

b) $\mu(r, \mu(r', v)) = \mu(rr', v)$

c) $\mu(1, v) = v$

d) $\mu(r, v) = v$ for $r \neq 1$ iff $v \in Im\, \hat{0}$

Ex. The product bundle (also called the trivial bundle)

$E = M \times \mathbb{R}^n$

Rem 1) π is called the (bundle) projection map

For $W \subset M$, write $E|_W = \pi^{-1}(W)$

For $p \in M$, $E|_p$ is called a fiber over p

λ_U defines a diffeo $\varphi_U : E|_U \to U \times \mathbb{R}^n$ by

$(\pi(v), \lambda_U(v))$ φ_U is called local trivialization of E

2) $\hat{0}$ or its image is called the zero section

$\lambda_U \circ \hat{0} = 0 \in \mathbb{R}^n$

3) μ corresponds to the scalar multiplication in the vector space

we usually write $\mu(r, v)$ as rv

Def 2: Let E, E' be two vector bundles over M.

A section of E is a smooth map $s : M \to E$ s.t. $\pi \circ s = Id$

A homomorphism $\phi : E \to E'$ is a smooth map s.t.

1) $\pi'(\phi(v)) = \pi'(v)$

2) $\phi(rv) = r\phi(v)$ (Hence $\phi(\hat{0}(p)) = \hat{0}'(p)$)

A bundle isomorphism is a homomorphism with an inverse $g : E' \to E$

Question: Is any bundle isomorphic to the trivial bundle?

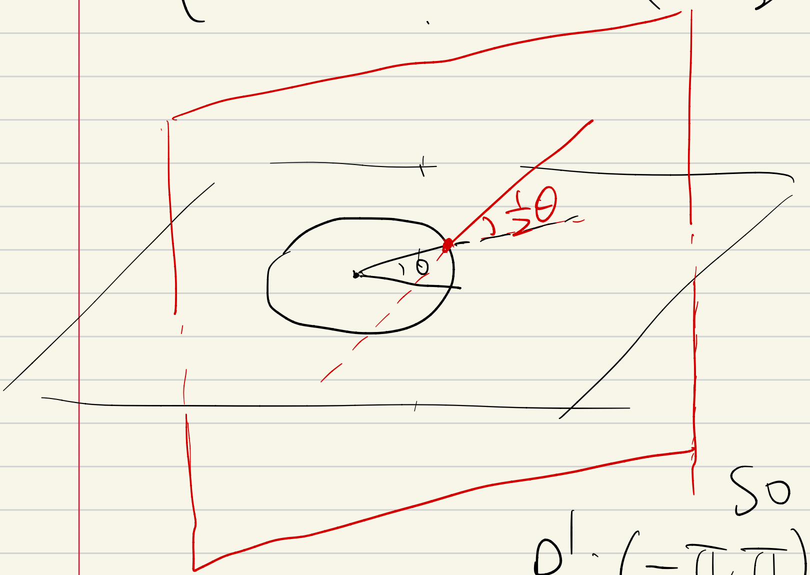
Counterexample: Möbius bundle over S^1

$$\left\{(\theta, v) \in S^1 \times \mathbb{R}^2 : \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} v = v \right\}$$

We have a smooth map

$$p : (0, 2\pi) \times \mathbb{R} \longrightarrow E$$
$$(\theta, t) \longmapsto (\theta, t\cos(\tfrac{1}{2}\theta), t\sin(\tfrac{1}{2}\theta))$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta \\ \sin\theta\cos\frac{1}{2}\theta - \cos\theta\sin\frac{1}{2}\theta \end{pmatrix}$$
$$= \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix}$$



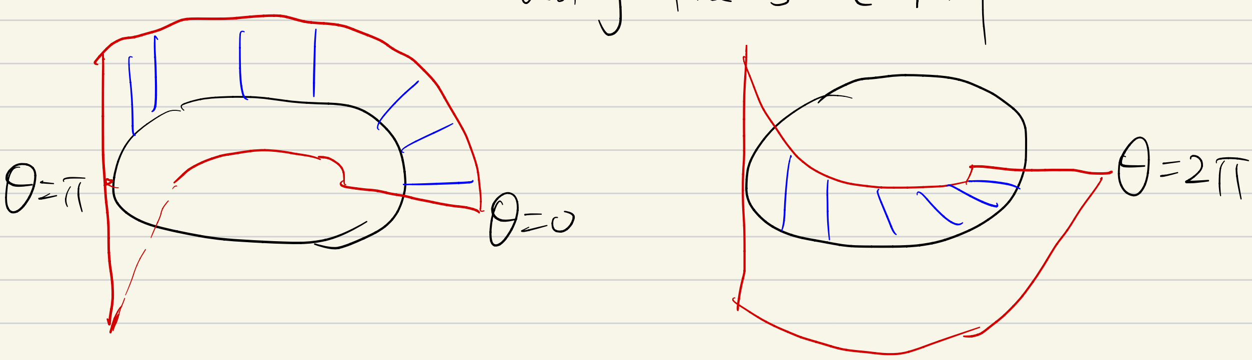
p is not continuous at

$\theta = 0 = 2\pi$

So we need the second chart

$$p' : (-\pi, \pi) \times \mathbb{R} \longrightarrow E$$

using the same map



We can find a nonzero section of $S^1 \times \mathbb{R} = E'$

(just use $s : S^1 \longrightarrow E'$ $\quad s(\theta) = (\theta, 1)$

But we cannot find a nonzero section of Möbius bundle

(Intuitively, when we move $s(0)$ smoothly to $s(2\pi)$, we will have $s(0) \neq s(2\pi)$)

More properties about sections:

Prop 1: 1) $\phi: E \to E'$ is a homomorphism. $s: M \to E$ is a section. Then $s': x \mapsto \phi(s(x))$ is a section of E' called the <u>pushforward</u> of s

2) $s_1, s_2: M \to E$ sections. Then $s_1 + s_2: x \mapsto s_1(x) + s_2(x)$ is also a section. Also rs_1 is a section.

Hence the space of sections is a vector space.

3) $f: M \to \mathbb{R}$, $s: M \to E$. Then fs is also a section.

4) On a nbhd U of any $p \in M$. we have a diffeo

$\varphi_U: E|_U = \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$

Suppose $e_1, \cdots, e_n$ are basis of $\mathbb{R}^n$. Then we can construct sections $s_1, \cdots, s_n: U \longrightarrow E|_U$ $\quad s_i(x) = \varphi_U^{-1}(x, e_i)$

i.e. locally, the vector bundle is always generated by sections.

Rem Globally (on M), we may not have sections (e.g. Möbius bundle), or the dim of the vector space generated by sections is less than dim of the fiber.

5) If we have sections $s_1, \cdots, s_n: M \to E$ so that in any nbhd U. $s_i|_U$ forms a basis

Then we can use them to trivialize E:

i.e. $\exists$ a bundle isomorphism $\phi: M \times \mathbb{R}^n \longrightarrow E$

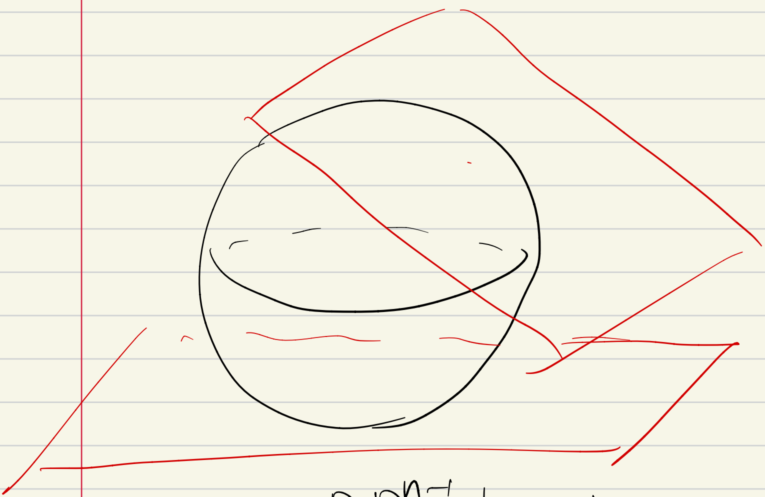
$$(x, \sum a_i e_i) \longmapsto \sum a_i S_i(x)$$

More examples of vector bundles.

- Let $S^2 = \{ x \in \mathbb{R}^3 \mid |x| = 1 \}$ $E = \{ (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \substack{|x|=1 \\ x \cdot v = 0} \}$

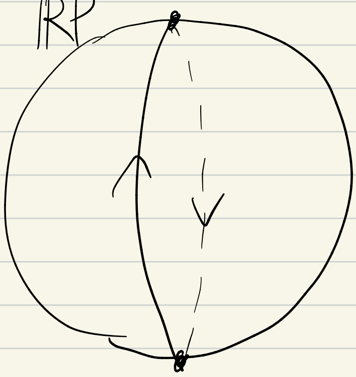
$\pi : E \to S^2$ $\pi(x,v) = x$ $r(x,v) = (x, rv)$ $r \in \mathbb{R}$

$\hat{0} : S^2 \to E$ $\hat{0}(x) = (x, 0)$



- Let $\mathbb{R}P^{n-1}$ be the space of lines through $0 \in \mathbb{R}^n$ (a pt is a line) It can also be regarded as $S^{n-1}/\{\pm 1\}$ where -1 acts on x by $-x$.

$\mathbb{R}P^2$ $E = \{ (\pm x, v) \in \mathbb{R}P^{n-1} \times \mathbb{R}^n \mid x // v \}$



Class 5 Tangent bundle

Recall the definition of vector bundle

Def 1: Let M be a smooth mfd of dim m.

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for each $p \in M$, $\exists\, U_p \subset M$ $\lambda_U : \pi^{-1}(U_p) \to \mathbb{R}^n$

s.t. for each $x \in U_p$. $\lambda_U : \pi^{-1}(x) \to \mathbb{R}^n$

is an diffeomorphism

2) There is a smooth map $\hat{0} : M \to E$ s.t. $\pi \circ \hat{0} = Id$

3) There is a smooth map $\mu : \mathbb{R} \times E \to E$ s.t.

a) $\pi(\mu(r,v)) = \pi(v)$

b) $\mu(r, \mu(r', v)) = \mu(rr', v)$

c) $\mu(1, v) = v$

d) $\mu(r,v) = v$ for $r \neq 1$ iff $v \in \mathrm{Im}\, \hat{0}$

We have an alternative way to define E

Def 2. Fix a locally finite open cover $\mathcal{U}$ of M

For $U, V \in \mathcal{U}$ s.t. $U \cap V \neq \phi$. choose a function

$g_{VU} : U \cap V \to GL(n, \mathbb{R})$ called <u>bundle transition function</u>

$$E = \bigcup_{U \in \mathcal{U}} U \times \mathbb{R}^n / (p,u) \in U \times \mathbb{R}^n \sim (p, g_{VU} \cdot u) \in V \times \mathbb{R}^n \quad p \in U \cap V$$

$\{g_{VU}\}_{U,V\in\mathcal{U}}$ need to satisfy the following condition

1) $g_{VU} = g_{UV}^{-1}$ $\quad U\cap V \neq \phi$

2) $g_{VU}\circ g_{UW}\circ g_{WU} = Id$ $\quad U\cap V\cap W \neq \phi$

called <u>cocycle condition</u>

The reason to introduce g_{VU} is the following

From Def I, for any $p\in M$, we have a nbhd $U\subset M$

s.t. $E|_U = \pi^{-1}(U) \xrightarrow{\cong} U\times\mathbb{R}^n$

$$v \longmapsto (\pi(v), \lambda_U(v))$$

If $v_1, v_2 \in E|_p$, then we can define

$$v_1+v_2 = (\lambda_U|_p)^{-1}(\underbrace{\lambda_U|_p(v_1)}_{\mathbb{R}^n} + \underbrace{\lambda_U|_p(v_2)}_{\mathbb{R}^n}) \in E|_p$$

The definition should be independent of the choice of U

if p is also in V, we set $e_i = \lambda_U|_p(v_i)$ $\quad i=1,2$

$$v_1+v_2 = (\lambda_U|_p)^{-1}(e_1+e_2) = (\lambda_V|_p)^{-1}(\lambda_V|_p\circ(\lambda_U|_p)^{-1}(e_1) + \lambda_V|_p\circ(\lambda_U|_p)^{-1}(e_2))$$

Let $\Psi_{VU} := \lambda_V\circ(\lambda_U)^{-1} : U\cap V\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$

Then $\Psi_{VU}|_p(e_1+e_2) = \Psi_{VU}|_p(e_1) + \Psi_{VU}|_p(e_2)$

Since $e_1, e_2 \in\mathbb{R}^n$ can be arbitary, $\Psi_{VU}|_p$ is linear.

i.e. $\Psi_{VU}|_p \in GL(n,\mathbb{R})$

Hence we have a map $g_{VU} : U \cap V \to GL(n, \mathbb{R})$

$g_{VU}(p) = \psi_{VU}|_p$ this map is smooth because ψ_{VU} is.

$g_{VU} \circ g_{UV} = Id \quad g_{VU} \circ g_{UW} \circ g_{WV} = Id$

by construction. Conversely. Given $\{g_{VU}\}$. we can also recover the vector bundle in Def 1.

The second def is useful to construct the tangent bundle:

Def M is a smooth mfd of dim n. The <u>tangent bundle</u> of M is a vector bundle TM constructed as follows.

Fix a locally finite coordinate atlas $\mathcal{U}$ for M

For $U, V \in \mathcal{U}$. with $U \cap V \neq \phi$, we have diffeo

$$\phi_U : U \to \mathbb{R}^n \quad \phi_V : V \to \mathbb{R}^n$$

$h_{VU} = \phi_V^{-1} \circ \phi_U : \mathbb{R}^n \to \mathbb{R}^n$ is the transition function.

Let $g_{VU} = (h_{VU})_* =$ the Jacobian of h_{VU}

$$\in GL(n, \mathbb{R})$$

$$h_{VU} \circ h_{UV} = Id \overset{\text{chain rule}}{\Longrightarrow} g_{VU} \circ g_{UV} = Id$$

$$h_{VU} \circ h_{UW} \circ h_{WV} = Id \Rightarrow g_{VU} \circ g_{UW} \circ g_{WV} = Id$$

Rem The construction is indeed independent of $\mathcal{U}$

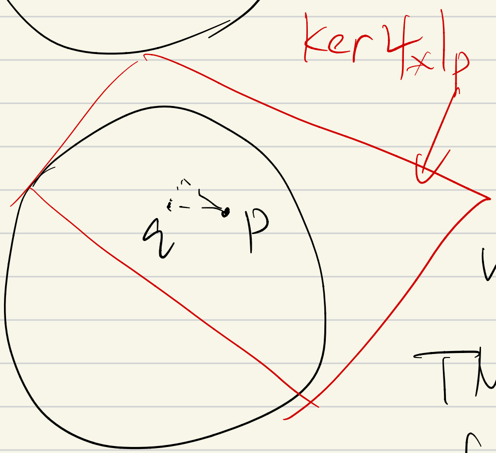
(see Cliff's book 3.4)

As a result. for any chart $U \subset M$. we have
$TM|_U \xrightarrow{\cong} U \times \mathbb{R}^n$, we denote the diffeo by $(\phi_U)_*$

Prop If $M \subset \mathbb{R}^N$ is a submfd of dim n. by definition, for each $p \in M$. $\exists$ nbhd $U \subset \mathbb{R}^N$, $\psi : U \to \mathbb{R}^{N-n}$ with 0 as the regular value i.e. the Jacobian ψ_* is surjective at each pt in $\psi^{-1}(0)$, and $M \cap U = \psi^{-1}(0)$

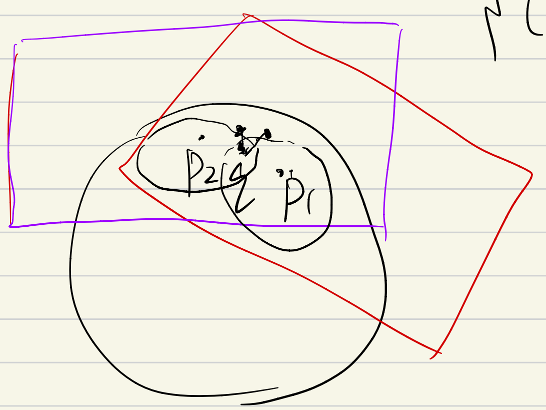
$N=3 \quad n=2$

$\ker \psi_*|_p$ is identified with the subspace $\mathbb{R}^n$ tangent to M at p. which is independent of the choice of U



For a pt $q \in M$ near p. the projection of $q-p$ to $\ker \psi_*|_p$ is a diffeomorphism to a ball $\subset \ker \psi_*|_p$ which gives a chart for M

TM can be regarded as subset $\{(p, v) \in M \times \mathbb{R}^N \mid p \cdot v = 0\}$



Def A section $s: M \to TM$ is called a <u>vector field</u>

Let $C^\infty(M;\mathbb{R})$ be the space of smooth functions $M \to \mathbb{R}$ with addition and multiplication

A <u>derivation</u> is a map $\mathcal{L}$ from $C^\infty(M;\mathbb{R})$ to itself

s.t. 1) $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$ 2) $\mathcal{L}(r) = 0$ for constant function r

3) $\mathcal{L}(fg) = (\mathcal{L}f)g + f(\mathcal{L}g)$ (Leibniz rule)

We can identify a derivation with a vector field

Given a vector field $s: M \to TM$. we construct a derivation $\mathcal{L}_s$ called <u>Lie derivative</u>

Let $U \subset M$ be a chart $\phi_U: U \to \mathbb{R}^n$

$$\phi_{U*}: TM|_U \xrightarrow{\cong} U \times \mathbb{R}^n$$

For $f: M \to \mathbb{R}$, define $f_U = f \circ \phi_U^{-1}: \mathbb{R}^n \to \mathbb{R}$

Suppose $\phi_{U*} \circ s|_U = (Id_U, v_1, v_2, \dots v_n)$

$v_k: U \to \mathbb{R}$

Define $$(L_s f)_U = \sum_{k=1}^n v_k \frac{\partial f_U}{\partial x^k} : U \to \mathbb{R}$$

Exer: check $L_s f: M \to \mathbb{R}$ is independent of the choice of U by the bundle transition function of TM

Because $(L_s f)_U = \sum_{k=1}^{n} V_k \frac{\partial f_U}{\partial x^k}$. We may write $S|_U = \sum_{k=1}^{n} V_k \frac{\partial}{\partial x^k}$ and write $\{\frac{\partial}{\partial x^k}\}$ as a basis of sections for $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$

This notation is important becauce in the future we usually do calculation locally and write a vector field as $\sum_{k=1}^{n} V_k \frac{\partial}{\partial x^k}$

Class 6 tangent bundle and Cotangent bundle.

Recall the tangent bundle TM is defined by the bundle transtion function $g_{VU} = (h_{VU})_*$

Where $h_{VU} = \phi_V^{-1} \circ \phi_U$ is the tranistion function of M

A section $s: M \to TM$ is called a vector field.

A derivation $\mathcal{L}$ is a map from $C^\infty(M, \mathbb{R})$ to itself

s.t. 1) $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$ 2) $\mathcal{L}(r) = 0$ for constant r

3) $\mathcal{L}(fg) = \mathcal{L}(f) g + f \mathcal{L}(g)$

From a vector field s, we define a derivation $\mathcal{L}_s$ called Lie derivative of s

Locally $U \subset M$ $\phi_U : U \to \mathbb{R}^n$ $\phi_{U*} : TM|_U \to U \times \mathbb{R}^n$

$\phi_{U*} \circ s|_U : U \to U \times \mathbb{R}^n$ $V_k : U \to \mathbb{R}$

$(Id, V_1, \dots, V_n)$

$(\mathcal{L}_s f)|_U = \sum_k V_k \frac{\partial (f \circ \phi_U^{-1})}{\partial x^k}$ indep of the choice of U.

So we usually write $s|_U = \sum V_k \frac{\partial}{\partial x^k}$

For another chart V. we write $\frac{\partial}{\partial y^k}$ as basis and

$s|_V = \sum V_k' \frac{\partial}{\partial y^k}$. we have

$$\frac{\partial(f\circ\phi_V^{-1})}{\partial y^k} = \frac{\partial(f\circ\phi_U^{-1}\circ\phi_U\circ\phi_V^{-1})}{\partial y^k} = \sum_l \frac{\partial(f\circ\phi_U^{-1})}{\partial x^l}\frac{\partial(x_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}$$

$\phi_U\circ\phi_V^{-1} : \mathbb{R}^n$ (with basis y^k) $\longrightarrow \mathbb{R}^n$ (with basis x^k)

So $$\frac{\partial}{\partial y^k} = \sum_l \frac{\partial(x_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}\frac{\partial}{\partial x^l}$$ ($x_l : \mathbb{R}^n \to \mathbb{R}$ projection to l-th coordinate)

and $$\sum_l V_l\frac{\partial}{\partial x_l} = \sum_k V_k\frac{\partial}{\partial x_k} = \sum_k\sum_l V_k'\frac{\partial(x_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}\frac{\partial}{\partial x_l}$$

$$\Rightarrow V_l = \sum_k V_k'\frac{\partial(x_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}$$

$$\Rightarrow \begin{pmatrix} V_1 \\ \vdots \\ V_n\end{pmatrix} = (h_{UV})_* \begin{pmatrix} V_1' \\ \vdots \\ V_n'\end{pmatrix}$$

Conversely, given a derivation $\mathcal{L}$. we construct a vector field as follows

Let $V_K : U \to \mathbb{R}$ be defined by

$\mathcal{L}(x_K\circ\phi_U)$ where $x_K : \mathbb{R}^n \to \mathbb{R}$ is the projection

Let $\frac{\partial}{\partial x^k} : U \to TM|_U \cong U\times\mathbb{R}^n$ be the section

correspondig to basis of $\mathbb{R}^n$ set $S = \sum_K V_K\frac{\partial}{\partial x^k}$

Exercise: for any $f : M\to\mathbb{R}$

$$(\mathcal{L}f)|_U = (L_S f)|_U = \sum_K V_K\frac{\partial(f\circ\phi_U^{-1})}{\partial x^k}$$

A function $f: M \to \mathbb{R}$ define a vector field ∇f as follows.

for a chart $U \subset M$, $\phi_U : U \to \mathbb{R}^n$

Define $f_U = f \circ \phi_U^{-1} : \mathbb{R}^n \to \mathbb{R}$ Consider $(f_U)_* = \left(\frac{\partial f_U}{\partial x_1}, \cdots \frac{\partial f_U}{\partial x_n}\right)$

Let $(\nabla f)|_U = \sum_K \frac{\partial f_U}{\partial x_K} \frac{\partial}{\partial x_K}$.

Def The <u>cotangent bundle</u> T^*M is defined by $g_{VU} = ((h_{VU})_*^{-1})^T$

A section $s: M \to T^*M$ is called a <u>1-form</u>

Similar to the definition of $\frac{\partial}{\partial x^k}$

Given $U \subset M$. we define $\{dx^k\}$ to be the sections corresponding to basis of $\mathbb{R}^n$.

$\phi_{U*} : T^*M|_U \to U \times \mathbb{R}^n$

Then for $f: M \to \mathbb{R}$, define a 1-form df by

$$df = \sum \frac{\partial f_U}{\partial x^k} dx^k$$

Algebra of vector bundles (Chap 4)

Def Given a vector bundle $E \to M$ with bundle transition functions $\{g_{VU}\}$. Then define the <u>dual</u> bundle $E^* \to M$ by $\{(g_{VU}^{-1})^T\}$

Rem T^*M is the dual bundle of TM.

$E^*|_p \cong \mathrm{Hom}(E|_p, \mathbb{R})$ $(E^*)^* = E$

The direct sum of two bundles $E_1, E_2 \to M$ is given by $E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 \mid \pi_1(v_1) = \pi_2(v_2)\}$

Rem. $(E_1 \oplus E_2)|_p = E_1|_p \oplus E_2|_p$

$$g_{VU} = g_{VU,1} \oplus g_{VU,2}$$

The bundle $\mathrm{Hom}(E_1, E_2)$ is defined by

$\mathrm{Hom}(E_1, E_2)|_p = \mathrm{Hom}(E_1|_p, E_2|_p)$

If $\dim E_1|_p = n$ $\dim E_2|_p = k$

then $\dim \mathrm{Hom}(E_1, E_2)|_p = nk$

$\mathrm{Hom}(E_1, E_2)|_U \cong U \times M(k, n)$

$M(k,n)$ $k \times n$ matrices

the bundle transition function sends $m_U \in M(k,n)$

to $m_V = \underset{k\times k}{g_{VU,2}} \, \underset{k\times n}{m_U} \, \underset{n\times n}{(g^{-1}_{VU,1})^T}$

$$\begin{bmatrix} & \end{bmatrix} \begin{bmatrix} & & \end{bmatrix} \begin{bmatrix} & & \\ & & \end{bmatrix}$$

The bundle $E_1 \otimes E_2$ can be defined by $\mathrm{Hom}(E_1^*, E_2)$

Class 7 Bundle algebra and bundle maps (Chap 4-5)

Last time, we introduce the tangent bundle and its dual bundle (called cotangent bundle). Today we introduce more constructions

Def. Given $E_1, E_2 \to M$. we define

vector spaces

1) $E_1 \oplus E_2 \to M$ by $(E_1 \oplus E_2)|_p = E_1|_p \oplus E_2|_p$

2) $\mathrm{Hom}(E_1, E_2) \to M$ by $\mathrm{Hom}(E_1, E_2)|_p = \mathrm{Hom}(E_1|_p, E_2|_p)$

3) $E_1 \otimes E_2 \to M$ by $\mathrm{Hom}(E_1^*, E_2)$

where the bundle tranistion function of E_1^* is $\{(g_{VU}^{-1})^T\}$

4) E_1 is a <u>sub bundle</u> of E_2 if E_1 is a submfd of E_2 and $E_1|_p$ is a subspace of $E_2|_p$

Fact: every vector bundle is a subbundle of a product bundle

equivalently. for any E, $\exists$ E' s.t. $E_1 \oplus E_2$ is isomorphic to a product bundle

5) If E_1 is a subbundle of E_2, then define the <u>quotient bundle</u> E_2/E_1 by $(E_2/E_1)|_p = E_2|_p / E_1|_p$.

Def. We introduce some operations on a vector space V and then generalize them to vector bundles

V^* is the dual of V : $V^* = \mathrm{Hom}(V, \mathbb{R})$

$V^* \otimes \cdots \otimes V^*$ is the set of multilinear functions $f: V \times V \times \cdots \times V \to \mathbb{R}$

i.e. when all variables except one are fixed. it is a linear function.

$Sym^k(V^*)$ is the set of symmetric, multilinear map

i.e. $f(\cdots, v_i, \cdots, v_j, \cdots) = f(\cdots, v_j, \cdots, v_i, \cdots)$

$\Lambda^k(V^*)$ is the set of anti-sym mul map

i.e. $f(\cdots, v_i, \cdots, v_j, \cdots) = -f(\cdots, v_j, \cdots, v_i, \cdots)$

$\dim (V^*)^{\otimes k} = (\dim V^*)^k = n^k$

$\dim Sym^k(V^*) = \binom{n+k-1}{k}$ $\quad \dim \Lambda^k V^* = \begin{cases} \binom{n}{k} & 0 \le k \le n \\ 0 & k > n \end{cases}$

$\dim \Lambda^n V^* = 1 \quad \Lambda^n V^* = \mathbb{R}$

We can also construct $Sym^k E^*$, $\Lambda^k E^*$ (also $Sym^k E$, $\Lambda^k E$)

If $\{g_{vu}\}$ is for E, then $\{\det(g_{vu})\}$ is for $\Lambda^n E$

An <u>orientation</u> of E is a nowhere zero section of $\Lambda^n E$.

which induces a bundle isomorphism $\Lambda^n E \cong M \times \mathbb{R}$

An orientation of M is an orientation of TM.

E (or M) is <u>orientable</u> if there is an orientation

E is orientable iff E^* is

Ex. The Mobius bundle is not orientable. TS^n is orientable.

Any two orientations $s_1, s_2 : M \to \Lambda^n E$ satisfy $s_1 = f s_2$

for $f: M \to \mathbb{R}$, $f(p) \neq 0$ for any $p \in M$.

If f is always positive, we say two orientations are the same

If f is always negative, we say two orientations are opposite

A section of $\Lambda^k T^*M$ is called a <u>k-form</u>,

(we will go back to study $\Lambda^k T^*M$ later in de Rham cohomology)

Def. $f: M \to N$ is a smooth map. $\pi: E \to N$ is a v.b.

Define $f^* E = \{(p,v) \in M \times E \mid f(p) = \pi(v)\}$

This is called the pull-back of E, $(f^*E)|_p = E|_{f(p)}$.

However, we DON'T have the "push-forward" construction for general bundle. For tangent bundle, we can define the tangent map $f_*: TM|_p \longrightarrow TN|_{f(p)} \ (= f^*TN|_p)$ as follows. For $U \subset M$, $V \subset N$, we have

$$\phi_{U*}: TM|_U \longrightarrow U \times \mathbb{R}^m \quad \phi_{V*}: TN|_V \longrightarrow V \times \mathbb{R}^n$$

f is smooth $\Longrightarrow$ $\phi_V \circ f \circ \phi_U^{-1}: \mathbb{R}^m \to \mathbb{R}^n$ is smooth

For $v \in TM|_U$, define $f_* v = \phi_{V*}^{-1} \circ (\phi_V \circ f \circ \phi_U^{-1})_* \circ \phi_{U*} v$

We can show it is independent of U, V because the bundle transition functions of TM, TN

Complex vector bundles (Chap 6)

Def. A complex v.b. E over M satisfies the following

1) There is a smooth map $\pi : E \to M$

for each $p \in M$, $\exists\, U_p \subset M$ $\lambda_U : \pi^{-1}(U_p) \to \mathbb{C}^n$

s.t. for each $x \in U_p$. $\lambda_U : \pi^{-1}(x) \to \mathbb{C}^n = \mathbb{R}^{2n}$

is an diffeomorphism

2) There is a smooth map $\hat{o} : M \to E$ s.t. $\pi \circ \hat{o} = Id$

3) There is a smooth map $\mu : \mathbb{C} \times E \to E$ s.t.

a) $\pi(\mu(r, v)) = \pi(v)$

b) $\mu(r, \mu(r', v)) = \mu(rr', v)$

c) $\mu(1, v) = v$

d) $\mu(r, v) = v$ for $r \neq 1$ iff $v \in Im\, \hat{o}$

Or we use the bundle transition function

$g_{VU} : U \cap V \to GL(n, \mathbb{C})$ called bundle transition function

$$E = \bigcup_{U \in \mathcal{U}} U \times \mathbb{C}^n / (p, u) \in U \times \mathbb{C}^n \sim (p, g_{VU} \cdot u) \in V \times \mathbb{C}^n \quad p \in U \cap V$$

$\{g_{VU}\}_{U, V \in \mathcal{U}}$ need to satisfy the following condition

1) $g_{VU} = g_{UV}^{-1}$ $\quad U \cap V \neq \phi$

2) $g_{VU} \circ g_{UW} \circ g_{WU} = Id$ $\quad U \cap V \cap W \neq \phi$

called cocycle condition

Rem. In the class of complex geometry, there is a def for complex manifold. We can get some natural complex v.b. similar to the natural real v.b. for real mfd (e.g. tangent bundle. cotangent bundle)

But we do have complex v.b. over <u>real</u> mfd

Ex. (Complexification): Let $E_{\mathbb{R}} \to M$ be a real v.b.

Let $E_{\mathbb{C}} = (E_{\mathbb{R}} \times \mathbb{C}) / (rv, c) \sim (v, rc) \quad r \in \mathbb{R}$

This is a cpx v.b. over M

We also have dual, $\oplus$, Hom, $\otimes$, Sym^k, Λ^k

subbundle, quotient bundle, pullback for complex v.b.