

Class 8 Metrics on vector bundles. (Chap 7)

Basic idea: For a real/complex vector bundle $E \rightarrow M$.

The fiber at each $p \in M$ is $\mathbb{R}^n$ or $\mathbb{C}^n$ with vector space structures.

A metric on E is a positive definite symmetric bilinear form for each fiber

Def 1 A metric g on a real vector bundle $E \rightarrow M$ is a section $M \rightarrow E^* \otimes E^*$, s.t. for each $p \in M$

1) $g|_p(v, v) > 0 \quad v \neq 0 \in E|_p$ (positive definite)

2) $g|_p(v, w) = g|_p(w, v)$ (symmetric)
 $v, w \in E$.

Rem (By 2), we can choose a basis of $E|_p \cong \mathbb{R}^n$ s.t. the matrix representing $g|_p$ is diagonal.

$$\text{i.e. } g|_p(v, w) = w^T \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} v = \sum_{i=1}^n \lambda_i v_i w_i$$

By 1): $\lambda_i > 0$

Def 2 An Hermitian metric g on a complex vector bundle $E \rightarrow M$ is a section $M \rightarrow \overline{E}^* \otimes E^*$ s.t. for $p \in M$

1) $g(v, v) \in (0, \infty) \quad v \neq 0 \in E|_p$

$$2) g(v, w) = c g(v, w)$$

$$3) g(c v, w) = \bar{c} g(v, w)$$

$$4) \overline{g(v, w)} = g(w, v)$$

Rem Similarly, at each pt, we can choose a basis s.t.

$$g|_p \text{ is diagonal } g|_p(v, w) = \sum_{i=1}^n \lambda_i \bar{v}_i w_i \quad \lambda_i \in (0, \infty)$$

Prop: Any real/cpx v.b. admits an (Hermitian) metric

We use partition of unity to construct it.

pf. For a chart $U \subset M$. $\lambda_U: \pi^{-1}(U) \rightarrow \mathbb{R}^n(\mathbb{C}^n)$

Given a map $g_U: U \rightarrow U \times (\mathbb{R}^* \oplus \mathbb{R}^*)$ satisfying the positive definite, symmetric condition on each fiber.

we obtain a metric on U by

$$g_U|_p(\lambda_U v, \lambda_U w) \quad v, w \in E|_p$$

We can choose a locally finite atlas $\mathcal{U}$ of M and construct the metric on each $U \in \mathcal{U}$

To put them together, we notice that if g_1, g_2 are two metrics, then $s g_1 + t g_2$ is still a metric for $s, t \geq 0$. s, t are not both 0.

(Just check Condition 1) & 2))

Lem. Suppose $\mathcal{U}$ is a locally finite atlas of M . Then there exists a set of smooth functions $\{\chi_\alpha: M \rightarrow [0, 1]\}_{U_\alpha \in \mathcal{U}}$ called a partition of unity, satisfying the following conditions

1) χ_α has support in U_α (i.e. $\{p \in M \mid \chi_\alpha(p) \neq 0\} \subset U_\alpha$)

2) $\sum_\alpha \chi_\alpha(p) = 1$ for any $p \in M$.

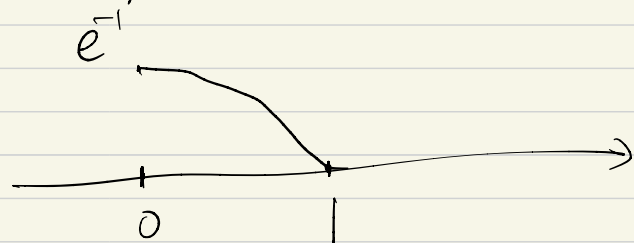
(Note that the locally finite condition + 1)

implies at each p , only finitely many χ_α is nonzero.

Pf: (See Cliff's book Appendix 1.2)

First we construct a function $\chi: [0, \infty) \rightarrow \mathbb{R}$

$$\chi(t) = \begin{cases} e^{\frac{1}{t-1}} & t < 1 \\ 0 & t \geq 1 \end{cases}$$



This function is smooth.

$$\lim_{t \rightarrow 1^-} \frac{d(e^{\frac{1}{t-1}})}{dt} = \lim_{t \rightarrow 1^-} e^{\frac{1}{t-1}} \cdot \left(-\frac{1}{(t-1)^2}\right) = 0$$

Similar for higher derivatives

Then for $\phi_{U_\alpha}: U_\alpha \rightarrow \mathbb{R}^n$, we construct $G_\alpha: M \rightarrow \mathbb{R}$

$$G_\alpha(p) = \begin{cases} \chi\left(\frac{|\phi_{U_\alpha}(p)|}{r_{U_\alpha}}\right) & p \in U_\alpha \\ 0 & p \notin U_\alpha \end{cases}$$

We can choose r_{U_α} s.t. $\{\phi_{U_\alpha}^{-1}(B(r_{U_\alpha}))\}_\alpha$ covers M .

Then $\sum_\alpha g_\alpha$ is nowhere zero

We define $\chi_\alpha = \left(\sum_\alpha g_\alpha\right)^{-1} g_\alpha$ $\square$

Using χ_α , we can define $g = \sum_\alpha \chi_\alpha g_{U_\alpha}$ as a metric on M $\square$

Recall a v.b. can be constructed by bundle transition

functions $\{g_{UV} : U \cap V \rightarrow GL(n, \mathbb{R})\}$ with some conditions

If we have a metric, then we can choose

$\text{Im } g_{UV} \subset O(n) \subset GL(n, \mathbb{R})$ as follows

First, locally we have $\varphi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$

and a basis of sections $\{S_i : U \rightarrow \pi^{-1}(U)\}$

Since we have a metric on each fiber, we can use

the Gram-Schmidt procedure to obtain another basis of sections $\{S'_i\}$ which are orthogonal with respect to

the metric at each fiber. Then we define a new diffeo

$\varphi'_U : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ by setting

$$(\varphi'_U)^{-1}(p, (v_1, \dots, v_n)) = \sum_{i=1}^n v_i S'_i(p)$$

Then the bundle transition function $g_{UV} = \varphi'_U \circ (\varphi'_V)^{-1}$
 sends orthogonal basis to orthogonal basis

so $g_{UV}: U \cap V \rightarrow O(n)$

If furthermore a real v.b. E is orientable, i.e.

$\wedge^n E$ is isomorphic to $M \times \mathbb{R}$, then we have

$\det g_{UV}(p) > 0$. so $g_{UV}: U \cap V \rightarrow SO(n)$

(The groups $GL(n, \mathbb{R})$, $O(n)$, $SO(n)$ for g_{UV}
 are called structure groups of the bundles)

The existence of metric $\Leftrightarrow$ str gp can be reduced
 to $O(n)$

— metric + orientation $\Leftrightarrow$ str. gp — $SO(n)$

— Hermitian metric for complex v.b. $\Leftrightarrow U(n) \subset GL(n, \mathbb{C})$
 (almost complex str) $\subset GL(2n, \mathbb{R})$

Hermitian metric + orientation $\Leftrightarrow SU(n)$

— symplectic str. $\Leftrightarrow$ some str gps.

- More facts: 1) a metric induces an isomorphism $E \xrightarrow{\cong} E^*$
- 2) an Hermitian metric induces a $\mathbb{C}$ anti-linear isomorphism $E \xrightarrow{\cong} E^*$
- 3) a metric on E induces a metric on $E^*, E^{\otimes k}$
 $\Lambda^k E, \text{Sym}^k E$. metrics on E_1, E_2 induce
 a metric on $E_1 \otimes E_2, \text{Hom}(E_1, E_2), E_1 \otimes E_2$
- 4) For a smooth map $f: M \rightarrow N$ and a pull-back $f^*E \rightarrow M$ of $E \rightarrow N$, the metric on E induces a metric on f^*E .

Def. A metric on TM is called a Riemannian metric.
 also called a metric of M .

Rem. Metrics on the submflds can be induced by the metric of the ambient space.

Any M can be embedded into $\mathbb{R}^N$; the metric of $\mathbb{R}^N$ also induces a metric on M .

Class 9. Riemannian metric.

A metric on a (real) v.b. associates a positive definite symmetric bilinear form on each fiber. i.e.

a section $M \rightarrow E^* \otimes E^*$ satisfies some conditions.

Basic properties

1) metrics always exist.

2) g_1, g_2 are metrics, then $sg_1 + tg_2$ are also metrics

$$s, t \geq 0 \quad s^2 + t^2 \neq 0$$

This implies the space of metrics is contractible because any metrics can be connected linearly.

$$\text{by } tg_1 + (1-t)g_2$$

3) The existence of metric reduces the str gp of the bundle transition functions from $GL(n, \mathbb{R})$ to $O(n)$ (Hermitian metric reduces $GL(n, \mathbb{C})$ to $U(n)$)

Def. A metric on TM is called a Riemannian metric.

also called a metric of M .

Recall that we write coordinates of

$$T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \text{ as } x_1, \dots, x_n, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$$

$$\text{and } T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \text{ as } x_1, \dots, x_n, dx_1, \dots, dx_n$$

Locally, a vector field (a section $M \rightarrow TM$)
is written as $\sum_{k=1}^n V_k \frac{\partial}{\partial x^k}$ for $V_k: U \rightarrow \mathbb{R}$

(Here we omit the function $\varphi_U: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$)

a 1-form (a section $M \rightarrow T^*M$)

is written as $\sum_{k=1}^n V_k dx^k$

Hence a metric (a section of $T^*M \otimes T^*M$)

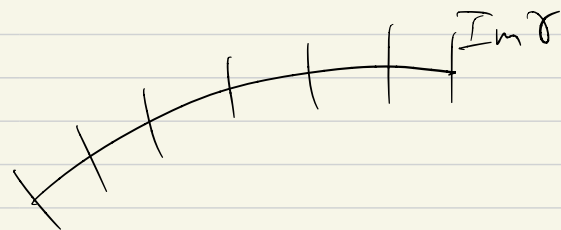
can be written as $\sum_{i,j=1}^n g_{ij} dx^i \otimes dx^j$

with $\{g_{ij}(p)\}$ a sym, posi definite matrix

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}$$

Intuition of length of a path in $\mathbb{R}^n$

Let $\gamma: I = [0,1] \rightarrow \mathbb{R}^n$



Step 1. Break I into N segments

i.e. consider $t_i = \frac{i}{N}$ and $\gamma(t_i)$

Approximate $\text{Im } \gamma|_{[t_i, t_{i+1}]}$ by $|\gamma(t_{i+1}) - \gamma(t_i)|$

$$\Delta t = \frac{1}{N}$$

Step 2. Use Taylor's thm to write

$$\frac{|\gamma(t_{i+1}) - \gamma(t_i)|}{\Delta t} = \left| \frac{d}{dt} \gamma \right|_{t=t_i} + \mathcal{O}(\Delta t)$$

$$\sum_{i=0}^{N-1} |\gamma(t_{i+1}) - \gamma(t_i)| = \left(\sum \left| \frac{d}{dt} \gamma \right|_{t=t_i} \right) \Delta t + \mathcal{O}(\Delta t)$$

Step 3 Take the limit $N \rightarrow \infty, \Delta t \rightarrow 0$

we get the length of γ

$$L_\gamma := \int_I \left| \frac{d}{dt} \gamma \right| dt \quad (\text{This is a definition})$$

Def. Given a Riemannian metric g on M and a path $\gamma: I \rightarrow M$ we define the length of γ as $L_\gamma := \int_I \left(g\left(\frac{d}{dt} \gamma, \frac{d}{dt} \gamma\right) \right)^{\frac{1}{2}} dt$

Rem. $\frac{d}{dt} \gamma$ means the pushforward of
(tangent map)

$$\frac{\partial}{\partial t} \in T_t I \quad \text{to} \quad \gamma_* \frac{\partial}{\partial t} \in T_{\gamma(t)} M$$

Explicitly $U \subset I \quad V \subset M$

$$\phi_U : U \rightarrow \mathbb{R} \quad \phi_V : V \rightarrow \mathbb{R}^n$$

$$\phi_{U*} : T I|_U \rightarrow U \times \mathbb{R} \quad \phi_{V*} : T M|_V \rightarrow V \times \mathbb{R}^n$$

$$\phi_V \circ \gamma \circ \phi_U^{-1} : \mathbb{R} \rightarrow \mathbb{R}^n$$

$$\gamma_* \frac{\partial}{\partial t} = \phi_{V*}^{-1} \circ (\phi_V \circ \gamma \circ \phi_U^{-1})_* \phi_{U*} \frac{\partial}{\partial t}$$

Since the notation $\frac{\partial}{\partial t}$ already means $\phi_{U*}^{-1}(\frac{\partial}{\partial t})$,
and $\phi_U = \text{Id}$,

we have

$$\begin{aligned} \frac{d}{dt} \gamma|_p &= \phi_{V*}^{-1} \circ (\phi_V \circ \gamma)_* \frac{\partial}{\partial t} |_p \\ &= \phi_{V*}^{-1} \left(p, \frac{d(x_1 \circ \phi_V \circ \gamma)}{dt}, \dots, \frac{d(x_n \circ \phi_V \circ \gamma)}{dt} \right) \end{aligned}$$

$\chi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ coordinate map

Def The distance between $p, q \in M$ is defined by

$$d(p, q) := \inf_{\gamma} L_{\gamma} \quad \gamma \text{ is a path with } \gamma(0) = p \quad \gamma(1) = q$$

Prop: 1) $d(p, p) = 0$ $d(p, q) \neq 0$ for $p \neq q$

$$2) d(p, q) = d(q, p)$$

$$3) d(p, q) \leq d(p, r) + d(r, q) \quad r \in M$$

Rem. 1). if $p \neq q$. we have a nbhd of p that is disjoint from q (Hausdorff). the path must intersect the boundary of the nbhd and in the nbhd the length is nonzero

Thm (Geodesic thm)

1) There is a smooth path γ from p to q with $l_\gamma = d(p, q)$, such path is called geodesic (or its image)

2) Any geodesic is an embedded, 1-dim submfd that can be reparameterized st. $g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = 1$

This is called a constant speed geodesic

(Repara means we get a new path by composition

$$I \xrightarrow{\psi} I \xrightarrow{\gamma} M)$$

3) A constant speed geodesic is characterized as follows

Let $U \subset M$, $\phi_U : U \rightarrow \mathbb{R}^n$ $g|_U = \phi_U^*(\sum g_{ij} dx^i \otimes dx^j)$

We write $\gamma_U : \gamma^{-1}(U) \cap I \rightarrow \mathbb{R}^n$

$$t \longmapsto \phi_U \circ \gamma(t)$$

$$= (\gamma_U^1(t), \gamma_U^2(t), \dots, \gamma_U^n(t))$$

Then
$$(*) \quad \frac{d^2}{dt^2} \gamma_U^i + \sum_{j,k=1}^n \Gamma_{jk}^i \frac{d}{dt} \gamma_U^j \frac{d}{dt} \gamma_U^k = 0$$

for any i where

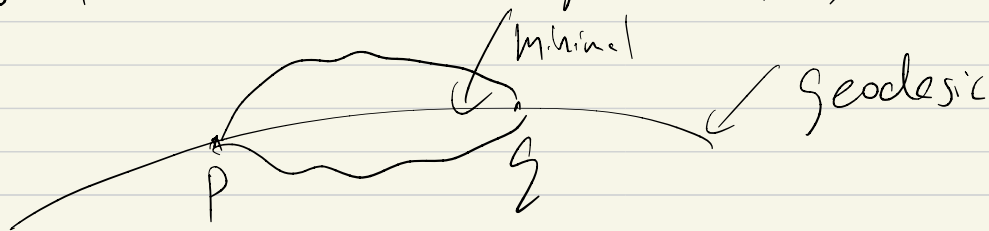
$$\Gamma_{jk}^i = \frac{1}{2} \sum_{l=1}^n g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$

Γ_{jk}^i is called Christoffel symbols

4) Any path satisfying (*) is locally length minimizing.

This means $\exists c_0 < 1$ s.t. when p, q on the curves with $d(p, q) \leq c_0$ then $\exists$ a segment of path connecting p and q with length $d(p, q)$



Class 10 Geodesic (Chap 8)

Recall a path in M is a smooth map

$\gamma: I = [0, 1] \rightarrow M$, the length of γ is

$$L_\gamma := \int_I \left| \frac{d\gamma}{dt} \right| dt = \int_I g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t})^{\frac{1}{2}} dt$$

$\in \mathbb{R}^n$ $\in M$

where $\frac{\partial}{\partial t}$ is the second coordinate of $TI \cong I \times \mathbb{R}$

and $\gamma_*: TI \rightarrow TM$ is the tangent map

(push forward) The distance $d(p, q) = \inf_{\substack{\gamma(0)=p \\ \gamma(1)=q}} L_\gamma$

Thm (Geodesic thm) M compact manifold

1) $\exists \gamma$ achieves $d(p, q)$, called geodesic

2) A geodesic is embedded, and can be reparameterized

Ex. so that $g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = 1$

3) Let $U \subset M$, $\phi_U: U \rightarrow \mathbb{R}^n$ $g|_U = \phi_U^*(\sum g_{ij} dx^i \otimes dx^j)$

We write $\gamma_U: \gamma^{-1}(U) \cap I \rightarrow \mathbb{R}^n$

$$t \mapsto \phi_U \circ \gamma(t)$$

$$= (\gamma_U^1(t), \gamma_U^2(t), \dots, \gamma_U^n(t))$$

$$\text{Then } \frac{d^2}{dt^2} \gamma_U^i + \sum_{j,k=1}^n \Gamma_{jk}^i \frac{d}{dt} \gamma_U^j \frac{d}{dt} \gamma_U^k = 0$$

for any i where

geodesic equation

$$\Gamma_{jk}^i = \frac{1}{2} \sum_{l=1}^n g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$

Γ_{jk}^i is called Christoffel symbols

4) Any path satisfying (*) is locally length minimizing.

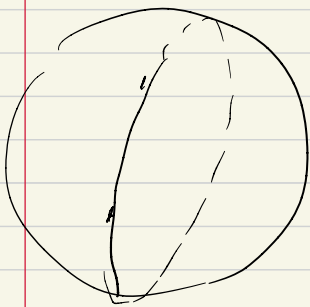
Ex

1) Geodesic in $\mathbb{R}^n$ with respect to the standard metric:

$$\partial_i g = 0. \quad (*) \Rightarrow \frac{d^2 x^i}{dt^2} = 0 \Rightarrow \partial_t(t) = at + b$$

i.e. geodesics in $\mathbb{R}^n$ are straight lines

2) Geodesic in S^2 with the induced metric from $\mathbb{R}^3$
the short segment of the great circle.



(see Cliff's book §.4.2 for Computations)

If p, q are two pole pts. then
geodesics are not unique

Pf of Thm.

Step 1. $\mathbb{R}^n$ with the standard metric g

Consider the balls centered at the origin and their boundary spheres. Then $g = dr \otimes dr + r^2 g_S \rightarrow \mathbb{R}^n$

g_S is the induced metric on S^{n-1} with radius 1.

Let $\gamma: I \rightarrow \mathbb{R}^n$ be a path in a ball with radius R .

$\gamma(0) = 0 \in \mathbb{R}^n$ $\gamma(1)$ is on the boundary

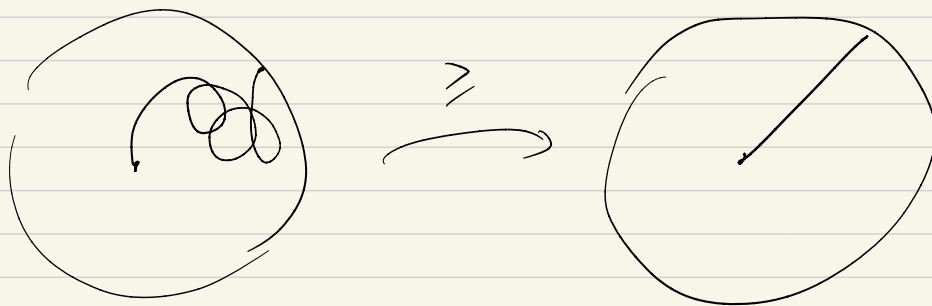
Write $\gamma(t)$ as $(r(t), \theta(t))$ $r: I \rightarrow \mathbb{R}_{\geq 0}$ $\theta: I \rightarrow S^{n-1}$

$$g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = |\frac{\partial r}{\partial t}|^2 + r^2 g_S(\theta_* \frac{\partial}{\partial t}, \theta_* \frac{\partial}{\partial t})$$

$$\geq |\frac{\partial r}{\partial t}|^2 \quad \text{the equation holds iff } \theta \text{ is constant}$$

$$L\gamma \geq \int_I |\frac{\partial r}{\partial t}| dt \geq \int_I \frac{\partial r}{\partial t} dt = R$$

the equation holds when γ is a straight line.



Moreover, we can reparameterize γ s.t. $g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = 1$

Step 2.

If M is compact, then $\exists R > 0$ s.t.

Lemma (Special coordinates) for each $p \in M$, $\exists$ chart $\phi_U: U \rightarrow \mathbb{R}^n$

s.t. $\phi_U(p) = 0$ and on a small ball B_R of $\mathbb{R}^n$, the metric g

on $\phi_U^{-1}(B_R)$ is $\phi_U^*(dr \otimes dr + r^2 g_r)$ g_r metric on S^{n-1}

Assuming the Lemma, we prove that for a fixed $p \in M$ and any $q \in M$ there exists a length minimizing curve from p to q .

a) $d(p, q) \leq \frac{R}{2}$, such curve exists because q is in B_R for p

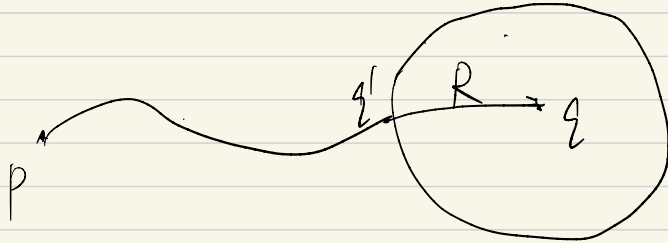
b) Suppose we already show the existence for q with

$d(p, q) \leq \frac{NR}{2}$, we prove the existence for $\frac{NR}{2} < d(p, q) \leq \frac{(N+1)R}{2}$

There is a path γ_0 from p, q s.t. $L_{\gamma_0} < d(p, q) + \frac{R}{2}$

Suppose $q' = \text{Im } \gamma_0 \cap \partial B_R(q)$

Then $d(p, q') \leq L_{\gamma_0} - R < d(p, q) - \frac{R}{2} \leq \frac{NR}{2}$



Let $q'' \in \partial B_R(q)$ s.t. $d(p, q'')$ is minimal.

we know $d(p, q'') \leq d(p, q') < \frac{NR}{2}$

Then $\exists$ geodesics from p to q'' , q'' to q .

the union of them is the geodesic from p to q

because it can't be shorten inside or outside the $B_R(q)$

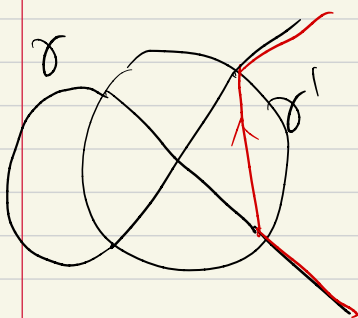
We can reparameterize it so that it has constant speed

Step 3. Nbd of each pt on the geodesic is a straight line

some B_R , so γ is an immersion (locally injective)

If $\text{Im } \gamma$ has self-intersection, then locally it is

two straight lines in B_R , if we replace γ by γ' ,



then $L_{\gamma'} < L_{\gamma}$ which contradicts the minimal length condition. Hence γ is an embedded curve (locally injective and no self-intersection)

we explain more on Lem 1.

Thm 2 (vector field thm) Let $S: X \rightarrow TX$ be a vector field. For any $q \in X$, $\exists \varepsilon > 0$ and a unique path $G_q: (-\varepsilon, \varepsilon) \rightarrow X$ s.t.

$$1) G_q(0) = q \quad 2) G_q \times \frac{\partial}{\partial t} = S(G_q(t))$$

Moreover, if $U \subset X$ is an open set with compact closure,

then $(q, t) \mapsto G_q(t)$ is a smooth map from $U \times [0, \varepsilon)$ to X s.t. for each t , $q \mapsto G_q(t)$ is a diffeo from U to an open set of X

Pf. Cliff's book Appendix 8.1 $\square$

We take $X = TM$ locally $q = (p, v) \in X$

$$\text{Define } S(p, v) = (v_i, -\Gamma_{jk}^i v_j v_k)$$

Then we obtain a unique path

$$G_q: (-\varepsilon, \varepsilon) \rightarrow TM \quad \text{s.t. } G_q(0) = q = (p, v)$$

$$G_q \times \frac{\partial}{\partial t} = S(G_q(t))$$

This implies a unique path $\gamma_{(p,v)}: (-\varepsilon, \varepsilon) \rightarrow M$

$$\text{s.t. } \gamma(0) = p \quad \gamma \times \frac{\partial}{\partial t} \Big|_{t=0} = v$$

$$\text{and } \frac{d^2 \gamma^i}{dt^2} + \sum_{j,k} \Gamma_{jk}^i \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0$$

(Here we write $\gamma = \gamma_U = \phi_U \circ \gamma_{(p,v)}: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$)

Class 11 Geodesic thm. (Chap 9)

If M is compact, then $\exists R > 0$ st.

Lem1 (Special coordinates) For each $p \in M$, $\exists$ chart $\phi_U: U \rightarrow \mathbb{R}^n$ s.t. $\phi_U(p) = 0$ and on a small ball B_R of $\mathbb{R}^n$, the metric g on $\phi_U^{-1}(B_R)$ is $\phi_U^*(dr \otimes dr + r^2 g_r)$

Last time we use Lem2 to prove 1), 2) in Thm 1
Today we use the following thm to prove Lem 1

Thm2 (Vector field thm) Let $S: X \rightarrow TX$ be a vector field. For any $q \in X$, $\exists \varepsilon > 0$ and a unique path $\gamma_q(-\varepsilon, \varepsilon) \rightarrow X$ s.t.

$$1) \gamma_q(0) = q \quad 2) \gamma_q \times \frac{\partial}{\partial t} = S(\gamma_q(t))$$

We take $X = TM$ locally $q = (p, v) \in X$

$$\text{Define } S(p, v) = (v_i, -\Gamma_{jk}^i v_j v_k)$$

$$\Gamma_{jk}^i = \frac{1}{2} \sum_{l=1}^n g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$$g|_U = \phi_U^*(\sum g_{ij} dx^i \otimes dx^j)$$

$\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$

Γ_{km}^j is called Christoffel symbols

Then we obtain a unique path

$$G_q: (-\varepsilon, \varepsilon) \rightarrow TM \quad \text{s.t.} \quad G_q(0) = q = (p, v)$$

$$G_q \times \frac{\partial}{\partial t} = S(G_q(t))$$

This implies a unique path $\gamma_{(p,v)}: (-\varepsilon, \varepsilon) \rightarrow M$

$$\text{s.t.} \quad \gamma(0) = p \quad \gamma \times \frac{\partial}{\partial t} \Big|_{t=0} = v$$

$$\text{and} \quad \frac{d^2 \gamma^i}{dt^2} + \sum_{j,k} \Gamma_{jk}^i \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0$$

(Here we write $\gamma = \gamma_U = \phi_U \circ \gamma_{(p,v)}: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$)

The reason to choose such s is because the following properties

Prop.

1) $g(\gamma \times \frac{\partial}{\partial t}, \gamma \times \frac{\partial}{\partial t})$ is constant

$$\text{Pf.} \quad \partial_t g(\gamma \times \frac{\partial}{\partial t}, \gamma \times \frac{\partial}{\partial t}) = \partial_t g_{ij} \dot{\gamma}^i \dot{\gamma}^j$$

(when an index appear twice, it means we take $\sum$ over the index)

(for simplicity, we write $\dot{\gamma}^i = \frac{d\gamma^i}{dt}$ $\ddot{\gamma}^i = \frac{d^2 \gamma^i}{dt^2}$)

$$\gamma_U = \phi_U \circ \gamma$$

$$\rightarrow = g_{ij} \ddot{\gamma}^i \dot{\gamma}^j + g_{ij} \dot{\gamma}^i \ddot{\gamma}^j + \dot{\gamma}^k \partial_k g_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0$$

Compute $\partial_t g_{ij} \dot{x}^i \dot{x}^j$

$$= \boxed{g_{ij} \ddot{x}^i \dot{x}^j + g_{ij} \dot{x}^i \ddot{x}^j} + \dot{x}^k \partial_k g_{ij} \dot{x}^i \dot{x}^j = 0$$

① $\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0$ ② $g_{ij} = g_{ji}$ ④ $\{g^{il}\} = \{g_{il}\}^{-1}$
 ③ $\Gamma_{jk}^i = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$ $g_{il} g^{il} = \delta_{il} = \begin{cases} 1 & i=l \\ 0 & i \neq l \end{cases}$

② $\rightarrow -2g_{ij} \dot{x}^i \ddot{x}^j \stackrel{\text{①}}{=} -2g_{ij} \Gamma_{mk}^i \dot{x}^m \dot{x}^k \dot{x}^j$

change index

$$\begin{aligned} m &\mapsto j \\ j &\mapsto m \end{aligned}$$

$$-2g_{im} \Gamma_{jk}^i \dot{x}^j \dot{x}^k \dot{x}^m$$

$$\stackrel{\text{③}}{=} -2g_{im} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}) \dot{x}^j \dot{x}^k \dot{x}^m$$

$$= -2\delta_{ml} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}) \dot{x}^j \dot{x}^k \dot{x}^m$$

$$= -2(\partial_j g_{mk} + \partial_k g_{jm} - \partial_m g_{jk}) \dot{x}^j \dot{x}^k \dot{x}^m$$

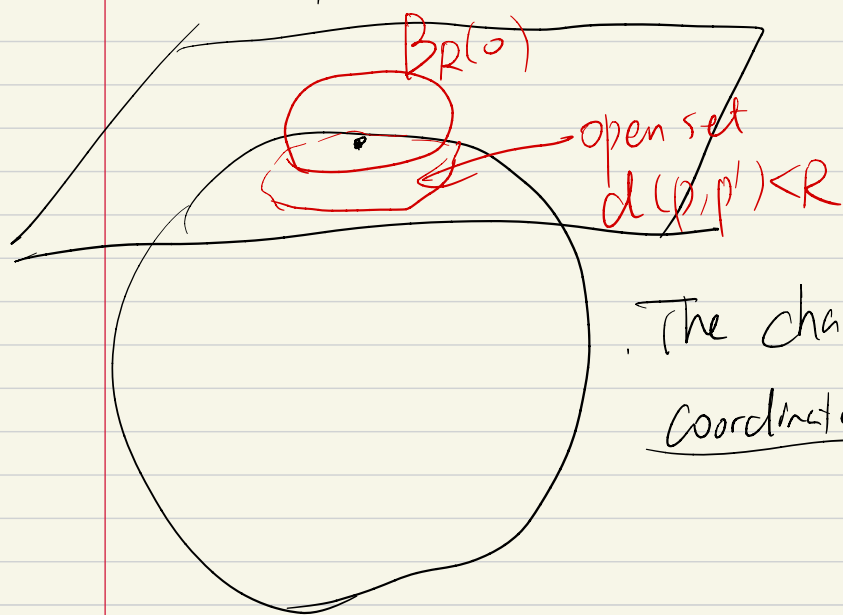
$$= -2\partial_k g_{ij} \dot{x}^i \dot{x}^j \dot{x}^k$$

2) $\dot{x} \cdot \frac{\partial}{\partial t} \Big|_{t=0} = v \Rightarrow g(\dot{x} \cdot \frac{\partial}{\partial t}, \dot{x} \cdot \frac{\partial}{\partial t}) = |v|^2$
 for all t (constant speed)

3) Fix $\mu > 0$, let $\eta(t) = \gamma_{(p,v)}(\mu t)$. Then

$\eta(0) = p$ $\eta \cdot \frac{\partial}{\partial t} = \mu v$ η also satisfy the geodesic equation. η is defined for $t \in (\mu^{-1}\epsilon, \mu^{-1}\epsilon)$

Then $\exists R$ s.t. the map $V \mapsto \gamma_{(p, \frac{2}{\varepsilon}V)}(\frac{\varepsilon}{2})$ defines a diffeo between $B_R(0) \subset TM|_p \cong \mathbb{R}^n$ and the open set of pts p' in M with $d(p, p') < R$. This map is called the exponential map at p $\exp_p: TM|_p \rightarrow M$



Note that this is an inverse map of a coordinate chart.

The charts are called Gaussian coordinates or normal coordinates

Claim: The pullback metric $\hat{g} = \exp^* g$ has the form $dr \otimes dr + r^2 g_r \leftarrow \begin{matrix} \text{metric on} \\ S^{n-1} \end{matrix}$

(If M is compact, we can use finitely many Gaussian charts to cover M , hence we have a minimal value of $\varepsilon > 0$. This will imply Lem 1

Pf of Claim: Cliff's book section 9.3

Note that the existence of the solution of the geodesic equation only for $t \in (-\varepsilon, \varepsilon)$ but we may extend the solution to a larger interval.

Def If all geodesics are defined on $\mathbb{R}$,
the manifold is called geodesically complete.

Fact all compact manifolds are geodesically compact

Ex. a manifold that is not geodesically compact

$$\mathbb{R}^2 - \{0\}$$

Understand the geodesic equation by Euler-Lagrange equation: (variation principle)

Consider a functional:

$$S: \left\{ \begin{array}{l} \text{the space of paths } \gamma \subset \mathbb{R}^n \\ \text{with } \gamma(0) = p \quad \gamma(1) = q \end{array} \right\} \longrightarrow \mathbb{R}$$

$$\gamma \longmapsto \int_I g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)^{\frac{1}{2}} dt$$

For $s \in (-\varepsilon, \varepsilon)$, consider the path

$$\gamma_s: I \longrightarrow \mathbb{R}^n$$

$$t \longmapsto \gamma(t) + s \eta(t) \quad \eta: I \longrightarrow \mathbb{R}^n \text{ any path} \\ \eta(0) = \eta(1) = 0$$

$$\left. \frac{dS(\gamma_s)}{ds} \right|_{s=0} = \int_I \left. \frac{d g\left(\frac{d\gamma_s}{dt}, \frac{d\gamma_s}{dt}\right)^{\frac{1}{2}}}{ds} dt \right|_{s=0}$$

$$= \int_I \left. \frac{d (g_{ij}(\gamma_s) \dot{\gamma}_s^i \dot{\gamma}_s^j)^{\frac{1}{2}}}{ds} dt \right|_{s=0} \left(\begin{array}{l} \frac{d\gamma_s}{ds} = \eta(t) \\ \frac{d\dot{\gamma}_s}{ds dt} = \dot{\eta}(t) \end{array} \right)$$

$$= \int_I \frac{1}{2} \cdot \frac{1}{\sqrt{g_{ij}(\gamma) \dot{\gamma}^i \dot{\gamma}^j}} \left(\partial_k g_{ij} \cdot \eta^k \dot{\gamma}^i \dot{\gamma}^j + 2 g_{ij} \dot{\eta}^i \dot{\gamma}^j \right) dt$$

Integrating by parts on the second term. the boundary terms vanish because $\eta(0) = \eta(1) = 0$

$$= \frac{1}{2} \int_I \left(\frac{\partial_k g_{ij} \dot{\gamma}^i \dot{\gamma}^j}{\sqrt{g_{ij}(\gamma) \dot{\gamma}^i \dot{\gamma}^j}} - 2 \frac{d}{dt} \left(\frac{g_{kj} \dot{\gamma}^j}{\sqrt{g_{ij}(\gamma) \dot{\gamma}^i \dot{\gamma}^j}} \right) \right) \eta^k dt$$

To obtain a minimal length path. we need

$$\left. \frac{dS(\gamma_s)}{ds} \right|_{s=0} = 0 \text{ for any } \eta(t).$$

$$\text{So } \frac{\partial_k g_{ij} \dot{\gamma}^i \dot{\gamma}^j}{\sqrt{g_{ij} \dot{\gamma}^i \dot{\gamma}^j}} - 2 \frac{d}{dt} \left(\frac{g_{kj} \dot{\gamma}^j}{\sqrt{g_{ij} \dot{\gamma}^i \dot{\gamma}^j}} \right) \text{ must vanish}$$

$$\text{Let } d\tau = dt \sqrt{g_{ij} \dot{\gamma}^i \dot{\gamma}^j}$$

Then we have

$$\partial_k g_{ij} \dot{\gamma}^i \dot{\gamma}^j - 2 \frac{d}{d\tau} (g_{kj} \dot{\gamma}^j) = 0 \quad (\text{Now } \dot{\gamma}^i = \frac{d\gamma^i}{d\tau})$$

$$\Rightarrow \partial_k g_{ij} \dot{x}^i \dot{x}^j - 2 \partial_i g_{kj} \dot{x}^i \dot{x}^j - 2 g_{kj} \ddot{x}^j = 0$$

$2 g_{kl} \ddot{x}^l$ (replace j by l)

$$\Rightarrow \ddot{x}^l + \frac{1}{2} g^{kl} (2 \partial_i g_{kj} - \partial_k g_{ij}) \dot{x}^i \dot{x}^j = 0$$

$\partial_i g_{kj} + \partial_j g_{ik}$ after times $\dot{x}^i \dot{x}^j$

$$\Rightarrow \ddot{x}^l + \Gamma_{ij}^l \dot{x}^i \dot{x}^j = 0$$

$$\Gamma_{jk}^l = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$