(Skip all materials about principal bundles and come back to Chap 10 later)

Class 12 de Rham cohomology (Chap 12)

Basic idea of homology / cohomology:

Given a graded abelian group $C = \bigoplus_i C_i$ and an endomorphism

$d: C \to C$ s.t. $d(C_i) \subset C_{i+1}$ (for cohomology)

$\subset C_{i-1}$ (for homology)

and $d^2 = 0$

Then $\mathrm{Im}\, d = \{ c \in C \mid \exists\, a \in C,\ da = c \}$

$\subset \ker d = \{ c \in C \mid dc = 0 \}$

$$H^i(C, d) = \ker(d: C_i \to C_{i+1}) / \mathrm{Im}(d: C_{i-1} \to C_i)$$

(Notation $H_i(C,d)$ for homology

$H^*(C,d) = \bigoplus_i H^i(C,d)$ $\quad H_*(C,d) = \bigoplus_i H_i(C,d)$)

Def. (C,d) is called a (co)chain complex. $H_*(C,d)$ or $H^*(C,d)$ is called the (co)homology group of (C,d)

elements in $\mathrm{Im}\, d$ are called (co) boundaries

elements in $\ker d$ are called (co) cycles

For an element $c \in \ker d$, the corresponding element in H^* is denoted by $[c]$

Rem

① The chain complex may not be an invariant (of manifold or bundle), but the homology and its rank are invariants (up to isomorphisms)

② When taking $\ker d / \mathrm{Im}\, d$, we may lose some information in (C,d), but that makes H^* simpler to study.

(3) Sometimes, we don't need the grading, just start with (C, d) with $d^2=0$ and take $H^*(C,d) = \ker d / \operatorname{Im} d$.

If C is a vector space, d can be understood as a matrix

Then let's define the de Rham cohomology

Let M be a smooth manifold of dim n

For $k = 0, 1, \dots, n$, let $\Omega^k = \Omega^k(M)$ be the space of sections $\omega : M \longrightarrow \Lambda^k T^* M$

(At each point $p \in M$, we know ω_p is an antisymmetric multilinear map

$$\omega_p : \underbrace{T_pM \times T_pM \times \cdots \times T_pM}_{k} \longrightarrow \mathbb{R}$$

Antisym means $\omega_p(\cdots v_i, \cdots, v_j, \cdots) = -\omega_p(\cdots, v_j, \cdots, v_i, \cdots)$

$$\dim (\Lambda^k T^* M)\Big|_p = \binom{n}{k}$$
)

Since sections are called k-form, Ω^k is the space of k-forms on M. In particular $\Omega^0 = C^\infty(M, \mathbb{R})$ is the space of smooth functions on M $(\Lambda^0 T_p^* M = \mathbb{R})$

Previously, for a smooth function (a 0-form) $f : M \to \mathbb{R}$ we construct a 1-form df by local chart $df = \sum_{i=1}^{n} \frac{\partial f_U}{\partial x^i} dx^i$

d is a map : $\Omega^0 \longrightarrow \Omega^1$

Goal: extend d to $\Omega^k \longrightarrow \Omega^{k+1}$ s.t. $d^2 = 0$

This map is called the exterior derivative

($\Lambda^k V$ is called the exterior algebra for V)

Fact: Locally, any k-form can be written as

$$\sum_{i_1<i_2<\cdots<i_k} f_{i_1\cdots i_k}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k} \qquad f_{i_1\cdots i_k}: U\to\mathbb{R}$$

($dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ are basis of sections on $\Lambda^k\mathbb{R}^n$ and we use $\phi_U: U\to\mathbb{R}$ to pull-back sections)

Define

$$d(\quad) = \sum_{i_1<\cdots<i_k}\ \sum_{m\notin\{i_1,\dots,i_k\}} \frac{\partial f_{i_1\cdots i_k}}{\partial x^m}\, dx^m\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$= \sum_{i_1<\cdots<i_k<i_{k+1}} \left(\frac{\partial f_{i_2 i_3\cdots i_{k+1}}}{\partial x^{i_1}} - \frac{\partial f_{i_1 i_3\cdots i_{k+1}}}{\partial x^{i_2}} + \cdots\right) dx^{i_1}\wedge\cdots\wedge dx^{i_{k+1}}$$

Ex $\omega = y\,dx$ on $\mathbb{R}^2$
$d\omega = dy\wedge dx$.

sign comes from anti-sym.

This generalizes $df = \sum_m \frac{\partial f}{\partial x^m} dx^m$ (Need to check it is independent of charts)

$$d(df) = d\left(\sum_m \frac{\partial f}{\partial x^m} dx^m\right) = \sum_{m,l} \frac{\partial^2 f}{\partial x^m \partial x^l} dx^m\wedge dx^l$$

Since $\frac{\partial^2 f}{\partial x^m\partial x^l} = \frac{\partial^2 f}{\partial x^l\partial x^m}$ and $dx^m\wedge dx^l = -dx^l\wedge dx^m$

$d(df) = 0$ In general, we can check $d^2 = 0: \Omega^k\to\Omega^{k+2}$

Prop 1) $d(\omega_1+\omega_2) = d\omega_1 + d\omega_2$

2) ω_1, ω_2 are k, l forms

$$d(\omega_1\wedge\omega_2) = d\omega_1\wedge\omega_2 + (-1)^k\omega_1\wedge d\omega_2$$

Def. ω is <u>closed</u> if $d\omega = 0$ (cocycle)

ω is <u>exact</u> if $\omega = d\alpha$ for some α. (coboundary)

The de Rham cohomology $H^*_{dR}(M) = \ker d / \operatorname{Im} d$

It is a vector space of $\mathbb{R}$.

(If you know singular cohomology $H^*(M)$ for a topological space M, then we have $H^*_{dR}(M) \cong H^*(M; \mathbb{R})$

in $\mathbb{R}$ coefficient

In general $H^*(M)$ can be defined over $\mathbb{Z}$.

$$\text{e.g. } H^i(\mathbb{R}P^3; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i=1 \\ \mathbb{Z}/2 & i=2 \\ \mathbb{Z} & i=3 \end{cases}$$

$$H^i_{dR}(\mathbb{R}P^3) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i=1 \\ 0 & i=2 \\ \mathbb{R}. & i=3 \end{cases}$$

If $\varphi: M \to N$ is a smooth map. φ induces a map

$\varphi^*: \Omega^k(N) \longrightarrow \Omega^k(M)$ by

$\varphi^*\omega(v_1, \dots, v_n) = \omega(\varphi_* v_1, \dots, \varphi_* v_n)$ $\quad v_i \in T_pM$ $\quad \varphi_*$ tangent map

which is well defined on $H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$ because

1) For $v \in T_pM$ $\quad df(v) = v(f) := \mathcal{L}_v f$

because $df = \frac{\partial f}{\partial x^i} dx^i$ $\quad v = v^i \frac{\partial}{\partial x^i}$ $\quad \mathcal{L}_v f = v^i \frac{\partial f}{\partial x^i}$

2) $\varphi^* df(v) = df(\varphi_* v) = (\varphi_* v)(f) = v(\varphi^* f) = d(\varphi^* f)(v)$

$\Longrightarrow \varphi^* df = d\varphi^* f$

3) $d(\varphi^*(dx^1 \wedge \dots \wedge dx^n)) = d(\varphi^*(dx^1) \wedge \dots \wedge \varphi^*(dx^n)) = 0$.

$\left(0 = \varphi^* d(dx^i) = d(\varphi^* dx^i)\right)$

$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$ $\quad d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$

4) $w = \sum_{i_1 < i_2 < \cdots < i_k} f_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

$$dw = \sum_{i_1 < \cdots < i_k} \underbrace{\sum_{m \notin \{i_1, \cdots, i_k\}} \frac{\partial f_{i_1 \cdots i_k}}{\partial x^m} dx^m}_{df} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$\psi^* w = \sum (\psi^* f) \, \psi^* (dx^{i_1} \wedge \cdots \wedge dx^{i_k})$$

$$\psi^* dw = \sum \psi^* (df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k})$$

$$= \sum \psi^*(df) \wedge \psi^* (dx^{i_1} \wedge \cdots \wedge dx^{i_k})$$

$$= \sum d(\psi^* f) \wedge \psi^* (\cdots)$$

$(d \, \psi^* (dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = 0)$

$$= d \left(\sum (\psi^* f) \wedge \psi^* (\cdots) \right)$$

$$= d \psi^* w$$

$\Rightarrow \psi^* d = d \psi^*$

5). If $dw = 0$, then $d(\psi^* w) = \psi^* dw = 0$

If $w_1 - w_2 = dw_0$ then $\psi^*(w_1) - \psi^*(w_2)$

$$= \psi^* dw_0 = d(\psi^* w_0)$$

$\Rightarrow \psi^*$ is well-defined on $\ker d / \operatorname{Im} d$

Prop. If ψ is a diffeomorphism, ψ^* induces an iso between $H^*_{dR}(N)$ and $H^*_{dR}(M)$ $\Rightarrow$ If $H^*_{dR}(N) \neq H^*_{dR}(M)$, then $M \not\cong_{diff} N$

If ψ and ϕ are homotopic map to N, i.e.

$\exists \, \Phi : [0,1] \times M \to N$ smooth, s.t. $\Phi(0, \cdot) = \phi$

and $\Phi(1, \cdot) = \psi$, then $\psi^* = \phi^* : H^*_{dR}(N) \to H^*_{dR}(M)$

Class 13 Exterior derivative and Lie derivative

Recall Ω^k is the space of k-form on M.

i.e. sections $M \to \Lambda^k T^*M$.

$\Omega^0 = C^\infty(M;\mathbb{R})$

Locally $w = \sum_{i_1<i_2<\cdots<i_k} f_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

$$dw = \sum_{i_1<\cdots<i_k} \sum_{m \notin \{i_1,\cdots,i_k\}} \frac{\partial f_{i_1\cdots i_k}}{\partial x^m} dx^m \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$= \sum_{i_1<\cdots<i_k} df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

$d: \Omega^k \to \Omega^{k+1}$ is independent of charts

$d^2 = 0$. $H^i_{dR}(M) = \dfrac{(\ker d: \Omega^i \to \Omega^{i+1})}{(\mathrm{Im}\, d: \Omega^{i-1} \to \Omega^i)}$

Prop 1) If $\psi: M \to N$ is a diffeomorphism. ψ^* induces an iso between $H^*_{dR}(N)$ and $H^*_{dR}(M)$ $\Longrightarrow$ If $H^*_{dR}(N) \neq H^*_{dR}(M)$, then $M \not\cong_{diff} N$

2) If ψ and ϕ are homotopic map to N, i.e.

$\exists\ \Phi: [0,1] \times M \to N$ smooth, s.t. $\Phi(0,\cdot) = \phi$

and $\Phi(1,\cdot) = \psi$, then $\psi^* = \phi^*: H^*_{dR}(N) \to H^*_{dR}(M)$

Pf: 1). Since $\psi^* d = d\psi^*$,

(This is called a (co)chain map)

ψ^* induces a well-defined map

$$\psi^*: H^*_{dR}(N) \longrightarrow H^*_{dR}(M)$$

If ψ is a diffeo, $\exists\, \phi: N \to M$

s.t. $\psi\circ\phi = Id \quad \phi\circ\psi = Id$. Then ψ^*, ϕ^*

induce isomorphism between $H^*_{dR}(N)$ $H^*_{dR}(M)$

2) Note that $T^*_{(t,p)}([0,1]\times M) = T^*_t[0,1] \times T^*_p M$

$$= \mathbb{R} \times T^*_p M$$

a 1-form on $T^*([0,1]\times M)$ can be written as

$$\alpha_t dt + \beta_t \quad \alpha_t \in \Omega^0(M) = C^\infty(M;\mathbb{R})$$
$$\beta_t \in \Omega^1(M)$$

$w \in \Omega^k(N)$ $\Phi^* w \in \Omega^k([0,1]\times M)$ can be written as

$$\Phi^* w = dt \wedge \alpha_t + \beta_t \quad \alpha_t \in \Omega^{k-1}(M) \quad \beta_t \in \Omega^k(M)$$

If $dw = 0$, then $0 = d(\Phi^* w)$

$$= -dt \wedge d^\perp \alpha_t + dt \wedge \frac{\partial}{\partial t}\beta_t + d^\perp \beta_t$$

$d(w_1\wedge w_2) = dw_1 \wedge w_2 + (-1)^k w_1 \wedge dw_2$

where $d^\perp$ is the exterior derivative on $\{t\}\times M \subset [0,1]\times M$

This implies $\frac{\partial}{\partial t}\beta_t = d^\perp \alpha_t$, $d^\perp \beta_t = 0$

$d^\perp \beta_t = 0$ means $\Phi(t,\cdot)^* w$ is closed for any t

$$\psi^* w - \phi^* w = \int_0^1 \frac{\partial}{\partial t}\beta_t = \int_0^1 d^\perp \alpha_t = d\int_0^1 \alpha_t$$

d on M

So $\psi^* w - \phi^* w \in \operatorname{Im} d$ $\quad [\psi^* w] = [\phi^* w]$ $\quad \square$

Cor. (Poincaré Lem) Let $U \subset M$ be a contractible open set, i.e. $\exists$ smooth map $\Phi : [0,1] \times U \to M$

s.t. $\Phi(1,\cdot)$ is the inclusion

$\Phi(0,\cdot)$ maps U to a pt $p \in M$

Then $H^k_{dR}(U) = 0$ for $k \geq 1$

i.e. any closed k-form ω ($d\omega = 0$) with $k \geq 1$

$\exists$ a $(k-1)$-form α s.t. $d\alpha = \omega$

Pf: $[\Phi(1,\cdot)^* \omega] = [\Phi(0,\cdot)^* \omega]$

$$\begin{array}{ccc} [\Phi(1,\cdot)^*\omega] & = & [\Phi(0,\cdot)^*\omega] \\ \| & & \| \\ \omega|_U & & \omega|_p \end{array}$$

$\Lambda^k T^*p = 0$ for $k \geq 1$ so $\omega|_p = 0$ $\square$

Lie derivative on k-forms.

Suppose M is compact and V is a vector field

From vector field thm, we have a smooth map

$\sigma : (-\varepsilon, \varepsilon) \times M \to M$ s.t.

1) $\sigma(0, p) = p$

2) $\sigma_* \frac{\partial}{\partial t} = V$

(We use the compactness to find $\varepsilon > 0$ for any pt $p \in M$

For $\omega \in \Omega^k(M)$, consider $\sigma^*\omega \in \Omega^k((-\varepsilon,\varepsilon)\times M)$

$$\sigma^*\omega = dt\wedge\alpha_t + \beta_t \qquad \alpha_t \in \Omega^{k-1}(M) \quad \beta_t \in \Omega^k(M)$$

Define $\mathcal{L}_V\omega = \frac{\partial}{\partial t}\beta_t\Big|_{t=0}$

Fact (Cartan formula) $\mathcal{L}_V\omega = (\iota_V d + d\iota_V)\omega$

where for a k form α, $\iota_V\alpha$ is a k-1 form defined by

$$\iota_V\alpha(v_1,\dots,v_{k-1}) = \alpha(V, v_1,\dots,v_{k-1})$$

So
$$\underbrace{\mathcal{L}_V\omega}_{k\text{-form}}(v_1,\dots,v_k) = \underbrace{d\omega}_{(k+1)\text{form}}(V, v_1,\dots,v_k)$$
$$+\, d\iota_V\omega(v_1,\dots,v_k)$$

Note that $\alpha_0 = \iota_V\omega$

because $\sigma^*\omega\left(\frac{\partial}{\partial t}, v_1,\dots,v_k\right)\Big|_{t=0}$

$$= \omega\left(\sigma_*\frac{\partial}{\partial t}, \sigma_* v_1,\dots, \sigma_* v_k\right)\Big|_{t=0}$$

$$= \omega(V, \sigma_* v_1,\dots,\sigma_* v_k)$$

$$= dt\wedge\alpha_t\left(\frac{\partial}{\partial t}, v_1,\dots,v_k\right)\Big|_{t=0} + \cancelto{0}{\beta_t\left(\frac{\partial}{\partial t}, v_1,\dots,v_k\right)}\Big|_{t=0}$$

$$= \alpha_0(v_1,\dots,v_k)$$

Recall that we define $\mathcal{L}_V$ as a derivation

$$\begin{array}{ccc} C^\infty(M;\mathbb{R}) & \longrightarrow & C^\infty(M;\mathbb{R}) \\ \shortparallel & & \shortparallel \\ \Omega^0 & & \Omega^0 \end{array}$$

The definition here $\mathcal{L}_V : \Omega^k \longrightarrow \Omega^k$ can be regarded as an extention of derivation on $\Omega^* = \bigoplus_{k=0}^{n} \Omega^k$

It satisfies the Leibniz rule

$$\mathcal{L}_V(\alpha \wedge \beta) = \mathcal{L}_V\alpha \wedge \beta + \alpha \wedge \mathcal{L}_V\beta$$

Ω^* is called the <u>exterior algebra</u> by wedge product $\wedge$

It is an example of <u>superalgebra</u>, which is a vector space with $\mathbb{Z}/2 = \{0,1\}$ grading and $\alpha\beta = (-1)^{\deg\alpha \deg\beta}\beta\alpha$ where $\deg\alpha, \deg\beta \in \{0,1\}$ for homogeneous α, β.

Note that from $dx^i \wedge dx^j = -dx^j \wedge dx^i$, we can show for α k-form, β l-form.

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

$$\Omega^* = \underbrace{\bigoplus_{k \text{ even}} \Omega^k}_{\text{grading } 0} \oplus \underbrace{\bigoplus_{k \text{ odd}} \Omega^k}_{\text{grading } 1}$$

Covariant derivative (Chap 11)

Let $\pi: E \to M$ be a vector bundle $s: M \to E$ a section

we want to find some derivative of s

Ex. If $E = M \times \mathbb{R}^n$, the section $s: M \to E$ is just

$S(p) = (p, S_1(p), \cdots, S_n(p)) \qquad S_i: M \to \mathbb{R}$

We can define $(S_i)_*: T_pM \to T_{S_i(p)}\mathbb{R} \cong \mathbb{R}$

$(S_i)_*$ can be regarded as a section of T^*M.

Hence we can construct $ds(p) := \left(p, (S_1)_*(p), \cdots, (S_n)_*(p)\right)$

as a section of $E \otimes T^*M = T^*M^{\otimes n}$

Then we extend the construction to general vector bundle

Def Let $C^\infty(M;E)$ be the space of sections $M \to E$.

A <u>covariant derivative</u> is a map

$\nabla: C^\infty(M;E) \to C^\infty(M; E \otimes T^*M)$ obeying the

Leibniz rule $\quad \nabla(fs) = s \otimes df + f\nabla s$

for any $f \in C^\infty(M;\mathbb{R})$, $s \in C^\infty(M;E)$

Rem. The extra factor T^*M means, if we specify a vector field v, we can define $\nabla_v: C^\infty(M;E) \to C^\infty(M;E)$

Construction of ∇

Take a locally finite open cover $\mathcal{U}$ of M and a partition of unity $\{\chi_\alpha: U_\alpha \to \mathbb{R}_{\geq 0}\}$ $\quad \varphi_\alpha: E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$

Set $\nabla s = \sum_{\alpha \in \mathcal{U}} \chi_\alpha \varphi_\alpha^{-1}(d\varphi_\alpha(s))$

where $\varphi_\alpha(s) : U_\alpha \to U_\alpha \times \mathbb{R}^n$

$\varphi_\alpha(s)(p) = (p, s_{\alpha 1}(p), \dots, s_{\alpha n}(p))$

and $d\varphi_\alpha(s) : U_\alpha \longrightarrow U_\alpha \times (\mathbb{R}^n \otimes \mathbb{R}^m)$ $m = \dim M$

$n = \dim E|_p$

$d\varphi_\alpha(s)(p) = (p, (s_{\alpha 1})_*(p), \dots (s_{\alpha n})_*(p))$

φ_α^{-1} extends to $U_\alpha \times (\mathbb{R}^n \otimes \mathbb{R}^m) \longrightarrow E \otimes T^*M|_U$

To check $\nabla fs = s \otimes df + f \nabla s$

it suffices to check over U_α

$d\varphi_\alpha(fs)(p) = (p, (fs_{\alpha 1})_*(p) \dots \quad)$

$s_{\alpha 1} : U_\alpha \longrightarrow \mathbb{R}$ $(s_{\alpha 1})_*|_p : T_p U_\alpha \longrightarrow T_{s_{\alpha 1}(p)} \mathbb{R}$

$f : U_\alpha \longrightarrow \mathbb{R}$

$(fs_{\alpha 1})_* = df \otimes s_{\alpha 1} + f \otimes ds_{\alpha 1}$

as section $U_\alpha \longrightarrow T^* U_\alpha$.

Class 14 Covariant derivative (Chap 11)

Def Let $C^\infty(M;E)$ be the space of sections $M \to E$.

A <u>covariant derivative</u> is a map

$$\nabla : C^\infty(M;E) \longrightarrow C^\infty(M; E \otimes T^*M)$$ obeying the

Leibniz rule $\nabla(fs) = s \otimes df + f\nabla s$

for any $f \in C^\infty(M;\mathbb{R})$, $s \in C^\infty(M;E)$

Construction of ∇

Take a locally finite open cover $\mathcal{U}$ of M and a partition of unity $\{\chi_\alpha : U_\alpha \to \mathbb{R}_{\geq 0}\}$ $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$

Set $\nabla s = \sum_{U_\alpha \in \mathcal{U}} \chi_\alpha \varphi_\alpha^{-1}(d\varphi_\alpha(s))$

where $\varphi_\alpha(s) : U_\alpha \to U_\alpha \times \mathbb{R}^n$

$$\varphi_\alpha(s)(p) = (p, s_{\alpha 1}(p), \dots, s_{\alpha n}(p))$$

and $d\varphi_\alpha(s) : U_\alpha \longrightarrow U_\alpha \times (\mathbb{R}^n \otimes \mathbb{R}^m)$ $m = \dim M$, $n = \dim E|_p$

$$d\varphi_\alpha(s)(p) = (p, (s_{\alpha 1})_*(p), \dots, (s_{\alpha n})_*(p))$$

φ_α^{-1} extends to $U_\alpha \times (\mathbb{R}^n \otimes \mathbb{R}^m) \longrightarrow E \otimes T^*M|_U$

Another construction of ∇

Consider E as a subbundle of $M \times \mathbb{R}^N$

and write $s : M \to E$ as a section of $s : M \to M \times \mathbb{R}^N$

($E|_p$ is a linear subspace of $p \times \mathbb{R}^N$)

Let $\nabla s = \Pi_E ds$, where ds is a section of

$M \times \mathbb{R}^N \otimes T^*M$ and Π_E is the orthogonal projection to $E \otimes T^*M$ by the standard inner product in $\mathbb{R}^N$

Since $\Pi_E s = s$, $\nabla fs = \Pi_E \, d(fs)$

$$= \Pi_E(f \otimes ds + df \otimes s)$$

$$= s \otimes df + f \nabla s$$

Rem. The covariant derivative is not unique.

Suppose ∇ and ∇' are two covariant derivatives

$$(\nabla - \nabla')(fs) = f(\nabla - \nabla')s$$

Claim $\nabla - \nabla'$ is a section of $\mathrm{End}(E) \otimes T^*M$
$:= \mathrm{Hom}(E,E) \otimes T^*M$.

Lem. Let E, E' be two vector bundles over M

If α is a map from $C^\infty(M;E)$ to $C^\infty(M;E')$ that is linear over $C^\infty(M;\mathbb{R})$, i.e. $f\alpha = \alpha f$, then α is a section of $\mathrm{Hom}(E,E')$

pf. Locally, fix a basis $\{e_1, \cdots, e_n\}$ for E and a basis $\{e_1', \cdots, e_m'\}$ for E'

Then we can write $\alpha e_j = \sum_i A_{ij} e_i'$

If s is a section of E $s = \sum_j s_j e_j$, $s_j : U \to \mathbb{R}$

$$\alpha s = \alpha\left(\sum_j s_j e_j\right) = \sum_j s_j\, \alpha(e_j) = \sum_{i,j} s_j A_{ij} e_i'$$

Then α is defined by $\{A_{ij}\}$ as a section of $\mathrm{Hom}(E, E')$, change basis of E or E' will change $\{A_{ij}\}$ by matrix multiplication. $\square$

we use α rather than $\mathfrak{a}$ (\mathfrak{a} in latex) just for writing simplicity.

Conversely, given $\alpha \in C^\infty(M; \mathrm{End}(E) \otimes T^*M)$, we can construct $\nabla' = \nabla + \alpha$. This is still a covariant derivative

Rem The space of covariant derivatives is affine over $C^\infty(M; \mathrm{End}(E) \otimes T^*M)$. This means these two spaces are isomorphic, but the isomorphism is canonical only when we pick an element ∇

Then we consider the transition of ∇ in different charts.

Locally. $\phi_U : U \to \mathbb{R}$ $\varphi_U : E|_U \to U \times \mathbb{R}^n$, $s : M \to E$

define the covariant derivative ∇^0 by

$$\nabla^0 \varphi_U s(p) = (p, ds_U(p))$$

$$s_U = \phi_U^{-1} \circ s : \mathbb{R}^n \to \mathbb{R}$$

For a general covariant derivative ∇, we know

$\varphi_U(\nabla s)(p) = (p, ds_U + \alpha_U s_U)$

where α_U is a section of $(End(E) \otimes T^*M)|_U$

For another chart V with $U \cap V \neq \phi$

we have $\varphi_V(\nabla s)(p) = (p, ds_V + \alpha_V s_V)$

$s_V = g_{VU} s_U$ for bundle transition function

$g_{UV}: V \cap U \longrightarrow GL(n, \mathbb{R})$

$ds_V + \alpha_V s_V = d(g_{VU} s_U) + \alpha_V (g_{VU} s_V)$

$= g_{VU} ds_U + dg_{VU} s_U + \alpha_V g_{VU} s_V.$

$= g_{VU}(ds_U + (g_{VU}^{-1} \alpha_V g_{VU} + g_{VU}^{-1} dg_{VU}) s_U))$

Since ∇s is a section of $E \otimes T^*M$.

this should equal to $g_{VU}(ds_U + \alpha_U s_U)$

Thus, $ds_U + (g_{VU}^{-1} \alpha_V g_{VU} + g_{VU}^{-1} dg_{VU}) s_U$

$= ds_U + \alpha_U s_U$

$\Rightarrow \alpha_U = g_{VU}^{-1} \alpha_V g_{VU} + g_{VU}^{-1} d g_{VU}$

$\Rightarrow \alpha_V = g_{UV}^{-1} \alpha_U g_{UV} + g_{UV}^{-1} d g_{UV}$

$= g_{VU} \alpha_U g_{VU}^{-1} + g_{VU} d g_{VU}^{-1}$

Note that the classes on

10/20 (Thu), 10/25 (Tue), and 10/27 (Thu)

will be online due to the travel.

The zoom link is ____________________

It will also be sent by email

The notes will be posted <u>before</u> the classes

Topics

10/20 Class 15 Curvature of a covariant derivative
(Chap 12.4–7)

10/25 Class 16 Metric compatible covariant derivative
(Chap 15.1–2)

10/27 Class 17 Levi-Civita connection
(Chap 15.3–4)

There are many computations. It is good to do it by yourself

Class 15 Curvature of a covariant derivative (Chap 12)

Def Let $C^\infty(M;E)$ be the space of sections $M \to E$.

A <u>covariant derivative</u> is a map

$\nabla : C^\infty(M;E) \longrightarrow C^\infty(M; E\otimes T^*M)$ obeying the

Leibniz rule $\nabla(fs) = s\otimes df + f\nabla s$

for any $f\in C^\infty(M;\mathbb{R})$, $s\in C^\infty(M;E)$

Prop. (last class)

1) Covariant derivative exists

2) ∇, ∇' are two covariant derivatives

iff $\nabla - \nabla' = \alpha$ is a section of $\mathrm{End}(E)\otimes T^*M$

Prop (last week) $\Omega^k = C^\infty(M;\Lambda^k T^*M)$

There exists map $d : \Omega^k \to \Omega^{k+1}$ s.t. $d^2 = 0$

$d(w_1+w_2) = dw_1 + dw_2$

$d(w_1\wedge w_2) = dw_1\wedge w_2 + (-1)^{\deg w_1} w_1 \wedge dw_2$

Today we combine ∇ and d together to define

exterior covariant derivative

$d_\nabla : C^\infty(M; E\otimes\Lambda^k T^*M) \longrightarrow C^\infty(M; E\otimes \Lambda^{k+1}T^*M)$

$k=0$ $\Lambda^0 T^*M = M\times\mathbb{R}$ $d_\nabla s = \nabla s$

$k>0$ If the section of $E\otimes\Lambda^k T^*M$ can be written as $s \otimes \omega$, then define

$$d_\nabla s \otimes \omega = \nabla s \wedge \omega + s \otimes d\omega$$

Then define $d_\nabla \left(\sum_i s_i \otimes \omega_i \right) = \sum_i d_\nabla (s_i \otimes \omega_i)$

In general, $d_\nabla^2 \neq 0$

For $s \in C^\infty(M; E)$, we write $d_\nabla^2 s = F_\nabla s$,

where F_∇ is called the <u>curvature</u>

$$F_\nabla(fs) = d_\nabla^2(fs) = d_\nabla(f d_\nabla s + s \otimes df)$$

$$= \underbrace{df \wedge d_\nabla s + f d_\nabla^2 s + d_\nabla s \wedge df}_{=0} + s \otimes \cancel{d^2 f}^{\,0}$$

$$= f d_\nabla^2 s = f F_\nabla s$$

Thus, F_∇ is a section of $\mathrm{End}(E) \otimes \Lambda^2 T^* M$

by Lem in the last class.

For $s \otimes \omega \in C^\infty(M; E \otimes \Lambda^k T^* M)$

section tensor 1-form

$$d_\nabla^2(s \otimes \omega) = d_\nabla(d_\nabla s \wedge \omega + s \otimes d\omega)$$

$(d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$ for 1-form $\alpha)$

$$= d_\nabla^2 s \wedge \omega - d_\nabla s \wedge d\omega$$

$$+ d_\nabla s \wedge d\omega + s \otimes \cancel{d^2 \omega}^{\,0}$$

$$= F_\nabla s \wedge \omega$$

Local description of F_∇

$\phi_U : U \to \mathbb{R}^m$ $\quad \varphi_U : E|_U \to U \times \mathbb{R}^n$ $\quad \dim M = m$ $\quad \dim E|_p = n$.

$S : M \to E$. $\quad S_U = \phi_U^{-1} \circ S : \mathbb{R}^n \to \mathbb{R}$

$$\nabla^0 \varphi_U S(p) = (p, dS_U(p))$$

$$\varphi_U(\nabla S)(p) = (p, dS_U + \alpha_U S_U)$$

where α_U is a section of $(\mathrm{End}(E) \otimes T^*M)|_U$

$$d_\nabla^2 S = d_\nabla(\nabla S) \qquad \varphi_U(d_\nabla^2 S)(p)$$

also have α_U part because we use ∇ rather than ∇^0.

$$= (p, \cancel{d^2 S_U}^{\,0} + (d\alpha_U) S_U - \alpha_U \wedge dS_U + \alpha_U \wedge (dS_U + \alpha_U S_U))$$

minus sign because α_U is 1-form.

$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg} \omega_1 \wedge d\omega_2$

$$= (p, (d\alpha_U + \alpha_U \wedge \alpha_U) \cdot S_U)$$

Thus, locally $(F_\nabla)_U = d\alpha_U + \alpha_U \wedge \alpha_U$

This is a matrix-value 2-form.

i.e. a section of $(\mathrm{End}(E) \otimes \Lambda^2 T^*M)|_U$

We look more carefully on the notation $\alpha_U \wedge \alpha_U$

because α_U is a matrix-value 1-form.

Write $dx^1, \dots, dx^n$ for basis of sections on $T^*M|_U$

$\alpha_U = \sum_k \alpha_{Uk} dx^k$ $\quad \alpha_{Uk}$ are matrices (depend on p)

$$\alpha_U \wedge \alpha_U = \sum_{i,j} \underbrace{\alpha_{U_i} \alpha_{U_j}}_{\text{matrix multiplication}} dx^i \wedge dx^j$$

$\left(-dx^j \wedge dx^i = dx^i \wedge dx^j\right)$

$$= \sum_{i<j} (\alpha_{U_i} \alpha_{U_j} - \alpha_{U_i} \alpha_{U_j}) dx^i \wedge dx^j$$

write $[\alpha_{U_i}, \alpha_{U_j}]$ for $\alpha_{U_i}\alpha_{U_j} - \alpha_{U_j}\alpha_{U_i}$,
called the <u>commutator</u>

Also, $d\alpha_U = \sum_{i<j} (\partial_i \alpha_{U_j} - \partial_j \alpha_{U_i})\, dx^i \wedge dx^j$

$$(F_\nabla)_U = \sum_{i<j} \underbrace{(\partial_i \alpha_{U_j} - \partial_j \alpha_{U_i} + [\alpha_{U_i}, \alpha_{U_j}])}_{\text{write as } (F_{\nabla ij})_U} dx^i \wedge dx^j$$

Check the result is independent of charts

$(F_\nabla)_V = d\alpha_V + \alpha_V \wedge \alpha_V$ From the last class

$$\alpha_V = g_{UV}^{-1} \alpha_U g_{UV} + g_{UV}^{-1} d g_{UV}$$
$$= g_{VU} \alpha_U g_{VU}^{-1} + g_{VU} d g_{VU}^{-1}$$

$$\Longrightarrow (F_\nabla)_U = g_{UV}^{-1} (F_\nabla)_V g_{UV}$$

$$d\alpha_V = -g_{UV}^{-1} dg_{UV} \wedge g_{UV}^{-1} dg_{UV} - g_{UV}^{-1} dg_{UV} \wedge g_{UV}^{-1} \alpha_U g_{UV}$$
$$+ g_{UV}^{-1} d\alpha_U g_{UV} - g_{UV}^{-1} \alpha_U g_{UV} \wedge g_{UV}^{-1} dg_{UV}$$

$$\alpha_V \wedge \alpha_V = g_{UV}^{-1} dg_{UV} \wedge g_{UV}^{-1} dg_{UV} + g_{UV}^{-1} \alpha_U g_{UV} \wedge g_{UV}^{-1} dg_{UV}$$
$$+ g_{UV}^{-1} dg_{UV} \wedge g_{UV}^{-1} \alpha_U g_{UV} + g_{UV}^{-1} \alpha_U \wedge \alpha_U g_{UV}$$

Meaning of F_∇ (for simplicity on a chart)

write $dx^1, \dots, dx^n$ for basis of sections of $T^*M|_U$.

∇S_U can be written as $\sum_k \nabla_k S_U dx^k$

$\nabla_k S_U$ is the covariant derivative of S_U along $\frac{\partial}{\partial x^k}$.

For the usual partial derivative, we have $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$

but for covariant derivative.

$$\nabla_i \nabla_j S_u - \nabla_j \nabla_i S_u = (F_{\nabla ij})_u S_u$$

where $(F_\nabla)_U = \sum_{i<j} (F_{\nabla ij})_u dx^i \wedge dx^j$

$$= \frac{1}{2} \sum_{i,j} (F_{\nabla ij})_u dx^i \wedge dx^j$$

Explicitly. $\nabla_i S_u = \partial_i S_u + \alpha_i S_u$

$$\nabla_i \nabla_j S_u = \partial_i \partial_j S_u + \partial_i(\alpha_j S_u) + \alpha_i \partial_j S_u + \alpha_i \alpha_j S_u$$

$$= \partial_i \partial_j S_u + (\partial_i \alpha_j) S_u + \alpha_j \partial_i S_u + \alpha_i \partial_j S_u + \alpha_i \alpha_j S_u$$

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) S_u = (\partial_i \alpha_j - \partial_j \alpha_i + \alpha_i \alpha_j - \alpha_j \alpha_i) S_u$$