Class 16 Metric compatible covariant derivative (Chap 15)

In weeks 7–8, we study ① the exterior derivative

$$d : \Omega^k \to \Omega^{k+1}, \text{ where } \Omega^k = C^\infty(M; \Lambda^k T^*M) = \{\text{sections in } \Lambda^k T^*M\}.$$

② the covariant derivative $\nabla : C^\infty(M; E) \to C^\infty(M; E \otimes T^*M)$

③ the exterior covariant derivative

$$d_\nabla : C^\infty(M; E \otimes \Lambda^k T^*M) \to C^\infty(M; E \otimes \Lambda^{k+1} T^*M)$$

and

④ the curvature F_∇ of d_∇ as a section of $\mathrm{End}(E) \otimes \Lambda^2 T^*M$.

In weeks 5-6, we study metric g on E, which is a section of $E^* \otimes E^*$ s.t. at each fiber, it is symmetric and positive definite.

Using partition of unity, we show the metric and the covariant derivative exist.

(the space of the metric is convex hence contractible and the space of covariant derivative is affine over $C^\infty(M; \mathrm{End}(E) \otimes T^*M)$)

In this week, we study metric compatible covariant derivative, i.e. for any sections u, v of E,

$$dg(u,v) = g(\nabla u, v) + g(u, \nabla v)$$

(some version of Leibniz rule)

Given $u, v \in C^\infty(M;E)$ $g(u,v)$ is a map $M \to \mathbb{R}$.

$dg(u,v)$ is a 1-form (we apply exterior derivative)

$\nabla u \in C^\infty(M; E \otimes T^*M)$, $\nabla u \otimes v \in C^\infty(M; E \otimes T^*M \otimes E)$

$g(\nabla u, v)$ means we apply g to two E components.

and hence also obtain a section in T^*M. i.e a 1-form.

Following the notation in previous class, we write everything on chart U and omit subscript U for s_U, α_U

Let $e_1, \cdots, e_n$ be a basis of orthonormal sections of E, where orthonormal means $g(e_i(p), e_j(p)) = \delta_{ij}$

It exists by Gram-Schmit procedure.

Then $s = \sum_{k=1}^{n} s_k e_k$ $\quad s_k : U \to \mathbb{R}$

(In weeks 5-6, we write $s_1, \cdots, s_n$ for a basis of sections and $v_1, \cdots, v_k$ as functions $U \to \mathbb{R}$

Now the notation follows from those in weeks 7-8)

$$\nabla s = \sum_{i,j} (ds_i + \alpha_i^j s_j) e_i$$

α_i^j is component of α in previous class. α_i^j is 1-form.

especially $\nabla e_i = \sum \alpha_i^j e_j$
$\qquad := \alpha_i^j e_j$

(write α_i^j rather than α_{ij} because we want to omit $\sum$ later when an index appears twice, we take sum over this index

From $dg(u,v) = g(\nabla u, v) + g(u, \nabla v)$

we have $0 = dg(e_i, e_j) = g(\nabla e_i, e_j) + g(e_i, \nabla e_j)$

$$= g(\alpha_i^k e_k, e_j) + g(e_i, \alpha_j^l e_l)$$

$$= \alpha_i^j + \alpha_j^i$$

i.e. the matrix value 1-form $\alpha = \{\alpha_j^i\}_{i,j}$

satisfies $\alpha^T + \alpha = 0$

Given ∇ on E, we can define ∇ on E^* as follows.

Let s be a section of E and s^* be a section of E^*, we have a pairing $\langle s, s^* \rangle : M \to \mathbb{R}$

$d\langle s, s^* \rangle$ is a 1-form

Then we define ∇s^* by the following equation

$$d\langle s, s^* \rangle = \langle \nabla s, s^* \rangle + \langle s, \nabla s^* \rangle \quad (1)$$

(this equation holds for any s and fixed s^*)

We can also extend ∇ to $E^* \otimes E^*$ by

$$\nabla(s_1^* \otimes s_2^*) = \nabla s_1^* \otimes s_2^* + s_1^* \otimes \nabla s_2^* \quad (2)$$

Then we can define $\nabla g \in C^\infty(M; E^* \otimes E^* \otimes T^*M)$

Lem ∇ is metric compatible iff $\nabla g = 0$

Pf: To define ∇g, we can first define ∇ on $E\otimes E$ by

$$\nabla(s_1\otimes s_2) = \nabla s_1\otimes s_2 + s_1\otimes \nabla s_2 \quad (3)$$

and then define ∇ on $(E\otimes E)^* = E^*\otimes E^*$ by defining $\nabla(s_1^*\otimes s_2^*)$

$$d\langle s_1\otimes s_2, s_1^*\otimes s_2^*\rangle = \langle \nabla(s_1\otimes s_2), s_1^*\otimes s_2^*\rangle + \langle s_1\otimes s_2, \nabla s_1^*\otimes s_2^*\rangle \quad (4)$$

for any $s_1\otimes s_2$ and then extend linearly over $\mathbb{R}$

Note that when multiply with a map $f: M\to\mathbb{R}$, we define

$$\nabla f(s_1^*\otimes s_2^*) = (s_1^*\otimes s_2^*)\otimes f + f\nabla(s_1^*\otimes s_2^*)$$
$$= fs_1^*\otimes fs_2^* + f\nabla(s_1^*\otimes s_2^*)$$

We need to check these two definitions are the same

i.e. (1–3) implies (4) and $(1,3,4)\Rightarrow(2)$

Use the second definition, we have

$$dg(u,v) = d\langle u\otimes v, g\rangle$$

$$= \langle \nabla(u\otimes v), g\rangle + \langle u\otimes v, \nabla g\rangle$$

$$= \langle \nabla u\otimes v + u\otimes\nabla v, g\rangle + \langle u\otimes v, \nabla g\rangle$$

$$= g(\nabla u, v) + g(u,\nabla v) + \langle u\otimes v, \nabla g\rangle \qquad \square$$

Then we consider the case $E = TM$. $E^* = T^*M$.

$$\nabla: C^\infty(M; T^*M) \longrightarrow C^\infty(M; T^*M\otimes T^*M)$$

Define A to be the antisymmetrization

from $C^\infty(M; T^*M \otimes T^*M) \longrightarrow C^\infty(M; \Lambda^2 T^*M)$

by $\mathbb{A}(\omega_1 \otimes \omega_2) = \frac{1}{2}(\omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1)$

and extending linearly.

For a 1-form $\omega \in C^\infty(M; T^*M)$

we can define $d\omega \in C^\infty(M; \Lambda^2 T^*M)$

and $\mathbb{A}(\nabla \omega) = \mathbb{A}(d_\nabla \omega) \quad \in C^\infty(M; \Lambda^2 T^*M)$

(Rem. previously we define d_∇ to be a map

$$C^\infty(M; E \otimes \Lambda^k T^*M) \to C^\infty(M; E \otimes \Lambda^{k+1} T^*M)$$

In the special case $E = T^*M$ we compose it with $\mathbb{A}$

Cliff still uses the same notation, but we write as $\mathbb{A}\nabla$

Def The torsion tensor $T_\nabla : C^\infty(M; T^*M) \to C^\infty(M; \Lambda^2 T^*M)$

is defined by $T_\nabla \omega = \mathbb{A}(\nabla \omega) - d\omega$

∇ (on T^*M) is called torsion free if $T_\nabla = 0$, i.e.

$$\mathbb{A}(\nabla \omega) = d\omega$$

Thm. Given a Riemannian metric g on TM, there is a unique metric compatible, torsion free covariant derivative ∇_{LC} called the Levi-Civita connection.

Class 17 Levi-Civita connection

Given a Riemannian metric g on TM, a covariant derivative ∇ on TM is compatible with g if

either 1) $dg(u,v) = g(\nabla u, v) + g(u, \nabla v)$

for any vector fields u, v

or 2) $\nabla g = 0$ for the covariant derivative on $T^*M \otimes T^*M$

induced by $\nabla(s_1^* \otimes s_2^*) = \nabla s_1^* \otimes s_2^* + s_1^* \otimes \nabla s_2^*$

$$d\langle s, s^* \rangle = \langle \nabla s, s^* \rangle + \langle s, \nabla s^* \rangle$$

∇ is called torsion-free if the torsion tensor

$T_\nabla = \mathcal{A}\nabla - d : C^\infty(M; T^*M) \longrightarrow C^\infty(M; \Lambda^2 T^*M)$ vanishes,

where $\mathcal{A}(w_1 \otimes w_2) = \frac{1}{2}(w_1 \otimes w_2 - w_2 \otimes w_1)$

Thm. Given a Riemannian metric g on TM, there is a unique metric compatible, torsion free covariant derivative ∇_{LC} called the <u>Levi-Civita connection</u>.

Here we don't distinguish ∇ on TM or T^*M because then can induce each other

Pf. 1 (in the next pages, using orthonormal basis)

In the last class, we compute in a locally orthonormal basis $e_1, \cdots, e_n$ Suppose

$$\nabla e_i = \sum_j e_j \otimes \alpha_i^j = \sum_j \alpha_i^j e_j = \alpha_i^j e_j \text{ for short}$$

α_i^j 1-forms, ∇ is compatible with g iff $\alpha_i^j = -\alpha_j^i$ (1)

consider the dual basis $e^i = e_i^*$

$\langle e_i, e^j \rangle = \delta_{ij} : M \to \mathbb{R}$ suppose

$$\nabla e^i = e^j \otimes \beta_j^i = e^j \otimes \beta_{jk}^i e^k = \beta_{jk}^i e^j \otimes e^k$$

β_j^i 1-form, $\beta_{jk}^i : M \to \mathbb{R}$

$$0 = d\langle e_i, e^j \rangle = \langle \nabla e_i, e^j \rangle + \langle e_i, \nabla e^j \rangle$$

$$= \langle \alpha_i^k e_k, e^j \rangle + \langle e_i, e^l \otimes \beta_l^j \rangle$$

$$= \alpha_i^j + \beta_i^j$$

$\Rightarrow \nabla$ is compatible with g iff $\beta_i^j = -\beta_j^i$ (1')

$$A \nabla e^i = A(\beta_{jk}^i e^j \otimes e^k)$$

$$= \beta_{jk}^i \left(\frac{1}{2}(e^j \otimes e^k - e^k \otimes e^j) \right)$$

$$= \frac{1}{2}(\beta_{jk}^i - \beta_{kj}^i) e^j \otimes e^k$$

Suppose $de^i = \sum_{j<k} \gamma^i_{jk} e^j \wedge e^k$

$= \sum_{j<k} \gamma^i_{jk} \frac{1}{2}(e^j \otimes e^k - e^k \otimes e^j) = \sum_{j,k} \frac{1}{2} \gamma^i_{jk} e^j \otimes e^k$

where we set $\gamma^i_{jk} = -\gamma^i_{kj}$ for $j>k$, $\gamma^i_{jj} = 0$

Then $T_\nabla e^i = A\nabla e^i - de^i = \frac{1}{2}(\beta^i_{jk} - \beta^i_{kj} - \gamma^i_{jk}) e^j \otimes e^k$

∇ is torsion free iff $\beta^i_{jk} - \beta^i_{kj} - \gamma^i_{jk} = 0$ (2)

Change indices, we have $\beta^j_{ki} - \beta^j_{ik} - \gamma^j_{ki} = 0$ (2')

$\beta^k_{ij} - \beta^k_{ji} - \gamma^k_{ij} = 0$ (2'')

Also from (1'), we have $\beta^j_{ik} = -\beta^i_{jk}$

(2)+(2')−(2'')

$\Longrightarrow$ $$\beta^i_{jk} = \frac{1}{2}(\gamma^i_{jk} + \gamma^j_{ki} - \gamma^k_{ij})$$

If there is another solution $(\beta')^i_{jk}$, suppose

$\eta^i_{jk} = \beta^i_{jk} - (\beta')^i_{jk}$, then (2) $\Rightarrow$ $\eta^i_{jk} = \eta^i_{kj}$

We have (1') $\Rightarrow$ $\eta^j_{ik} = -\eta^i_{jk}$

$$\eta^j_{ik} \overset{(1')}{=} -\eta^i_{jk} \overset{(2)}{=} -\eta^i_{kj} \overset{(1')}{=} \eta^k_{ij} \overset{(2)}{=} \eta^k_{ji} \overset{(1')}{=} -\eta^j_{ki} \overset{(2)}{=} -\eta^j_{ik}$$

$\Rightarrow$ $\eta^j_{ik} = 0$, so the solution is unique.

In the next two pages

Pf 2 (Using coordinate basis $\frac{\partial}{\partial x^i}$ and dx^i,)

Note that we don't have $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \delta_{ij}$ in this case

But $d(dx^i) = 0$, so we don't have γ^i_{jk} terms.

Suppose $\nabla \frac{\partial}{\partial x^i} = \Gamma^j_{ik} \frac{\partial}{\partial x^j} \otimes dx^k$

(We use Γ^j_{ik} because we will show this is indeed the Christoffel symbol

$$\Gamma^j_{ik} = \frac{1}{2} g^{jl} (\partial_i g_{lk} + \partial_k g_{ik} - \partial_l g_{ik})$$)

From the pairing $\nabla \langle \frac{\partial}{\partial x^i}, dx^j \rangle$, we have

$$\nabla dx^i = -\Gamma^i_{jk} dx^j \otimes dx^k$$

$$T_\nabla = 0 \Rightarrow \Gamma^i_{jk} = \Gamma^i_{kj}$$

Then we need to show ∇ is compatible with g

Suppose $g = g_{ij} dx^i \otimes dx^j$ $\quad g_{ij} = g_{ji} : U \to \mathbb{R}$

$\{g^{ij}\} = (g_{ij})^{-1}$ $\quad 0 = \nabla g = \nabla (g_{ij} dx^i \otimes dx^j)$

$$= (\nabla g_{ij}) dx^i \otimes dx^j + g_{ij} (\nabla dx^i) \otimes dx^j + g_{ij} dx^i \otimes (\nabla dx^j)$$

For simplicity, we can consider $\nabla_\cdot g = \langle \frac{\partial}{\partial x}, \nabla g \rangle$

Note that $\nabla_l g_{ij} = \frac{\partial g_{ij}}{\partial x^l} = \partial_l g_{ij}$

$$\nabla_l dx^i = -\Gamma^i_{lk} dx^k$$

Hence we have

$$0=\partial_l g_{ij}\, dx^i\otimes dx^j - g_{ij}\left(\Gamma^i_{lp}\, dx^p\otimes dx^j + \Gamma^j_{lq}\, dx^i\otimes dx^q\right)$$

We can change indices and solve Γ^i_{jk} as in the first proof, and also prove the uniqueness similarly.

Here for simplicity, I just show the Christoffel symbol satisfies the equation.

Recall $\Gamma^j_{ik} = \frac{1}{2} g^{jl}\left(\partial_i g_{lk} + \partial_k g_{il} - \partial_l g_{ik}\right)$

$$g_{ij}\Gamma^i_{lp} = \frac{1}{2} g_{ij}\, g^{ir}\left(\partial_l g_{rp} + \partial_p g_{lr} - \partial_r g_{lp}\right)$$

$$= \frac{1}{2}\, \partial_l g_{jp} + \partial_p g_{lj} - \partial_j g_{lp}$$

because $g_{ij}\, g^{ir} = \delta_{jr}$

Then $g_{ij}\left(\Gamma^i_{lp}\, dx^p\otimes dx^j + \Gamma^j_{lq}\, dx^i\otimes dx^q\right)$

$$= \frac{1}{2}\left(\partial_l g_{jp} + \partial_p g_{lj} - \partial_j g_{lp}\right) dx^p\otimes dx^j$$

$g_{ij}=g_{ji}$

$$+ \frac{1}{2}\left(\partial_l g_{iq} + \partial_q g_{li} - \partial_i g_{lq}\right) dx^i\otimes dx^q$$

$$\overset{\substack{p\mapsto i\\ q\mapsto j}}{=} \frac{1}{2}\left(\partial_l g_{ji} + \cancel{\partial_i g_{lj}} - \cancel{\partial_j g_{li}} + \partial_l g_{ij} + \cancel{\partial_j g_{li}}\right.$$

$$\left. - \cancel{\partial_i g_{lj}}\right) dx^i\otimes dx^j$$

$$= \partial_l g_{ij}\, dx^i\otimes dx^j$$

Class 18 Covariantly constant section and curvature

Last time, we show there is a unique covariant derivative on TM and T^*M that is compatible with a given metric g and torsion free, which is called Levi-Civita connection ∇_{LC}, we proved the existence and uniqueness using two different bases.

① orthonormal basis $e_1, \cdots, e_n$, $e^i = e_i^*$
metric compatible condition is easier, but
de^i may not be zero

② coordinate basis $\frac{\partial}{\partial x^i}$, dx^i.
$d(dx^i) = 0$, so the torsion free condition is easier,
but $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ may not be δ_{ij}

In the second proof, we notice that

$$\nabla \frac{\partial}{\partial x^i} = \Gamma^j_{ik} \frac{\partial}{\partial x^j} \otimes dx^k, \quad \nabla dx^i = -\Gamma^i_{jk} dx^j \otimes dx^k$$

where $\Gamma^j_{ik} = \frac{1}{2} g^{jl} (\partial_i g_{lk} + \partial_k g_{il} - \partial_l g_{ik})$

is the Christoffel symbol. Recall it appears in the geodesic equation (weeks 5-6)

$$\frac{d^2\gamma^i}{dt^2} + \Gamma^i_{jk} \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0 \quad \text{or} \quad \ddot{\gamma}^i + \Gamma^i_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0$$

What is the relation between these two?

First, we give a definition for general v.b.

Def Let $\gamma : [0,1] \longrightarrow M$ be a smooth path

A section of $E|_\gamma = \pi^{-1}(\gamma)$ is <u>parallel</u> if

$\nabla_{\dot\gamma} S := \langle \gamma^* \frac{\partial}{\partial t}, \nabla S \rangle$ vanishes, i.e.

the covariant derivative at tangent direction of γ vanishes. For a basis e_i of E

we write $\nabla S = (ds_i + \alpha^{ij} s_j) e_i$

$$\gamma^* \frac{\partial}{\partial t} = \frac{d\gamma^k}{dt} \frac{\partial}{\partial x^k} \quad \alpha^{ij} = \alpha^{ij}_k \, dx^k \quad \alpha^{ij}_k : U \to \mathbb{R}$$

Then $$\nabla_{\dot\gamma} S = \left(\frac{d s_i \circ \gamma}{dt} + \alpha^{ij}_k \frac{d\gamma^k}{dt} s_j \circ \gamma \right) e_i \quad (**)$$

This is a section of $E|_U$

$\nabla_{\dot\gamma} S = 0$ iff $\langle \nabla_{\dot\gamma} S, e^i \rangle = 0$ for any i

In particular if $E = TM$ $e_i = \frac{\partial}{\partial x^i}$ $S = \gamma_* \frac{\partial}{\partial t}$

$\nabla = \nabla_{LC}$ Then $s_i \circ \gamma = \frac{d\gamma^i}{dt}$

$$\nabla \frac{\partial}{\partial x^l} = \Big(\cancelto{0}{d\left(\frac{\partial}{\partial x^l}\right)_i} + \alpha^{ij} \left(\frac{\partial}{\partial x^l}\right)_j \Big) \frac{\partial}{\partial x^i} = \alpha^{il} \frac{\partial}{\partial x^i}$$

$$\left(\left(\frac{\partial}{\partial x^l}\right)_i = \delta_{il} \right) \Longrightarrow \alpha^{il}_k = \Gamma^l_{ik}$$

$$\nabla_{\dot\gamma} \dot\gamma = 0 \iff \frac{d^2\gamma^i}{dt^2} + \Gamma^i_{jk} \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0$$

i.e. For a geodesic γ, $\gamma_* \frac{\partial}{\partial t}$ is a parallel section with respect to ∇_{LC}

Recall in weeks 5-6, we use vector field thm to show that given any point $p = \gamma(0)$ $V_0 = TM|_{\gamma(0)}$ we can always find a unique solution of γ satisfying the geodesic equation.

Similarly, for a general vector bundle, given a path γ and and $v_0 \in E|_{\gamma(0)}$, we can find a unique parallel section s on $E|_\gamma$ (Solving $(**)$)

Let $v_1 = s(\gamma(1))$, it is called the parallel transportation of v_0 along γ

Moreover, if $v_0^1, v_0^2, \cdots, v_0^n$ a basis of $E|_{\gamma(0)}$, we obtain a basis $v_1^1, \cdots, v_1^n$ of $E|_{\gamma(1)}$ after parallel transportation.

i.e. ∇ and γ induces an isomorphism between $E|_{\gamma(0)}$ and $E|_{\gamma(1)}$

Question. does the isomorphism depend on γ for given $\gamma(0) = p$, $\gamma(1) = q$?

Answer: This is true when the curvature $F_\nabla = 0$.
We will come back to this topic after introducing principal bundle.

Then let's discuss curvature for metric compatible covariant derivative

Let $e_1, \dots, e_n$ be an orthonormal basis of E.

$$\nabla e_i = \alpha_i^j e_j \quad \alpha_i^j = \alpha_{ik}^j dx^k \quad \alpha = \{\alpha_i^j\}$$

$\alpha_i^j = -\alpha_j^i$ if ∇ is compatible with g

Recall $F_\nabla = d\alpha + \alpha \wedge \alpha \in C^\infty(M; \mathrm{End}(E) \otimes \Lambda^2 T^*M)$

$$F_\nabla e_i = e_j \otimes d\alpha_i^j + e_k \otimes \alpha_i^l \wedge \alpha_l^k$$

$$= e_j \otimes (d\alpha_i^j + \alpha_i^l \wedge \alpha_l^j)$$

$$(F_\nabla)_i^j = d\alpha_i^j + \alpha_i^l \wedge \alpha_l^j \qquad \alpha_j^l = -\alpha_l^j$$

$$(F_\nabla)_j^i = d\alpha_j^i + \alpha_j^l \wedge \alpha_l^i = -d\alpha_i^j + (-\alpha_l^j) \wedge (-\alpha_i^l)$$

$$= -d\alpha_i^j - \alpha_i^l \wedge \alpha_l^j = -(F_\nabla)_i^j$$

For the dual basis e^i, we have $\nabla e^i = -\alpha_j^i e^j$

$$F_\nabla e^i = e^j \otimes (-d\alpha_j^i + \alpha_l^i \wedge \alpha_j^l)$$

$$(F_\nabla)_j^i = -d\alpha_j^i + \alpha_l^i \wedge \alpha_j^l = d\alpha_i^j + \alpha_i^l \wedge \alpha_l^j$$

This is compatible with previous notation.

For $\nabla = \nabla_{LC}$, write $(F_\nabla)^i_j = \frac{1}{2} R^i_{jkl}\, e^k \wedge e^l$

$R^i_{jkl} : U \to \mathbb{R}$

$R^j_{ikl} = -R^i_{jkl} \quad R^i_{jlk} = -R^i_{jkl}$

Since ∇_{LC} is torsion free $d_\nabla \omega = d\omega$ for 1-form ω

where d_∇ denote $\mathcal{A}(\nabla\omega)$ and $\mathcal{A}$ is the antisymmetrization

We can generalize $\mathcal{A}$ for k-tensor s.t.

$\mathcal{A}(dx^1 \otimes \cdots \otimes dx^n) = dx^1 \wedge \cdots \wedge dx^n$

Define ∇ on $\Lambda^k T^*M$ by $\nabla(\omega_1 \wedge \omega_2) \doteq \nabla\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge \nabla\omega_2$

$\Rightarrow \mathcal{A}\nabla(\omega_1 \wedge \omega_2) = (\mathcal{A}\nabla\omega_1) \wedge \omega_2 + (-1)^{\deg\omega_1} \omega_1 \wedge \mathcal{A}\nabla\omega_2$

Note that $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg\omega_1} \omega_1 \wedge d_\nabla \omega_2$

So $\mathcal{A}\nabla\omega = d\omega$ for any form

We have $F_\nabla e^i = d_\nabla^2 e^i = e^j \otimes (F_\nabla)^i_j$

From $\mathcal{A}(F_\nabla e^i) = \mathcal{A} d_\nabla^2 e^i = (\mathcal{A}\nabla)^2 e^i = d^2 e^i = 0$

We have $\mathcal{A}(e^j \otimes F_{\nabla j}^{\;i}) = e^j \wedge (F_\nabla)^i_j = 0$

This means $R^i_{jkl}\, e^j \wedge e^k \wedge e^l = 0$

So $R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$ (1)

Change indices we have

$R^j_{kli} + R^j_{lik} + R^j_{ikl} = 0$ (2) $\quad R^k_{ijl} + R^k_{jli} + R^k_{lij} = 0$ (3)

$R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0$ (4) $\quad$ Since $R^i_{klj} = -R^k_{ilj} = R^k_{ijl}$

(1) − (2) − (3) + (4) $\Longrightarrow R^i_{jkl} = R^k_{lij}$

From the definition of R^i_{jkl},

we have $F_\nabla = \frac{1}{2} R^i_{jkl}\, e_i \otimes e^j \otimes (e^k \wedge e^l)$

Since e_i are orthonormal basis, we write

$R_{ijkl} = R^i_{jkl}$ (In $\frac{\partial}{\partial x^i}$ basis, $R_{ijkl} = g_{im} R^m_{jkl}$)

Define the Riemannian tensor

$$\mathrm{Riem} = \frac{1}{2} R_{ijkl}\, e^i \otimes e^j \otimes (e^k \wedge e^l)$$

$$= \frac{1}{4} R_{ijkl} (e^i \wedge e^j) \otimes (e^k \wedge e^l)$$

$$\nabla \mathrm{Riem} = \frac{1}{4} \nabla_m R_{ijkl} (e^i \wedge e^j) \otimes (e^k \wedge e^l) \otimes e^m$$

Since $\nabla e^i = \alpha^i_j e^j$, we have

$$\nabla_m R_{ijkl} = \partial_m R_{ijkl} + \alpha^{in} R_{njkl} + \alpha^{jn} R_{inkl} + \alpha^{kn} R_{ijnl} + \alpha^{ln} R_{ijkn}$$

Define the Ricci tensor by $\mathrm{Ric} = \mathrm{Ric}_{ik}\, e^i \otimes e^k$

$$\mathrm{Ric}_{ik} = \sum_j R_{ijkj}$$

Define the scalar curvature $R : M \to \mathbb{R}$ by trace of Ric:

$$R = \sum_i \mathrm{Ric}_{ii} = \sum R_{ijij}$$

$\left(\mathrm{Ric}_{ik} - \frac{1}{2} R\, \delta_{ik}\right) e^i \otimes e^k$ is called the Einstein tensor (However, the Einstein equation in physics doesn't use Riemannian metric i.e. not positive definite)

Class 19. Bianchi identity and Chern class. (Ch 14)

Last time, we compute the curvature for ∇_{LC}

Suppose $e^1, \dots, e^n$ are orthonormal basis of T^*M (locally)

Write $F_\nabla e^i = d_\nabla^2 e^i = e^j \otimes (F_\nabla)^i_j$

$(F_\nabla)^i_j = \frac{1}{2} R^i_{jkl} e^k \wedge e^l \quad R^i_{jkl} : U \to \mathbb{R}$

We proved the following identities

Prop. 1) $R^j_{ikl} = -R^i_{jkl}$ (from metric compatible condition)

2) $R^i_{jlk} = -R^i_{jkl}$ (from definition)

3) $R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$ (from torsion free condition)

4) $R^i_{jkl} = R^k_{lij}$ (from 1)–3))

Today we first prove 5) $\nabla_m R^i_{jkl} + \nabla_k R^i_{jlm} + \nabla_l R^i_{jmk} = 0$

This is equivalent to $d_\nabla F_\nabla = 0$ (we omit the pf of equivalence)

Recall $F_\nabla = d\alpha + \alpha \wedge \alpha$

This means $F_\nabla s = d_\nabla^2 s = (p, (d\alpha_U + \alpha_U \wedge \alpha_U) \cdot s_U)$

$d_\nabla(d_\nabla^2 s) = (p, d((d\alpha_u + \alpha_u \wedge \alpha_u) s_u) + \alpha_u \wedge ((d\alpha_u + \alpha_u \wedge \alpha_u) s_u)$

$\hookrightarrow d(d\alpha_u + \alpha_u \wedge \alpha_u) s_u + (d\alpha_u + \alpha_u \wedge \alpha_u) d s_u$

$+ \alpha_u \wedge (d\alpha_u + \alpha_u \wedge \alpha_u) s_u \quad$ Note $d_\nabla s_u = (d + \alpha_u) s_u$

$$= \Big(d(d\alpha_u + \alpha_u \wedge \alpha_u) + \alpha_u \wedge (d\alpha_u + \alpha_u \wedge \alpha_u)$$

$$- \alpha_u \wedge (d\alpha_u + \alpha_u \wedge \alpha_u) \Big) s_u + (d\alpha_u + \alpha_u \wedge \alpha_u) \wedge d_\nabla s_u$$

$$= \Big(d^2 \alpha_u + d(\alpha_u \wedge \alpha_u) + \alpha_u \wedge d\alpha_u + \alpha_u \wedge \alpha_u \wedge \alpha_u$$

$$- \alpha_u \wedge d\alpha_u - \alpha_u \wedge \alpha_u \wedge \alpha_u \Big) s_u$$

$$+ (d\alpha_u + \alpha_u \wedge \alpha_u) \wedge d_\nabla s_u$$

Then we have $d_\nabla (F_\nabla s) = F_\nabla \wedge d_\nabla s_u$

Note $d_\nabla (F_\nabla s) = d_\nabla F_\nabla + (-1)^{\deg F_\nabla} F_\nabla \wedge d_\nabla s_u$

So $d_\nabla F_\nabla = 0$ (Note $d_\nabla (F_\nabla s) \neq d_\nabla (F_\nabla s)$)

Chern classes.

For a complex vector bundle E with $E|_p \cong \mathbb{C}^n$

we can also define covariant derivative ∇

and the curvature F_∇

Define the k-th Chern class to be

$$C_k(E) = \frac{1}{(2\pi\sqrt{-1})^k} \operatorname{tr}(\underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{k \text{ times}})$$

We define the trace. First, choose a basis

$e_1, \dots, e_n$ for E, and dual basis $e^1, \dots, e^n$

For any section S of $\overset{End(E)}{=} E \otimes E^*$, we can

write $S = S_i^j e_i \otimes e^j$, define $\operatorname{tr}(S) = S_i^i$

It is independent of basis.

If $\omega \in C^\infty(End(E) \otimes \Lambda^k T^*M)$,

we can check $d(\operatorname{tr}\omega) = \operatorname{tr}(d_\nabla \omega)$

If $\omega_i \in C^\infty(End(E) \otimes \Lambda^{k_i} T^*M)$

define $\omega_1 \wedge \omega_2 \in C^\infty(End(E) \otimes \Lambda^{k_1+k_2} T^*M)$

by $(\omega_1 \wedge \omega_2) e_1 = \omega_1 \wedge (\omega_2 e)$

Then $\operatorname{tr}(\omega_1 \wedge \omega_2) \in C^\infty(\Lambda^{k_1+k_2} T^*M)$

Note $F_\nabla \in C^\infty(\operatorname{End}(E) \otimes \Lambda^2 T^*M)$

$$\operatorname{tr}(\underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{k}) \in C^\infty(\Lambda^{2k} T^*M)$$

$$d(\operatorname{tr}(\underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{k})) = k \operatorname{tr}(d_\nabla F_\nabla \wedge \cdots \wedge F_\nabla)$$

$$= 0 \quad \text{(Bianchi identity)}$$

Thm. the de Rham cohomology class

$[\operatorname{tr}(\underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{k})]$ doesn't depend on

the choice of ∇, i.e. c_k only depends on E

pf. Recall any two covariant derivative ∇, ∇'

are differed by $\alpha \in C^\infty(\operatorname{End}(E) \otimes T^*M)$

Consider $\nabla^t = \nabla + t\alpha$

$$F_{\nabla^t} = F_\nabla + t\, d_\nabla \alpha + t^2 \alpha \wedge \alpha$$

$$d_{\nabla^t} \alpha = d_\nabla \alpha + (t\alpha) \wedge \alpha + \alpha \wedge (t\alpha)$$

$$\frac{\partial}{\partial t} F_{\nabla^t} = d_{\nabla^t} \alpha$$

$$\frac{\partial}{\partial t} tr(F_{\nabla^t} \wedge \cdots \wedge F_{\nabla^t})$$

$$= k\, tr(d_{\nabla^t}\alpha \wedge F_{\nabla^t} \cdots \wedge F_{\nabla^t})$$

$$= k\, d\, tr(\alpha \wedge F_{\nabla^t} \wedge \cdots \wedge F_{\nabla^t}) \quad \left(\text{Since } d_{\nabla^t} F_{\nabla^t} = 0\right)$$

Hence $C_k(F_{\nabla'}) - C_k(F_\nabla)$

$$= d \int_0^1 \frac{k}{(2\pi\sqrt{-1})^k} tr(\alpha \wedge F_{\nabla^t} \wedge \cdots \wedge F_{\nabla^t})$$

Cor. If $C_k(E) \neq 0$ for some k, then $F_\nabla \neq 0$ for any ∇ on E

Note that the product bundle $M \times \mathbb{C}^n$ with $\nabla = d$ has $F_\nabla = 0$, so $C_k(M \times \mathbb{C}^n) = 0$

Since $C_k(E)$ is independent of the choice of ∇, we can compute $tr(F_\nabla \wedge \cdots \wedge F_\nabla)$ by any simple ∇

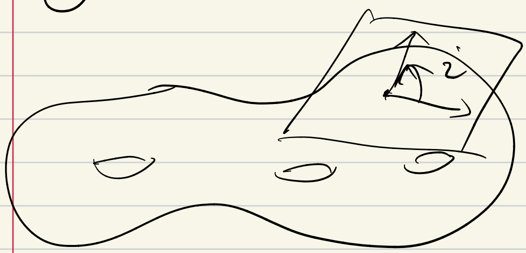
Rem. Indeed Chern classes can be defined in $H^{2k}_{\text{singular}}(M; \mathbb{Z})$,

i.e. it is an integer class,

if we intergrate the class over submanifolds, we will get integers. that's why we need $\frac{1}{(2\pi\sqrt{-1})^k}$ in the def

Ex ① $C_1(E) \neq 0$ for tautological $\mathbb{C}$-bundle over $\mathbb{CP}^n$

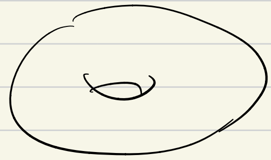
② Σ is an orientable surface in $\mathbb{R}^3$ of genus g



$T\Sigma$ can be regarded as a complex vector bundle of dim 1 if we consider multiplying by i as rotation $90°$

$$C_1(T\Sigma) = (2-2g)\cdot \text{generator in } H^2_{\text{Singular}}(\Sigma; \mathbb{Z})$$

$C_1(T\Sigma) \neq 0$ unless $g=1$



The tautological bundles over complex Grassmanians have nonvanishing $C_1, \cdots, C_{\dim E}$

For real vector bundle E, we can define pontryagin class

$$p_k(E) = C_{2k}(\underbrace{E \otimes_{\mathbb{R}} \mathbb{C}}_{\text{roughly } E \oplus \overline{E}}) \in H^{4k}_{dR}(M; \mathbb{R})$$

The odd Chern class of complexification is determined by Stiefel-Whitney class of the original bundle

For more discussion of characteristic classes, see [Hatcher vector bundles and K-theory Chap 3.]

Roughly speaking. Char classes are obstructions of the triviality of the bundle: Chern class $C_k \in H^{2k}(M; \mathbb{Z})$, pontryagin class $P_k \in H^{4k}(M; \mathbb{Z})$

Stiefel-Whitney $W_k \in H^k(M; \mathbb{Z}/2\mathbb{Z})$. Euler class $e \in H^2(M; \mathbb{Z})$

Property of Chern class

$f: M \to N \quad \pi: E \to N \quad c_k(f^*E) = f^* c_k(E)$

$c_k(E^*) = (-1)^k c_k(E)$

$c(E) = 1 + c_1(E) + c_2(E) + \cdots$

$c(E_1 \oplus E_2) = c(E_1) \wedge c(E_2)$