211BR Problem Set 1

Due Wednesday March 1 at 3:00 PM (hard copy or email submissions accepted).

Sourced Maxwell Equations in Lorenz Gauge

Consider a massless scalar field, which can be expanded in $\frac{1}{r}$ in retarded Bondi coordinates near $\mathcal{I}^+$ as

$$\phi(u,r,z,\bar{z}) = \sum_{n=1}^{\infty} \frac{\phi^{(n)}(u,z,\bar{z})}{r^n}. \tag{1}$$

The matter current of a charged massless scalar is constructed as $j_\mu = iQ(\bar{\phi}\partial_\mu\phi - \phi\partial_\mu\bar{\phi})$.

a) Write down the leading order in $\frac{1}{r}$ of each component of j_μ in retarded Bondi coordinates.

b) Consider the Maxwell equations $\nabla_\mu F^{\mu\nu} = e^2 j^\nu$ in Lorenz gauge $\nabla^\mu A_\mu = 0$. In order to consistently solve the sourced Maxwell equations in this gauge, we will see that we must allow logarithmic falloffs in the gauge fields. Using the $\frac{1}{r}$ expansion of A_μ,

$$\begin{aligned} A_u &= \sum_{n=1}^{\infty} \frac{A_u^{(n)}}{r^n} + \sum_{m=1}^{\infty} \frac{\tilde{A}_u^{(m)}}{r^m} \log r, \\ A_r &= \sum_{n=2}^{\infty} \frac{A_r^{(n)}}{r^n} + \sum_{m=2}^{\infty} \frac{\tilde{A}_r^{(m)}}{r^m} \log r, \\ A_z &= \sum_{n=0}^{\infty} \frac{A_z^{(n)}}{r^n} + \sum_{m=1}^{\infty} \frac{\tilde{A}_z^{(m)}}{r^m} \log r, \end{aligned} \tag{2}$$

and respectively for $A_{\bar{z}}$, find the $\mathcal{O}(\log r)$ and $\mathcal{O}(1)$ terms in the expansion of the Maxwell equations in Lorenz gauge at large r in retarded Bondi coordinates. Show that the expansion is inconsistent without the logarithmic terms in the gauge field, and that the falloffs (2) are consistent with finite total charge and energy flux.

c) Lorenz gauge leaves unfixed residual gauge transformations of the form $A_\mu \to A_\mu + \partial_\mu\varepsilon$ with $\Box\varepsilon = 0$. Write down the large-r expansion (including logarithmic terms) of this equation in retarded Bondi coordinates, and show that it has independent pieces of free data at $\mathcal{O}(1)$ and $\mathcal{O}(r^{-1})$. Show that the latter can be used to fix the residual gauge condition $A_u^{(1)} = 0$.

d) Working in Lorenz gauge with residual gauge condition $A_u^{(1)} = 0$, find the relation between $A_z^{(1)}$ and the boundary data $A_z^{(0)}$. Show that $A_z^{(1)}$ contains an undetermined integration function on the sphere. This is related to the Goldstone boson of the subleading soft symmetry, which we will discuss from another perspective later in the course.

e) Now consider the charge current in a worldline formalism:

$$j_\mu(y^\nu) = Q \int d\tau u_\mu \frac{\delta^{(4)}\left(y^\nu - x^\nu(\tau)\right)}{\sqrt{-g}}, \tag{3}$$

where $u_\mu = \frac{\partial x_\mu}{\partial \tau}$. Consider the trajectory of a massless point particle

$$x^\mu(\tau) = \frac{p^\mu}{\omega}\tau + b^\mu, \tag{4}$$

with four-momentum p^μ ($p^\mu p_\mu = 0$), energy ω, and impact parameter $b^\mu = (0, b^i)$. The angular momentum of this trajectory is

$$L^{\mu\nu} = b^\mu p^\nu - p^\mu b^\nu. \tag{5}$$

Find the terms through $\mathcal{O}(r^{-2})$ in the large-r expansion of $j_\mu(u, r, z, \bar{z})$ in retarded Bondi coordinates. What are the corresponding components of the Lienard-Weichart potential A_μ?