211BR Problem Set 2

Due Wednesday March 22 at 3:00 PM (hard copy or email submissions accepted).

Soft Graviton Theorems

Denote the tree-level scattering amplitude involving n massless scalar states by

$$\mathcal{A}_n = \langle \text{out}|\mathcal{S}|\text{in}\rangle, \tag{1}$$

We use a convention in which incoming states are described as CPT conjugate outgoing states with negative p^0 so that momentum conservation implies $\sum_{k=1}^{n} p_k^\mu = 0$.

Let $\mathcal{A}_{n+1}^{\pm}(q)$ be an amplitude involving a graviton of momentum q^μ, energy $\omega = q^0$, and polarization $\varepsilon_{\mu\nu}^{\pm}(q)$ as well as n other massless asymptotic scalar states

$$\mathcal{A}_{n+1}^{\pm}(q) = \langle \text{out}|a_{\pm}(q)\mathcal{S}|\text{in}\rangle. \tag{2}$$

The soft $\omega \to 0$ limit of this amplitude is governed by the leading [1] and sub-leading [2, 3, 4, 5] soft-graviton theorems

$$\mathcal{A}_{n+1}^{\pm}(q) \to \left[S_0^{\pm} + S_1^{\pm} + \mathcal{O}(\omega)\right]\mathcal{A}_n\ , \tag{3}$$

where $S_0^{\pm} \sim \omega^{-1}$, $S_0^{\pm} \sim \omega^0$, $\mathcal{A}_n$ is the original amplitude without the soft graviton and

$$S_0^{\pm} = \frac{\kappa}{2}\sum_{k=1}^{n}\frac{p_k^\mu p_k^\nu \varepsilon_{\mu\nu}^{\pm}(q)}{p_k\cdot q}\ , \qquad S_1^{\pm} = -\frac{i\kappa}{2}\sum_{k=1}^{n}\frac{\varepsilon_{\mu\nu}^{\pm}(q)p_k^\mu q_\lambda}{p_k\cdot q}\mathcal{L}_k^{\lambda\nu}\ , \qquad \kappa = \sqrt{32\pi G}\ . \tag{4}$$

Here, where we are restricting to the scattering of hard scalars, $\mathcal{L}_{k\mu\nu}$ is the angular momentum operator acting on the kth outgoing state,

$$\mathcal{L}_{k\mu\nu} = -i\left[p_{k\mu}\frac{\partial}{\partial p_k^\nu} - p_{k\nu}\frac{\partial}{\partial p_k^\mu}\right]. \tag{5}$$

1. Leading Soft Graviton Theorem

In this problem, you will show that the leading soft graviton theorem for a single outgoing graviton is equivalent to the Ward identity for a supertranslation current.

a) Near $\mathcal{I}^+$, consider the free field mode expansion for $h_{\mu\nu}^{\text{out}}$:

$$h_{\mu\nu}^{\text{out}}(x) = \sum_{\alpha=\pm}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\left[\varepsilon_{\mu\nu}^{\alpha *}(\vec{q})a_\alpha^{\text{out}}(\vec{q})e^{iq\cdot x} + \varepsilon_{\mu\nu}^{\alpha}(\vec{q})a_\alpha^{\text{out}}(\vec{q})^\dagger e^{-iq\cdot x}\right], \tag{6}$$

where $\varepsilon^{\mu\nu}_{\pm}(q) = \varepsilon^{\mu}_{\pm}(q)\varepsilon^{\nu}_{\pm}(q)$, and consider the definitions in retarded Bondi coordinates

$$\begin{aligned} C_{zz}(u,z,\bar{z}) &\equiv \kappa \lim_{r\to\infty} \frac{1}{r} h^{\text{out}}_{zz}(r,u,z,\bar{z}), \\ N^{\omega}_{zz}(z,\bar{z}) &\equiv \int_{-\infty}^{\infty} du e^{i\omega u}\partial_u C_{zz}(u,z,\bar{z}), \\ N^{0}_{zz}(z,\bar{z}) &\equiv \lim_{\omega\to 0^+} \frac{1}{2}\left(N^{\omega}_{zz} + N^{-\omega}_{zz}\right), \end{aligned} \tag{7}$$

and graviton polarization definitions

$$\varepsilon^{\mu}_{+}(q) = \frac{1}{\sqrt{2}}(\bar{z},1,-i,-\bar{z}), \quad \varepsilon^{\mu}_{-}(q) = \frac{1}{\sqrt{2}}(z,1,i,-z). \tag{8}$$

Using the above, show that the zero-energy mode of the Bondi news tensor at large r (i.e. on $\mathcal{I}^+$) is

$$N^{0}_{zz}(z,\bar{z}) = -\frac{\kappa}{8\pi}\hat{\varepsilon}_{zz} \lim_{\omega\to 0^+}\left[\omega a^{\text{out}}_{+}(\omega\hat{x}) + \omega a^{\text{out}}_{-}(\omega\hat{x})^{\dagger}\right] \tag{9}$$

where $\hat{\varepsilon}_{zz} = r^{-2}\partial_z x^{\mu}\partial_z x^{\nu}\varepsilon^{-}_{\mu\nu}(\hat{x})$ and $\hat{x}$ is the unit three-vector:

$$\hat{x} \equiv \left(\frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i(z-\bar{z})}{1+z\bar{z}}, \frac{(1-z\bar{z})}{1+z\bar{z}}\right). \tag{10}$$

Hint: use the saddle point approximation in the calculation of C_{zz} at large r.

b) Using the parametrization of massless momenta $q^{\mu}(z,\bar{z})$, $p^{\mu}_k(z_k,\bar{z}_k)$ as

$$\begin{aligned} q^{\mu} &= \omega\left(1, \frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i(z-\bar{z})}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}\right), \\ p^{\mu}_k &= \omega_k\left(1, \frac{z_k+\bar{z}_k}{1+z_k\bar{z}_k}, \frac{-i(z_k-\bar{z}_k)}{1+z_k\bar{z}_k}, \frac{1-z_k\bar{z}_k}{1+z_k\bar{z}_k}\right), \quad k = 1,\cdots,n \end{aligned} \tag{11}$$

and the graviton polarizations (8), rewrite the leading soft graviton theorem with soft factor S^{+}_{0} in terms of $(z,\bar{z})$, $(z^{\text{out}}_k, \bar{z}^{\text{out}}_k)$, and obtain an expression for $\langle \text{out}|N^{0}_{zz}\mathcal{S}|\text{in}\rangle$ as a factor multiplying $\langle \text{out}|\mathcal{S}|\text{in}\rangle$.

c) Show that upon defining

$$P^{+}_{z} \equiv \frac{1}{2G}\gamma^{z\bar{z}}\partial_{\bar{z}}N^{0}_{zz}. \tag{12}$$

the expression of the leading soft theorem from the previous part resembles the Ward identity for a current in a 2D conformal field theory.

2. Subleading Soft Graviton Theorem

Next, you will study the action of subleading soft gravitons on massless external scalar particles.

a) Using the definitions (7), show that

$$\begin{aligned} N^{(1)}_{zz} &\equiv -\frac{i}{2}\lim_{\omega\to 0^+}\partial_\omega\left[N^{\omega}_{zz} - N^{-\omega}_{zz}\right] \\ &= \frac{i\kappa}{8\pi}\hat{\varepsilon}_{zz}\lim_{\omega\to 0^+}\left(1+\omega\partial_\omega\right)\left[a^{\text{out}}_{+}(\omega\hat{x}) - a^{\text{out}}_{-}(\omega\hat{x})^{\dagger}\right] , \end{aligned} \tag{13}$$

and similarly for $N^{(1)}_{\bar{z}\bar{z}}$. Note that $N^{(1)}_{zz}$ involves one less factor of ω than $N^{(0)}_{zz}$, but has the leading soft pole projected out by the factor of $1+\omega\partial_\omega$. Hence it has nonzero finite scattering amplitudes.

b) Using the definitions (4) and (5), rewrite the subleading soft factor S_1^{-} in terms of (z,z) and $(z_k,\bar{z}_k)$ and obtain an expression for $\langle \text{out}|N^{(1)}_{\bar{z}\bar{z}}\mathcal{S}|\text{in}\rangle$ as an action on $\langle\text{out}|\mathcal{S}|\text{in}\rangle$.

c) Show that upon defining

$$T_{zz} \equiv \frac{i}{8\pi G}\int d^2w \frac{1}{z-w}D^2_w D^{\bar{w}} N^{(1)}_{\bar{w}\bar{w}}, \tag{14}$$

the expression from the previous part takes the form of a stress tensor Ward identity for a 2D conformal field theory on a curved background, explicitly

$$\langle\text{out}|T_{zz}\mathcal{S}|\text{in}\rangle = \sum_{k=1}^{n}\left[\frac{h_k}{(z-z_k)^2} + \frac{\Gamma^{z_k}_{z_k z_k}}{z-z_k}h_k + \frac{\partial_{z_k}}{z-z_k}\right]\langle\text{out}|\mathcal{S}|\text{in}\rangle, \tag{15}$$

where Γ^{z}_{zz} is the connection for the sphere metric $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$ and $h_k \equiv -\frac{\omega_k\partial_{\omega_k}}{2}$.

References

[1] S. Weinberg, "Infrared photons and gravitons," *Phys. Rev.* **140** (1965) B516–B524.

[2] D. J. Gross and R. Jackiw, "Low-Energy Theorem for Graviton Scattering," *Phys. Rev.* **166** (1968) 1287–1292.

[3] R. Jackiw, "Low-Energy Theorems for Massless Bosons: Photons and Gravitons," *Phys. Rev.* **168** (1968) 1623–1633.

[4] C. D. White, "Factorization Properties of Soft Graviton Amplitudes," *JHEP* **05** (2011) 060, `arXiv:1103.2981 [hep-th]`.

[5] F. Cachazo and A. Strominger, "Evidence for a New Soft Graviton Theorem," `arXiv:1404.4091 [hep-th]`.