# 211BR Problem Set 3

Due Wednesday April 5 at 3:00 PM (hard copy or email submissions accepted).

## 1. "Flat Bondi" Coordinates

In this problem, we will explore an alternative coordinate system known as "flat Bondi" coordinates, in which the Minkowski line element becomes

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -du dr + r^2 dz d\bar{z}. \tag{1}$$

where $-\infty < u, r < \infty$ and $z \in \mathbb{C}$ (note that this is in contrast to the usual Bondi coordinates, in which $z$ parametrizes the Riemann sphere $\mathbb{C} \cup \{\infty\}$).

a) Verify that the coordinate transformation from Cartesian to flat Bondi coordinates takes the form

$$\begin{aligned} x^0 &= \frac{1}{2}\left(u + r\left(1 + z\bar{z}\right)\right), \\ x^1 &= \frac{r}{2}(z + \bar{z}), \\ x^2 &= -\frac{ir}{2}(z - \bar{z}), \\ x^3 &= -\frac{1}{2}\left(u - r\left(1 - z\bar{z}\right)\right). \end{aligned} \tag{2}$$

Write the expressions for each of the **retarded Bondi** coordinates $(u_{\rm rB}, r_{\rm rB}, z_{\rm rB}, \bar{z}_{\rm rB})$ and **advanced Bondi** coordinates $(v_{\rm aB}, r_{\rm aB}, z_{\rm aB}, \bar{z}_{\rm aB})$ in flat Bondi coordinates. Write out the leading order in $r$ parts of the retarded Bondi coordinates as $r \to +\infty$, as well as the leading order in $r$ part of the advanced Bondi coordinates as $r \to -\infty$. Use your answer to argue what region of flat Bondi coordinates corresponds to each of $\mathcal{I}^+$ and $\mathcal{I}^-$, and verify that the antipodal map $z_{rB} = -\frac{1}{\bar{z}_{\rm aB}}$ holds in this case.

b) Determine the Killing vector $\zeta$ which generates Lorentz transformations in "flat Bondi" $(u, r, z, \bar{z})$ coordinates.

Then, using the fact that the global conformal Killing vectors defined on the sphere must take the form

$$Y^z(z) = a + bz + cz^2, \quad a, b, c \in \mathbb{C}, \tag{3}$$

Show that $\zeta$ can be parametrized in terms of a global CKV $Y^A$ with $A \in \{z, \bar{z}\}$ and find $\zeta(Y)$.

c) Show that $[\zeta(Y_1), \zeta(Y_2)] = \zeta([Y_1, Y_2])$. This shows that the Lorentz algebra is isomorphic to $\mathrm{SL}(2, \mathbb{C})$, the algebra of global conformal killing vectors on the 2d sphere.

d) ***Optional (for 5% extra credit):*** A *conformal Killing vector* (CKV) on $S^2$ is defined as a vector $Y^A$ with $A, B \in \{1,2\}$ satisfying

$$D_A Y_B + D_B Y_A = \gamma_{AB} D_C Y^C. \tag{4}$$

Taking $\gamma_{AB}$ to be the usual metric on the round sphere,

$$ds^2_{S^2} = \gamma_{AB} dx^A dx^B = \frac{4dwd\bar{w}}{(1+w\bar{w})^2}, \tag{5}$$

show that the conformal Killing vectors that are defined globally on the 2D sphere $S^2$ must take the form

$$Y^w(w) = a + bw + cw^2, \quad a, b, c \in \mathbb{C}. \tag{6}$$

## 2. Soft Factorization (in Flat Bondi Coordinates)

In this problem, we will learn how to represent massless particles in terms of operators on the celestial sphere and describe soft factorization of the gravitational $\mathcal{S}$-matrix using a supertranslation current algebra.

We consider four-dimensional perturbative gravity coupled to massless matter on a Minkowski background in the coordinates (1), with null momenta parametrized as

$$p_k^\mu = \eta_k \omega_k \hat{q}^\mu(z_k, \bar{z}_k), \quad \hat{q}^\mu(z_k, \bar{z}_k) = (1 + z_k\bar{z}_k, z_k + \bar{z}_k, -i(z_k - \bar{z}_k), 1 - z_k\bar{z}_k), \tag{7}$$

where $\omega_k \geq 0$, $\eta_k = \pm 1$ for outgoing and incoming particles respectively. Spinning massless particles are constructed from the polarization vectors

$$\varepsilon^\mu_{k+} = \frac{1}{\sqrt{2}} \partial_{z_k} \hat{q}^\mu(z_k, \bar{z}_k), \qquad \varepsilon^\mu_{k-} = \frac{1}{\sqrt{2}} \partial_{\bar{z}_k} \hat{q}^\mu(z_k, \bar{z}_k), \tag{8}$$

and the polarization tensor for a positive helicity outgoing graviton is $\varepsilon^{\mu\nu}_{k+} = \varepsilon^\mu_{k+}\varepsilon^\nu_{k+}$.

As in Problem Set 2, we can define (note now in flat Bondi coordinates)

$$C_{zz}(u, z, \bar{z}) \equiv \kappa \lim_{r\to\infty} \frac{1}{r} h^{\text{out}}_{zz}(r, u, z, \bar{z}). \tag{9}$$

Now, using the coordinates (1) and parametrization (7), define soft graviton currents $P_z^\pm$ by

$$4GP_z^+ = \partial_{\bar{z}} C_{zz}|_{\mathcal{I}^+_+} - \partial_{\bar{z}} C_{zz}|_{\mathcal{I}^+_-} = -\frac{\kappa}{8\pi} \lim_{\omega\to 0} \partial_{\bar{z}} \left(\omega a^{\text{out}}_+(\omega, z, \bar{z}) + \omega a^{\dagger\text{out}}_-(\omega, z, \bar{z})\right), \tag{10}$$

where $\kappa = \sqrt{32\pi G}$. Similar expressions on $\mathcal{I}^-$ hold for $P_z^-$.

The leading soft graviton theorem for a scattering process involving $n$ hard massless particles and one soft graviton of momentum $q^\mu = \omega\hat{q}^\mu(z, \bar{z})$ and polarization $\varepsilon^{\mu\nu}$ resembles a 2D Ward identity for a conserved current:

$$\langle \text{out}|P_z^+\mathcal{S} - \mathcal{S}P_z^-|\text{in}\rangle = -\lim_{\omega\to 0}\partial_{\bar{z}}\left[\omega\sum_{k=1}^{n}\frac{\varepsilon^+_{\mu\nu}p_k^\mu p_k^\nu}{p_k\cdot q}\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle = \sum_{k=1}^{n}\frac{\eta_k\omega_k}{z-z_k}\langle \text{out}|\mathcal{S}|\text{in}\rangle. \tag{11}$$

Supertranslation invariance of the gravitational $\mathcal{S}$-matrix follows from taking the derivative of (11) with respect to $\bar{z}$ and then integrating the result against an arbitrary function $f(z,\bar{z})$:

$$\begin{aligned}\int\frac{d^2z}{2\pi}f(z,\bar{z})\partial_{\bar{z}}\langle \text{out}|P_z^+\mathcal{S} - \mathcal{S}P_z^-|\text{in}\rangle &= \int d^2z f(z,\bar{z})\sum_{k=1}^{n}\eta_k\omega_k\delta^{(2)}(z_k-z)\langle \text{out}|\mathcal{S}|\text{in}\rangle \\ &\equiv \langle \text{out}|Q_H^+\mathcal{S} - \mathcal{S}Q_H^-|\text{in}\rangle.\end{aligned} \tag{12}$$

The hard charges $Q_H^\pm$ implement the action of the supertranslation symmetry on matter. For the full charge $Q^\pm = Q_H^\pm + Q_S^\pm$, the left-hand side of (12) is identified as $-\langle \text{out}|Q_S^+\mathcal{S} - \mathcal{S}Q_S^-|\text{in}\rangle$. The soft charges $Q_S^\pm$ act by adding soft gravitons, performing a non-trivial vacuum transformation.

a) Under infinitesimal supertranslations, a hard massless asymptotic state transforms as

$$\delta_f|p_k\rangle = iQ_H^\pm|p_k\rangle, \tag{13}$$

where $Q_H^\pm$ acts on states with $\eta_k = \pm 1$, respectively. Massless particles in momentum eigenstates are associated to unique points on the celestial sphere and resemble local 2D operators $|p_k\rangle \leftrightarrow \mathcal{O}_k(p_k)$, which transform under supertranslations according to (13). Find the explicit transformation $\delta_f\mathcal{O}_k(p_k)$ in terms of $\omega_k, z_k, \bar{z}_k$.

b) Supertranslation symmetry is spontaneously broken in the standard Minkowski vacuum, giving rise to a Goldstone boson, denoted $C$. $C$ is canonically paired with the soft graviton $P_z^+$ and related to a boundary component of the asymptotic metric:

$$C_{zz}|_{\mathcal{I}^+_-} = -\partial_z^2 C. \tag{14}$$

The Goldstone boson transforms under an infinitesimal supertranslation by an inhomogeneous shift: $\delta_f C = f$. The supertranslation transformation properties of $\mathcal{O}(p_k)$ can be isolated with operators $\mathcal{W}_k(p_k)$, which transform as

$$\delta_f\mathcal{W}_k(z_k,\bar{z}_k) = i\eta_k\omega_k f(z_k,\bar{z}_k)\mathcal{W}_k(z_k,\bar{z}_k). \tag{15}$$

Then we can decompose

$$\mathcal{O}_k = \mathcal{W}_k\mathcal{O}'_k, \tag{16}$$

where the transformation of $\mathcal{W}_k$ accounts for the full transformation of $\mathcal{O}_k(p_k)$ under $\delta_f$, implying that $\mathcal{O}'_k$ is invariant under supertranslations. Note that neither $\mathcal{W}_k$ nor $\mathcal{O}'_k$ alone create physical scattering states, which created by the composite operator $\mathcal{O}_k$. Find an explicit form of $\mathcal{W}_k$ in terms of the operator $C$.

c) An *operator product expansion* (OPE) is an expansion of a product of operators into a sum of single operators at a point:

$$\mathcal{O}_i(z_1,\bar{z}_1)\mathcal{O}_j(z_2,\bar{z}_2) = \sum_k c_{ij}^k(z_1-z_2,\bar{z}_1-\bar{z}_2)\mathcal{O}_k(z_2,\bar{z}_2), \tag{17}$$

where $c_{ij}^k(z_1-z_2,\bar{z}_1-\bar{z}_2)$ is coefficient function of $z_1-z_2,\bar{z}_1-\bar{z}_2$. In the following we will use the symbol $\sim$ to denote "equal up to nonsingular terms in $z_1-z_2$ and $\bar{z}_1-\bar{z}_2$." The soft graviton theorem can be recast as the insertion of a current

$$P_z \equiv P_z^+ - P_z^- \tag{18}$$

in a correlation function of operators:

$$\langle P_z\mathcal{O}_1\cdots\mathcal{O}_n\rangle = \sum_{k=1}^n \frac{\eta_k\omega_k}{z-z_k}\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle. \tag{19}$$

Use the soft theorem to deduce the singular terms in the OPEs $P_z\mathcal{W}_k$ and $P_zC$.

d) In analogy with the supertranslation current, define a Goldstone current

$$\widetilde{P}_z = i\partial_z C. \tag{20}$$

Using the previous part, find the singular terms in the OPE $P_z\widetilde{P}_w$. Deduce the form of the singular terms in the OPE $\widetilde{P}_z\widetilde{P}_w$ up to a numerical constant, which we will determine in the next part. Note that soft gravitons have vanishing energy and do not couple at leading order in a low-energy expansion, implying that there are no singular terms in the supertranslation current OPE, i.e. $P_zP_w \sim 0$.

e) In addition to singularities arising from the emission and absorption of soft gravitons, which are captured by the soft graviton theorem, scattering amplitudes in four-dimensional theories of gravity contain IR divergences arising from virtual soft gravitons exchanged between external legs. As Weinberg first explained [1], virtual graviton exchange contributes a universal soft factor to the $\mathcal{S}$-matrix. It can be shown that this factor takes the form

$$\langle \text{out}|\mathcal{S}|\text{in}\rangle = \exp\left[-\frac{1}{\epsilon}\frac{G}{\pi}\sum_{i\neq j}^n \eta_i\eta_j\omega_i\omega_j|z_{ij}|^2\ln|z_{ij}|^2\right]\widehat{\langle \text{out}|\mathcal{S}|\text{in}\rangle}, \tag{21}$$

in $d = 4+2\epsilon$ dimensional regularization, where $\widehat{\langle \text{out}|\mathcal{S}|\text{in}\rangle}$ is IR finite. Recast this statement in terms of correlation functions of the operators introduced in the preceding parts, and use it to find the numerical coefficients of the singular terms in the OPE $\widetilde{P}_z\widetilde{P}_w$.

f) Now, we will recompute the gravitational memory effect using the current algebra. We will show that the correlation function of soft gravitons and Goldstone modes determines the gravitational memory formula.

Using crossing symmetry $\langle \text{out}|P_z^+\mathcal{S}|\text{in}\rangle = -\langle \text{out}|\mathcal{S}P_z^-|\text{in}\rangle$, we have

$$\frac{1}{2}\frac{\langle P_z\mathcal{O}_1\cdots\mathcal{O}_n\rangle}{\langle \mathcal{O}_1\cdots\mathcal{O}_n\rangle} = \frac{\langle \text{out}|P_z^+\mathcal{S}|\text{in}\rangle}{\langle \text{out}|\mathcal{S}|\text{in}\rangle}. \tag{22}$$

The right-hand side can be interpreted as the expectation value of the change in the asymptotic metric $\Delta h_{\mu\nu}$ induced by the scattering of $n$ hard particles. Show that the left-hand side can be written in terms of correlation functions involving $P_z$ and a Goldstone operator. Use the form of these correlation functions to obtain the Braginsky-Thorne formula [2] for gravitational memory due to the scattering of massive bodies:

$$\lim_{r\to\infty} r\varepsilon_+^{\mu\nu}\Delta h_{\mu\nu}(r,z,\bar{z}) = -\sqrt{\frac{G}{2\pi}}\sum_{j=1}^{n}\frac{\varepsilon_{\mu\nu}^+ p_j^\mu p_j^\nu}{p_j\cdot\hat{q}(z,\bar{z})}. \tag{23}$$

Note that gravitational memory is IR safe observable: it involves a *ratio* of scattering amplitudes that precisely cancels the IR divergences due to virtual gravitons.

## References


[1] S. Weinberg, "Infrared photons and gravitons," *Phys. Rev.* **140** (1965) B516–B524.

[2] V. B. Braginsky and K. S. Thorne, "Gravitational-wave bursts with memory and experimental prospects," *Nature* **327** (1987) 123–125.