211BR Problem Set 4

Due Wednesday April 19 at 3:00 PM (hard copy or email submissions accepted).

Supertranslations and Superrotations in Harmonic Gauge

We will now consider perturbations to an asymptotically Minkowski metric in harmonic gauge. The metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{1}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$ and $h_{\mu\nu}$ is known as the "trace-reversed" perturbation. Harmonic gauge is the condition that

$$\nabla^{\mu}h_{\mu\nu} = 0. \tag{2}$$

In this gauge, the linearized Einstein equations are

$$\Box h_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{3}$$

Harmonic gauge leaves unfixed a set of residual diffeomorphisms ξ that obey $\Box\xi = 0$.

a) We will first consider perturbations in harmonic gauge in retarded Bondi coordinates. Show that the harmonic gauge condition implies that it is possible to consistently set (in the usual large$-r$ notation) $h^{(1)}_{ur}$ and $h^{(1)}_{rr}$ to zero, and that residual gauge freedom can be used to fix $h^{(1)}$, $h^{(1)}_{uu}$, and $h^{(0)}_{uz}$ to zero. Using harmonic gauge with this residual gauge choice, derive the leading order in r terms of each of the residual diffeomorphism components in the large r limit, and show that they are characterized by a free function $f(z,\bar{z})$ and vector field Y (with $Y^z(z), Y^{\bar{z}}(\bar{z})$) on the sphere.

b) Now consider hyperbolic coordinates, in which the Minkowski line element becomes

$$ds^2 = -d\tau^2 + \tau^2 d\sigma^2, \quad d\sigma^2 = \frac{dy^2 + dz d\bar{z}}{y^2}, \tag{4}$$

with $-\infty < \tau < \infty$, $0 < y < \infty$, and $z, \bar{z} \in \mathbb{C}$. $d\sigma^2$ is the line element on a hyperbolic slice (Euclidean AdS_3). Consider the residual diffeomorphisms in harmonic gauge in hyperbolic coordinates. Starting from the ansatz

$$\xi = \hat{\xi}^{\mu}(y, z, \bar{z})\partial_{\mu} + \mathcal{O}(\tau^{-1}), \tag{5}$$

use the harmonic gauge conditions and compatibility with the residual gauge fixing in retarded Bondi coordinates in the previous part to derive the form of $\hat{\xi}^{\mu}$. Show that it is again characterized by a free function $f(z,\bar{z})$ and vector field $Y^z(z)$.

Hint: Use the condition $h^{(1)} = 0$ in retarded Bondi coordinates from the previous part to argue that $h \sim \mathcal{O}(\tau^{-2})$ at large τ.