211BR Problem Set 5

Due Wednesday May 3 at 3:00 PM (hard copy or email submissions accepted).

Celestial Three-Point Amplitudes

Consider an n-particle momentum space amplitude $\mathcal{A}_n(q_1,\ldots,q_{n_1},p_1,\ldots,p_{n_2})$, with n_1 massless particles and $n_2 = n-n_1$ massive particles. It can be translated to a celestial amplitude, $\widetilde{\mathcal{A}}_n(\{\Delta_k,z_k,\bar{z}_k\})$, with $k=1,\ldots,n$ by Mellin transforming the massless energies ω_i, and integrating the massive momenta against the bulk-to-boundary propagator $G_{\Delta_i}(\hat{p}_i;z_i,\bar{z}_i)$ as follows:

$$\widetilde{\mathcal{A}}_n(\Delta_k,z_k,\bar{z}_k) = \prod_{i=1}^{n_1}\left(\int_0^\infty d\omega_i\omega_i^{\Delta_i-1}\right)\prod_{j=1}^{n_2}\left(\int_0^\infty \frac{dy_j}{y_j^3}\int d^2w_j\; G_{\Delta_j}(y_j,w_j,\bar{w}_j;z_j,\bar{z}_j)\right)\mathcal{A}_n, \quad (1)$$

where we have used the following parameterizations for massless and massive momenta

$$q_i^\mu = \epsilon_i\omega_i\hat{q}_i^\mu = \epsilon_i\omega_i(1+z_i\bar{z}_i, z_i+\bar{z}_i, -i(z_i-\bar{z}_i), 1-z_i\bar{z}_i), \quad (2)$$
$$p_j^\mu = \epsilon_j m_j\hat{p}_j^\mu = \frac{\epsilon_j m_j}{2y_j}\left(1+y_j^2+w_j\bar{w}_j, w_j+\bar{w}_j, -i(w_j-\bar{w}_j), 1-y_j^2-w_j\bar{w}_j\right) = \frac{\epsilon_j m_j}{2y_j}\left(n^\mu y_j^2+\hat{q}_j^\mu\right),$$

where the signs $\epsilon_k=\pm1$ distinguish incoming/outgoing momenta and $n^\mu = \partial_z\partial_{\bar{z}}q^\mu(z,\bar{z})$. We use the definition of the bulk-to-boundary propagator

$$G_\Delta(y,w,\bar{w};z,\bar{z}) \equiv \left(\frac{y}{y^2+|z-w|^2}\right)^\Delta. \quad (3)$$

In this problem, we will study the celestial amplitude corresponding to a tree-level three-point interaction between two massless scalars and one massive scalar.

a) The momentum-space amplitude for a tree-level three-point interaction between two massless scalars and one massive scalar takes the form

$$\mathcal{A}_3(q_1,q_2,p_3) = g\delta^{(4)}(q_1+q_2+p_3) \quad (4)$$

where g is the coupling constant for the interaction, the massless particles have momentum $q_i=\epsilon_i\omega_i\hat{q}_i$, and the massive particle has momentum $p_3=\epsilon_3 m\hat{p}$. Using (1), explicitly calculate the corresponding celestial amplitude, $\widetilde{\mathcal{A}}_3(\{\Delta_k,z_k,\bar{z}_k\}_{k=1,2,3})$ and verify that it takes the form of a three-point correlation function in 2D CFT. That is, show that

$$\widetilde{\mathcal{A}}_3(\Delta_k,z_k,\bar{z}_k) = \frac{C(\Delta_1,\Delta_2,\Delta_3)}{|z_1-z_2|^{\Delta_1+\Delta_2-\Delta_3}|z_1-z_3|^{\Delta_1+\Delta_3-\Delta_2}|z_2-z_3|^{\Delta_2+\Delta_3-\Delta_1}}, \quad (5)$$

and give the expression for $C(\Delta_1,\Delta_2,\Delta_3)$.

Hint: Use $SL(2,\mathbb{C})$ transformations to take the three points (z_1,z_2,z_3) to $(0,1,\infty)$.

b) In this problem, we will show that the celestial amplitude from part a) can be fully fixed by 4D Poincaré invariance. Translation invariance of momentum space amplitudes implies that

$$\sum_{k=1}^{n} P_{\mu,k}\mathcal{A}_n = 0. \tag{6}$$

where in momentum space, the translation generators for massless particles with momenta $q_{\mu,k}$ act as

$$P_{\mu,k}\mathcal{A}_n\left(q_1,\ldots,q_n\right) = \epsilon_k\omega_k\hat{q}_{\mu,k}\mathcal{A}_n\left(q_1,\ldots,q_n\right). \tag{7}$$

i) Show that for massless particles, the translation generators act on celestial amplitudes as weight-shifting operators

$$P_{\mu,k}\ \widetilde{\mathcal{A}}_n = \epsilon_k\hat{q}_{\mu,k}\left(z_k,\bar{z}_k\right)e^{\partial_{\Delta_k}}\widetilde{\mathcal{A}}_n, \tag{8}$$

ii) By a similar (but more tedious) process as in part i), one finds that for massive particles with momentum $p_k^\mu = \frac{\epsilon_k m_k}{2y_k}\left(n^\mu y_k^2 + \hat{q}_k^\mu\right)$, the action of translation generators on celestial amplitudes takes the form

$$P_{\mu,k} = \frac{\epsilon_k m_k}{2}\left[\left(\left(\partial_{z_k}\partial_{\bar{z}_k}\hat{q}_{\mu,k}\right) + \frac{\left(\partial_{\bar{z}_k}\hat{q}_{\mu,k}\right)\partial_{z_k} + \left(\partial_{z_k}\hat{q}_{\mu,k}\right)\partial_{\bar{z}_k}}{\Delta_k - 1} + \frac{\hat{q}_{\mu,k}\partial_{z_k}\partial_{\bar{z}_k}}{\left(\Delta_k - 1\right)^2}\right)e^{-\partial_{\Delta_k}} + \frac{\Delta_k\hat{q}_{\mu,k}}{\Delta_k - 1}e^{\partial_{\Delta_k}}\right]. \tag{9}$$

For the three-point amplitude discussed in part a), (6) takes the form

$$\left(P_{\mu,1} + P_{\mu,2} + P_{\mu,3}\right)\widetilde{\mathcal{A}}_3\left(\Delta_k, z_k, \bar{z}_k\right) = 0 \tag{10}$$

where $P_{\mu,1}, P_{\mu,2}$ are as in (8) and $P_{\mu,3}$ is as in (9). 4D Lorentz invariance (equivalently, 2D global conformal invariance) fixes that the celestial amplitude must take the form (5). Starting from (5), show that the momentum conservation constraint (10) yields a set of recursion relations for $C\left(\Delta_1, \Delta_2, \Delta_3\right)$. Verify that your answer for problem a) satisfies these relations.

iii) *Optional (for 5% extra credit):* Prove that (9) holds. The definition of the bulk-to-boundary propagator (3) may be helpful.