Proofs of Propensity Score Results

Theorem: If $E = P(A = 1|X)$ then $P(X|A = 1, E) = P(X|A = 0, E)$ i.e. $X \coprod A|E$

Proof: $P(X|A = 1, E(X) = e)$
$= \frac{P(X, A=1|E(X)=e)}{P(A=1|E(X)=e)}$
$= \frac{P(A=1|X, E(X)=e) P(X|E(X)=e)}{P(A=1|E(X)=e)}$
$= \frac{eP(X|E(X)=e)}{e}$
$= P(X|E(X) = e)$

Theorem: If $Y_a \coprod A|X$ then $Y_a \coprod A|E$

Proof: Suppose $Y_a \coprod A|X$ then $P(Y_a|A, X) = P(Y_a|X)$

Thus, $P(Y_a|A, E) =$

$$= \sum_x P(Y_a|A, E(X), x) P(x|A, E(X))$$

$$= \sum_x P(Y_a|A, x) P(x|A, E(X))$$

$$= \sum_x P(Y_a|x) P(x|E(X))$$

$$= \sum_x P(Y_a|E(X), x) P(x|E(X))$$

$= P(Y_a|E)$

Proof for Propensity Score Weighting

Theorem: If $Y_a \coprod A|X$ then $\mathbb{E}[Y_1] = \mathbb{E}[\frac{YA}{E}]$and $\mathbb{E}[Y_0] = \mathbb{E}[\frac{Y(1-A)}{1-E}]$.
Proof: $\mathbb{E}[\frac{YA}{E}] = \mathbb{E}[\mathbb{E}[\frac{YA}{E}|X]]$
$= \mathbb{E}[\mathbb{E}[\frac{(Y)(1)}{E}|A=1,X]P(A=1|X) + \mathbb{E}[\frac{(Y)(0)}{E}|A=0,X]P(A=0|X)]]$
$= \mathbb{E}[\mathbb{E}[\frac{Y}{E}|A=1,X]E(X)]$
$= \mathbb{E}[\mathbb{E}[Y|A=1,X]\frac{E}{E}]$
$= \mathbb{E}[\mathbb{E}[Y|A=1,X]]$
$= \mathbb{E}[\mathbb{E}[Y_1|A=1,X]]$
$= \mathbb{E}[\mathbb{E}[Y_1|X]]$
$= \mathbb{E}[Y_1]$.
Similarly, $\mathbb{E}[\frac{Y(1-A)}{1-E}] = \mathbb{E}[\mathbb{E}[\frac{Y(1-A)}{1-E}|X]]$
$= \mathbb{E}[\mathbb{E}[\frac{(Y)(0)}{1-E}|A=1,X]P(A=1|X) + \mathbb{E}[\frac{(Y)(1)}{1-E}|A=0,X]P(A=0|X)]]$
$= \mathbb{E}[\mathbb{E}[\frac{Y}{1-E}|A=0,X]\{1-E(X)\}]$
$= \mathbb{E}[\mathbb{E}[Y|A=0,X]\frac{1-E}{1-E}]$
$= \mathbb{E}[\mathbb{E}[Y|A=0,X]]$
$= \mathbb{E}[\mathbb{E}[Y_0|A=0,X]]$
$= \mathbb{E}[\mathbb{E}[Y_0|X]]$
$= \mathbb{E}[Y_0]$.

Theorem: If $Y_a \coprod A|X$ then $\mathbb{E}[Y_1|A=1] - \mathbb{E}[Y_0|A=1] = \frac{1}{P(A=1)}(\mathbb{E}[YA] - \mathbb{E}[\frac{Y(1-A)E}{1-E}])$ and $\mathbb{E}[Y_1|A=0] - \mathbb{E}[Y_0|A=0] = \frac{1}{P(A=0)}(\mathbb{E}[\frac{YA(1-E)}{E}] - \mathbb{E}[Y(1-A)])$.

Proof:
First, we have $\mathbb{E}[YA] = \mathbb{E}[YA|A=1]P(A=1) + \mathbb{E}[YA|A=0]P(A=0) = \mathbb{E}[Y|A=1]P(A=1) = \mathbb{E}[Y_1|A=1]P(A=1)$ and thus $\mathbb{E}[Y_1|A=1] = \frac{1}{P(A=1)}\mathbb{E}[YA]$
Next, $\mathbb{E}[Y_0|A=1]$
$= \mathbb{E}[\mathbb{E}[Y_0|A=1,X]|A=1]$
$= \mathbb{E}[\mathbb{E}[Y_0|A=1,X]\frac{P(A=1|X)}{P(A=1)}]$
$= \frac{1}{P(A=1)}\mathbb{E}[\mathbb{E}[Y_0|A=1,X]E]$
$= \frac{1}{P(A=1)}\mathbb{E}[\mathbb{E}[Y_0|A=0,X]E]$
Furthermore, $\mathbb{E}[\frac{Y(1-A)E}{1-E}] = \mathbb{E}[\mathbb{E}[\frac{Y(1-A)E}{1-E}|X]]$
$= \mathbb{E}[\mathbb{E}[\frac{Y(1-0)E}{1-E}|A=0,X]P(A=0|X) + \mathbb{E}[\frac{Y(1-1)E}{1-E}|A=1,X]P(A=1|X)]]$
$= \mathbb{E}[\mathbb{E}[\frac{YE}{1-E}|A=0,X](1-E)]$
$= \mathbb{E}[\mathbb{E}[Y|A=0,X]E\frac{1-E}{1-E}]$
$= \mathbb{E}[\mathbb{E}[Y|A=0,X]E]$
Thus $\mathbb{E}[Y_0|A=1] = \frac{1}{P(A=1)}(\mathbb{E}[\frac{Y(1-A)E}{1-E}])$. This completes the proof.
The proof of $\mathbb{E}[Y_1|A=0] - \mathbb{E}[Y_0|A=0] = \frac{1}{P(A=0)}(\mathbb{E}[\frac{YA(1-E)}{E}] - \mathbb{E}[Y(1-A)])$ is similar.