Introduction to Factor Analysis

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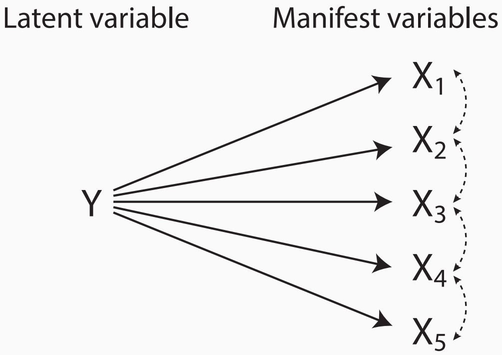
Latent Variable

Recall our discussion from PHS2000A about scales that aim to measure a **latent construct**, where the **latent variable** is

- not directly observable
- varies between individuals and/or over time
- Examples:
  - parents' aspirations for their children's achievement
  - depression
  - perceived stress
  - socioeconomic position
  - neighborhood cohesion
  - social capital

Scales

A **scale** is a collection of items combined into a composite score intended to reveal levels of a theoretical variable not directly observable. The items of a scale share a common cause (the latent variable). We distinguish this from an **index**, which is a collection of items which share a common effect.



Latent Variable

The latent variable is regarded as a *cause* of the item score. That is, the value of the latent variable (its "true score") is presumed to cause an item or set of items to take on a certain value.

- If the item value is caused by a latent variable, there should be a correlation between that value and the true score of the latent variable.
- Each of the items should also correlate with each other by virtue of their shared correlation with the latent variable.
- Because we cannot directly assess the true score, we cannot compute a correlation between it and an individual item.
- However, we can examine the relationships between items to learn something about **reliability.**

Factor Analysis

In PHS2000A we focused on scales that measure a **unidimensional latent variable**, and we discussed Cronbach's α as a measure of reliability.

- What do we do if we think that the items that make up our scale may be reflecting **multiple latent variables** (or we're not sure)?
- The multiple constructs may be correlated with each other, but it's still possible to sort items based on which latent construct they map most strongly onto
- We could just look at the scale items and see which ones we think go together, but that could be a bit arbitrary.
- Factor analysis is essentially a more empirical (though still theory-driven!) way of doing this

Factor analysis is a theory driven statistical **data reduction** technique used to explain **covariance** among observed random variables in terms of **fewer unobserved** random variables named factors.

Today

- We will focus on the "big picture" and introduce **exploratory factor analysis (EFA)**
- How can we learn about the latent variables that give rise to observed correlations between manifest variables?
- The general factor model is a model for the **covariance matrix** of the manifest variables
- What information do we get out of a factor analysis?
- What is factor rotation and how does it help us?
- We'll also introduce the idea of EFA vs. confirmatory factor analysis (CFA)

Why factor analysis?

- Testing of theory
  - Explain covariation among multiple observed variables by
  - Mapping variables to latent constructs (called factors)
- Understanding the structure underlying a set of measures
  - Gain insight into the dimensions
  - Construct validation
- Scale development
  - Exploit redundancy to improve scale's validity and reliability

A brief note on the history of factor analysis

- Factor analysis was originally developed by Spearman (1904) for the case of one common factor, and then later generalized by Thurstone (1947) and others to the case of multiple factors.
- It is important to note that the origins of factor analysis were in the eugenics movement of the early 20th century and attempts to theorize a construct *"g"*, "general intelligence", that would justify the supposed "superiority" of particular racial and class groups. *
- Over a century of development, researchers have approached the technique with many different perspectives, varying vocabulary, and proposals for model fitting. This can make it hard to decide which of the many options to use in SAS, Stata, or R.
- Fits into the broader modern context of statistical learning algorithms, especially algorithms for dimension reduction.

* See: Wintroub, M. Sordid genealogies: a conjectural history of Cambridge Analytica's eugenic roots. Humanit Soc Sci Commun 7, 41 (2020). https://doi.org/10.1057/s41599-020-0505-5

An example: Centers for Epidemiologic Studies Depression Scale (CES-D)

(direct quotes from Radloff 1977)

- The CES-D scale is a short self-report scale designed to measure **depressive symptomatology** in the general population.
- The items of the scale are symptoms associated with depression which have been used in previously validated longer scales.
- The new scale was tested in household interview surveys and in psychiatric settings.
- It was found to have **very high internal consistency** and adequate **test-retest repeatability**.
- **Validity** was established by patterns of correlations with other self-report measures, by correlations with clinical ratings of depression, and by relationships with other variables which support its construct validity.
- **Reliability, validity,** and **factor structure** were similar across a wide variety of demographic characteristics in the general population samples tested.

* Radloff LS. The CES-D scale: a self-report depression scale for research in the general population. *Applied Psychological Measurement*. 1977;1:385-401 (cited over 40,000 times)

Factor analysis of the CESD

The 20 questions ask about the subject's experiences in the past week and are scored on a four point scale:

- 0 = Rarely or none of the time (less than 1 day)
- 1 = Some or a little of the time (1-2 days)
- 2 = Occasionally or a moderate amount of time (3-4 days)
- 3 = Most or all of the time (5-7 days)

The items include:

1. I was bothered by things that usually don't bother me.
2. I did not feel like eating; my appetite was poor.
3. I felt that I could not shake off the blues even with help from my family or friends.
4. I felt that I was just as good as other people.
5. I had trouble keeping my mind on what I was doing.
6. I felt depressed.
7. I felt that everything I did was an effort.
8. I felt hopeful about the future.
9. I thought my life had been a failure.
10. I felt fearful.
11. My sleep was restless.
12. I was happy.
13. I talked less than usual.
14. I felt lonely.
15. People were unfriendly.
16. I enjoyed life.
17. I had crying spells.
18. I felt sad.
19. I felt that people disliked me.
20. I could not get going.

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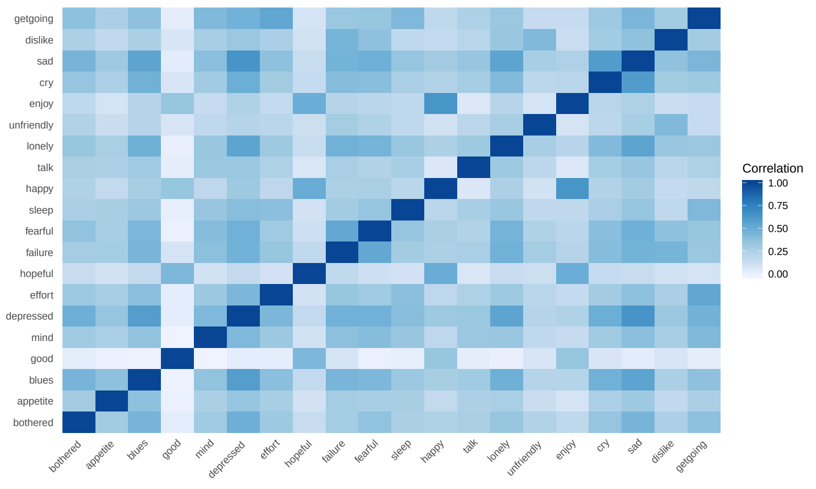
Correlation matrix of CES-D items



Is this a “lumpy” correlation matrix?

“Lost a planet, Master Obi-Wan has”

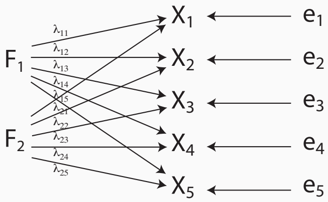
https://youtu.be/G67eMq1YwmI?t=27

General factor model

- We want something more parsimonious than just considering each item separately, but more flexible than assuming that all the items measure the same construct.
- How can we empirically see which items "go together"?
  - We want to take p observed items and simplify them into q factors, where $q < p$
  - The underlying math is a bit complicated (requires some matrix algebra)
  - Today we're focusing more on the concepts than the mathematical detail (but see supplemental slides)
  - Conceptually, we take the correlation matrix of all p items in the scale and extract q factors based on the relative strength or weakness of the items' correlations with each other
  - These q factors, together, should explain the covariation among the items.

General factor model

When we run a factor analysis model, what can we get out of it?



- **Factor loadings** (λs)
  - Characterizes how strongly each item correlates with, or "loads onto," each factor
  - Let you see which items "go together"
- **Communalities** (h_i) vs. **uniqueness** (u_i) of each item
  - let you see what fraction of the variation in each item is/is not explained by the factors
- **Eigenvalues** (sum of squared factor loadings)
  - Let you see how much value each additional factor adds (how much additional variance it explains)
  - Help you decide how many factors to keep in your model

Latent variable... or unique error?

Each manifest variable (theoretically) has two "causes": the **latent variable** (which represents *shared* variance) and **measurement error** (which represents *unique* variance). Let's think again about some of the items that make up the CESD

- "I did not feel like eating; my appetite was poor."
  - Were you depressed... or did you get a stomach bug?
- "I felt I was just as good as other people."
  - Were you depressed... or did you just do badly on a test?
- "I had trouble keeping my mind on what I was doing."
  - Were you depressed... or are you working from home with a 2 year-old?

Factor analysis

Suppose that we observe p random variables (i.e. manifest variables; indicators; items), $x_1, \ldots, x_p$ with means $\mu_1, \ldots, \mu_p$. We have collected these data on n individuals, so for each individual i, we observe a vector of item responses: $\mathbf{X_i} = X_{i1}, X_{i2}, \ldots, X_{ip}$.

The normal linear factor model assumes that

$$\mathbf{X}|\mathbf{F} \sim MVN_p(\boldsymbol{\mu} + \boldsymbol{\Lambda}\mathbf{F}, \boldsymbol{\Psi})$$

and

$$\mathbf{F} \sim MVN_q(\mathbf{0}, \mathbf{I})$$

where $\mathbf{F} = \mathbf{F}_1, \ldots, \mathbf{F}_q$ represents q unobserved *factors*, $\boldsymbol{\Lambda}$ is a $p \times q$ matrix of *factor loadings*, $\boldsymbol{\Psi}$ is a $p \times p$ diagonal matrix of *specific variances*, and $q < p$.

Factor analysis

If we mean center the everyone's responses, taking $\mathbf{X_i} - \boldsymbol{\mu} = X_{i1} - \mu_1, X_{i2} - \mu_2, \ldots, X_{ip} - \mu_p$, we can re-write this as

$$\mathbf{X} - \boldsymbol{\mu} = \boldsymbol{\Lambda}\mathbf{F} + \mathbf{e}$$

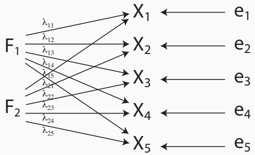
where $\mathbf{F} \sim MVN_q(\mathbf{0}, \mathbf{I})$, $\mathbf{e} \sim MVN_p(\mathbf{0}, \boldsymbol{\Psi})$, and $\mathbf{F}$ is independent of $\mathbf{e}$.

Note that $\boldsymbol{\Psi}$ being diagonal means $\text{Cov}(e_i, e_j) = 0$ for $i \neq j$.

Also, note that the observed $\mathbf{x}$'s are *conditionally independent* given the underlying factors.

General factor model

- In the example figure we looked at earlier, we had $q = 2$ factors, F_1 and F_2.
- Each of these factors had some relationship to each of the items, $\mathbf{X} = X_1, X_2, \ldots, X_p$, represented by an arrow on the diagragm.
- These relationships are called **factor loadings** and are designated λ_{kj}, where j indexes the item and k indexes the factor.
    - So, for example, λ_{24} is the factor loading relating factor 2 (F_2) to item 4 (X_4).
- Factor loadings may be standardized or unstandardized, but we'll be working with standardized factor loadings here.
- Standardized factor loadings range from -1 to 1 and can be interpreted much like a correlation – so factor loadings close to 1 or -1 indicate a stronger relationship between the factor and the item



General factor model

In matrix form, this is written as

$$\mathbf{X}_{p\times n} - \boldsymbol{\mu}_{p\times n} = \boldsymbol{\Lambda}_{p\times q}\mathbf{F}_{q\times n} + \mathbf{e}_{p\times n}$$

(You can compare this with our usual set up for linear regression)

$$\mathbf{Y}_{n\times 1} = \mathbf{X}_{n\times p}\boldsymbol{\beta}_{p\times 1} + \mathbf{e}_{n\times 1}$$

This is just a very compact way of writing out the relationships between each item and the underlying factors across the population. So in the example with two underlying factors, for item *j* and person *i*, we can say that

$$X_{ij} - \mu_j = \lambda_{j1}F_{i1} + \lambda_{j2}F_{i2} + e_{ij}$$

In words: person *i*'s mean-centered value for item *j* is that person's value of factor 1 times the factor loading to factor 1 and item *j*, plus that person's value of factor 2 times the factor loading specific to factor 2 and item *j*, plus some random error specific to the person and item.

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[*] Recall that we can add and subtract matrices as long as their dimensions match, and we can multiply matrices as long as their inner dimensions match, which produces a matrix with dimension equal to their outer dimensions.

General factor model

- For item j and person i, we had

$$X_{ij} - \mu_j = \lambda_{j1}F_{i1} + \lambda_{j2}F_{i2} + e_{ij}$$

- What if we want to represent this for all p items?
- For person i:

$$\begin{bmatrix} X_{i1} \\ X_{i2} \\ \vdots \\ X_{ip} \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \\ \vdots & \vdots \\ \lambda_{31} & \lambda_{32} \end{bmatrix} \begin{bmatrix} F_{i1} \\ F_{i2} \end{bmatrix} + \begin{bmatrix} e_{i1} \\ e_{i2} \\ \vdots \\ e_{i3} \end{bmatrix}$$

- We can further extend this to q factors for $q > 2$ and write this in matrix notation:

$$\mathbf{X_i}_{p\times 1} - \boldsymbol{\mu}_{p\times 1} = \boldsymbol{\Lambda}_{p\times 1}\mathbf{F_i}_{q\times 1} + \mathbf{e_i}_{p\times 1}$$

- If we do this over all n subjects,

$$\mathbf{X}_{p\times n} - \boldsymbol{\mu}_{p\times n} = \boldsymbol{\Lambda}_{p\times q}\mathbf{F}_{q\times n} + \mathbf{e}_{p\times n}$$

General factor model

Note that so far, we have been writing this model in terms of latent variables, F_1 and F_2, that are fundamentally unobservable. What **do** we observe?

- We observe $\mathbf{X_i}$ and $\boldsymbol{\mu}$ (the scale items and their means)
- We don't observe $F_i = F_{i1}, \ldots, F_{iq}$ (the true values of the underlying latent factors)
- We want to estimate $\boldsymbol{\Lambda}$ (the factor loadings)

But how are we supposed to estimate $\boldsymbol{\Lambda}$ when we have so many unknowns in the model?

- With the model in this form, we can't!
- But: we saw heuristically that we could get a sense of which items seem to "go together" by examining the covariance or correlation matrix.
- What if we take the covariance of both sides of this equation?
- It also helps to standardize everything (i.e. subtract the mean from each item and divide by the standard deviation) so that we can work with the correlation matrix (makes the math simpler)

General factor model

Suppose that $\text{Cov}(\mathbf{X} - \boldsymbol{\mu}) = \boldsymbol{\Sigma}$. Then, from the conditions imposed on $\mathbf{F}$ ($\mathbf{F} \perp\!\!\!\perp \mathbf{e}$, $\mathbb{E}(\mathbf{F}) = \mathbf{0}$, $\text{Cov}(F_a, F_b) = 0$)

$$\text{Cov}(\mathbf{x} - \boldsymbol{\mu}) = \text{Cov}(\boldsymbol{\Lambda}\mathbf{F} + \mathbf{e}),$$

or

$$\begin{aligned} \boldsymbol{\Sigma} &= \boldsymbol{\Lambda}\text{Cov}(\mathbf{F})\boldsymbol{\Lambda}^T + \text{Cov}(\mathbf{e}) \\ &= \boldsymbol{\Lambda}\boldsymbol{\Lambda}^T + \boldsymbol{\Psi} \end{aligned}$$

Thus, the factor analysis model is a model for the covariance matrix of the manifest variables that partitions the covariance matrix into that which is predicted by *common factors*, $\boldsymbol{\Lambda}\boldsymbol{\Lambda}^T$, and that which is unique, $\boldsymbol{\Psi}$ (the diagonal matrix of *uniquenesses*).

Note that this model no longer is written in terms of things we can't see (the factors), so now we can estimate $\boldsymbol{\Lambda}$!

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* Note that $\text{Cov}(\mathbf{F}_i) = \mathbf{I}$ (the identity matrix) because the factors are assumed to be uncorrelated and so $\text{Cov}(\mathbf{F}_i)$ drops out. We can relax this assumption later.

* Remember that $\boldsymbol{\Lambda}$ is just the matrix of all factor loadings, i.e. the λ_{kj}s.

Factor analysis

Wait… what just happened?

- Don't worry to much about the matrix algebra.
- The important thing to understand is that our observed item scores $\mathsf{X_i}$, are a **linear combination** of the true factor values, $\mathsf{F_i}$, and the factor loadings, $\mathbf{\Lambda}$.
- And even though $\mathsf{F_i}$ is unobserved, we can still estimate $\mathbf{\Lambda}$ using the covariance matrix of the Xs
  - much like how we estimated Cronbach's α only using the covariance matrix (for a single latent construct)
  - or how we could intuitively look at a "lumpy" correlation matrix and group items together

Non-uniqueness of the factor solution

Note that for any orthogonal matrix $\mathbf{Q}$, if we set $\mathbf{\Lambda}^* = \mathbf{\Lambda}\mathbf{Q}$ and $\mathbf{F}^* = \mathbf{Q}^T\mathbf{F}$, the criteria for being factors and factor loadings still hold. I.e.

$$\mathbf{x} - \boldsymbol{\mu} = \mathbf{\Lambda}\mathbf{F} + \mathbf{e} = \mathbf{\Lambda}\mathbf{Q}\mathbf{Q}^T\mathbf{F} + \mathbf{e}$$

(since $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$).

Thus, the set of factors and factor loadings is unique only up to an orthogonal transformation. That is, for a given matrix of factor loadings $\mathbf{\Lambda}$ and set of factors $\mathbf{F}$, we can apply an orthogonal transformation and obtain the same correlation matrix $\mathbf{\Sigma}$. We can make factors more easily interpretable by **rotating** the solution while keeping the number of factors and communalities of X's fixed.

Note that rotation does NOT improve model fit!

---

*Recall that a matrix $\mathbf{Q}$ is orthogonal if its transpose is equal to its inverse, $\mathbf{Q}^T = \mathbf{Q}^{-1}$.

Factor rotation

We redefine factors such that loadings (or pattern matrix coefficients) on various factors tend to be very high (-1 or 1) or very low (0). This makes for sharper distinctions in the meanings of the factors.

In effect, we make use of the non-uniqueness of solution to make interpretation more simple.

Some criteria for evaluating rotated solutions

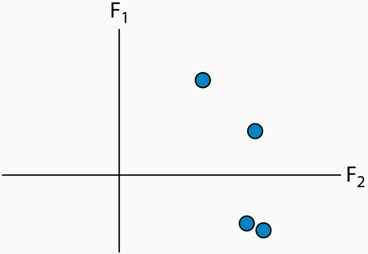
Thurstone (1947) proposed and argued for five criteria that needed to be met for "simple structure" to be achieved:

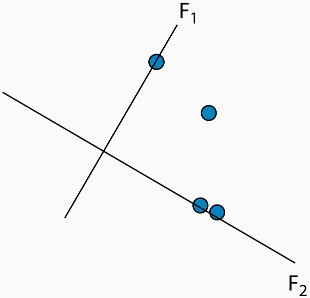
1. Each variable should produce at least one zero loading on some factor.
2. Each factor should have at least as many zero loadings as there are factors.
3. Each pair of factors should have variables with significant loadings on one and zero loadings on the other.
4. Each pair of factors should have a large proportion of zero loadings on both factors (if there are say four or more factors total).
5. Each pair of factors should have only a few complex variables.

Some helpful hints:

- What's a zero loading? One rule of thumb (after Gorsuch, 1983, p. 180) is that zero loadings includes any that fall between -0.10 and +0.10.
- What's a significant loading? With a sample size of say 100 participants, loadings of 0.30 or higher can be considered significant, or at least salient (see discussion in Kline, 2002, pp. 52-53).
- What are complex variables? Simply put, these are variables with loadings of 0.30 or higher on more than one factor.

Factor rotation





	Factor 1	Factor 2		Factor 1	Factor 2
x_1	0.4	0.69	x_1	-0.8	0
x_2	0.4	0.69	x_2	-0.8	0
x_3	0.65	0.32	x_3	-0.6	0.4
x_4	0.69	-0.4	x_4	0	0.8
x_5	0.61	-0.35	x_5	0	0.7

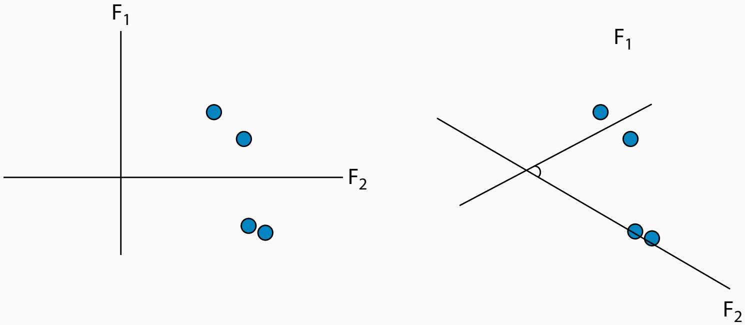
Factor rotation

- **Orthogonal**: factors are *independent*
  - **varimax:** maximize variance of squared loadings across variables (sum over factors)
    Goal: the simplicity of interpretation of factors
  - **quartimax:** maximize variance of squared loadings across factors (sum over variables)
    Goal: the simplicity of interpretation of variables
  - **Intuition:** on the previous slide, there is a right angle between axes (factors)
  - **Note: "uniquenesses" remain the same**

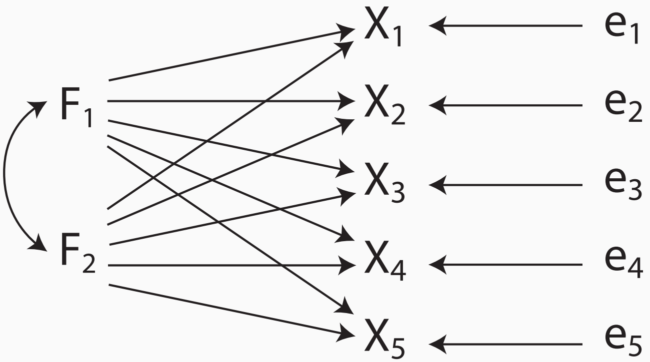
Factor rotation

- **Oblique**: Factors are **not** independent. (Not a 90 degree angle)
    - **oblimin:** minimize covariance of squared loadings between factors.
    - **promax:** simplify orthogonal rotation by making small loadings even closer to zero.
    - **Target matrix:** choose “simple structure” a priori.
    - **Intuition:** from previous figure, angle between axes is not necessarily a right angle.
    - **Note: “Uniquenesses” remain the same!**

Oblique rotation



Correlated factors



Pattern vs. Structure Matrix

- In oblique rotation, one typically presents both a **pattern matrix** and a **structure matrix**
- Also need to report correlation between the factors (ϕ)
- The **pattern matrix** presents the usual factor loadings ($\boldsymbol{\Lambda}$)
- The **structure matrix** presents correlations between the variables and the factors ($\boldsymbol{\Gamma} = \boldsymbol{\Lambda}\phi$)
- The modeled correlation matrix is the pattern matrix times the structure matrix plus the the uniquenesses

$$\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Gamma}^T + \boldsymbol{\Psi} = \boldsymbol{\Lambda}\phi\boldsymbol{\Lambda}^T + \boldsymbol{\Psi}$$

- For orthogonal factors, pattern matrix=structure matrix

Factor analysis of the CES-D

Table 11. Factor Loadings of CES Items for All Q1 Whites, All Q2 Whites and All Q3 Whites

	Factor I Depressed			Factor II Positive			Factor III Somatic			Factor IV Interpersonal			Item Communalities		
Item	Q1	Q2	Q3	Q1	Q2	Q3	Q1	Q2	Q3	Q1	Q2	Q3	Q1	Q2	Q3
1. Bothered	-.35	.24	.23	.01	-.11	.09	-.46	.43	-.51	-.01	-.02	-.10	.34	.26	.33
2. Appetite	-.16	.20	.12	.07	-.09	.00	-.54	.45	-.50	-.00	-.05	.13	.32	.25	.28
3. Blues	-.58	.61	.60	.19	-.11	.15	-.36	.44	-.41	.12	-.09	-.13	.52	.59	.57
4. Good	-.17	.11	.11	-.66	.63	-.68	.10	.01	.01	-.13	.28	.11	.49	.48	.48
5. Mind	-.21	.28	.24	.05	-.31	.10	-.58	.37	-.59	.17	-.21	-.11	.41	.36	.43
6. Depressed	-.63	.66	.64	.23	-.24	.18	-.43	.40	-.43	.16	-.12	-.15	.66	.66	.65
7. Effort	-.09	.06	.15	.05	.14	.07	-.67	.65	-.64	.07	-.12	-.06	.46	.45	.44
8. Hopeful	.09	-.12	-.10	-.70	.70	-.68	-.01	-.04	.06	.03	.05	-.01	.49	.51	.47
9. Failure	-.38	.29	.44	.22	-.18	.28	-.12	.30	-.07	.45	-.29	-.11	.41	.29	.29
10. Fearful	-.41	.55	.31	.03	-.03	.19	-.23	.25	-.26	.33	-.23	-.13	.33	.42	.22
11. Sleep	-.30	.08	.21	.05	-.13	-.01	-.47	.63	-.55	-.05	.06	.07	.31	.42	.35
12. Happy	.32	-.46	-.38	-.66	.52	-.62	.22	-.25	.25	-.06	-.13	.05	.59	.56	.59
13. Talk	-.12	.23	.00	.09	.01	.10	-.44	.39	-.54	.21	-.30	-.20	.25	.30	.35
14. Lonely	-.65	.67	.72	.09	-.11	.06	-.21	.20	-.18	.20	-.07	-.09	.51	.51	.57
15. Unfriendly	-.00	.04	.15	.01	-.10	.07	-.12	.14	-.07	.78	-.74	-.84	.62	.57	.74
16. Enjoy	.31	-.42	-.35	-.70	.55	-.68	.08	-.27	.14	-.09	-.12	-.02	.60	.56	.60
17. Cry	-.70	.76	.65	.03	.02	.01	-.09	.01	-.15	.04	-.04	.04	.50	.57	.45
18. Sad	-.73	.75	.78	.14	-.18	.09	-.28	.21	-.20	.18	-.19	-.15	.67	.68	.68
19. Dislike	-.26	.20	.15	.09	-.05	.04	-.08	.03	-.08	.72	-.79	-.83	.61	.66	.73
20. Get Going	-.07	.18	.14	.10	-.14	.11	-.69	.66	-.66	.19	-.16	-.07	.52	.51	.48
SS Cols.	3.116	3.504	3.164	2.036	1.784	1.997	2.757	2.682	2.921	1.701	1.651	1.607	9.610	9.622	9.688

* Radloff LS. The CES-D scale: a self-report depression scale for research in the general population. *Applied Psychological Measurement*. 1977;1:385-401 (cited over 40,000 times)

Factor analysis of the CES-D

- **Depressed affect** (blues, depressed, lonely, cry sad)
- **Positive affect** (good, hopeful, happy, enjoy)
- **Somatic and retarded activity** (bothered, appetite, effort, sleep, get going)
- **Interpersonal** (unfriendly, dislike)

"Including items with loadings of at least 0.35 in at least two groups would add the items *failure, fearful, happy,* and *enjoy* to the depressed affect factor and the items *blues, mind, depressed,* and *talk* to the somatic factor."

Factor analysis of the CES-D

"In all three groups, the depressed affect factor shares the largest proportion of the variance (about 16%) and the interpersonal factor, the smallest proportion (about 8%)."

Factor analysis of the CES-D

"The factors found in the general population are consistent with the components of depression built into the scale. However, the high internal consistency of the scaler found in all groups argues against undue emphasis on separate factors. The items are all symptoms related to depression. For epidemiologic research, a simple total score is recommended as an estimate of the degree of depressive symptomatology."

EFA and CFA

Exploratory Factor Analysis (EFA) vs. Confirmatory Factor Analysis CFA)

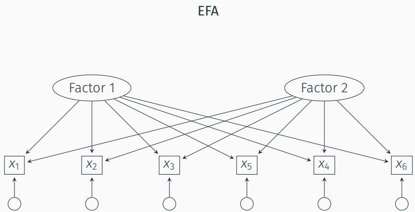
Both are *data reduction* techniques (taking many indicators and ideally extracting fewer underlying factors):

- **Exploratory Factor Analysis (EFA)** explores how items are related to potential underlying latent variables.
- **Confirmatory Factor Analysis (CFA)** allows you to test and compare how well different *a priori* hypothesized factor structures fit your data.

* Special thanks to Gabe Schwartz for these slides

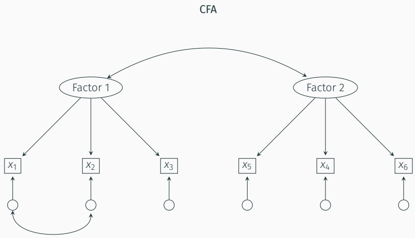
Why EFA?

- Measurement theory (or substantive theory) in an area is sparse.
- Not sure whether a given construct is unidimensional, two-dimensional, etc.
- Not sure whether the indicators you have chosen to measure a construct are good measures of a single thing (whatever that thing might be) – that they "hang together"
- This can happen if we are:
    - developing a new scale
    - testing an existing scale in a completely new context
    - adding new items to an existing scale
    - drastically adapting an existing scale

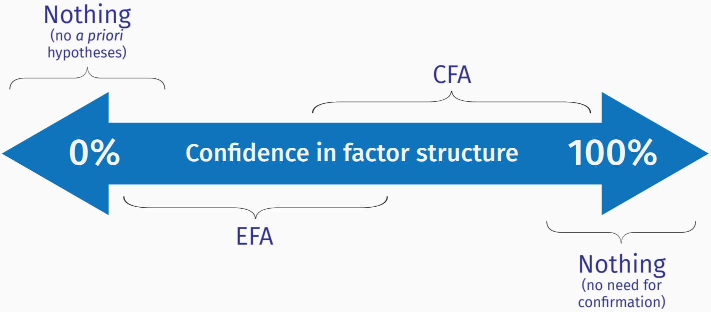


Why CFA?

- We have a candidate model we have hypothesized from EFA results and want to confirm that this model fits well in a different sample
- We have a small set of competing models we want to compare
- We think we already know how indicators fit together to form latent constructs, but are not sure if it is a good fit for *our* data or in *our study's context*
    - Original confirmatory studies were run on idiosyncratic data
    - e.g. "most work on X construct was done on predominantly white undergraduates at a private Midwestern university"



EFA vs. CFA


Nothing
(no *a priori* hypotheses)
CFA
0%
Confidence in factor structure
100%
EFA
Nothing
(no need for confirmation)

A quick note on CFA

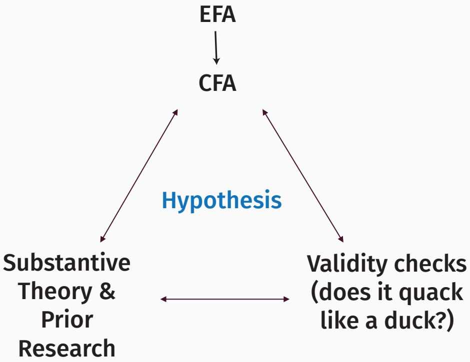
- "confirmatory" factor analysis is a bit of a misnomer
- CFA goodness of fit statistics can tell you whether your model is **A** good fit to the data, and allow you to compare whether one of several candidate models fits the data better than the other(s)
- It does not tell you whether it is the **CORRECT** model

If your model fits well $\rightarrow$ Evidence is consistent with your hypothesis (*but does not prove it*).

If your model fits poorly $\rightarrow$ Evidence against your hypothesis

This is an art as much as a science

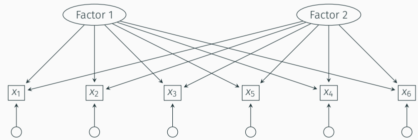
- This does not mean measurement models aren't real science!
- But the fact is: we will never be able to prove, in a counterfactual sense, that specific theoretical constructs exist; we can simply build up an evidence base for them, and use quantitative methods to support or refute the hypotheses we develop.
- Measurement models are triangulation tools.



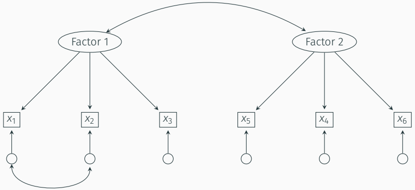
EFA vs. CFA

- EFA & CFA are quite similar mechanistically.
- Core differences:
    1. EFA allows indicators to load freely on all factors, whereas CFA fixes factor loadings to zero when not specified.
    2. EFA usually includes solution rotation to try to get at a simpler factor structure, whereas CFA does not.
    3. EFA is often run assuming factors are uncorrelated (at least at first), whereas CFA always estimates a correlation.
    4. EFA does not permit error correlations, whereas CFA does.

EFA



CFA



Steps

EFA:

1. Get all items to be on the same scale (rescale & reverse-code as necessary).
2. Examine correlation matrix.
    - are items positively, substantively correlated?
    - if not, can theory guide you?
3. Fit EFA and choose number of factors: examine eigenvalues, scree plot, and factor loadings
4. Rotate (this is almost always a good idea)
5. Evaluate whether your results make sense; re-evaluate as necessary

CFA:

1. Specify candidate model(s) (often only one if you are trying to reproduce what you hypothesized from an EFA)
2. Choose standardizing method
3. Fit; examine fit statistics
4. Compare model fits across candidate models; reject all models that do not fit the data well
5. If your hypothesized EFA model doesn't fit well, think: is there information in my EFA results or background theory that could provide insight?

References

1. Bartholomew D, Knott M, Moustaki I. (2011) *Latent variable models and factor analysis: a unified approach, 3rd edition*. New York: John Wiley & Sons.
2. DeVellis RF. (2017) *Scale Development: Theory and Applications (4th Edition)* Thousand Oaks: Sage
3. Revelle, W. (2016) *An introduction to psychometric theory with applications in R*, Chapter 6. http://personality-project.org/r/book/.

Supplemental Material: Eigen Decomposition

Motivation

A factor analytic model is a model for the $p \times p$ correlation matrix $\mathbf{R}$

$$\mathbf{R} = \mathbf{\Lambda}\mathbf{\Lambda}^T + \mathbf{\Psi}$$

where $\mathbf{\Lambda}$ is a $p \times q$ matrix of factor loadings (coefficients that describe how the q latent factors are related to the p observed items) and $\mathbf{\Psi}$ is a diagonal matrix of uniquenesses (i.e. the contributions of the item-specific errors).

Let's for the moment focus on the problem of figuring out what the factor loadings are. To do that, let's ignore the $\mathbf{\Psi}$ term for the moment and focus on the idea of *decomposing* the correlation matrix.

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*I'm using $\mathbf{R}$ to denote the correlation matrix instead of $\mathbf{\Sigma}$ to match the notation in Revelle (2016)

Eigen Decomposition

Eigen decomposition provides us with a way of factoring the correlation matrix into a form that looks like what we are after in terms of the $\boldsymbol{\Lambda}\boldsymbol{\Lambda}^T$ part of the factor model (i.e. the product of a matrix and its transpose).

Given a $p \times p$ matrix R, each eigenvector, $\mathsf{x_i}$ solves the equation

$$\mathsf{x_i R} = \lambda_i \mathsf{x_i}.$$

The set of p eigenvectors are solutions to the equation

$$\mathsf{RX} = \mathsf{X}\boldsymbol{\lambda}$$

where X is a matrix of orthogonal eigenvectors and $\boldsymbol{\lambda}$ is a diagonal matrix of the eigenvalues, λ_i. Equivalently,

$$\mathsf{R} = \mathsf{X}\boldsymbol{\lambda}\mathsf{X}^{-1}$$

Eigen Decomposition

That the vectors making up $\mathbf{X}$ are *orthogonal* means that

$$\mathbf{X}\mathbf{X}^T = \mathbf{I}$$

Since $\mathbf{X}$ is orthogonal, its transpose is equal to its inverse, so we can also write

$$\mathbf{R} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^T$$

Thus, it is possible to recreate the correlation matrix $\mathbf{R}$ in terms of an orthogonal set of vectors (the eigenvectors) scaled by their associated eigenvalues.

Eigen Decomposition

If we define $\mathsf{C} = \mathsf{X}\sqrt{\boldsymbol{\lambda}}$ then we can write

$$\mathsf{R} = \mathsf{C}\mathsf{C}^T$$

and thus C is a matrix of loadings.

Principal Components

This is exactly the approach taken by **principal components analysis** (PCA), a popular method for data reduction that has similarities to factor analysis. Typically we don't want to reproduce exactly the original $p \times p$ correlation matrix since we would not gain parsimony if $\mathbf{C}$ is the same rank as the rank of the original $\mathbf{R}$ matrix. Rather, we want to approximate $\mathbf{R}$ with a matrix of lower rank ($k < p$). This can be done by using just the first k principal components so that

$$\mathbf{R} \approx \mathbf{C}\mathbf{C}^T$$

Principal Components

Note that, unlike the factor analytic model, this model for the correlation matrix $\mathbf{R}$ does not involve the contribution of item-specific error terms. Thus, principal components analysis is a method for decomposing the observed correlation matrix that does not posit the existence of underlying latent factors or independent sources of error.

A Simple Example

Let's look at a simple eigen decomposition of a 2×2 matrix. Let's say that we have a 2×2 matrix $\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$ upon which we would like perform eigen decomposition into a matrix of eigenvectors $\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a diagonal matrix of eigenvalues $\boldsymbol{\lambda}$.

$$\mathbf{R}\mathbf{X} = \mathbf{X}\boldsymbol{\lambda}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

A Simple Example

The above equation can be decomposed into 2 simultaneous equations:

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} a\lambda_1 \\ c\lambda_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} b\lambda_2 \\ d\lambda_2 \end{bmatrix}$$

A Simple Example

Factoring out the eigenvalues λ_1 and λ_2,

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \lambda_1 \begin{bmatrix} a \\ c \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \lambda_2 \begin{bmatrix} b \\ d \end{bmatrix}$$

Letting $\vec{a} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b \\ d \end{bmatrix}$ gives us two vector equations,

$$\mathsf{R}\vec{a} = \lambda_1 \vec{a}$$
$$\mathsf{R}\vec{b} = \lambda_2 \vec{b}$$

A Simple Example

This can be represented by a single vector equation involving 2 solutions as eigenvalues:

$$\mathbf{R}\mathbf{u} = \lambda \mathbf{u}$$

where λ represents the two eigenvalues and $\mathbf{u}$ represents the vectors $\vec{a}$ and $\vec{b}$. Shifting $\lambda\mathbf{u}$ to the left hand side and factorizing $\mathbf{u}$ out,

$$(\mathbf{R} - \lambda \mathbf{I})\mathbf{u} = 0$$

This only has a non-zero solution $\mathbf{u}$ if the determinant of the matrix $(\mathbf{R} - \lambda \mathbf{I})$ is zero.

A Simple Example

The eigenvalues of $\mathbf{R}$ are thus the values of λ that satisfy the equation

$$\begin{aligned}
\det(\mathbf{R} - \lambda \mathbf{I}) &= 0 \\
\begin{vmatrix} 1-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} &= 0 \\
(1-\lambda)(3-\lambda) &= 0
\end{aligned}$$

This gives us the solutions of the eigenvalues for the matrix $\mathbf{R}$ as $\lambda = 1$ or $\lambda = 3$, and the resulting diagonal matrix from the eigendecomposition of R is thus $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

(Note that for a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(\mathbf{A}) = ad - bc$).

A Simple Example

Substituting these values of λ_1 and λ_2 into the above simultaneous equations,

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = 1 \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = 3 \begin{bmatrix} b \\ d \end{bmatrix}$$

Multiplying through, we get

$$\begin{bmatrix} a \\ a + 3c \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\begin{bmatrix} b \\ b + 3d \end{bmatrix} = \begin{bmatrix} 3b \\ 3d \end{bmatrix}$$

Solving the equations, we have $a = -2c, c \in \mathbb{R}$ and $b = 0, d \in \mathbb{R}$. Thus, the matrix X required for the eigendecomposition of R is $\begin{bmatrix} -2c & 0 \\ c & d \end{bmatrix}$, $[c, d] \in \mathbb{R}$.

Principal Axis Factor Analysis

Now let's return to our factor analytic problem where our model for the correlation matrix is

$$\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^T + \boldsymbol{\Psi}$$

While principal components attempts to reproduce all of the $\boldsymbol{\Sigma}$ matrix, factor analysis attempts to model just the common parts of the matrix, which means all of the off-diagonal elements and the common part of the diagonal (the *communalities*). The non-common part, the *uniquenesses*, are the part which is left over after we take out the communalities.

Principal Axis Factor Analysis

In **principal axis factor analysis**, the communalities are found iteratively by doing eigen decomposition of the correlation matrix and replacing the original 1s on the diagonal with the value of $1 - u_i^2$ where

$$\mathrm{U}^2 = diag(\mathbf{\Sigma} - \mathbf{\Lambda}\mathbf{\Lambda}^T)$$

Principal Axis Factor Analysis

More formally (using m to index iterations of this algorithm):

1. Begin with the eigen decomposition of the original correlation matrix $\mathbf{\Sigma}$, which we will designate $\mathbf{\Sigma}_{(0)}$.
2. Obtain current estimates of the factor loadings, $\mathbf{\Lambda}_{(m)}$ from the first k eigenvectors and eigenvalues.
3. Use the current estimates of the factor loadings to predict the correlation matrix

$$\mathbf{\Sigma}^*_{(m)} = \mathbf{\Lambda}_{(m)}\mathbf{\Lambda}^T_{(m)}$$

4. Replace the diagonal of $\mathbf{\Sigma}_{(0)}$ with the diagonal of $\mathbf{\Sigma}^*_{(m)}$ to get a new $\mathbf{\Sigma}_{(m)}$.
5. Perform the eigen decomposition of $\mathbf{\Sigma}_{(m)}$.
6. Repeat steps 2-5 until the change in the sum of the communalities $\left[\text{trace}(\mathbf{\Sigma}^*_{(m-1)}) - \text{trace}(\mathbf{\Sigma}^*_{(m)})\right]$ is negligible.

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*(Remember that the trace of a square matrix $\mathbf{A}$, denoted trace($\mathbf{A}$), is the sum of its diagonal elements).