Q: What is QFT?

A: QM + locality

local ("UV complete") QFT

- QM w/ Poincaré sym (and necessarily ∞-ly many d.o.f.)
- local "field" operators
- micro causality

Examples:

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- ϕ^4 theory in $D=2,3$
- Statistical Ising model in $D=2,3$ near criticality

253b

- Yang-Mills / QCD

Effective Field Theory

- defined perturbatively based on a path integral over space of fields
- captures long-distance low-energy observables

renormalizable

- ϕ^4 theory in $D=4$
- QED
- standard model

- chiral perturbation theory
- GR

non-renormalizable

Plan of 253a

(1) Lagrangian formulation of QM

- path integral, regularization
- perturbation theory, Feynman diagrams
- renormalization and counter terms

(2) Relativistic particles and fields

- ϕ^4 theory
- Green functions
- asymptotic states
- S-matrix, LSZ reduction

(3) Particles and fields with spin

- classification of relativistic particles
- fermions and gauge bosons
- QED
- Applications: electron g-factor, Lamb shift.

Prelude: why do we need "field theory" to describe QM of particles with relativistic symmetry?

- Hilbert space $\mathcal{H}$
  - vacuum $|\Omega\rangle$, $H|\Omega\rangle = 0$.
  - single particle $|\vec{p}\rangle$

$p^\mu \equiv (p^0, \vec{p})$
D-dimensional

$$H|\vec{p}\rangle = \sqrt{\vec{p}^2 + m^2}\,|\vec{p}\rangle.$$

  - multi-particle $|\vec{p}_1, \cdots, \vec{p}_n\rangle$
(non-interacting)

$$H = \sum_{i=1}^{n} \sqrt{\vec{p}_i^{\,2} + m^2}.$$

Equivalently, we can introduce

$$\left.\begin{array}{ll} \text{creation} & a^\dagger_{\vec{p}} \\ \text{annihilation} & a_{\vec{p}} \end{array}\right\} \text{ operators.}$$

$$|\vec{p}\rangle = a^\dagger_{\vec{p}}|\Omega\rangle$$

$$|\vec{p}_1, \vec{p}_2\rangle = a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} |\Omega\rangle, \quad \text{etc.}$$

$$[a, a] = [a^\dagger, a^\dagger] = 0.$$

$$[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = \delta^{(D-1)}(\vec{p} - \vec{p}')$$

so that $\langle \vec{p} | \vec{p}' \rangle = \delta^{(D-1)}(\vec{p} - \vec{p}')$

We can then express the Hamiltonian as

$$H = \int d^{D-1}\vec{p} \, \sqrt{\vec{p}^2 + m^2} \, a^\dagger_{\vec{p}} a_{\vec{p}}.$$

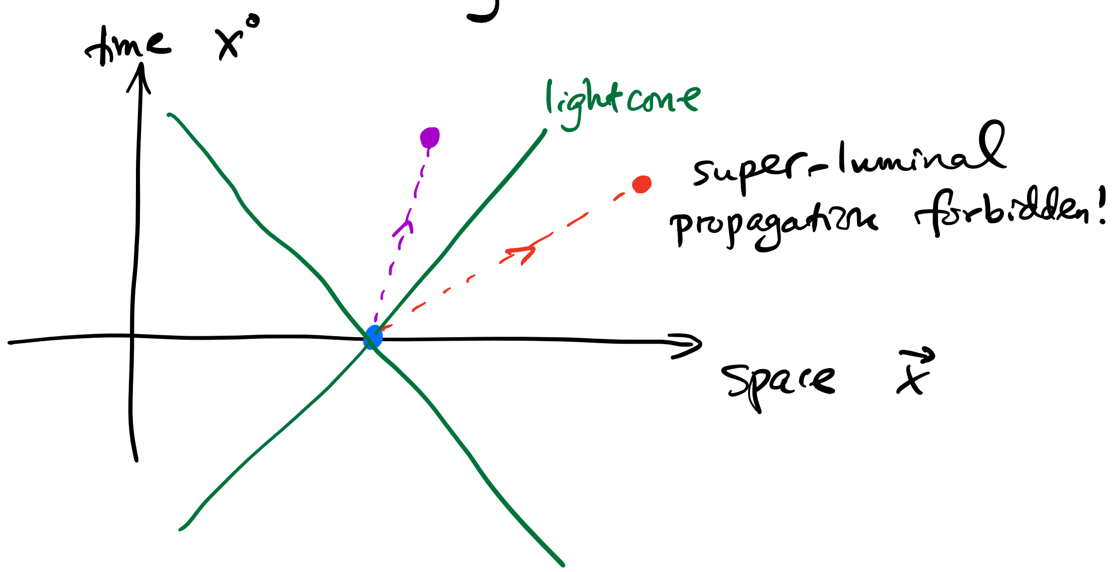
- a QM system of free relativistic particles. ✓

- What about interacting particles?

$$H = H_0 + \underbrace{H_{int}}_{\substack{\cup \\ a^\dagger a^\dagger a, \; a a a^\dagger, \; \cdots}}$$

- not easy to find H_{int} that respects relativistic symmetry!

Issue: Causality



In a generic QM system, signal propagation can be instantaneous ....

In a relativistic system, expect local disturbance to be represented by "field operator" $\hat{\phi}(x)$

$x^\mu = (x^0, \vec{x})$

x^μ are merely parameters, not themselves operators.

. $\hat{\phi}(x)$ and $\hat{\phi}(x')$ are related by Poincaré symmetry.

$$x^\mu \rightsquigarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu .$$

$$\eta^{\alpha\beta}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta^{\mu\nu}, \quad \eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

In QM, represented by a unitary operator $U(\Lambda, a)$ such that

$$\hat\phi(\Lambda x + a) = U(\Lambda, a)\,\hat\phi(x)\,\big(U(\Lambda, a)\big)^{-1}$$

It suffizes to study infinitesimal version

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu , \qquad a^\mu = \epsilon^\mu .$$

small, keep only 1st order

$$U(\Lambda, a) = 1 - i\epsilon^\mu \hat P_\mu + \frac{i}{2}\omega_{\mu\nu}\hat J^{\mu\nu}$$

energy-momentum

boost + angular momentum

$$U(\Lambda', a')\,U(\Lambda, a) = U(\Lambda'\Lambda, \Lambda' a + a')$$

$$[P, P] = 0.$$

$$[P^\mu, J^{\rho\sigma}] = -i(\eta^{\mu\rho}P^\sigma - (\rho \leftrightarrow \sigma))$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = -i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - (\rho \leftrightarrow \sigma))$$

Microcausality:

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0$$ whenever x, y are spacelike-separated.

i.e.

$$(x-y)^2 \equiv -(x^0-y^0)^2 + (\vec{x}-\vec{y})^2 > 0.$$

- What about our model of free relativistic particles?

$$H = \int d^{D-1}\vec{p}\, \sqrt{\vec{p}^2+m^2}\, a_{\vec{p}}^\dagger a_{\vec{p}},$$

$$\vec{P} = \int d^{D-1}\vec{p}\, \vec{p}\, a_{\vec{p}}^\dagger a_{\vec{p}}.$$

$$\hat{P}^\mu = (H, \vec{P}).$$

Does $\hat{\phi}(x)$ exist? Yes!

$$\hat{\phi}(x) = \int \frac{d^{D-1}\vec{p}}{\sqrt{(2\pi)^{D-1} 2\omega_{\vec{p}}}} \left(a_{\vec{p}}\, e^{ip\cdot x} + a_{\vec{p}}^\dagger\, e^{-ip\cdot x} \right)$$

$$p\cdot x \equiv \vec{p}\cdot\vec{x} - \omega_{\vec{p}} x^0.$$

$$\omega_{\vec{p}} \equiv \sqrt{\vec{p}^2 + m^2}.$$

Check micro causality:

$$[\hat{\phi}(x), \hat{\phi}(y)] = \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1} 2\omega_{\vec{p}}} \cdot \left[e^{ip\cdot(x-y)} - e^{-ip\cdot(x-y)} \right]$$

$$= \int \underbrace{\frac{d^D p^\mu}{(2\pi)^{D-1}} \theta(p^0)\, \delta(p^2+m^2)}_{\text{invariant under } p^\mu \to \Lambda^\mu{}_\nu p^\nu.} \left[e^{ip\cdot(x-y)} - e^{-ip\cdot(x-y)} \right]$$

$$\underbrace{\phantom{\theta(p^0)\,\delta(p^2+m^2)\left[e^{ip\cdot(x-y)} - e^{-ip\cdot(x-y)} \right]}}_{\text{invt under } x\to\Lambda x,\ y\to\Lambda y.}$$

result a function of $(x-y)^2$ only.

If $(x-y)^2 > 0$, WLOG can take $x^0 - y^0 = 0$.

$\vec{x} - \vec{y} \neq 0$.

integrand odd under $\vec{p} \to -\vec{p}$.

thus $\int \cdots = 0$,

i.e. $[\hat{\phi}(x), \hat{\phi}(y)] = 0$ for $(x-y)^2 > 0$. ✓

- Can also verify that
$$U(\Lambda, a)\, \hat{\phi}(x)\, (U(\Lambda, a))^{-1} = \hat{\phi}(\Lambda x + a).$$

Postulate that the properties

- Hilbert space $\mathcal{H}$
- Poincaré sym $U(\Lambda, a)$
- Poincaré-invt vacuum $|\Omega\rangle$
- local "field" operators $\hat{\phi}(x)$
  that obey Poincaré-covariance and
  microcausality.

hold for system of interacting relativistic particles as well!

- How to construct such theories?
- useful to work with a formalism in which Poincaré sym is manifest.

Lagrangian formalism

Recall in classical mechanics:

	Lagrangian	vs	Hamiltonian
"state"	$q, \dot{q}$		$q, p.$
EOM	extremize $S = \int dt\, L(q,\dot{q})$		$\dot{f} = \{H, f\}_{\text{Poisson}}$ e.g. $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$.

related by

$$H(p,q) = p\dot{q} - L(q,\dot{q})\Big|_{\frac{\partial L}{\partial \dot{q}} = p}.$$

QM:

	Hamiltonian	vs	Lagrangian
state	$\lvert\psi\rangle \in \mathcal{H}$ $H(p,q) \rightsquigarrow \hat{H}$		$S = \int dt\, L(q,\dot{q})$

time evolution

Schrödinger:

$$|\psi\rangle \mapsto U(t)|\psi\rangle,$$

$$U(t) = e^{-\frac{i}{\hbar}\hat{H}t}$$

Heisenberg:

$$\mathcal{O} \mapsto \underbrace{(U(t))^{-1}\mathcal{O}\,U(t)}_{\equiv\ \mathcal{O}(t)}.$$

$$\dot{\mathcal{O}}(t) = \frac{i}{\hbar}[\hat{H}, \mathcal{O}(t)]$$

$$\langle q_f | U(T) | q_i \rangle$$

$$= \int_{q(0)=q_i}^{q(T)=q_f} [Dq]\; e^{\frac{i}{\hbar}S}$$

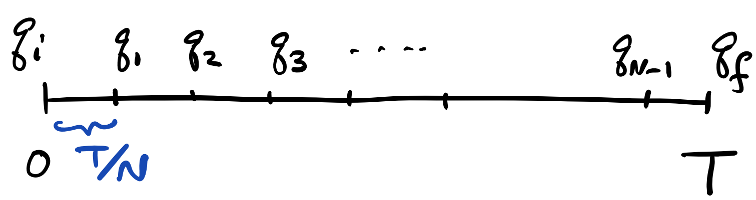
what is this ?

- Much of the subtleties with the Lagrangian formulation of QM have to do with the path integral measure $[Dq]$.

Let us revisit its derivation:

consider time evolution by T.

in N steps,



$$\langle q_f | U(T) | q_i \rangle$$

$$= \int d^D q_1 \cdots d^D q_{N-1} \langle q_f | e^{-\frac{i}{\hbar}\hat{H}\cdot\frac{T}{N}} | q_{N-1} \rangle$$

$$\cdots\cdots \langle q_1 | e^{-\frac{i}{\hbar}\hat{H}\frac{T}{N}} | q_i \rangle$$

[denote $q_f \equiv q_N$, $q_i \equiv q_0$]

$$= \int \prod_{n=1}^{N-1} d^D q_n \cdot \prod_{n=0}^{N-1} \underbrace{\langle q_{n+1} | e^{-\frac{i}{\hbar}\hat{H}\frac{T}{N}} | q_n \rangle}$$

$$\int \frac{d^D p_n}{(2\pi\hbar)^D} \underbrace{\langle q_{n+1} | p_n \rangle}_{= e^{\frac{i}{\hbar} p_n \cdot q_{n+1}}} \underbrace{\langle p_n | e^{-\frac{i}{\hbar}\hat{H}\frac{T}{N}} | q_n \rangle}_{\approx e^{-\frac{i}{\hbar} H(q_n, p_n)\frac{T}{N}} e^{-\frac{i}{\hbar} p_n \cdot q_n}}$$

classical Hamiltonian

$$= \int \prod_{n=1}^{N-1} d^D q_n \prod_{n=0}^{N-1} \frac{d^D p_n}{(2\pi\hbar)^D}$$

$$\cdot \exp\left\{ \frac{i}{\hbar} \sum_{n=0}^{N-1} \left[p_n \cdot (q_{n+1} - q_n) - H(q_n, p_n)\frac{T}{N} + \mathcal{O}(\hbar, (\tfrac{T}{N})^2) \right] \right\}$$

formally, in $N \to \infty$ "limit"

$$= \int \mathcal{D}q\, \mathcal{D}p \; e^{\frac{i}{\hbar}\int_0^T dt\,(p\cdot\dot{q} - H(q,p))}$$

defined via regularization

e.g. discretizing time

Contains $\mathcal{O}(\hbar)$ ambiguities

- "counter terms"
- these ambiguities are tied to the scheme of regularization.

An example of alternative time-lattice regularization:

$$\int_0^T dt\, p\,\dot{q} \;\rightsquigarrow\; \sum_{n=0}^{N-1} p_n\cdot(q_{n+1}-q_n)$$

$$\text{versus} \quad \sum_{n=0}^{N-1} p_{n+1}\cdot(q_{n+1}-q_n)$$

The exact representation of the time evolution is

$$\cdots\langle q_{n+1}|p_n\rangle \underbrace{\langle p_n| e^{-\frac{i}{\hbar}\hat{H}\frac{T}{N}}|q_n\rangle}\cdots$$

in the former case, and

$$\langle p_n|\hat{H}|q_n\rangle = H(p_n,q_n)\,\langle p_n|q_n\rangle$$

$$\cdots\langle p_{n+1}|q_n\rangle \underbrace{\langle q_n| e^{-\frac{i}{\hbar}\hat{H}\frac{T}{N}}|p_n\rangle}\cdots$$

in the latter case

$$\langle q_n|\hat{H}|p_n\rangle = H'(p_n,q_n)\,\langle q_n|p_n\rangle$$

$H(p_n,q_n)$ and $H'(p_n,q_n)$ can differ at order $\mathcal{O}(\hbar)$.

If $H(q,p)$ is quadratic in p.

e.g. $H(q,p) = \frac{p^2}{2m} + V(q)$.

we can integrate out $p(t)$ as Gaussian, leaving

$$\int [Dq]\, e^{\frac{i}{\hbar}\int_0^T dt\, \underbrace{(p\dot{q} - H)\Big|_{\dot{q} = \frac{\partial H}{\partial p}}}_{= L(q,\dot{q})}}$$

absorbs Gaussian determinant of $p(t)$

More generally, Lagrangian in generalized coordinates typically takes the form

$$L = \frac{1}{2} G_{\alpha\beta}(q)\, \dot{q}^\alpha \dot{q}^\beta - V(q) \qquad \left[\begin{array}{l}\text{e.g. rotating}\\ \text{3D rigid body}\end{array}\right]$$

$$\longleftrightarrow \quad H = \frac{1}{2} (G^{-1}(q))^{\alpha\beta} P_\alpha P_\beta + V(q) + \mathcal{O}(\hbar)$$

path integral $\displaystyle\int Dp\, Dq\; e^{\frac{i}{\hbar}\int dt\, (P_\alpha \dot{q}^\alpha - H(p,q))}$

$$= \int [Dq]\, e^{\frac{i}{\hbar}\int dt\, L(q,\dot{q})}$$

$$[Dq] \rightsquigarrow \prod_{n=1}^{N} \frac{dq_n^\alpha}{\sqrt{2\pi i \hbar \cdot \frac{T}{N}}} \cdot \sqrt{\det G_{\alpha\beta}(q_n)}.$$

- Consider correlation function in ground state $|0\rangle$, e.g.

$$G(t) \equiv \langle 0| \hat{q}(t) \, \hat{q}(0) |0\rangle$$

$$= \sum_n \langle 0| \hat{q}(t) |n\rangle \langle n| \hat{q}(0) |0\rangle$$

a basis of $\hat{H}$ - eigenstates

$\hat{H}|n\rangle = E_n |n\rangle$.

$$= \sum_n \langle 0| e^{\frac{i}{\hbar}\hat{H}t} \, \hat{q}(0) \, e^{-\frac{i}{\hbar}\hat{H}t} |n\rangle \langle n| \hat{q}(0) |0\rangle$$

$$= \sum_n \underbrace{\langle 0| \hat{q}(0) |n\rangle \langle n| \hat{q}(0) |0\rangle}_{\equiv \, \rho_n \text{, some number}} \cdot e^{-\frac{i}{\hbar}(E_n - E_0)t} .$$

As $E_n - E_0 \geq 0$, can analytically continue $G(t)$ as a function of complex t, provided $\mathrm{Im}\, t < 0$. so that $\sum_n$ converges.

[Paley - Wiener theorem]

e.g. $t = -i\tau$, $\tau \in \mathbb{R}_+$

"Wick rotation"

$$\langle 0 | \hat{q}(t)\, \hat{q}(0) | 0 \rangle = \sum_n \langle 0 | \hat{q}(0) | n \rangle \langle n | \hat{q}(0) | 0 \rangle \cdot e^{-\frac{1}{\hbar}(E_n - E_0)\tau}$$

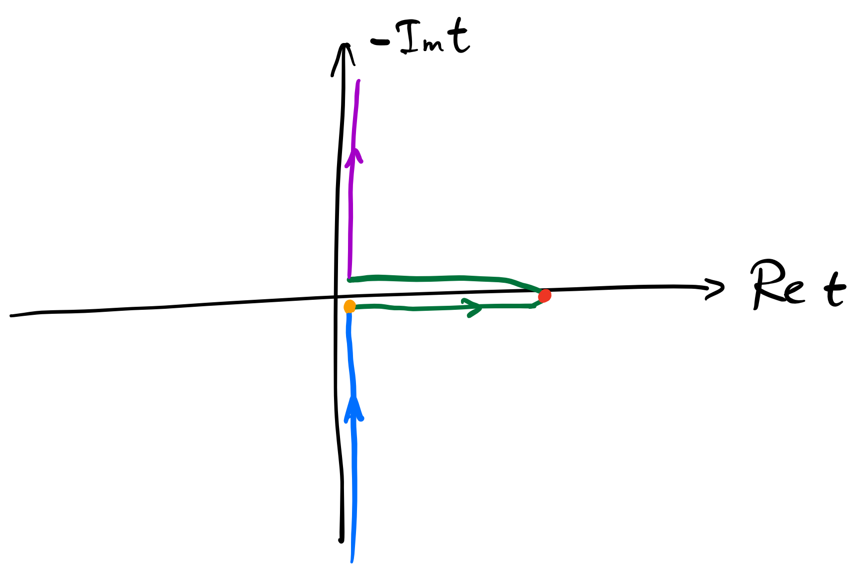
Assuming a gap $E_1 - E_0 > 0$, can replace $|0\rangle$ by $\lim\limits_{T \to \infty} e^{-\frac{1}{\hbar}(H - E_0)T} |\psi\rangle$,

up to overall normalization,

for any generic state $|\psi\rangle$.

(that has nonzero overlap with $|0\rangle$)

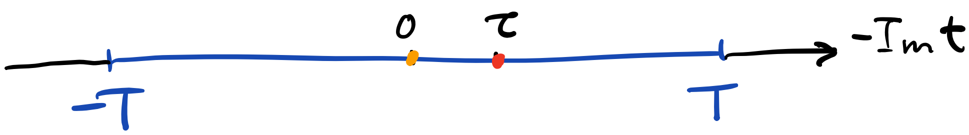
$$\langle 0 | \hat{q}(t)\, \hat{q}(0) | 0 \rangle = \lim_{T \to \infty} \frac{\langle \psi | \overbrace{e^{-\frac{1}{\hbar}HT}}\, \hat{q}(t)\, \hat{q}(0)\, \overbrace{e^{-\frac{1}{\hbar}HT}} | \psi \rangle}{\langle \psi | e^{-\frac{2}{\hbar}HT} | \psi \rangle}$$



Alternatively,

$$G(t=-i\tau) = \langle 0|\, e^{\frac{1}{\hbar}H\tau}\, \hat{q}(0)\, e^{-\frac{1}{\hbar}H\tau}\, \hat{q}(0)\,|0\rangle$$

$$= \lim_{T\to\infty} \frac{\langle\psi|\, e^{-\frac{1}{\hbar}H(T-\tau)}\, \hat{q}(0)\, e^{-\frac{1}{\hbar}H\tau}\, \hat{q}(0)\, e^{-\frac{1}{\hbar}HT}\,|\psi\rangle}{\langle\psi|\, e^{-\frac{2}{\hbar}HT}\,|\psi\rangle}$$



From path integral:

$$\langle 0|\, \hat{q}(t)\, \hat{q}(0)\,|0\rangle$$

$$= \lim_{\substack{\mathrm{Im}\, t_f \to -\infty \\ \mathrm{Im}\, t_i \to +\infty}} \frac{\int [Dq]\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} L[q,\dot{q}]dt}\, q(t)\, q(0)}{\int [Dq]\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} L[q,\dot{q}]}}.$$

Wick rotation:

$$q(t) \xrightarrow{t=-i\tau} q(\tau)$$

still use the same notation

$$\partial_t q = +i\,\partial_\tau q .$$

Define $\quad L[q, \dot{q}] \equiv -L^E[q, \partial_\tau q]$

so that

$$e^{\frac{i}{\hbar}\int L[q,\dot{q}]\,dt} \rightsquigarrow e^{-\frac{1}{\hbar}\overbrace{\int L^E[q,\partial_\tau q]\,d\tau}^{S^E}}.$$

e.g. $\quad L[q, \dot{q}] = \frac{1}{2}\dot{q}^2 - V(q)$

$$L^E[q, \partial_\tau q] = \frac{1}{2}(\partial_\tau q)^2 + V(q)$$

- How do we evaluate the path integral?

the key is to find a convenient regularized expression of the measure $[Dq]$.

e.g. $\quad q(\tau) = \sum_n q_n \underbrace{f_n(\tau)}$

a suitable basis of functions

$$[Dq] \rightsquigarrow \prod_{n=1}^{\infty} dq_n \quad ?$$

Example 1:

$$H = \frac{p^2 + \omega^2 q^2}{2} \longleftrightarrow L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2.$$

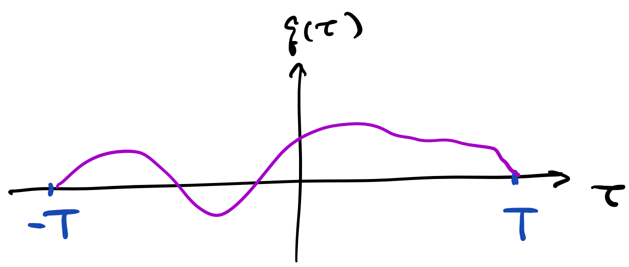
Wick rotation $t = -i\tau$

To evaluate

$$\int [Dq]\, e^{-\frac{1}{\hbar} S^E} \ldots$$

choose boundary condition in imaginary time

e.g. $q(-T) = q(T) = 0$.



Expand $q(\tau)$ on a Fourier basis

$$q(\tau) = \sum_{n=1}^{\infty} q_n f_n(\tau)$$

$$f_n(\tau) = \frac{1}{\sqrt{T}} \sin \frac{n\pi(\tau + T)}{2T}$$

• $f_n(\tau)$ is a basis of functions that obeys the prescribed boundary condition, orthonormal with respect to a suitable inner product, e.g.

$$(f_n, f_m) := \int_{-T}^{T} f_n^*(\tau)\, f_m(\tau)\, d\tau = \delta_{nm}.$$

and, for convenience, diagonalize the kinetic term in the action:

$$S^E = \int_{-T}^{T} \left[\tfrac{1}{2}(\partial_\tau q)^2 + \tfrac{1}{2}\omega^2 q^2 \right] d\tau$$

$$= \sum_{n=1}^{\infty} \left[\tfrac{1}{2} \left(\tfrac{n\pi}{2T}\right)^2 + \tfrac{1}{2}\omega^2 \right] q_n^2$$

$$\int [Dq]\, e^{-\frac{1}{\hbar} S^E} \cdots$$

$$\rightsquigarrow \underbrace{\int \prod_{n=1}^{\infty} dq_n\, e^{-\frac{1}{\hbar}\sum_{n=1}^{\infty} \frac{1}{2}\left[\left(\frac{n\pi}{2T}\right)^2 + \omega^2\right] q_n^2}}_{\text{Gaussian}} \cdots$$

write

$$\langle \cdots \rangle = \frac{\int [Dq]\, e^{-\frac{1}{\hbar} S^E} \cdots}{\int [Dq]\, e^{-\frac{1}{\hbar} S^E}},$$

Easy to evaluate using Gaussian integral:

$$\langle q_n q_m \rangle = \delta_{nm} \cdot \frac{\hbar}{\left(\frac{n\pi}{2T}\right)^2 + \omega^2}.$$

So $\langle q(\tau_1)\, q(\tau_2) \rangle$

$$= \sum_{n,m=1}^{\infty} \langle q_n q_m \rangle f_n(\tau_1) f_m(\tau_2)$$

$$= \sum_{n=1}^{\infty} \frac{\hbar}{\left(\frac{n\pi}{2T}\right)^2 + \omega^2} \cdot \frac{1}{T} \sin \frac{n\pi(\tau_1+T)}{2T} \sin \frac{n\pi(\tau_2+T)}{2T}$$

Take limit $T \to \infty$,

$$\frac{n\pi}{2T} \equiv k, \quad \sum_n \frac{\pi}{2T} \cdots \rightsquigarrow \int dk \cdots$$

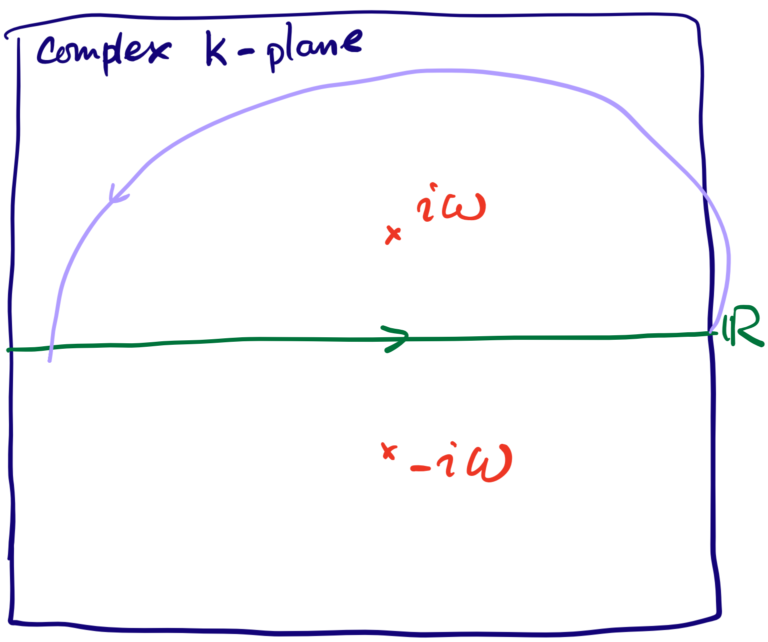
$$= \frac{2\hbar}{\pi} \int_0^\infty dk \cdot \frac{\sin k(\tau_1+T) \cdot \sin k(\tau_2+T)}{k^2 + \omega^2}$$

$$= \frac{\hbar}{\pi} \int_{-\infty}^{\infty} dk \cdot \frac{1}{k^2+\omega^2} \Big[\tfrac{1}{4} e^{ik(\tau_1-\tau_2)} + \tfrac{1}{4} e^{-ik(\tau_1-\tau_2)}$$
$$- \tfrac{1}{4} e^{ik(\tau_1+\tau_2+2T)} - \tfrac{1}{4} e^{-ik(\tau_1+\tau_2+2T)} \Big]$$

Consider $\displaystyle\int_{-\infty}^{\infty} dk \, \frac{e^{ik\tau}}{k^2+\omega^2}$

integrand as an analytic function in k

has poles at $k = \pm \omega i$



If $\tau > 0$,

$|e^{ik\tau}| = e^{-\mathrm{Im}\,k \cdot \tau}$

exponentially suppressed at large $\mathrm{Im}\,k$, can close contour via large semi-circle on UHP.

$$\Rightarrow \int_{-\infty}^{\infty} dk \, \frac{e^{ik\tau}}{k^2+\omega^2} \overset{\tau>0}{=\!=} 2\pi i \operatorname*{Res}_{k\to i\omega} \frac{e^{ik\tau}}{k^2+\omega^2}$$

$$= \frac{\pi}{\omega} e^{-\omega\tau}.$$

Similarly, for $\tau<0$,
the result is $\frac{\pi}{\omega} e^{\omega\tau}$.

Back to
$\langle q(\tau_1)\, q(\tau_2) \rangle$

$$= \frac{\hbar}{\pi} \int_{-\infty}^{\infty} dk \cdot \frac{1}{k^2+\omega^2} \Big[\frac{1}{4} e^{ik(\tau_1-\tau_2)} + \frac{1}{4} e^{-ik(\tau_1-\tau_2)} - \frac{1}{4} e^{ik(\tau_1+\tau_2+2T)} - \frac{1}{4} e^{-ik(\tau_1+\tau_2+2T)} \Big]$$

$$= \frac{\hbar}{\omega} \cdot \Big(\frac{1}{2} e^{-\omega|\tau_{12}|} - \frac{1}{2} e^{-\omega(\tau_1+\tau_2+2T)} \Big)$$

$\to 0$ as $T \to \infty$

$$= \frac{\hbar}{2\omega} e^{-\omega|\tau_{12}|}.$$

✓

Consider a general Gaussian integral

$$Z[\vec{J}] = \int d^N\vec{q}\ e^{-\frac{1}{2} q^T A q + \vec{J}\cdot\vec{q}}$$

symmetric matrix

$$= (2\pi)^{\frac{N}{2}} \cdot (\det A)^{-\frac{1}{2}} \cdot e^{\frac{1}{2} J^T A^{-1} J}.$$

$$\langle F(\vec{q})\rangle := \frac{1}{Z[0]} \int d^N\vec{q}\ e^{-\frac{1}{2} q^T A q} \cdot F(\vec{q})$$

$$= \frac{1}{Z[0]} F\left(\frac{\partial}{\partial \vec{J}}\right) \cdot Z[\vec{J}] \Big|_{J=0}.$$

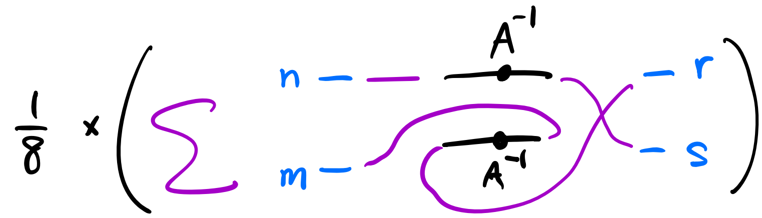
e.g. $\langle q_n\rangle = 0.$

$$\langle q_n q_m\rangle = \frac{\partial}{\partial J_n}\frac{\partial}{\partial J_m} e^{\frac{1}{2} J_p (A^{-1})_{pq} J_q} \Big|_{J=0}$$

$$= (A^{-1})_{nm}.$$

$$\langle q_n q_m q_r q_s\rangle = \frac{\partial}{\partial J_n}\frac{\partial}{\partial J_m}\frac{\partial}{\partial J_r}\frac{\partial}{\partial J_s} e^{\frac{1}{2} J^T A^{-1} J}\Big|_{J=0}$$

$$= \frac{\partial}{\partial J_n}\frac{\partial}{\partial J_m}\frac{\partial}{\partial J_r}\frac{\partial}{\partial J_s} \cdot \frac{1}{2}\left(\frac{1}{2}J^T A^{-1} J\right)^2 .$$



all possible ways to connect $\rightsquigarrow$ $4! = 24$ terms

$$= (A^{-1})_{nm}(A^{-1})_{rs} + (A^{-1})_{nr}(A^{-1})_{ms} + (A^{-1})_{ns}(A^{-1})_{mr} .$$

- Wick contraction rule:

$$\langle q_n q_m q_r \cdots\cdots \rangle$$

$$= \sum_{\substack{\text{all inequivalent} \\ \text{Wick contractions}}} q_n q_m q_r \cdots\cdots$$

$$\overline{q_n q_m} := (A^{-1})_{nm} .$$

- Generalization to Gaussian functional integral

$$\vec{q} \rightsquigarrow q(\tau) \in \text{linear space of real functions.}$$

(defined with a suitable measure)

$$\vec{q} \cdot \vec{J} \rightsquigarrow \int d\tau \, q(\tau) J(\tau).$$

Consider the linear operator A.

$$A \cdot q(\tau) := \left(-\partial_\tau^2 + \omega^2\right) q(\tau).$$

$$A^{-1} \cdot q(\tau) = \int d\tau' \, G(\tau, \tau') \, q(\tau').$$

G obeys

$$(-\partial_\tau^2 + \omega^2) \, G(\tau, \tau') = \delta(\tau - \tau').$$

$$S^E = \frac{1}{2}(q, Aq) = \frac{1}{2} \int d\tau \, q(\tau) \, A \cdot q(\tau)$$

$$\langle q(\tau_1) \, q(\tau_2) \rangle = \frac{\int Dq \, e^{-\frac{1}{2}(q, Aq)} \, q(\tau_1) \, q(\tau_2)}{\int Dx \, e^{-\frac{1}{2}(q, Aq)}}$$

To avoid confusion with normalization in discretizing the integral in τ, we can obtain the answer using an $\int$-by-part trick:

$$\int [Dq]\, e^{-\frac{1}{2}\int d\tau\, q(\tau)(Aq)(\tau)}\, q(\tau_1)\, q(\tau_2)$$

functional derivative

$$= \int [Dq]\, q(\tau_2) \left(-A^{-1}\cdot \frac{\delta}{\delta q(\tau_1)}\right) e^{-\frac{1}{2}\int d\tau\, q(\tau) A q(\tau)}$$

$$\overset{\int \text{by part}}{=\!=} \int [Dq]\, e^{-\frac{1}{2}\int d\tau\, q(\tau) A q(\tau)}\, \underbrace{A^{-1}\cdot \frac{\delta}{\delta q(\tau_1)} q(\tau_2)}$$

$$\begin{aligned} &= A^{-1}\cdot \delta(\tau_1 - \tau_2) \\ &= \int d\tau'\, G(\tau_1, \tau')\, \delta(\tau' - \tau_2) \\ &= G(\tau_1, \tau_2). \end{aligned}$$

$$\Rightarrow \quad \langle q(\tau_1)\, q(\tau_2) \rangle = \overbrace{q(\tau_1)\, q(\tau_2)} = G(\tau_1, \tau_2)$$

Alternatively, can work with the Fourier transformed variable

$$q(\tau) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{q}(k)\, e^{ik\tau}$$

$$\tilde{q}^*(k) = \tilde{q}(-k).$$

$$\widetilde{A\cdot q}\,(k) = (k^2+\omega^2)\,\tilde{q}(k).$$

$$\widetilde{A^{-1}\cdot q}\,(k) = \frac{1}{k^2+\omega^2}\,\tilde{q}(k).$$

$$\Rightarrow \quad G(\tau,\tau') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{k^2+\omega^2} e^{ik(\tau-\tau')}$$

$$\overset{\text{residue}}{=} \frac{1}{2\omega} e^{-\omega|\tau_{12}|}.$$

In agreement with earlier direct evaluation (having set $\hbar = 1$)

Since A is diagonal in $\tilde{x}(k)$ basis,

$$S^E = \frac{1}{2}(q, Aq) = \frac{1}{2}\int_{-\infty}^{\infty} d\tau \; q(\tau)\, A \cdot q(\tau)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi} \cdot \tilde{q}(-k)\,(k^2+\omega)\,\tilde{q}(k).$$

Using the same $\int$-by-part trick, we find

$$\langle \tilde{q}(k_1)\, \tilde{q}(k_2) \rangle = \frac{2\pi\,\delta(k_1+k_2)}{k_1^2+\omega^2}. \quad ✓$$

Example 2 "anharmonic oscillator"

$$H = \frac{p^2}{2} + \frac{q^2}{2} + \frac{1}{4!} g q^4$$

$$\leftrightarrow \quad L = \frac{1}{2}\dot{q}^2 - \frac{q^2}{2} - \frac{1}{4!} g q^4 .$$

Study its spectrum using

$$\langle 0 | \hat{q}(\tau)\, \hat{q}(0) | 0 \rangle \qquad (\tau > 0)$$

$$= \sum_n |\langle n | \hat{q}(0) | 0 \rangle|^2 e^{-E_n \tau}$$

$$= \frac{1}{Z} \int [Dq]\, e^{-\int d\tau\, L^E} q(\tau)\, q(0),$$

where $Z = \int [Dq]\, e^{-\int d\tau\, L^E}$

$$L^E = \frac{1}{2}(\partial_\tau q)^2 + \frac{q^2}{2} + \frac{1}{4!} g q^4$$

Expand in powers of g,

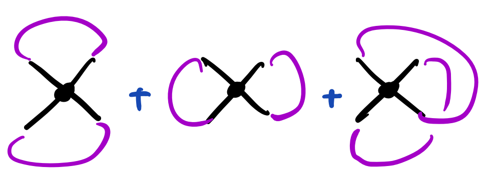
$$Z = \int [Dq]\, e^{-\int d\tau \left(\frac{1}{2}(\partial_\tau q)^2 + \frac{1}{2} q^2\right)} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{g}{4!} \int d\tau'\, q^4(\tau')\right)^n$$

Assume that we can exchange the order of $\int [Dq]$ with the expansion in g [actually problematic, will revisit later]

$$\frac{Z_g}{Z_0} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{g}{4!}\right)^n \int d\tau_1 \dots d\tau_n \cdot \underbrace{\langle (q(\tau_1))^4 \cdots (q(\tau_n))^4 \rangle_0}_{\text{computed using "free" Wick contractions}}$$

e.g.

order g: $-\frac{g}{4!} \int d\tau \, \langle (q(\tau))^4 \rangle_0$



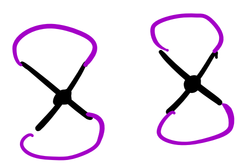
$$= -\frac{g}{8}\int d\tau \,\Big(\underbrace{G_0(\tau,\tau)}_{=\,\text{const}}\Big)^2$$

diverges as $\tau \to \pm\infty$,
will interpret later.

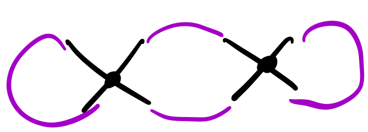
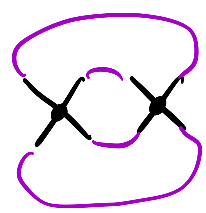
At order g^2.

$$\frac{1}{2}\left(-\frac{g}{4!}\right)^2 \int d\tau_1\, d\tau_2 \;\left\langle (q(\tau_1))^4\, (q(\tau_2))^4 \right\rangle_0$$

Two types of Wick contractions:

disconnected  + ...

and connected

 +  + ...

More generally, at order g^n,
each set of Wick contractions can be
represented by a graph that consists of

K connected components, among them there are m_ℓ connected graphs that contain ℓ vertices each, $\ell = 1, 2, 3, \cdots$

$$\sum_{\ell \geq 1} m_\ell = K, \qquad \sum_{\ell \geq 1} m_\ell \cdot \ell = n.$$

For a given such partition of n, specified by m_ℓ's, the number of ways of grouping the n vertices in this pattern

is $$\frac{n!}{\prod_{\ell \geq 1} (\ell!)^{m_\ell} \cdot m_\ell!}$$

Total contribution is

$$\frac{1}{n!} \frac{n!}{\prod_{\ell \geq 1} (\ell!)^{m_\ell} \cdot m_\ell!} \prod_{\ell \geq 1} \left(G^{(\ell)}\right)^{m_\ell}$$

$$= \prod_{\ell \geq 1} \frac{1}{m_\ell!} \left(\frac{1}{\ell!} G^{(\ell)} \right)^{m_\ell}$$

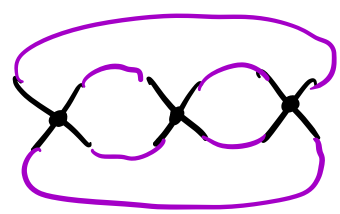
$G^{(\ell)}$ is the contribution from connected Wick contraction of ℓ vertices

$$\frac{Z_g}{Z_0} = \sum_{\{m_\ell\}} \prod_{\ell \geq 1} \frac{1}{m_\ell!} \left(\frac{1}{\ell!} G^{(\ell)} \right)^{m_\ell}$$

$$= \prod_{\ell \geq 1} \exp\left(\frac{1}{\ell!} G^{(\ell)} \right)$$

$$= \exp\Big(\underbrace{\sum_{\ell \geq 1} \frac{1}{\ell!} G^{(\ell)}}_{\text{contribution from all connected graphs}} \Big)$$

e.g.



$$= \frac{1}{3!} \left(-\frac{g}{4!} \right)^3 \int d\tau_1 d\tau_2 d\tau_3 \, \left(G_0(\tau_1, \tau_2) \right)^2 \cdot \left(G_0(\tau_2, \tau_3) \right)^2 \left(G_0(\tau_3, \tau_1) \right)^2 .$$

still diverges like $\int d\tau \cdot (\text{const})$.

Interpretation of

$$Z = \int [Dq]\, e^{-\int d\tau\, L^E}$$

$$= \langle q_f |\, e^{-HT}\, | q_i \rangle$$

boundary condition at initial and final imaginary time $\tau = 0, T$, unspecified as we take $T \to \infty$

$$\log Z = -E_0 \cdot T + O(1)$$

ground state energy

Indeed, we have seen in perturbation theory,

$$\log \frac{Z_g}{Z_0} = \sum_{l=1}^{\infty} \frac{1}{l!}\, G^{(l)}$$

sum of all Wick contractions of l vertices via <u>connected</u> graphs

* One often absorbs the combinatorical

factor $\frac{1}{\ell!}$ into the "symmetry factor" of the graph itself.

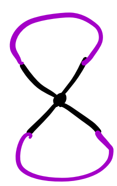
* each $G^{(\ell)}$ is of the form

$$\int d\tau \, (\text{const}) \propto T.$$

Thus $E_0 = -\lim_{T\to\infty} \frac{1}{T} \log Z$

is well-defined.

As already seen, at order g, $\ell = 1$.

$G^{(1)} = 3 \cdot$ 

$$= -\frac{g}{8} \cdot \int d\tau \, \left(G_0(\tau,\tau)\right)^2$$

Recall $G_0(\tau_1,\tau_2) = \frac{1}{2\omega} e^{-\omega|\tau_{12}|}$

$G_0(\tau,\tau) = \frac{1}{2\omega}$

(have set $\omega = 1$)

$$\Rightarrow \Delta E_0 = \frac{g}{8} \cdot \left(\frac{1}{2\omega}\right)^2 + O(g^2)$$

correction to ground state energy

Comparison to Hamiltonian perturbation theory:

$$H = \underbrace{\frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2}}_{\substack{\parallel \\ H_0}} + \underbrace{\frac{1}{4!} g \, \hat{q}^4}_{\substack{\parallel \\ H_1}}$$

Let $|\psi_0\rangle$ be ground state of H_0,

to leading order in g,

$$\Delta E_0 \approx \langle \psi_0 | H_1 | \psi_0 \rangle$$

$$= \frac{1}{4!} g \underbrace{\langle \hat{q}^4 \rangle_0}_{= \frac{3}{4}} = \frac{g}{32} .$$

✓

Next we turn to

$$\langle 0|\hat{q}(\tau)\,\hat{q}(0)|0\rangle \qquad (\tau>0)$$

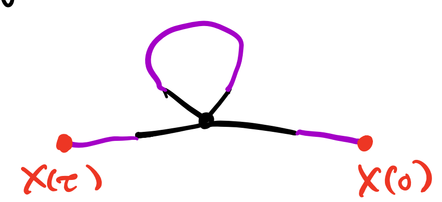
$$= \frac{1}{Z}\int [Dq]\; e^{-\int d\tau\, L^E} \cdot q(\tau)\, q(0).$$

expand in g.

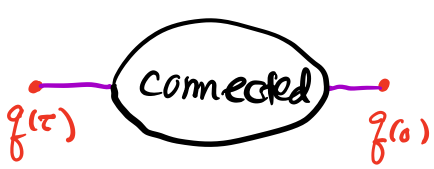
$$= \frac{1}{Z}\left\langle q(\tau)\, q(0) \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{g}{4!}\int d\tau'\, q^4(\tau')\right)^n \right\rangle_0$$

Computed by summing up all possible Wick contractions, consisting of terms like

$$q(\tau)\, q(0)\left(-\frac{g}{4!}\int d\tau'\, q^4(\tau')\right)\underbrace{(\quad)(\quad)(\quad)\cdots}_{\text{other disconnected component of the graph}}$$



The sum factorizes as



$$\times\ \underbrace{(\text{disconnected graphs})}$$

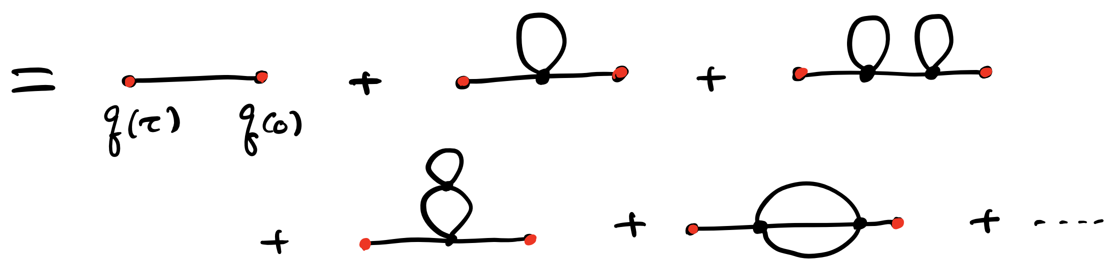
''

Z

cancels against $\frac{1}{Z}$.

leaving the result

$\langle 0|\hat{q}(\tau)\,\hat{q}(0)|0\rangle$



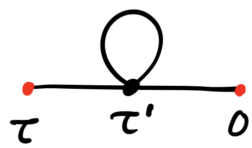
Note: same path integral expression computes $\langle 0|\hat{q}(0)\,\hat{q}(\tau)|0\rangle$ for $\tau < 0$.

We can write the Euclidean Green function

$$\langle q(\tau)\,q(0)\rangle = \begin{cases} \langle 0|\hat{q}(\tau)\,\hat{q}(0)|0\rangle, & \tau \geq 0 \\ \langle 0|\hat{q}(0)\,\hat{q}(\tau)|0\rangle, & \tau < 0 \end{cases}$$

$$= G_0(\tau) = \frac{1}{2}e^{-|\tau|}$$

$$= \int \frac{dk}{2\pi}\,\tilde{G}_0(k)\,e^{ik\tau}, \quad \tilde{G}_0(k) = \frac{1}{k^2+1}$$



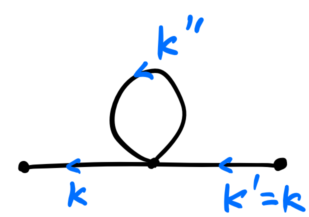
$$= -\frac{g}{2}\int d\tau'\, G_0(\tau-\tau')\, G_0(\tau')\, G_0(0)$$

convenient to evaluation using Fourier transf.

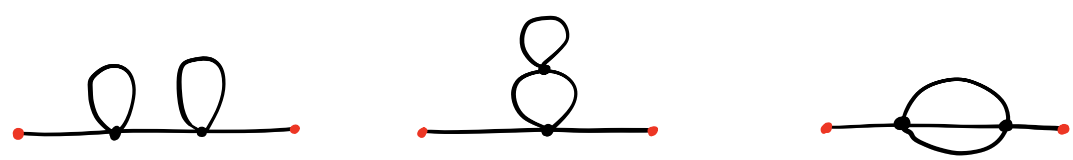
$$= -\frac{g}{2}\int d\tau' \int \frac{dk}{2\pi}\frac{dk'}{2\pi}\, \tilde{G}_0(k)\, \tilde{G}_0(k')\, e^{ik(\tau-\tau')+ik'\tau'}$$

$$\cdot \int \frac{dk''}{2\pi}\cdot \tilde{G}_0(k'')$$

$$= -\frac{g}{2}\cdot\frac{1}{2}\int \frac{dk}{2\pi}\cdot \frac{e^{ik\tau}}{(k^2+1)^2}$$



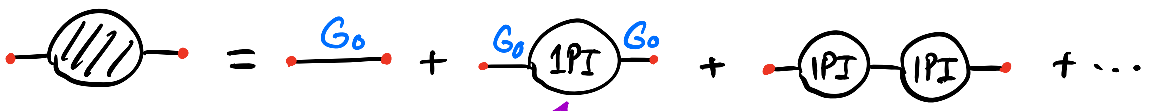
At order g^2, we have



[calculate these in Pset 2]

The full (perturbative) 2-point function

$\langle q(\tau)\, q(0) \rangle$ has the structure



"1-particle-irreducible"

[would remain connected after cutting one propagator, i.e.

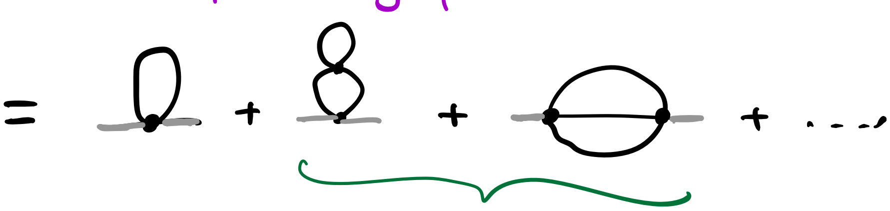

]

$$= \int \frac{dk}{2\pi}\, e^{ik\tau}\, \tilde{G}_0(k) \sum_{n=0}^{\infty} \left(\tilde{G}_0(k)\cdot\Sigma(k)\right)^n$$

$$= \int \frac{dk}{2\pi}\, e^{ik\tau}\, \frac{1}{k^2+1-\Sigma(k)}$$

$\Sigma(k) =$

not including external propagators
"amputated graph"



homework

$$= -\frac{g}{4} + \frac{g^2}{8}\left(\frac{1}{4} + \frac{1}{k^2+9}\right) + O(g^3).$$

In conclusion:

$$\langle q(\tau)\, q(0)\rangle \equiv G(\tau) = \int \frac{dk}{2\pi}\, \tilde{G}(k)\, e^{ik\tau}$$

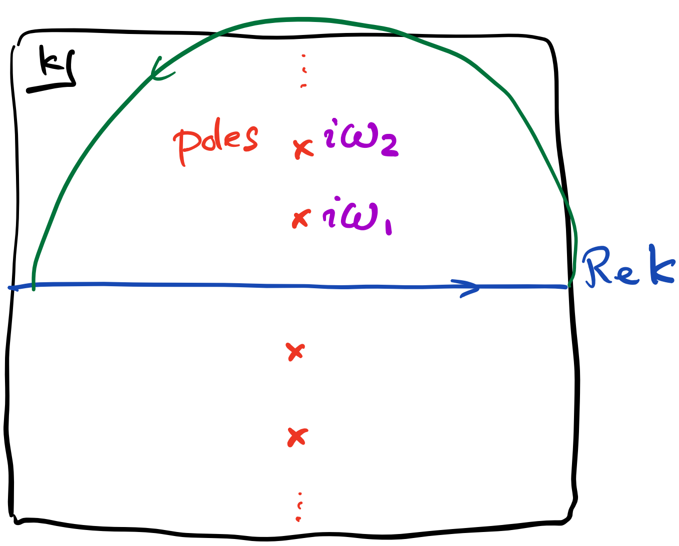
$$\tilde{G}(k) = \frac{1}{k^2 + 1 + \frac{g}{4} - \frac{g^2}{8}\left(\frac{1}{4} + \frac{1}{k^2+9}\right) + \mathcal{O}(g^3)}$$

On the other hand, we know

$$\langle 0|\, \hat{q}(\tau)\, \hat{q}(0)\, |0\rangle \qquad (\text{for } \tau > 0)$$

$$= \sum_n \left|\langle n|\, \hat{q}(0)\, |0\rangle\right|^2 e^{-(E_n - E_0)\tau}$$

Consider the analytic continuation of $\tilde{G}(k)$ to the complex k-plane:



Closing the contour at infinity on UHP.

$$\int \frac{dk}{2\pi} \tilde{G}(k)\, e^{ik\tau} = i \sum_n e^{-\omega_n \tau} \operatorname*{Res}_{k \to i\omega_n} \tilde{G}(k)$$

$$\boxed{\omega_n = E_n - E_0}$$

e.g. from $\tilde{G}(k) = \dfrac{1}{k^2+1+\frac{g}{4} - \frac{g^2}{8}\left(\frac{1}{4}+\frac{1}{k^2+9}\right)+\cdots}$,

we expect the first pole to be near $k \simeq i$,

$$\omega_1 = 1 + \frac{g}{8} - \frac{g^2}{32} + O(g^3).$$

✓

[agrees with Hamiltonian perturbation theory at second order]

Example 3: $\quad L = \frac{1}{2}\dot{q}^2 - \frac{1}{2} q^2 - \frac{g}{8} q^2 \dot{q}^2$

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}\left(1 - \frac{g}{4} q^2\right)$$

$$H = \dot{q} p - L$$

$$= \frac{1}{2}\left(1 - \frac{g}{4} q^2\right)^{-1} p^2 + \frac{1}{2} q^2.$$

↗

ordering in quantization ?

* Can also consider a change of variable

$$\dot{y} = \sqrt{1 - \frac{g}{4} q^2}\ \dot{q}$$

$$y = q - \frac{g}{24} q^3 + \dots$$

and rewrite $L \rightsquigarrow L(y, \dot{y})$

$$= \frac{\dot{y}^2}{2} - V(y).$$

$$V(y) = \frac{1}{2} y^2 + \frac{g}{24} y^4 + \dots$$

Same as anharmonic oscillator up to $O(g^2)$ corrections to the potential.

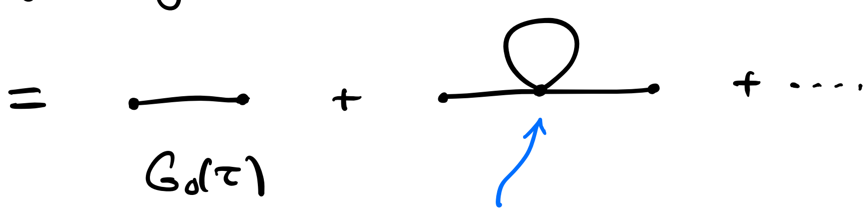
But, we will insist on working with the q-variable in the path integral, and see what happens ...

$$L^E = \frac{1}{2}(\partial_\tau q)^2 + \frac{1}{2} q^2 \underbrace{- \frac{1}{8} g\, q^2 (\partial_\tau q)^2}_{= \; \underbrace{-\frac{g}{24}\partial_\tau (q^3 \partial_\tau q)}_{\text{total derivative}} \; + \; \underbrace{\frac{g}{24} q^3 \partial_\tau^2 q}_{\text{can keep just this}}}$$

does not contribute to S^E

Up to first order in g,

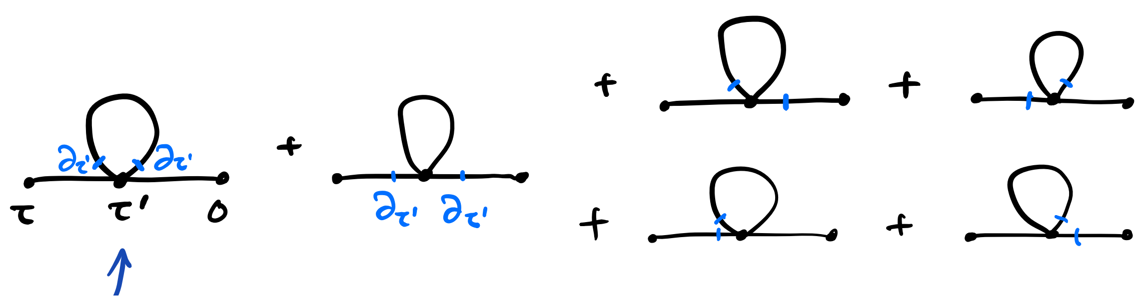
$$\langle q(\tau)\, q(0) \rangle$$



$$q(\tau)\, q(0) \cdot \frac{g}{8} \int d\tau' \, (q(\tau'))^2 \, (\partial_{\tau'} q(\tau'))^2$$

+ various other contractions

can represent as several diagrams



$$2 \times \; q(\tau)\, q(0) \cdot \frac{g}{8}\int d\tau' \; q(\tau')\, q(\tau')\, \partial q(\tau')\, \partial q(\tau')$$

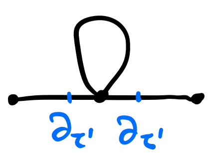
$$= \frac{g}{4} \cdot \int d\tau' \; G_0(\tau-\tau')\, G_0(\tau')\, \left(-\partial^2 G_0(0)\right)$$

$$= \frac{g}{4} \int \frac{dk}{2\pi} \frac{e^{ik\tau}}{(k^2+1)^2} \underbrace{\int \frac{dk'}{2\pi} \frac{k'^2}{k'^2+1}}_{\text{divergent ?!}}$$

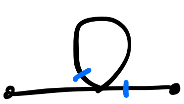
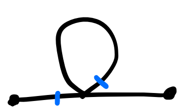
$$G_0(\tau) = \tfrac{1}{2} e^{-|\tau|}.$$

$$-\partial^2 G_0(0) = \infty \qquad ?!$$

The other contributions are finite:

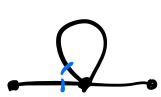
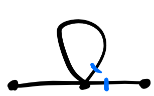


$$= \frac{g}{4} \int \frac{dk}{2\pi} \frac{e^{ik\tau}}{(k^2+1)^2} \cdot k^2 \int \frac{dk'}{2\pi} \frac{1}{k'^2+1}.$$

 = 

$$= \frac{g}{2} \int \frac{dk}{2\pi} \frac{e^{ik\tau}}{(k^2+1)^2} \cdot ik \int \frac{dk'}{2\pi} \frac{ik'}{k'^2+1} = 0$$

by symmetry

Same for  and 

The divergence is due to $\int$ at large k.

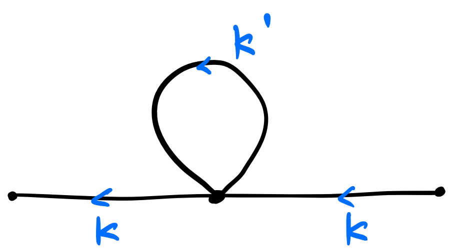
The origin of this divergence lies in the path integral measure

$$[Dq] = \prod_k d\tilde{q}_k$$

$$\left(q(\tau) \equiv \int \frac{dk}{2\pi} \tilde{q}(k) e^{ik\tau} \right)$$

$$\Big\downarrow \text{regularization}$$

$$\prod_{|k|<\Lambda} d\tilde{q}_k \, .$$



Effectively cuts off the k' "loop" integral by $|k'| < \Lambda$,

[will eventually take $\Lambda \to \infty$ ...]

$$\langle q(\tau)\, q(0) \rangle \equiv G(\tau) = \int \frac{dk}{2\pi} \frac{e^{ik\tau}}{k^2+1-\Sigma(k)}$$

With the cutoff regularization,

$$\Sigma(k) = \frac{g}{4}\int_{|k'|<\Lambda} \frac{dk'}{2\pi}\,\frac{k'^2}{k'^2+1} + \frac{g}{4}k^2 \overbrace{\int \frac{dk'}{2\pi}\,\frac{1}{k'^2+1}}^{\frac{1}{2}} + O(g^2)$$

$$= \frac{g}{4}\left(\frac{k^2}{2} + \frac{\Lambda}{\pi} - \frac{1}{2} + O(\Lambda^{-1})\right) + O(g^2)$$

* What is the interpretation of $\Lambda \to \infty$?

  - Nothing wrong with QM w/

$$H = \frac{1}{2}\left(1 - \frac{g}{4}q^2\right)^{-1} p^2 + \frac{1}{2}q^2$$

(at least for $g<0$)

except for potential ordering ambiguity ....

To understand what's going on,

let us restore $\hbar$ - dependence :

$$Z = \int [Dq]\, \exp\left[-\frac{1}{\hbar}\int d\tau\, L^E\right]$$

$$L^E = \frac{(\partial_\tau q)^2}{2} + \frac{q^2}{2} - \frac{1}{8} g\, q^2 (\partial_\tau q)^2 + \mathcal{O}(\hbar)\ ?$$

"counter terms"

possible corrections needed for agreement with quantum Hamiltonian?

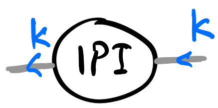
$$\tilde{G}(k) = \frac{\hbar}{k^2 + 1 - \Sigma(k)},$$

$$\Sigma(k) = \frac{\hbar g}{4}\left(\frac{k^2}{2} + \frac{\Lambda}{\pi} - \frac{1}{2} + \mathcal{O}(\Lambda^{-1})\right) + \mathcal{O}(\hbar^2 g^2)$$

Indeed, if we include

$$\Delta L^E = c \cdot g\, \hbar\, \frac{q^2}{2}.$$

$\Sigma(k) =$



receives an extra contribution $-c \cdot g\, \hbar$ at order g, giving the result

$$\Sigma(k) = \hbar g \left(\frac{k^2}{8} + \frac{\Lambda}{4\pi} - \frac{1}{8} - c + \mathcal{O}(\Lambda^{-1})\right) + \mathcal{O}(g^2)$$

* The path integral is only defined provided regularization via the cutoff Λ.

Choosing $c = \frac{\Lambda}{4\pi} + \text{finite}$, and take the limit $\Lambda \to \infty$ in the end, we find a finite result for $\Sigma(k)$.

* The finite part of c represents an ambiguity in quantization, i.e. the possibility of adjusting the quantum Hamiltonian by a term of order $\hbar$.

$$\tilde{G}(k) = \frac{\hbar}{k^2+1-\Sigma(k)}$$ has pole at $k = i\omega_1$,

$$\omega_1 = 1 + \hbar g\left(-\frac{\Lambda}{8\pi} + \frac{c}{2} + \frac{1}{8}\right) + \mathcal{O}(g^2).$$

$$= E_1 - E_0$$

- energy gap, unambiguous physical observable regardless of the interpretation of $q(\tau)$.

* The counter terms, together with regularization scheme, should be viewed as part of the definition of the path integral.

To fix the finite part of the counter terms require additional physical input, e.g. symmetries of the quantum Hamiltonian

In general QM, anything goes in H

In QFT, Poincaré sym + causality highly restrictive!

A manifestly relativistic theory of "fields"

- classical mechanics,

  where the dynamical variables

$$q^i(t) \rightsquigarrow \phi(t, \vec{x})$$

$\vec{x} \in \mathbb{R}^{D-1}$, viewed as parameters labeling the continuous family of variables

we will also write $x^\mu \equiv (x^0, \vec{x})$. $x^0 = t$ interchangably.

Action $$S = \int \underbrace{dt\, d^{D-1}\vec{x}}_{\substack{\equiv \\ d^Dx}} \, \mathcal{L}[\phi(x), \partial_\mu \phi(x), \dots]$$

Lagrangian density

Equation of motion: $$\frac{\delta S}{\delta \phi(x)} = 0$$

Assume for now that $\mathcal{L}$ depends on $\phi(x)$ only as a function of $\phi(x)$ and its first-order derivative $\partial_\mu \phi(x)$.

$$\delta S = \int d^D x \left[\delta\phi(x) \frac{\partial \mathcal{L}}{\partial \phi(x)} + \underbrace{\delta \partial_\mu \phi(x)}_{\equiv\ \partial_\mu \delta\phi(x)} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} \right]$$

$$\overset{\int \text{by part}}{=\!=} \int d^D x\, \delta\phi(x) \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} \right]$$

Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} = 0.$$

<u>Example:</u> (free massive scalar field)

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}\sum_{i=1}^{D-1} (\partial_{x^i} \phi)^2 - \frac{1}{2} m^2 \phi^2$$

$$\equiv -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2,$$

convention:

$$(\partial_\mu \phi)^2 \equiv \eta^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi.$$

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

E-L equation:

$$\partial^\mu \partial_\mu \phi - m^2 \phi = 0.$$

general solution

$$\phi(x) = \int d^D k \; f(k) \, e^{i \underbrace{k \cdot x}_{\vec{k}\cdot\vec{x} - k^0 x^0}}$$

$$(k^2 + m^2) f(k) = 0,$$

i.e. $f(k)$ is a distribution supported at

k^μ: $k^2 + m^2 = 0$ "mass-shell"

i.e. $k^0 = \pm \sqrt{\vec{k}^2 + m^2}$.

Can write

$$\phi(x) = \int d^{D-1}\vec{k} \left[\tilde{f}(\vec{k}) \, e^{i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}} x^0} + \tilde{f}^*(\vec{k}) \, e^{-i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}} x^0} \right]$$

$\omega_{\vec{k}} \equiv \sqrt{\vec{k}^2 + m^2}$.

Initial value problem: Given $\phi(t=0, \vec{x})$ and $\partial_t \phi(t=0, \vec{x})$,

determine $\phi(t>0, \vec{x})$?

$$\phi(t=0,\vec{x}) = \int d^{D-1}\vec{k}\; e^{i\vec{k}\cdot\vec{x}} \left[\tilde{f}(\vec{k}) + \tilde{f}^*(-\vec{k}) \right]$$

$$\partial_t\phi(t=0,\vec{x}) = \int d^{D-1}\vec{k}\; e^{i\vec{k}\cdot\vec{x}} (-i\omega_{\vec{k}}) \left[\tilde{f}(\vec{k}) - \tilde{f}^*(-\vec{k}) \right].$$

Together, they determine $\tilde{f}(\vec{k})$ and $\tilde{f}^*(-\vec{k})$, and hence $\phi(x)$ at any $t\,(\equiv x^0)$.

- A relativistic quantum theory may be constructed via "quantization" of the classical field theory.

Two approaches:

(1) canonical quantization

- construct Hilbert space using wave functions in variable $\phi(\vec{x}, t=0)$ and the Hamiltonian H.

(2) Path integral

$$Z = \int [D\phi(x)]\, e^{\frac{i}{\hbar}\int d^D x\, \mathcal{L}[\phi, \partial_\mu \phi]}$$

to make sense of the functional measure typically requires regularization

---

(1) canonical formalism

Lagrangian

$$L = \int d^{D-1}\vec{x}\, \mathcal{L}[\phi, \partial_\mu \phi]$$

for free scalar,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 .$$

Dynamical variable $\phi(t, \vec{x})$

time just some label

canonical coordinates

* (conjugate) canonical momentum density

$$\Pi(t,\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(t,\vec{x})}$$

$$= \dot{\phi}(t,\vec{x})$$ in the present example.

(classical) Hamiltonian

$$H = \int d^{D-1}\vec{x}\, \mathcal{H}$$

density

$$\mathcal{H} = \dot{\phi}(t,\vec{x})\,\Pi(t,\vec{x}) - \mathcal{L}$$

e.g. free scalar field

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\partial_{\vec{x}}\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$\xlongequal{\text{in terms of } (\phi,\Pi)} \underbrace{\frac{1}{2}\Pi^2}_{\text{"kinetic"}} + \underbrace{\frac{1}{2}(\partial_{\vec{x}}\phi)^2 + \frac{1}{2}m^2\phi^2}_{\text{"potential"}}.$$

Poisson bracket: (omit time)

$$\{\phi(\vec{x}), \phi(\vec{x}')\} = 0 = \{\Pi(\vec{x}), \Pi(\vec{x}')\}$$

$$\{\Pi(\vec{x}), \phi(\vec{x}')\} = \delta^{D-1}(\vec{x}-\vec{x}').$$

$$\{F, G\} := \int d^{D-1}\vec{x} \left[\frac{\delta F}{\delta \Pi(\vec{x})} \frac{\delta G}{\delta \phi(\vec{x})} - \frac{\delta F}{\delta \phi(\vec{x})} \frac{\delta G}{\delta \Pi(\vec{x})} \right]$$

EOM $\Rightarrow$ $\dot{F} = \{H, F\}$ as usual.

Quantization: promote $\phi(t,\vec{x})$, $\Pi(t,\vec{x})$

to operators

$$\hat{\phi}(t,\vec{x}), \quad \hat{\Pi}(t,\vec{x}),$$

that obey equal time commutators

$$(*)\begin{cases} [\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{x}')] = 0 = [\hat{\Pi}(t,\vec{x}), \hat{\Pi}(t,\vec{x}')] \\ [\hat{\phi}(t,\vec{x}), \hat{\Pi}(t,\vec{x}')] = i\hbar\, \delta^{D-1}(\vec{x}-\vec{x}'). \end{cases}$$

Convenient to perform spatial Fourier transform, say at $t=0$:

$$\hat{\phi}(t=0,\vec{x}) = \int \frac{d^{D-1}\vec{k}}{\sqrt{(2\pi)^{D-1}\, 2\,\nu(\vec{k})}} \underbrace{\left(a_{\vec{k}} + a^{\dagger}_{-\vec{k}}\right)} e^{i\vec{k}\cdot\vec{x}}$$

for now, just a norm. convention

view as definition of $a, a^{\dagger}$.

$$\hat{\Pi}(t=0,\vec{x}) = \int \frac{d^{D-1}\vec{k}}{\sqrt{(2\pi)^{D-1}\, 2\,\nu(\vec{k})}} \cdot (-i\,\nu(\vec{k})) \overbrace{\left(a_{\vec{k}} - a^{\dagger}_{-\vec{k}}\right)} e^{i\vec{k}\cdot\vec{x}}$$

* It follows from the canonical commutation relation $\circledast$ that

$$\left[a_{\vec{k}} + a^{\dagger}_{-\vec{k}},\ a_{\vec{k}'} + a^{\dagger}_{-\vec{k}'}\right] = 0$$

$$\left[a_{\vec{k}} - a^{\dagger}_{-\vec{k}},\ a_{\vec{k}'} - a^{\dagger}_{-\vec{k}'}\right] = 0.$$

$$\left[a_{\vec{k}} + a^{\dagger}_{-\vec{k}},\ a_{\vec{k}'} - a^{\dagger}_{-\vec{k}'}\right] = -2\,\delta^{D-1}(\vec{k}+\vec{k}').$$

and also $\left[a_{\vec{k}} - a^{\dagger}_{-\vec{k}},\ a_{\vec{k}'} + a^{\dagger}_{-\vec{k}'}\right] = 2\,\delta^{D-1}(\vec{k}+\vec{k}')$

Note that $a^{\dagger}_{\vec{k}}$ being the Hermitian conjugate of $a_{\vec{k}}$ ensures that $\hat{\phi}$, $\hat{\Pi}$ are Hermitian.

These commutation relations are equivalent to

$$[a_{\vec{k}}, a_{\vec{k}'}] = 0 = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}]$$

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta^{D-1}(\vec{k}-\vec{k}') .$$

These are precisely the commutation relations obeyed by particle annihilation and creation operators.

However, this doesn't mean $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ are actually ann. & crea. operators... we haven't used the Hamiltonian nor specified $v(\vec{k})$.

Note: had we defined $b_{\vec{k}}, b^\dagger_{\vec{k}}$, via

$$b_{\vec{k}} + b^\dagger_{-\vec{k}} = \lambda(\vec{k}) \left(a_{\vec{k}} + a^\dagger_{-\vec{k}} \right),$$

$$b_{\vec{k}} - b^\dagger_{-\vec{k}} = \frac{1}{\lambda(\vec{k})} \left(a_{\vec{k}} - a^\dagger_{-\vec{k}} \right),$$

$b, b^\dagger$ also obey comm. rel. of ann. & creat. operators!

$$b_{\vec{k}} = \frac{\lambda+\lambda^{-1}}{2} a_{\vec{k}} + \frac{\lambda-\lambda^{-1}}{2} a^\dagger_{-\vec{k}}$$

- "Bogoliubov transformation"

Q: Is the vacuum state $|\Omega\rangle$ annihilated by $a_{\vec{k}}$?
If so, $b_{\vec{k}} |\Omega\rangle \neq 0$ (for $\lambda \neq 1$)

- Hamiltonian of free scalar field

$$\hat{H} = \int d^{D-1}\vec{x}\left[\frac{1}{2}\hat{\Pi}^2 + \frac{1}{2}(\partial_{\vec{x}}\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2\right]$$

$$= \int d^{D-1}\vec{k}\left[\frac{1}{2}\cdot\frac{\nu(\vec{k})}{2}(a_{\vec{k}} - a^{\dagger}_{-\vec{k}})(a^{\dagger}_{\vec{k}} - a_{-\vec{k}}) + \frac{1}{2}\cdot\frac{1}{2\nu(\vec{k})}\cdot(\vec{k}^2+m^2)(a_{\vec{k}} + a^{\dagger}_{-\vec{k}})(a^{\dagger}_{\vec{k}} + a_{-\vec{k}})\right]$$

$$\underset{\omega_{\vec{k}} \equiv \sqrt{\vec{k}^2+m^2}}{\overset{\text{set } \nu(\vec{k}) = \omega_{\vec{k}}}{=\!=\!=}} \int d^{D-1}\vec{k}\;\omega_{\vec{k}}\left(a^{\dagger}_{\vec{k}}a_{\vec{k}} + \frac{1}{2}\delta^{D-1}(0)\right)$$

??

ordering ambiguity (of order $\hbar$)

* The true quantum Hamiltonian must be a well defined (self-adjoint) operator.

We will remove the normal ordering constant by a constant shift of $\hat{H}$. This will be justified as a consequence of the assumption of Poincaré invariance of the vacuum $|\Omega\rangle$.

using $\quad \dot{F} = \frac{i}{\hbar}[\hat{H}, F]$,

we can determine the t-dependence of $\hat{\phi}(t,\vec{x})$ and $\hat{\Pi}(t,\vec{x})$.

e.g.

$$\hat{\phi}(t,\vec{x}) = \int \frac{d^{D-1}\vec{k}}{\sqrt{(2\pi)^{D-1} 2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{ik\cdot x} + a_{\vec{k}}^{\dagger} e^{-ik\cdot x} \right)$$

$k\cdot x \equiv \vec{k}\cdot\vec{x} - \omega_{\vec{k}} t$

- As we have seen previously, $\hat{\phi}(x)$ is a Poincaré-covariant local operator.

$$\hat{\phi}(\Lambda x + a) = U(\Lambda, a)\,\hat{\phi}(x)\,(U(\Lambda, a))^{-1}.$$

- vacuum $|\Omega\rangle$ obeys

$$a_{\vec{k}}|\Omega\rangle = 0, \quad \forall \vec{k}.$$

(and $H|\Omega\rangle = 0$.)

Symmetry of a classical field theory

$S[\phi]$ invariant under

$$\phi(x) \rightsquigarrow \phi'(x) = \phi(x) + \epsilon \underbrace{\delta\phi(x)}$$

infinitesimal parameter

determined by $\phi(x), \partial_\mu\phi(x), \cdots$

e.g. Poincaré symmetry acts on scalar field $\phi(x)$ as $\phi(x) \rightsquigarrow \phi'(x)$, where $\phi'(x)$ is such that

$$\phi'(\Lambda\cdot x + a) = \phi(x)$$

Infinitesimal form:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

$\omega_{\mu\nu} \equiv \eta_{\mu\rho}\,\omega^\rho{}_\nu$ is anti-symmetric in $[\mu\nu]$.

$\omega_{\mu\nu}$ and a^μ are infinitesimal,

$$\phi'(\Lambda\cdot x + a) = \phi'(x) + (\omega^\mu{}_\nu x^\nu + a^\mu)\,\partial_\mu\phi'(x)$$

$$\Rightarrow \quad \delta\phi(x) = -(\omega^\mu{}_\nu x^\nu + a^\mu)\,\partial_\mu\phi(x).$$

$$S[\phi'] = \int d^Dx\, \mathcal{L}[\phi'(x), \partial_\mu \phi'(x), \dots]$$

$$\overset{\text{rename variable}}{=\!=} \int d^Dx'\, \mathcal{L}\left[\phi'(x'), \frac{\partial \phi'(x')}{\partial x'^\mu}, \dots\right]$$

$$\overset{x' = \Lambda x + a}{=\!=} \int \underbrace{d^Dx}_{\text{measure inv't under Poincaré}}\, \mathcal{L}\underbrace{\left[\phi(x), (\Lambda^{-1})_\mu{}^\nu \frac{\partial \phi(x)}{\partial x^\nu}, \dots\right]}_{\text{// covariance}}$$

$$\mathcal{L}[\phi, \partial_\mu \phi, \dots]$$

$= S[\phi]$, provided that $\mathcal{L}$ is covariant, e.g.

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi).$$

Noether's theorem:

Continuous symmetry

$\Rightarrow$ conserved current $j^\mu(x)$.

- $\partial_\mu j^\mu(x) = 0$ will be a consequence of EOM
- $Q = \int d^{D-1}\vec{x}\, j^0(x)$ is the conserved charge

Idea: consider a non-symmetry transformation

$$\tilde{\delta}\phi(x) = \phi(x) + \varepsilon(x)\,\underbrace{\delta\phi(x)}_{\text{as in symmetry variation}}$$

coord. dep.
no longer a sym.

$$\tilde{\delta}S[\phi] \equiv S[\phi + \tilde{\delta}\phi] - S[\phi]$$

is not expected to vanish,

but should take the form

$$= \int d^D x \, \partial_\mu \varepsilon(x) \, j^\mu(x).$$

so that it vanishes
when $\varepsilon(x) = \text{const}$

Now assume $\varepsilon(x)$ has compact support,

e.g. $\varepsilon(x) = 0$ for $x \notin \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^{1,D-1}$
a compact domain.

can $\int$ by part and write

$$\tilde{\delta}S = -\int_{\mathcal{D}} d^D x \, \varepsilon(x) \, \partial_\mu j^\mu(x)$$

By construction, $\hat{j}^\mu(x)$ is built out of $\phi(x)$, $\partial_\mu \phi(x)$, $\cdots$,

Now if we restrict to $\phi(x)$ that obeys E.O.M., i.e. $\delta' S = 0$ for any var. $\delta'\phi$.

$\Rightarrow \tilde{\delta} S = 0$. for any $\varepsilon(x)$

i.e. $\partial_\mu j^\mu(x) = 0$

– conservation law.

$$Q = \int d^{D-1}\vec{x}\ \hat{j}^0(x) \text{ is conserved:}$$

$$\partial_t Q = \int d^{D-1}\vec{x}\ \partial_0 \hat{j}^0$$

$$= -\int d^{D-1}\vec{x} \sum_{i=1}^{D-1} \frac{\partial j^i}{\partial x^i} = 0.$$

* There is more: if we express Q in terms of the canonical variables (coordinates $\phi(\vec{x})$ and conjugate momenta $\Pi(\vec{x})$) then Q is also the generator of the symmetry

in the sense that:

$$\delta\phi(x) = \{Q, \phi(x)\}$$

Poisson bracket.

Proof: $\tilde{\delta} S = \int d^D x \left[\frac{\partial \mathcal{L}}{\partial \phi} \varepsilon(x) \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\mu (\varepsilon(x) \delta\phi) \right]$

$$= \int d^D x \left[\partial_\mu \varepsilon(x) \cdot \delta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} + \varepsilon(x) \Big(\underbrace{\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu \delta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}}_{= \partial_\mu \delta \mathcal{U}^\mu} \Big) \right]$$

$$j^\mu = \delta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \delta \mathcal{U}^\mu$$

$$Q = \int d^{D-1} \vec{x} \, [\delta\phi \cdot \Pi - \delta \mathcal{U}^0]$$

- If $\delta\phi$ is independent of $\dot{\phi}$, and thus of Π, $\delta \mathcal{U}^0$ must also vanish, and we have

$$\{Q, \phi(x)\} = \frac{\delta Q}{\delta \Pi(x)} = \delta\phi(x), \quad \checkmark$$

- If $\delta\phi$ depends on $\dot{\phi}$, say $\delta\phi \propto \dot{\phi}$ in the case of time-translation symmetry, for $\mathcal{L} = \frac{1}{2} G_{\alpha\beta}(\phi) \dot{\phi}^\alpha \dot{\phi}^\beta - V(\phi)$, we have

$$\delta \mathcal{U}^0 = \frac{1}{2} \delta\phi^\alpha \Pi_\alpha + (\text{independent of } \dot{\phi} \text{ or } \Pi)$$

$\{Q, \phi\} = \delta\phi$ still holds.

The Noether current associated with translation symmetry

$$\delta\phi(x) = -a^\mu \partial_\mu \phi(x)$$

is $\quad j^\mu(x) = a^\nu T^\mu{}_\nu(x)$

stress-energy tensor

Example: scalar field theory with action

$$S[\phi] = \int d^D x \left[-\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \right]$$

under $\tilde{\delta}\phi(x) = -\varepsilon^\mu(x)\,\partial_\mu \phi(x)$.

$$\tilde{\delta} S = \int d^D x \Big[\partial^\mu \phi\, \partial_\mu (\varepsilon^\nu(x) \partial_\nu \phi) + \varepsilon^\mu(x) \partial_\mu \phi \cdot \frac{\partial V}{\partial \phi} \Big]$$

$$= \int d^D x \left[\partial_\mu \varepsilon^\nu \cdot \partial^\mu \phi \partial_\nu \phi + \varepsilon^\nu \cdot \partial_\nu \left(\tfrac{1}{2} (\partial^\mu \phi \partial_\mu \phi) + V \right) \right]$$

$$= \int d^D x \;\; \partial_\mu \varepsilon^\nu(x)\; T^\mu{}_\nu(x),$$

where

$$T^\mu{}_\nu(x) = \partial^\mu\phi\,\partial_\nu\phi - \delta^\mu{}_\nu\left[\frac{1}{2}\partial^\rho\phi\,\partial_\rho\phi + V(\phi)\right].$$

In particular,

$$T^{00} = (\partial_0\phi)^2 + \frac{1}{2}\partial^\rho\phi\,\partial_\rho\phi + V(\phi)$$
$$= \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}\sum_{i=1}^{D-1}(\partial_i\phi)^2 + V(\phi) = \mathcal{H}$$

The Hamiltonian

$H = \int d^{D-1}\vec{x}\, T^{00}$ is the Noether charge associated with time-translation

Similarly,

$$P^i = \int d^{D-1}\vec{x}\, T^{0i}, \quad i = 1, \cdots, D-1$$

are the conserved (spatial) momenta.

* What about Lorentz symmetry?

For scalar field $\phi(x)$,

$$\delta\phi(x) = -\omega^\mu{}_\nu\, x^\nu\,\partial_\mu\phi(x).$$

corresponding Noether current

$$j^\mu(x) = \omega^\nu{}_\rho x^\rho T^\mu{}_\nu(x)$$

Conservation:

$$\partial_\mu j^\mu = \omega^\nu{}_\rho x^\rho \underbrace{\partial_\mu T^\mu{}_\nu}_{\substack{\text{// translation sym} \\ 0}} + \omega^\nu{}_\mu T^\mu{}_\nu$$

$$= \omega_{[\mu\nu]} T^{\mu\nu} = 0$$

provided that $T^{\mu\nu} = T^{\nu\mu}$, ✓

Upon quantization, the Noether current $j^\mu(x)$ becomes a local (field) operator, just like $\hat{\phi}(x)$.

$Q = \int d^{D-1}\vec{x}\, j^0(x)$ is now a Hermitian operator that generates the symmetry via

$$\delta\hat{\phi}(x) = -\frac{i}{\hbar}[Q, \hat{\phi}(x)]$$ ← commutator

* The stress-energy tensor $T^{\mu\nu}(x)$ is always a well-defined local operator in a QFT.

$$\hat{P}^{\mu} = \int d^{D-1}\vec{x}\, T^{0\mu}(x)$$

is the energy-momentum operator,

and

$$\hat{J}^{\mu\nu} = \int d^{D-1}\vec{x} \left[x^{\mu} T^{0\nu}(x) - (\mu \leftrightarrow \nu) \right]$$

is the boost-angular-momentum operator as we have encountered in Lecture 1

(recall $U(\Lambda, a) = 1 - i\epsilon^{\mu}\hat{P}_{\mu} + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}$)

Path integral quantization of scalar field theory

$$Z = \int [D\phi]\, e^{i \int d^D x\, \mathcal{L}(\phi, \partial_\mu \phi)}$$

Formally, extending path integral of QM in the case of finitely many degrees of freedom

state $\Psi \in \mathcal{H}$ ← Hilbert space

$\updownarrow$

wave functional $\Psi[\phi(\vec{x}, t_0)]$

↑ fixed time

$$\int [D\phi] \Big|_{\phi(\vec{x}, t_i) = \phi_i(\vec{x})}^{\phi(\vec{x}, t_f) = \phi_f(\vec{x})} e^{i \int_{t_i}^{t_f} dx^0 \int d^{D-1}\vec{x}\, \mathcal{L}(\phi, \partial_\mu \phi)}$$

$$= \langle \phi_f | \, U(t_f, t_i) \, | \phi_i \rangle$$

$$U(t_f, t_i) = e^{-i \hat{H} \cdot (t_f - t_i)}$$

$|\phi_i\rangle$ is an eigenstate w.r.t. the operator $\hat{\phi}(x)$ defined via insertion of $\phi(x)$ in the path integral,

$$\hat{\phi}(\vec{x},0)\cdot \Psi[\phi(\vec{x},0)] = \phi(\vec{x},0)\,\Psi[\phi(\vec{x},0)]$$

$$|\phi_i\rangle \longleftrightarrow \Psi_i[\phi(\vec{x},0)] \propto \delta[\phi(\vec{x},0)-\phi_i(\vec{x})]$$

"functional δ-distribution"

- As always, to make all of this precise requires regularization, e.g. discretizing space or cut off on spatial wave number of $\phi(\vec{x},t)$

- We will mostly avoid directly working with the wave functional $\Psi[\phi(\vec{x},0)]$, which is harder to regularize.

  It will be more convenient to extract physical observable from vacuum expect. values, e.g

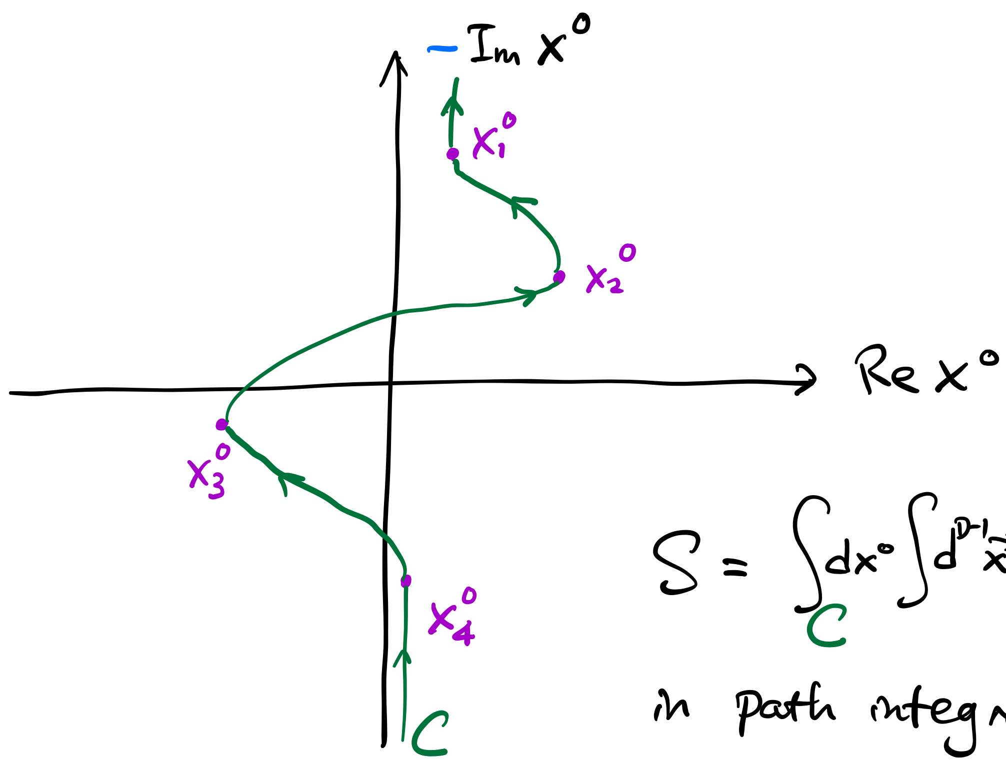
$$\langle\Omega|\,\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\,|\Omega\rangle.$$

To ensure convergence of path integral, it is useful to analytically continue $x_1^0, x_2^0, \cdots, x_n^0$ to complex time, with the specific ordering of imaginary time

$$\mathrm{Im}\, x_1^0 < \mathrm{Im}\, x_2^0 < \cdots < \mathrm{Im}\, x_n^0$$

so that analyticity is maintained.

- The real parts $\mathrm{Re}\, x_1^0, \cdots, \mathrm{Re}\, x_n^0$ are not subject to any ordering a priori



$$S = \int_C dx^0 \int d^{D-1}\vec{x}\, \mathcal{L}$$

in path integral.

It is most convenient to start with the Euclidean correlator

$$\langle \Omega | \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | \Omega \rangle$$

$$\equiv \langle \hat{\phi}_E(x_1^E) \cdots \hat{\phi}_E(x_n^E) \rangle$$

where $x_i^0 = -i\,(x_i^E)^D$, $\vec{x}_i \equiv \vec{x}_i^E$.

$$(x_1^E)^D > (x_2^E)^D > \cdots > (x_n^E)^D$$

$\hat{\phi}(x) \equiv \hat{\phi}_E(x^E)$. later we will typically omit the subscript "E" and write $\hat{\phi}_E$ simply as $\hat{\phi}$ in the context of Euclidean correlators.

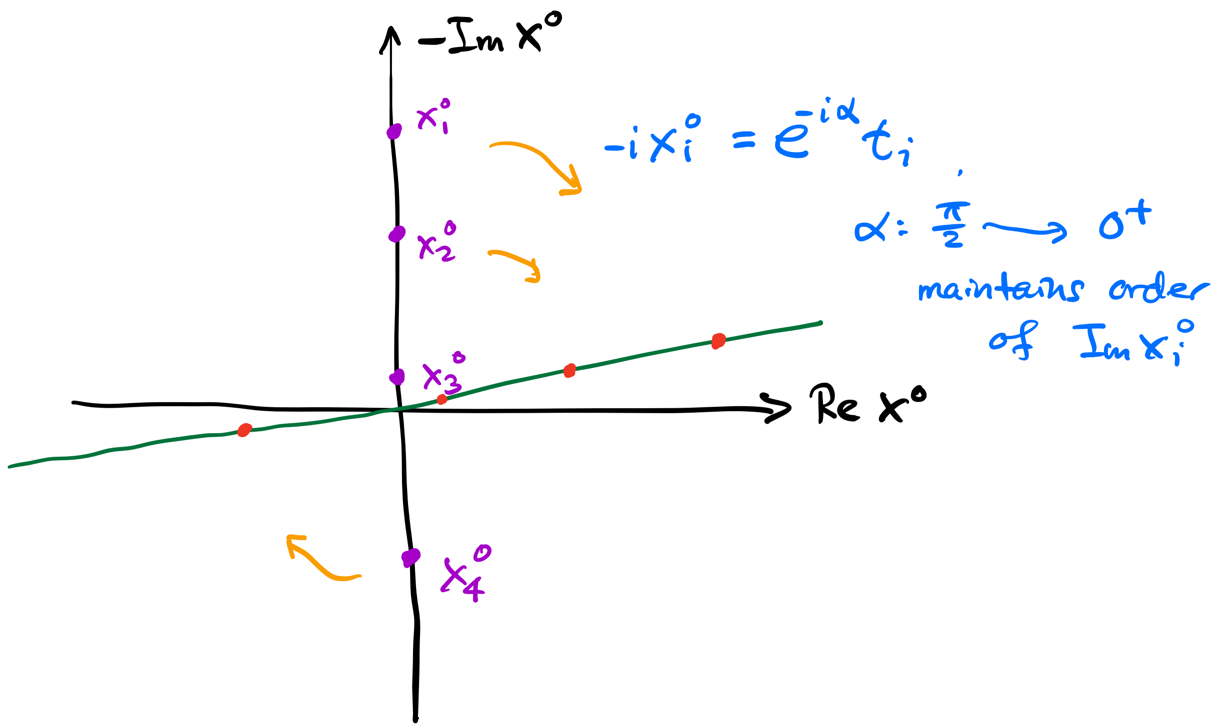
$$\langle \hat{\phi}(x_1^E) \cdots \hat{\phi}(x_n^E) \rangle$$

$$= \frac{1}{Z} \int [D\phi]\, e^{-S^E[\phi]} \phi(x_1^E) \cdots \phi(x_n^E)$$

where $S^E = \int d^D x^E \mathcal{L}^E[\phi, \partial_\mu \phi]$

$$\mathcal{L}^E[\phi(x^E), \partial_\mu \phi(x^E)] = -\mathcal{L}[\phi(x), \partial_\mu \phi(x)]$$

as in QM before.



A uniform Wick rotation on all $i = 1, \cdots, n$ simultaneously leads to

$$\langle \Omega | \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | \Omega \rangle$$

and in the limit $\alpha \to 0^+$ we have a real-time correlator that is time-ordered, i.e.

$$x_1^0 > x_2^0 > \cdots > x_n^0$$

Conclusion: the time-ordered real-time correlator is obtained from the Euclidean one by uniform Wick rotation on all x_i^0's.

Example: free scalar field theory

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2$$

$$\mathcal{L}^E = \frac{1}{2}\sum_{\mu=1}^{D}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2$$

Euclidean 2-pt function

$$\langle\phi(x)\,\phi(y)\rangle = \frac{1}{Z}\int[D\phi]\,e^{-S^E}\phi(x)\,\phi(y)$$

$$= G(x,y)$$

$$(-\Box_x + m^2)\,G(x,y) = \delta^D(x-y)$$

$$G(x,y) = \int\frac{d^Dk}{(2\pi)^D}\,\frac{e^{ik\cdot(x-y)}}{k^2+m^2}$$ Euclidean

- Continuation to real time

set $-ix^D = e^{-i\alpha}t$.

$-iy^D = e^{-i\alpha}t'$, $\quad t > t'$

$\alpha \to 0^+$

- to maintain analyticity of $G(x,y)$,
also Wick rotate k^D-integration contour

set $-i k^D = e^{+i\alpha} k^0$.

$$G(x,y) = -i e^{i\alpha} \int \frac{dk^0 d^{D-1}\vec{k}}{(2\pi)^D} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y}) - i k^0 (t-t')}}{\underbrace{-e^{2i\alpha}(k^0)^2 + \vec{k}^2 + m^2}}.$$

orientation

denominator never vanishes
as α ranges from $\frac{\pi}{2}$ to 0^+.

What about the $\alpha \to 0$ limit?

- if we set $\alpha = 0$ in the integrand, the denominator would vanish when k^0 hits $\pm\sqrt{\vec{k}^2 + m^2}$, $\int$ ill-defined.

- to maintain analyticity, we can add $-i\epsilon$ to the denominator, so that its imaginary part never vanishes, and take $\epsilon \to 0^+$ limit after evaluating the k-integrals.

This results in the time-ordered

This results in the time-ordered 2-point function

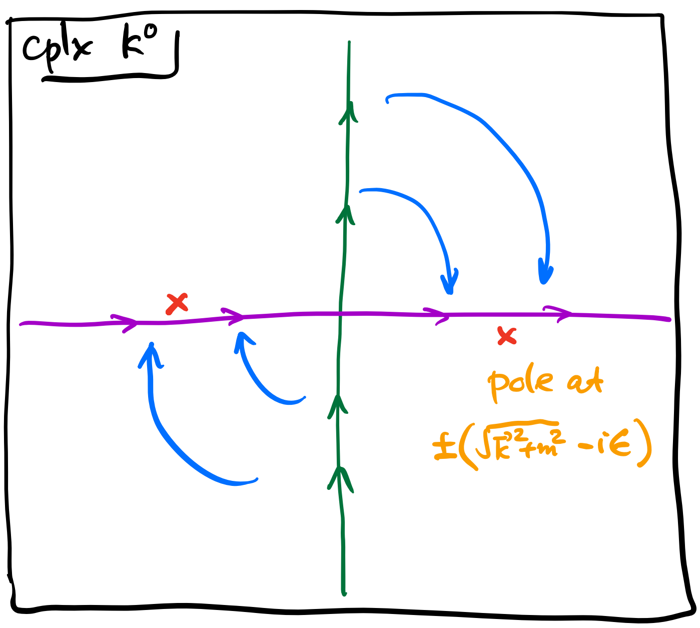
$$\langle \Omega | T \, \hat{\phi}(x) \, \hat{\phi}(y) | \Omega \rangle$$

$$= -i \int \frac{dk^0 \, d^{D-1}\vec{k}}{(2\pi)^D} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y}) - ik^0(x^0 - y^0)}}{-(k^0)^2 + \vec{k}^2 + m^2 - i\epsilon}$$

$$\equiv -i \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\cdot(x-y)}}{k^2 + m^2 - i\epsilon}$$

Lorentzian

Lorentzian



Three useful types of Green functions (in a general QFT)

- Wightman function $\qquad x_a = (t_a, \vec{x}_a) \in \mathbb{R}^{1,D-1}$

$$W(x_1,\dots,x_n) = \langle \Omega | \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | \Omega \rangle$$

- Time-ordered Green function

$$G(x_1,\dots,x_n) = \langle \Omega | \underbrace{T\, \hat{\phi}(x_1) \cdots \hat{\phi}(x_n)} | \Omega \rangle$$

$\hat{\phi}(x_{i_1}) \cdots \hat{\phi}(x_{i_n})$

if $x^0_{i_1} \geq x^0_{i_2} \geq \cdots \geq x^0_{i_n}$

for $(i_1 \cdots i_n)$ a permutation of $(1 \cdots n)$.

- Euclidean Green function (Schwinger function)

$x^E_a = (\tau_a, \vec{x}_a) \in \mathbb{R}^D$

$$G_E(x^E_1,\dots,x^E_n) = W(x_{i_1},\dots,x_{i_n})\Big|_{t_a \to -i\tau_a}$$

analytic continuation

for $\tau_{i_1} \geq \tau_{i_2} \geq \cdots \geq i_n$

$(i_1,\dots,i_n)$ a perm. of $(1,\dots,n)$

- The Wightman functions are most directly related to the spectrum:

e.g. $\langle\Omega|\hat{\phi}(x)\,\hat{\phi}(y)|\Omega\rangle$

$$= \int d\alpha\ \langle\Omega|\hat{\phi}(x)|\alpha\rangle\langle\alpha|\hat{\phi}(y)|\Omega\rangle$$

$\int d\alpha\ |\alpha\rangle\langle\alpha| = 1$,

$\hat{P}^\mu|\alpha\rangle = P^\mu_\alpha|\alpha\rangle$

$$= \int d\alpha\ \langle\Omega|e^{-i\hat{P}\cdot x}\hat{\phi}(0)\,e^{i\hat{P}\cdot x}|\alpha\rangle \cdot \langle\alpha|e^{-i\hat{P}\cdot y}\hat{\phi}(0)\,e^{i\hat{P}\cdot y}|\Omega\rangle$$

$$= \int d\alpha\ e^{iP_\alpha\cdot(x-y)}\,|\langle\Omega|\hat{\phi}(0)|\alpha\rangle|^2$$

assumed $\hat{\phi}(x)$ Hermitian

$$= \int d^Dk\ e^{ik\cdot(x-y)}\underbrace{\int d\alpha\ \delta^D(k-P_\alpha)\,|\langle\Omega|\hat{\phi}(0)|\alpha\rangle|^2}_{\text{invariant under } k^\mu\to\Lambda^\mu{}_\nu k^\nu \text{ Lorentz transf.}}$$

[Proof: Lorentz transf. acts as unitary operator $U(\Lambda)$:

$$U(\Lambda)\,\hat{\phi}(x)\,(U(\Lambda))^{-1} = \hat{\phi}(\Lambda\cdot x)$$ (scalar field)

$$U(\Lambda)\,|\alpha\rangle = |\alpha^\Lambda\rangle, \qquad P^\mu_{\alpha^\Lambda} = \Lambda^\mu{}_\nu P_\alpha^\nu$$

$$\int d\alpha\, \delta^D(\Lambda\cdot k - P_\alpha)\, |\langle\Omega|\hat{\phi}(0)|\alpha\rangle|^2$$

$$= \int d\alpha\, \underbrace{\delta^D(\Lambda\cdot k - P_{\alpha^\Lambda})}_{=\,\delta^D(k-P_\alpha)}\, |\underbrace{\langle\Omega|\hat{\phi}(0)|\alpha^\Lambda\rangle}_{=\,\langle\Omega|(U(\Lambda))^{-1}\hat{\phi}(0)U(\Lambda)|\alpha\rangle}|^2$$

$$\left[\langle\Omega|(U(\Lambda))^{-1}\hat{\phi}(0)U(\Lambda)|\alpha\rangle = \langle\Omega|\hat{\phi}(0)|\alpha\rangle\right]$$

Can write

$$\int d\alpha\, \delta^D(k-P_\alpha)\, |\langle\Omega|\hat{\phi}(0)|\alpha\rangle|^2$$

$$:= \frac{\theta(k^0)}{(2\pi)^{D-1}}\, \rho(-k^2),$$

$k^2 \equiv k^\mu k_\mu = -(k^0)^2 + \vec{k}^2$

for some non-negative function $\rho(-k^2)$

- the appearance of $\theta(k^0) := \begin{cases} 1, & k^0 \geq 0 \\ 0, & k^0 < 0 \end{cases}$ is due to assumption of $P^0 \geq 0$ for all physical states.

Thus, we have the 2-pt Wightman function

$$\langle\Omega|\,\hat{\phi}(x)\,\hat{\phi}(y)\,|\Omega\rangle$$

$$= \int d^D k \; e^{ik\cdot(x-y)} \, \frac{\theta(k^0)}{(2\pi)^{D-1}} \, \rho(-k^2)$$

$$= \int_0^\infty d(\mu^2)\; \rho(\mu^2) \underbrace{\int \frac{d^D k}{(2\pi)^{D-1}}\, \theta(k^0)\, \delta(k^2+\mu^2)\, e^{ik\cdot(x-y)}}_{\substack{\parallel\!\parallel\!\parallel \\ \Delta_+(x-y;\,\mu^2)}}$$

$$\Delta_+(x;\mu^2) = \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \, \frac{1}{2\sqrt{\vec{k}^2+\mu^2}} \, e^{i\vec{k}\cdot\vec{x} - i\sqrt{\vec{k}^2+\mu^2}\,x^0}$$

Have already seen in lecture 1 that

$$\Delta_+(x;\mu^2) = \Delta_+(-x;\mu^2) \quad \text{for } x^2>0 \quad (\text{spacelike } x^\mu)$$

$$\neq \qquad \text{for } x^2<0 \quad (\text{timelike } x^\mu)$$

- The time-ordered Green function will be closely related to the scattering amplitude of particle (via LSZ reduction).

For now, let us inspect

$$G(x,y) := \langle \Omega | T \hat{\phi}(x) \hat{\phi}(y) | \Omega \rangle$$

$$\equiv \theta(x^0 - y^0) \langle \Omega | \hat{\phi}(x) \hat{\phi}(y) | \Omega \rangle + \theta(y^0 - x^0) \langle \Omega | \hat{\phi}(y) \hat{\phi}(x) | \Omega \rangle$$

$$= \int_0^\infty d\mu^2 \, \rho(\mu^2) \Big[\underbrace{\theta(x^0 - y^0) \Delta_+(x - y; \mu^2) + \theta(y^0 - x^0) \Delta_+(y - x; \mu^2)}_{\equiv\, \Delta_F(x-y;\mu^2)} \Big]$$

$$\Delta_F(x;\mu^2) = \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \frac{e^{i\vec{k}\cdot\vec{x}}}{2\sqrt{\vec{k}^2 + \mu^2}} \Big[\theta(x^0) e^{-i\sqrt{\vec{k}^2 + \mu^2}\, x^0} + \theta(-x^0) e^{i\sqrt{\vec{k}^2 + \mu^2}\, x^0} \Big]$$

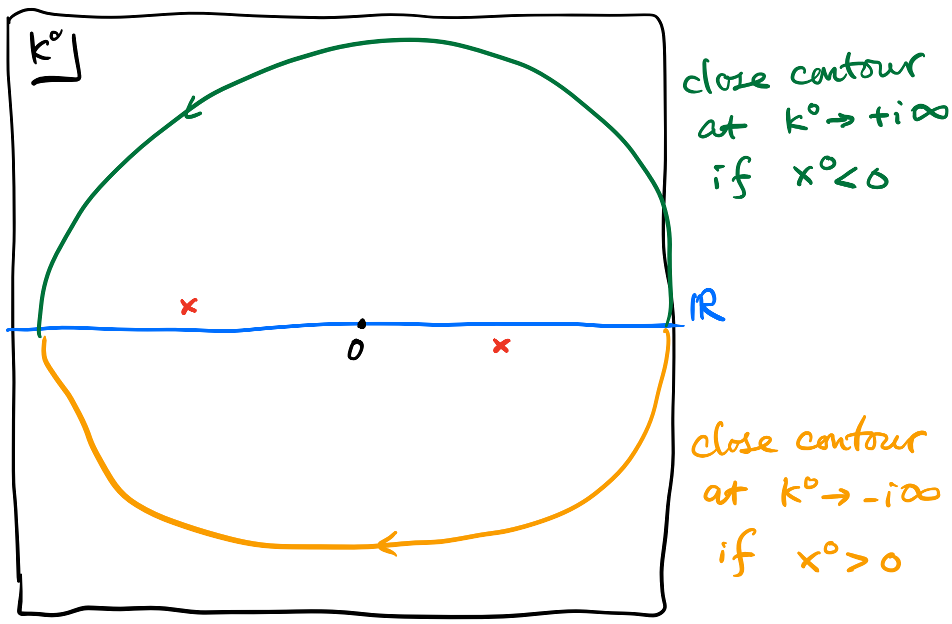
Consider $$\int \frac{dk^0}{2\pi i} \, \frac{e^{-ik^0x^0}}{-(k^0)^2 + \vec{k}^2 + \mu^2 - i\epsilon}$$

viewed as a contour integral on complex k^0- plane.

The integrand has poles at

$$k^0 = \pm \sqrt{\vec{k}^2 + \mu^2 - i\epsilon}$$

$$\approx \pm \sqrt{\vec{k}^2 + \mu^2}\left(1 - \frac{i\epsilon}{2(\vec{k}^2+\mu^2)}\right)$$



result: $$\int \frac{dk^0}{2\pi i} \, \frac{e^{-ik^0x^0}}{k^2 + \mu^2 - i\epsilon}$$

$$= -\theta(x^0) \operatorname*{Res}_{k^0 \to \sqrt{\vec{k}^2+\mu^2-i\epsilon}} \frac{e^{-ik^0x^0}}{k^2+\mu^2-i\epsilon} + \theta(-x^0) \operatorname*{Res}_{k^0 \to -\sqrt{\vec{k}^2+\mu^2-i\epsilon}} \frac{e^{-ik^0x^0}}{k^2+\mu^2-i\epsilon}$$

contour orientation

$$= -\theta(x^0)\left(\frac{e^{-i\sqrt{\vec{k}^2+\mu^2}\,x^0}}{-2\sqrt{\vec{k}^2+\mu^2}}\right) + \theta(-x^0)\left(\frac{e^{i\sqrt{\vec{k}^2+\mu^2}\,x^0}}{2\sqrt{\vec{k}^2+\mu^2}}\right)$$

Compare to the definition of Δ_F above, we can write equivalently

$$\Delta_F(x;\mu^2) = -i\int \frac{d^D k}{(2\pi)^D}\,\frac{e^{ik\cdot x}}{k^2+\mu^2-i\epsilon}$$

"Feynman propagator"

- As for the Euclidean Green function

  - begin with Wightman function

$$W(x,y) = \langle\Omega|\hat{\Phi}(x)\,\hat{\Phi}(y)|\Omega\rangle$$

$$= \int d\alpha\, \underbrace{e^{iP_\alpha\cdot(x-y)}}_{\parallel \atop e^{i\vec{P}_\alpha\cdot(\vec{x}-\vec{y}) - iP_\alpha^0(x^0-y^0)}}\, |\langle\Omega|\hat{\Phi}(0)|\alpha\rangle|^2$$

$(P_\alpha^0 \geq 0)$

analytic continuation to complex x^0, y^0 ?

- $W(x,y)$ is an analytic function of complex x^0, y^0 provided $e^{-iP_\alpha^0(x^0-y^0)}$ is exponentially suppressed at large P_α^0, i.e. $\mathrm{Im}\,(x^0-y^0)<0$.

- possible to maintain $\mathrm{Im}\,(x^0-y^0)<0$ in going from real x^0, y^0 to

$$x^0 = -i\tau_1, \quad y^0 = -i\tau_2, \quad \text{with } \tau_1>\tau_2$$

$$x^E \equiv (\tau_1, \vec{x}), \quad y^E \equiv (\tau_2, \vec{y})$$

Euclidean Green function

$G_E(x^E, y^E) :=$ analytic continuation of $W(x,y)$ to $x^0=-i\tau_1$, $y^0=-i\tau_2$ for $\tau_1>\tau_2$

For $\tau_1<\tau_2$,

$G_E(x^E, y^E) =$ a.c. of $W(y,x)$ instead.

Result:

$$G_E(x^E, y^E) = \int_0^\infty d\mu^2\, \rho(\mu^2)\, \Delta_E(x^E - y^E; \mu^2)$$

$$\Delta_E(x^E; \mu^2) = \int \frac{d^D k^E}{(2\pi)^D} \frac{e^{i k^E \cdot x^E}}{(k^E)^2 + \mu^2}$$

$$(k^E)^2 := \sum_{\mu=1}^{D} (k^E_\mu)^2$$

$$k^E \cdot x^E = \sum_{\mu=1}^{D} k^E_\mu x^E_\mu .$$

The Euclidean Green function has the simplest analytic structure (only singular when points collide) and will be easiest to evaluate (e.g. with numerical approx.)

• More on the structure of Green functions

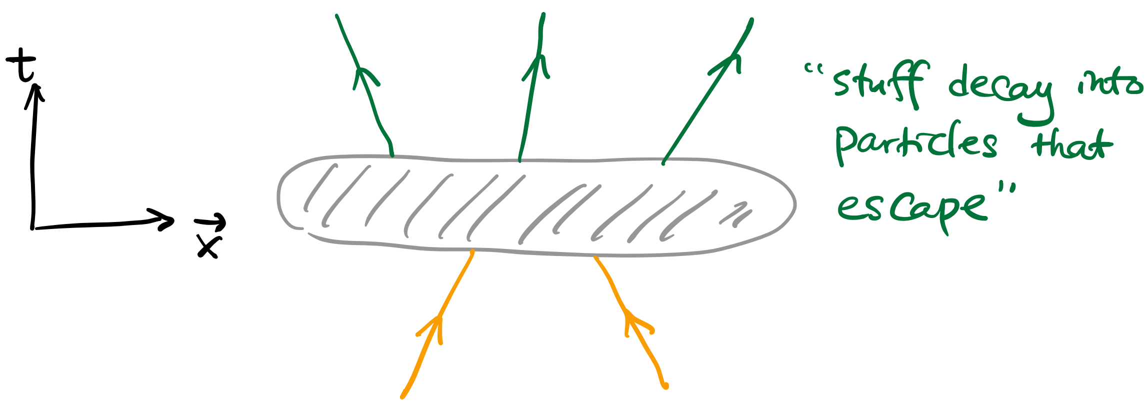
Wightman function

$$\langle\Omega|\,\hat{\phi}(x)\,\hat{\phi}(y)\,|\Omega\rangle = \int_0^\infty d\mu^2\, \rho(\mu^2)\, \Delta_+(x-y;\mu^2)$$

$$\rho(\mu^2) = (2\pi)^{D-1}\int d\alpha\, \delta^D(k-P_\alpha)\,\left|\langle\alpha|\,\hat{\phi}(0)\,|\Omega\rangle\right|^2$$

$k^2 = -\mu^2, \quad k^0 > 0$

What is the content of $\hat{\phi}(x)\,|\Omega\rangle$?



At a given instance of time, a physical state may not admit any obvious particle interpretation, but it is expected to evolve into a superposition of states that consist of far-separated wave packets of individual particles as $t \to +\infty$ or $t \to -\infty$.

- an underlying assumption is that "particles" are weakly interacting at long distance
- the particles could be strongly interacting at finite distances, but the invariant mass squared (i.e. $-P_\mu P^\mu$) of any state would be identical to that of its decay product

- Suppose there is only 1 species of particle in our theory, of some mass m

possible invariant masses of $|\alpha\rangle$:

- vacuum $|\Omega\rangle$ $\quad -P^2 = 0$
- 1-particle state $\quad -P^2 = m^2$
- states that decay into 2 or more particles $\quad -P^2 \geq (2m)^2$ (a continuum of possible values)

WLOG, can shift $\hat{\phi}(x)$ by a constant so that $\langle\Omega|\hat{\phi}(x)|\Omega\rangle = 0$

$-P^2 = m^2$

$$\hat{\phi}(x)|\Omega\rangle = |1\text{-particle}\rangle + |\text{multi-particle stuff}\rangle$$

$-P^2 \geq (2m)^2$

- Assuming no internal ("spin") degree of freedom, a basis of 1-particle states is labeled by the spatial momentum

$|\vec{k}\rangle$. with

$$P^i|\vec{k}\rangle = k^i|\vec{k}\rangle$$

$$i=1,\cdots,D-1$$

$$P^0|\vec{k}\rangle = \sqrt{\vec{k}^2+m^2}\,|\vec{k}\rangle.$$

Choose normalization such that

$$\langle\vec{k}|\vec{k}'\rangle = \delta^{D-1}(\vec{k}-\vec{k}').$$

Q: how does Lorentz sym $U(\Lambda)$ act on $|\vec{k}\rangle$?

Expect

$$U(\Lambda)\,|\vec{k}\rangle = C(\Lambda,\vec{k})\,|\overrightarrow{\Lambda k}\rangle$$

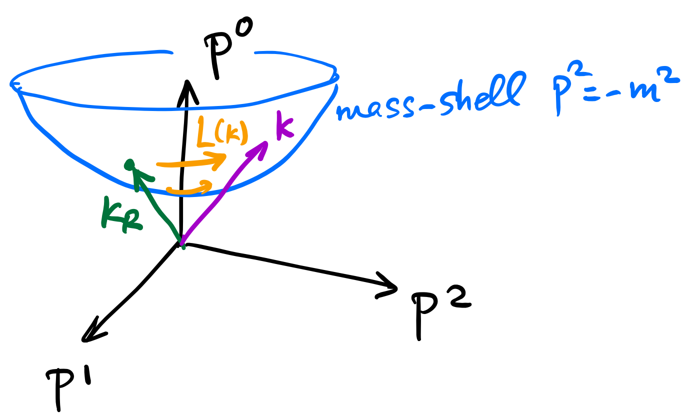
the spatial component of $(\Lambda k)^\mu = \Lambda^\mu{}_\nu k^\nu$.

To proceed, pick a reference momentum

$$k_R^\mu\,, \qquad k_R^2 \equiv k_R^\mu k_{R\mu} = -m^2.$$

For every possible momentum k^μ of the particle, choose a "standard boost" $L^\mu{}_\nu(\vec{k})$, such that

$$k^\mu = L^\mu{}_\nu(\vec{k})\cdot k_R^\nu$$



We can specify the phase of $|\vec{k}\rangle$ by defining

$$|\vec{k}\rangle := \underbrace{N(\vec{k})}_{\substack{\text{normalization} \\ \text{factor, TBD}}} U(L(\vec{k}))\,|\vec{k}_R\rangle$$

For any Lorentz transf. Λ,

$$U(\Lambda)\,|\vec{k}\rangle = N(\vec{k})\,U(\Lambda)\,U(L(\vec{k}))\,|\vec{k}_R\rangle$$

$$= N(\vec{k}) \cdot U(\Lambda L(\vec{k}))\,|\vec{k}_R\rangle$$

$$= N(\vec{k})\,U(L(\vec{\Lambda k}))\,U(\underbrace{(L(\vec{\Lambda k}))^{-1}\Lambda L(\vec{k})}_{\substack{\cong \\ W}})\,|\vec{k}_R\rangle$$

Note: W fixes $\vec{k}_R$

$$k_R \xrightarrow{L(\vec{k})} k \xrightarrow{\Lambda} \Lambda k \xrightarrow{(L(\vec{\Lambda k}))^{-1}} k_R$$

Thus

$$U(W)\,|\vec{k}_R\rangle \propto |\vec{k}_R\rangle$$

. $U(W)$ acts on $|\vec{k}_R\rangle$ by multiplying a phase $D(W)$.

- $$U(\Lambda)|\vec{k}\rangle = N(\vec{k})\, U(L(\vec{\Lambda k}))\, D(w)|\vec{k}_R\rangle$$

$$= \underbrace{N(\vec{k})\, D(w)\, (N(\vec{\Lambda k}))^{-1}}_{\text{this is the } C(\Lambda, \vec{k}) \text{ we were looking for.}}\, |\vec{\Lambda k}\rangle$$

- What is $D(w)$?

For now, assume mass $m > 0$,

W.L.O.G. can choose

$$k_R^\mu = (m, \vec{0})$$

W is a Lorentz transf. that fixes k_R^μ, i.e. a spatial rotation

$$W \in SO(D-1)$$

$$D(W_1 W_2) = D(W_1)\, D(W_2)$$

$$|D(W)|^2 = 1.$$

For $D \geqslant 4$, $SO(D-1)$ is non-Abelian,

furthermore, every element of $SO(D-1)$

can be written as a "commutator"

$$W = X Y X^{-1} Y^{-1} \quad \text{for some} \quad X, Y \in SO(D-1).$$

It follows that

$$D(W) = D(X)\, D(Y)\, \big(D(X)\big)^{-1} \big(D(Y)\big)^{-1} = 1.$$

- Such particles are called "scalar particles" (no obvious relation to "scalar fields")

Conclusion:

$$U(\Lambda)\, |\vec{k}\rangle = \frac{N(\vec{k})}{N(\overrightarrow{\Lambda k})}\, |\overrightarrow{\Lambda k}\rangle$$

Compare: $\langle \vec{k}' | \big(U(\Lambda)\big)^{\dagger} U(\Lambda)\, |\vec{k}\rangle$

$$= \frac{N(\vec{k}')}{N(\overrightarrow{\Lambda k}')} \frac{N(\vec{k})}{N(\overrightarrow{\Lambda k})} \langle \overrightarrow{\Lambda k}' | \overrightarrow{\Lambda k} \rangle$$

$$= \left(\frac{N(\vec{k})}{N(\overrightarrow{\Lambda k})} \right)^2 \delta^{D-1}(\overrightarrow{\Lambda k}' - \overrightarrow{\Lambda k})$$

$$\langle \vec{k}' | \vec{k} \rangle = \delta^{D-1}(\vec{k}' - \vec{k}).$$

Using
$$\underbrace{\delta^D(k'-k)}_{} = 2k^0 \underbrace{\delta(k'^2-k^2)}_{} \delta^{D-1}(\vec{k}'-\vec{k})$$

invariant
under $k \to \Lambda k$, $k' \to \Lambda k'$

$$\Rightarrow \quad \frac{\delta^{D-1}(\vec{k}'-\vec{k})}{\delta^{D-1}(\overrightarrow{\Lambda k}'-\overrightarrow{\Lambda k})} = \frac{2(\Lambda k)^0}{2k^0} = \left(\frac{N(\vec{k})}{N(\overrightarrow{\Lambda k})}\right)^2$$

$$\Rightarrow \quad N(\vec{k}) \propto \frac{1}{\sqrt{k^0}}.$$

Since $N(\vec{k}_R) = 1$ by definition,

we conclude that

$$N(\vec{k}) = \sqrt{\frac{m}{k^0}}.$$

Back to

$$\hat{\phi}(x)|\Omega\rangle = |1\text{-particle}\rangle + |\text{multi-particle}\rangle$$

Let us study the overlap

$$\langle \vec{k} |\, \hat{\phi}(x) \,| \Omega \rangle = \langle \vec{k} |\, e^{-i\hat{P}\cdot x}\, \hat{\phi}(0)\, e^{i\hat{P}\cdot x} \,| \Omega \rangle$$

$$= e^{-ik\cdot x} \underbrace{\langle \vec{k} |\, \hat{\phi}(0) \,| \Omega \rangle}$$

$$\langle \vec{k} |\, \hat{\phi}(0) \,| \Omega \rangle = \langle \vec{k} |\, (U(\Lambda))^{-1}\, \hat{\phi}(0)\, U(\Lambda) \,| \Omega \rangle$$

$$= \underset{\displaystyle \sqrt{\frac{(\Lambda k)^0}{k^0}}}{\underset{\shortparallel}{\boxed{\left(\frac{N(\vec{k})}{N(\vec{\Lambda k})}\right)^*}}} \langle \vec{\Lambda k} |\, \hat{\phi}(0) \,| \Omega \rangle$$

Conclusion :

$$\langle \vec{k} |\, \hat{\phi}(0) \,| \Omega \rangle \propto \frac{1}{\sqrt{k^0}} = \frac{1}{\sqrt{\omega_{\vec{k}}}} \qquad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$$

Can write

$$\langle \vec{k} |\, \hat{\phi}(0) \,| \Omega \rangle \equiv \left[\frac{Z}{(2\pi)^{D-1} \cdot 2\omega_{\vec{k}}} \right]^{\frac{1}{2}} .$$

Z (≥ 0) is a constant.

Remark: Z is often called "field renormalization constant"

(for free scalar field, we have seen that $Z = 1$)

It is sometimes asserted that Z is "unphysical"; this is false:

Z is intrinsic to the field operator $\hat{\phi}(x)$, and is unambiguously defined provided that $\hat{\phi}(x)$ is defined.

Recall the spectral function

$$\rho(\mu^2) = (2\pi)^{D-1} \int d\alpha \, \delta^D(p - P_\alpha) \, |\langle\alpha|\hat{\phi}(0)|\Omega\rangle|^2$$

$p^2 = -\mu^2, \quad p^0 > 0$

$$= \underbrace{\int d^{D-1}\vec{k} \, \frac{Z}{2\omega_{\vec{k}}} \, \delta^D(p-k)}\Big|_{k^0 = \omega_{\vec{k}}}$$

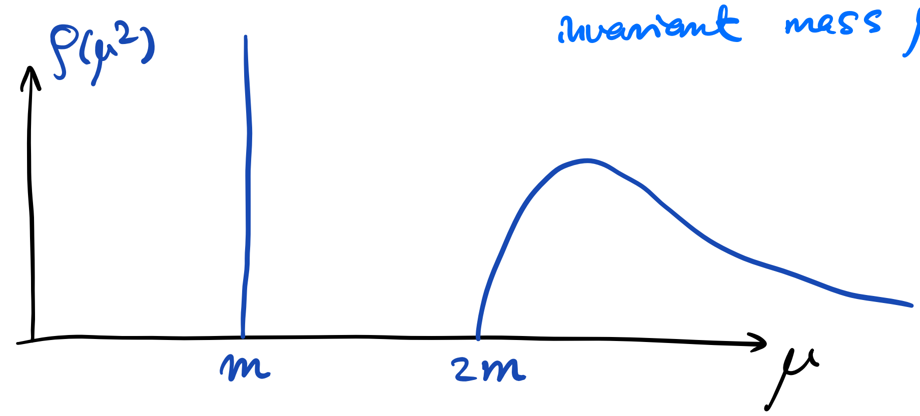
$+$ (multi-particle $|\alpha\rangle$ contribution)

$$\frac{Z}{2\omega_{\vec{k}}}\,\delta\left(p^0-\sqrt{\vec{p}^{\,2}+m^2}\right) = Z\cdot\delta(p^2+m^2)$$

$$= Z\cdot\delta(\mu^2-m^2)$$

$$\rho(\mu^2) = Z\cdot\delta(\mu^2-m^2) + \underbrace{\sigma(\mu^2)}$$

multi-particle contri.
supported at
invariant mass $\mu \geq 2m$.



It follows that

$$\langle\Omega|\, T\,\hat{\phi}(x)\,\hat{\phi}(y)\,|\Omega\rangle$$

$$= \int_0^\infty d\mu^2\, \rho(\mu^2)\, \Delta_F(x-y;\mu^2)$$

$$= Z\cdot\Delta_F(x-y;m^2) + \int_{4m^2}^\infty d\mu^2\, \sigma(\mu^2)\, \Delta_F(x-y;\mu^2)$$

Källén – Lehmann spectral representation

Computation of correlation functions in perturbation theory

- illustrate with scalar ϕ^4 theory

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}g\phi^4$$

$$\updownarrow$$

$$\mathcal{L}^E = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g\phi^4$$

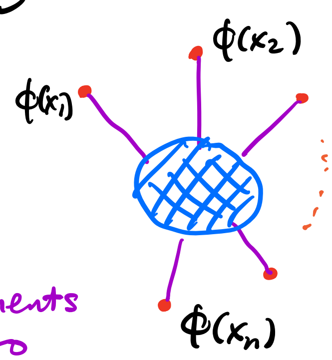
Euclidean:

$$\langle \phi(x_1^E)\cdots\phi(x_n^E)\rangle$$

$$= \frac{\int [D\phi]\, e^{-S^E[\phi]}\,\phi(x_1^E)\cdots\phi(x_n^E)}{\int [D\phi]\, e^{-S^E[\phi]}}$$

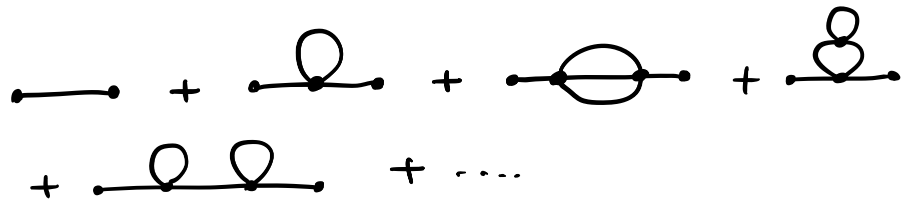
$$= \sum$$

graphs whose connected components are connected to external ϕ-lines



2-pt function

$\langle \phi(x)\,\phi(y) \rangle =$ 

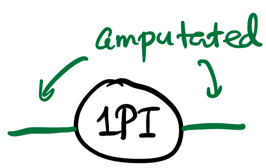
$=$ 

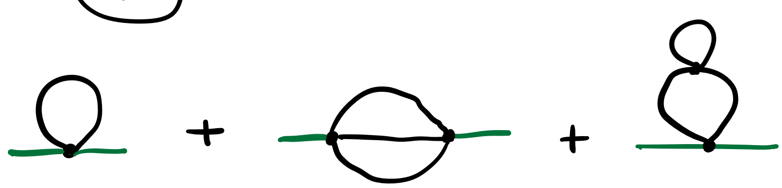
$+ \dots$

$$G_E(x,y) = \int \frac{d^D k^E}{(2\pi)^D}\, e^{i k^E\cdot(x^E - y^E)}\, \tilde{G}_E(k)$$

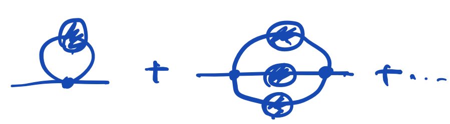
$$= \int_0^\infty d\mu^2\, \rho(\mu^2) \int \frac{d^D k^E}{(2\pi)^D}\, \frac{e^{i k^E\cdot(x^E-y^E)}}{k^2+\mu^2}$$

$$\tilde{G}_E(k) = \frac{1}{k^2+m^2-\Sigma(k)} = \int_0^\infty d\mu^2\, \frac{\rho(\mu^2)}{k^2+\mu^2}$$

$\Sigma(k) =$ 

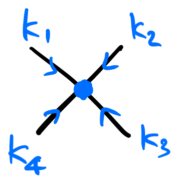
$=$ 

$+ \dots$

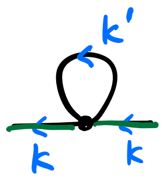
[can also organize more efficiently as $=$ ]

propagator 

$= \dfrac{1}{k^2+m^2}$

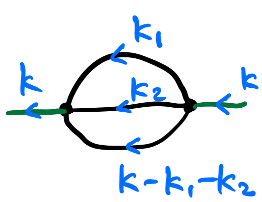
vertex 

$= -\dfrac{g}{4!} \cdot (2\pi)^D \delta^D(k_1+k_2+k_3+k_4)$

e.g. 

$$= 4\cdot 3 \cdot \left(-\frac{g}{4!}\right) \int \frac{d^D k'}{(2\pi)^D} \frac{1}{k'^2+m^2}$$

$$= -\frac{g}{2} \int \frac{d^D k'}{(2\pi)} \frac{1}{k'^2+m^2}$$

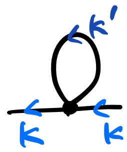


$$= 4\cdot 4! \left(-\frac{g}{4!}\right)^2 \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D}$$

$$\cdot \frac{1}{(k_1^2+m^2)(k_2^2+m^2)((k-k_1-k_2)^2+m^2)}$$

$$= \frac{g^2}{6} \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{1}{(k_1^2+m^2)(k_2^2+m^2)((k-k_1-k_2)^2+m^2)}$$

Let us interpret the order g^1 contribution to $\Sigma(k)$,



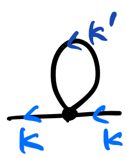
$$= -\frac{g}{2} \underbrace{\int \frac{d^D k'}{(2\pi)^D} \frac{1}{k'^2+m^2}}_{\text{divergent for } D \geq 2}$$

divergent for $D \geq 2$??

but independent of k

First, regularize the path integral

$$\int [D\phi] \rightarrow \int \prod_{|k|<\Lambda} d\tilde{\phi}(k)$$



$$= -\frac{g}{2} \int_{|k'|<\Lambda} \frac{d^D k'}{(2\pi)^D} \frac{1}{k'^2+m^2}$$

$$= -\frac{g}{2} \cdot \frac{\mathrm{vol}(S^{D-1})}{(2\pi)^D} \int_0^{\Lambda} dk \, \frac{k^{D-1}}{k^2+m^2}$$

$$\equiv -g \cdot C_1$$

[$\mathrm{vol}(S^{D-1}) = \dfrac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$ is the area of unit $(D-1)$-sphere]

$$\int_0^{\Lambda} dk \, \frac{k^{D-1}}{k^2+m^2} = \begin{cases} \frac{1}{2}\log \frac{\Lambda^2+m^2}{m^2} \,, & D=2 \\ \frac{\Lambda^{D-2}}{D-2} + \cdots \,, & D \geq 3 \end{cases}$$

Recall, Källén – Lehmann spectral rep:

$$\tilde{G}_E(k) = \frac{1}{k^2+m^2-\Sigma(k)} = \frac{Z}{k^2+m_*^2} + \int_{\mu^2 \geq 4m_*^2} d\mu^2 \, \frac{\sigma(\mu^2)}{k^2+\mu^2}$$

m_* is the actual mass of the particle, corresponding to the pole in k^2 (or just k^D at fixed $\vec{k} = (k^1, \cdots, k^{D-1})$) closest to the origin.

A constant term at order g in $\Sigma(k)$ shifts the pole position:

$$m_*^2 = m^2 + g c_1 + \mathcal{O}(g^2)$$

$$c_1 \sim \begin{cases} \log \frac{\Lambda}{m} \,, & D=2 \\ \Lambda^{D-2} \,, & D \geq 2 \end{cases}$$

Interpretation:

1. m_* is the mass of the particle

2. m is just some parameter in the Lagrangian; its relation to the physical mass is dependent on the regularization scheme.

3. it is standard terminology to refer to m as the "bare mass", and to the Lagrangian that enter the regularized path integral as the "bare Lagrangian".

4. One might be slightly disturbed by the fact that while the difference between m^2 and m_*^2 is formally suppressed by the coupling g, the coefficient C_1 diverges as $\Lambda \to \infty$ and so gC_1 is not exactly "small".

One way to handle this in a consistent

manner in perturbation theory is to write

$$m^2 = \underbrace{m_R^2}_{\text{a finite "renormalized" mass } m_R \text{ close to } m_*} + \underbrace{\delta m^2}_{\text{viewed as "counter term", diverges as } \Lambda\to\infty\text{, but appears at order } g}$$

Generally, $\delta m^2 = \sum_{n=1}^{\infty} (\delta m^2)^{(n)} g^n$

$(\delta m^2)^{(n)}$ is chosen to cancel the divergent contribution to pole position m_*^2 at order g^n in perturbation theory.

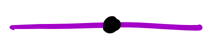
e.g. $$m_*^2 = m_R^2 + \delta m^2 + g c_1 + \mathcal{O}(g^2)$$

choose $(\delta m^2)^{(1)} = -c_1 + \text{finite}$

Correspondingly, we split the bare Lagrangian as

$$\mathcal{L}^E = \underbrace{\mathcal{L}_R^E}_{\text{contains } \frac{1}{2} m_R^2 \phi^2} + \underbrace{\Delta\mathcal{L}^E}_{\Delta\mathcal{L}^E = \frac{1}{2}\delta m^2 \phi^2 .}$$

and treat $\Delta\mathcal{L}$ as producing interaction vertizes at order g^n :


$$-(\delta m^2)^{(n)} g^n \phi^2$$

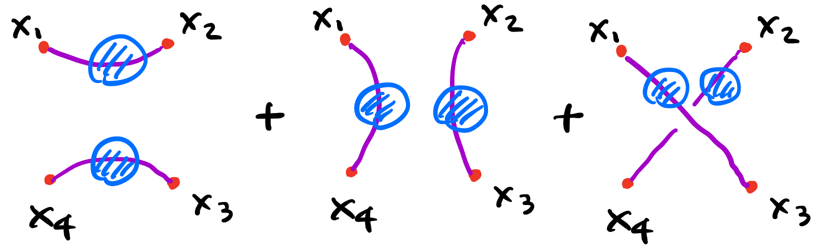
The separation of m^2 into m_R^2 and δm^2 is entirely ad hoc. Its only purpose is to (re-)organize perturbation theory so that when we take $\Lambda \to \infty$, divergences cancel manifestly at each order in g (as opposed to cancelling divergences between different orders in g, e.g. between m^2 at 0^{th} order and $c_1 g$ at 1^{st} order)

To make meaningful physical predictions, we should pick a regularization scheme, find the relation between m and m_*, compute observables using Lagrangian that involves m^2 $(= m_R^2 + \delta m^2)$, and express the result in terms of m_* (the physical mass).

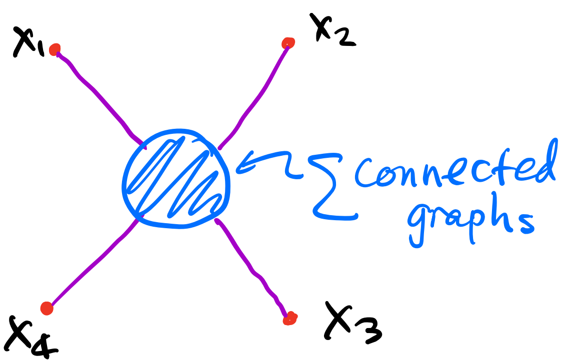
A 4-point function

$$\langle \phi(x_1^E)\, \phi(x_2^E)\, \phi(x_3^E)\, \phi(x_4^E) \rangle$$

=



+



$$= \langle \phi(x_1^E)\, \phi(x_2^E) \rangle \langle \phi(x_3^E)\, \phi(x_4^E) \rangle$$

$$+ \ (2 \leftrightarrow 4) \ + \ (2 \leftrightarrow 3)$$

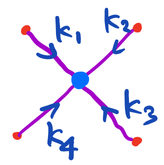
$$+ \ \langle \phi(x_1^E)\, \phi(x_2^E)\, \phi(x_3^E)\, \phi(x_4^E) \rangle^{\text{conn.}}$$

"connected correlator"

$$G_E^{\text{conn}}(x_1^E, \ldots, x_4^E) = \langle \phi(x_1^E) \cdots \phi(x_4^E) \rangle^{\text{conn}}$$

$$\equiv \int \frac{d^D k_1^E}{(2\pi)^D} \cdots \frac{d^D k_4^E}{(2\pi)^D} \, e^{i k_1^E \cdot x_1^E + \ldots + i k_4^E \cdot x_4^E}$$

$$\cdot \, \tilde{G}_E^{\text{conn}}(k_1^E, \ldots, k_4^E)$$

$$\tilde{G}_E^{\text{conn}}(k_1^E, \ldots, k_4^E) = $$  $$+ \cdots$$

$$= (2\pi)^D \delta^D(k_1^E + k_2^E + k_3^E + k_4^E) \cdot$$

$$\times \left[-g \prod_{j=1}^{4} \frac{1}{(k_j^E)^2 + m^2} + \mathcal{O}(g^2) \right]$$

can write $m^2 = m_R^2 + \delta m^2$, and expand in δm^2, and regroup the terms according to powers of g.

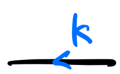
(a convenient choice is $m_R = m_*$)

Everything we said so far about the perturbative computation of

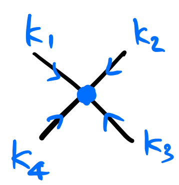
$G_E(x_1^E, \dots, x_n^E)$ can be equally applied to computing the time-ordered Green function

$$G(x_1, \dots, x_n) = \langle \Omega | T\, \hat{\phi}(x_1) \cdots \phi(x_n) | \Omega \rangle ,$$

with the modification of Feynman rules:

propagator 

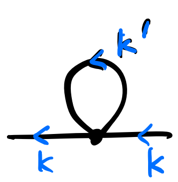
$$= \frac{-i}{k^2 + m^2 - i\epsilon}$$

vertex 

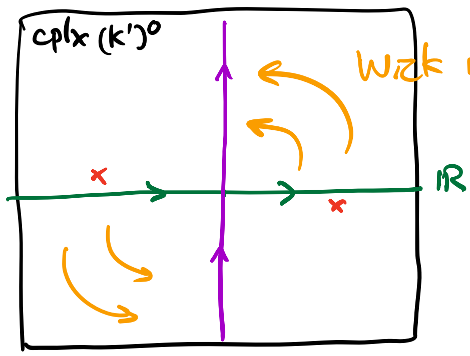
$$= -\frac{i g}{4!} \cdot (2\pi)^D \delta^D(k_1 + k_2 + k_3 + k_4)$$

All momenta k^μ are now Lorentzian

e.g. $k^2 \equiv \eta_{\mu\nu} k^\mu k^\nu$.



$$= -\frac{i g}{2} \int \frac{d^D k'}{(2\pi)^D} \frac{-i}{k'^2 + m^2 - i\epsilon}$$



$$\Big\| \quad (k'^0) \equiv i k'^D$$

$$-\frac{ig}{2}\cancel{i}\int \frac{d^D k^E}{(2\pi)^D}\,\frac{\cancel{-i}}{(k^E)^2+m^2}$$

contributes to $i\Sigma(k)$

[Consistent with Euclidean computation]

The full time-ordered 2-pt function takes the form

$$G(x,y) = \int \frac{d^D k}{(2\pi)^D}\, e^{ik\cdot(x-y)}\,\tilde{G}(k)$$

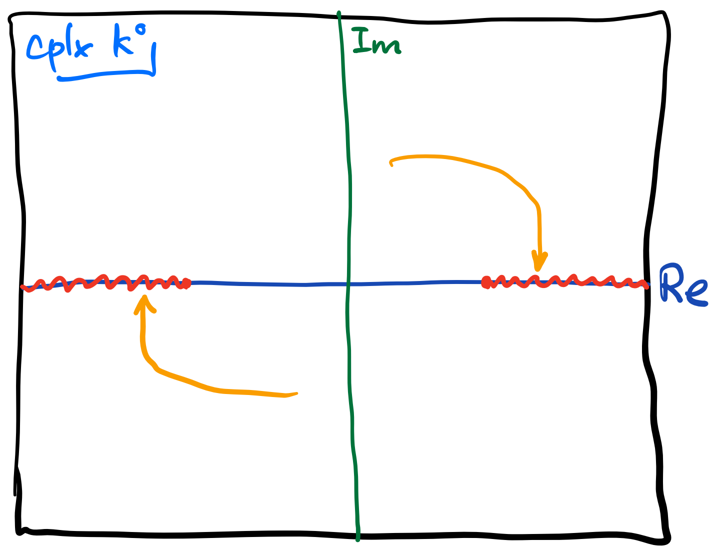
$$\tilde{G}(k) = \frac{-i}{k^2+m^2-i\epsilon-\Sigma(k)}$$

Note: $\Sigma(k)$ is expected to have a pair of branch cuts in k^0, starting at the multi-particle threshold $-k^2 = M^2$.

M = invariant mass of lightest multi-particle state

These branch cuts may be chosen to extend along real k^0-axis:

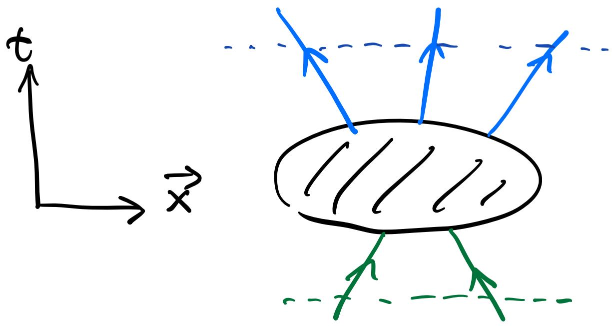
$$k^0 > \sqrt{\vec{k}^2+M^2} \quad \text{and} \quad k^0 < -\sqrt{\vec{k}^2+M^2}$$



The actual value of $\Sigma(k)$ at real k^0 is determined by our Wick rotation procedure relating G_E to G.

e.g. for real $k^0 > \sqrt{\vec{k}^2 + M^2}$,
$\Sigma(k)$ is given by analytic continuation from $k^0 \in i\mathbb{R}$ (or more generally, cplx k^0 on "1st sheet") to positive real k^0-axis from above.

Scattering Theory



out-basis: ~~states~~ that look like far-separated (wave packets of) particles in the far future

in-basis: states that look like far-separated particles in the far past

Can label out- or in- basis states as if they consist of free particles:

$$|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\rangle^{out}$$

and

$$|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\rangle^{in}$$

(assuming one species for now, otherwise also add labels of particle type and helicity etc.)

Either the out-basis states $|\alpha\rangle^{out}$ or the in-basis states $|\alpha\rangle^{in}$ span the entire Hilbert space $\mathcal{H}$.

In an interacting QFT, the in- and out-basis are different, and are related by a linear change of basis.

$$^{out}\langle\beta|\alpha\rangle^{in} \equiv S_{\beta\alpha} \qquad \text{"S-matrix"}$$

Equivalently,

$$|\alpha\rangle^{in} = \int d\beta\, S_{\beta\alpha} |\beta\rangle^{out}.$$

$\int d\beta\, |\beta\rangle^{out}\, {}^{out}\langle\beta| = 1.$

- Next we will give an explicit construction of in- or out-basis states using local field operator $\hat{\phi}(x)$, thereby relating the S-matrix to correlator of $\hat{\phi}$.

[Haag 1958, Ruelle 1962]

Recall

$\hat{\phi}(x)|\Omega\rangle$

$$= \int d^{D-1}\vec{k}\, e^{-i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}} x^0} \left[\frac{Z}{(2\pi)^{D-1}} \frac{1}{2\omega_{\vec{k}}}\right]^{\frac{1}{2}} |\vec{k}\rangle$$

$$+ \int_{\text{multi-particle}} d\alpha \quad e^{-iP_\alpha\cdot x} \langle\alpha|\hat{\phi}(0)|\Omega\rangle \; |\alpha\rangle$$

The multi-particle states are separated from the 1-particle states by a gap in their invariant masses.

Consider a smearing function $f(x)$.

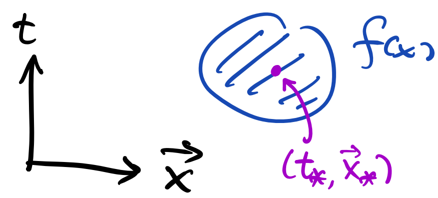
$x \in \mathbb{R}^{1,D-1}$,

$$f(x) = \int \frac{d^D k}{(2\pi)^D} \tilde{f}(k)\, e^{ik\cdot x}$$

Such that $\tilde{f}(k)$ is supported near the mass-shell: $k^0 > 0$, $k^2 + m^2 = 0$,

and at the same time $f(x)$ is

concentrated in a finite domain in the spacetime.



- An example is the Gaussian profile

$$f(x) = \frac{1}{(2\pi)^{\frac{D}{2}} L^D} \exp\Big[-\frac{1}{2L^2}(x^0 - t_*)^2 - \frac{1}{2L^2}(\vec{x} - \vec{x}_*)^2 + i\,\vec{p}_* \cdot (\vec{x} - \vec{x}_*) - i E_* (x^0 - t_*) \Big].$$

Fourier

for some (large) constant distance $L > 0$.

$$\tilde{f}(k) = \exp\Big[-\frac{L^2}{2}(k^0 - E_*)^2 - \frac{L^2}{2}(\vec{k} - \vec{p}_*)^2 - i\,\vec{k} \cdot \vec{x}_* + i\,k^0 t_* \Big].$$

peaked near $(k^0, \vec{k}) \approx (E_*, \vec{p}_*)$

But still knows about $(t_*, \vec{x}_*)$

- A more convenient choice would be to make $\tilde{f}(k)$ compactly supported (within a momentum domain of size $\sim \frac{1}{L}$) but still smooth, in which case $f(x)$ spreads to a blob of size $\sim L$ and diminishes outside the blob faster than $|x|^{-N}$ for any $N > 0$ as $|x| \to \infty$.

We will further choose $(E_*, \vec{P}_*)$ to be such that

$$E_*^2 - \vec{P}_*^{\,2} = m^2.$$

Now consider the "smeared operator"

$$\hat{\phi}_f \equiv \int d^D x \, f(x) \, \hat{\phi}(x) .$$

$$\hat{\phi}_f |\Omega\rangle = \int d^{D-1}\vec{k} \; \tilde{f}(\omega_{\vec{k}}, \vec{k}) \left[\frac{Z}{(2\pi)^{D-1}} \frac{1}{2\omega_{\vec{k}}} \right]^{\frac{1}{2}} |\vec{k}\rangle$$

$$+ \int_{m.p.} d\alpha \; \boxed{\tilde{f}(P_\alpha)} \langle \alpha | \hat{\phi}(0) | \Omega \rangle \; |\alpha\rangle$$

suppressed
by our choice of $\tilde{f}(k)$.

only 1-particle state is retained.

<u>Define</u> : the transformation $f(x) \mapsto f^{(T)}(x)$

by setting

$$\tilde{f}^{(T)}(k) := e^{i(k^0 - \omega_{\vec{k}})T} \, \tilde{f}(k)$$

$$\omega_{\vec{k}} \equiv \sqrt{\vec{k}^2 + m^2} .$$

In particular, $\tilde{f}^{(T)}(\omega_{\vec{k}}, \vec{k}) = \tilde{f}(\omega_{\vec{k}}, \vec{k})$
independent of T.

therefore, the 1-particle part of
$\hat{\phi}_{f^{(T)}}|\Omega\rangle$ is independent of T.

With the above prescribed $f(x)$, the multi-particle component of $\hat{\phi}_f|\Omega\rangle$ is negligible, and therefore

$$\hat{\phi}_{f^{(T)}}|\Omega\rangle \approx \hat{\phi}_f|\Omega\rangle$$

[the approximation is uniformly valid for all, including arbitrarily large, T]

What does $f^{(T)}(x)$ look like?

$$f^{(T)}(x) = \int \frac{d^D k}{(2\pi)^D} e^{ik\cdot x + i(k^0 - \omega_{\vec{k}})T} \tilde{f}(k)$$

$$= \int d^D y \int \frac{d^D k}{(2\pi)^D} e^{ik\cdot(x-y) + i(k^0 - \omega_{\vec{k}})T} f(y)$$

$$= \int d^D y \; \delta(x^0 - y^0 - T) \underbrace{\int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} e^{i\vec{k}\cdot(\vec{x}-\vec{y}) - i\omega_{\vec{k}} T}}_{K(\vec{x}-\vec{y};T)} f(y)$$

$$= \int d^{D-1}\vec{y} \; K(\vec{x}-\vec{y};T) \, f(x^0 - T, \vec{y})$$

$K(\vec{x};T)$ obeys the Klein-Gordon eqn

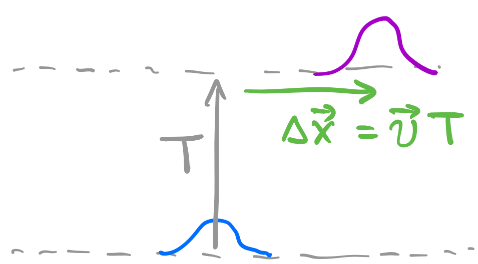
$$(\partial_{\vec{x}}^2 - \partial_T^2 - m^2) K(\vec{x};T) = 0$$

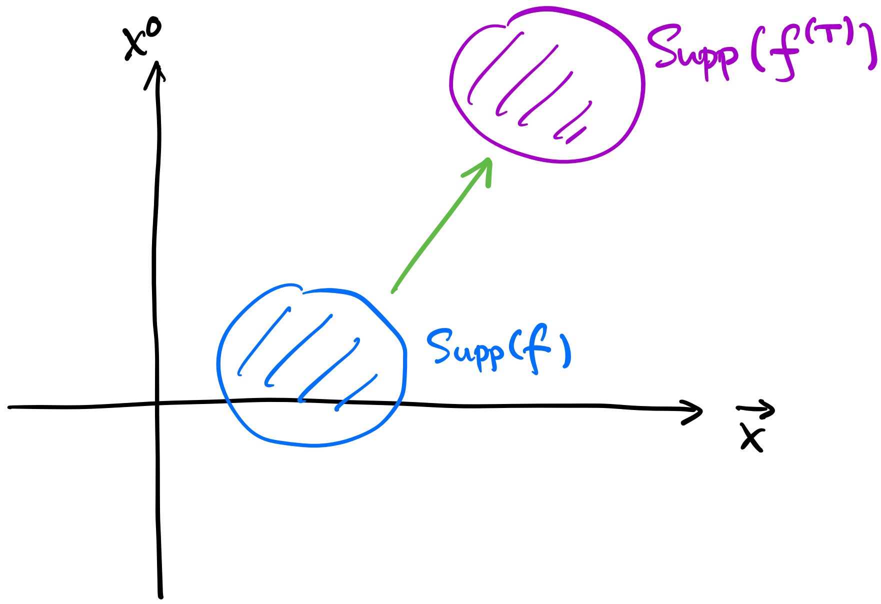
and the initial condition

$$K(\vec{x};0) = \delta^{D-1}(\vec{x})$$

If $\tilde{f}(k)$ is peaked at $k \approx (\underset{\omega_{\vec{p}_*}}{\overset{\shortparallel}{E_*}}, \vec{p}_*)$

then $f^{(T)}(x^0, \vec{x})$ looks like the wave packet $f(x^0 - T, \vec{x})$ having moved with group velocity $\vec{v} = \dfrac{\partial \omega_{\vec{p}_*}}{\partial \vec{p}_*}$ for time T.





Evolving $f \rightsquigarrow f^{(T)}$, the support of $f(x)$, say a region in spacetime of large size $\sim L$ centered around $(t_*, \vec{x}_*)$, moves to a region centered around $(t_* + T, \vec{x}_* + \vec{v}\,T)$, and yet

$$\hat{\phi}_{f^{(T)}} |\Omega\rangle \approx \hat{\phi}_f |\Omega\rangle$$

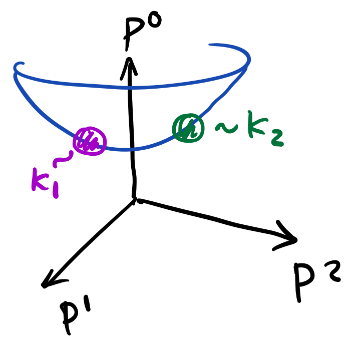
same 1-particle state!

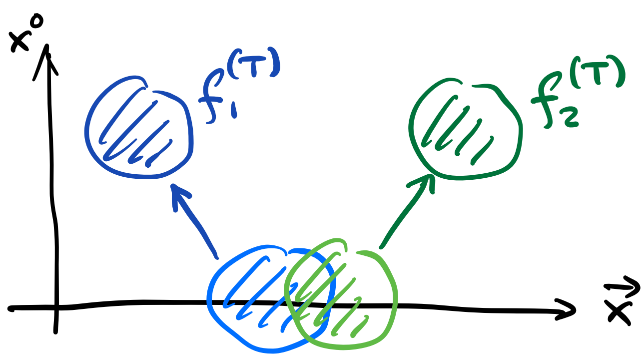
[This may seem counter intuitive, but there is no contradiction: "there are a lot more operators than states"!]

Now consider a pair of functions $f_1(x)$, $f_2(x)$ of the above prescribed form, whose Fourier transforms $\tilde{f}_1(k)$, $\tilde{f}_2(k)$ are supported near $k \approx k_1, k_2$ respectively.

$$k_1^2 = k_2^2 = -m^2 .$$

Also assume $\vec{k}_1 \neq \vec{k}_2$





The regions of concentration of $f_1^{(T)}$ and $f_2^{(T)}$ move apart as $T \to \pm\infty$, and become far space-like separated.

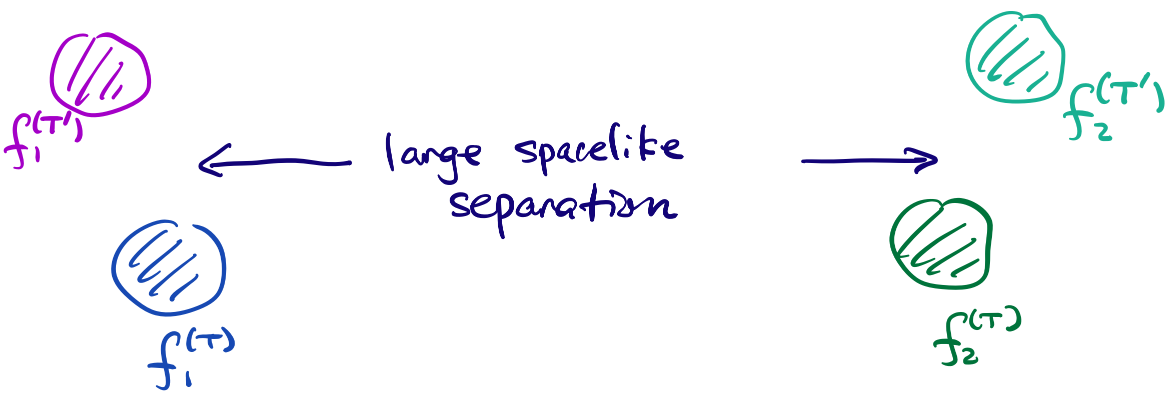
Consider the state

$$|\Psi^{(T)}\rangle = \hat{\Phi}_{f_1^{(T)}} \hat{\Phi}_{f_2^{(T)}} |\Omega\rangle .$$

The overlap between $|\Psi^{(T)}\rangle$ and $|\Psi^{(T')}\rangle$

$$\langle \Psi^{(T)} | \Psi^{(T')} \rangle$$
$$= \langle \Omega | \hat{\Phi}_{f_2^{(T)*}} \hat{\Phi}_{f_1^{(T)*}} \hat{\Phi}_{f_1^{(T')}} \hat{\Phi}_{f_2^{(T')}} |\Omega\rangle \qquad \star$$

Fix $T'-T \equiv \Delta T$, send $T \to \pm\infty$



"cluster property"

$$\star \longrightarrow \underbrace{\langle \Omega | \hat{\Phi}_{f_1^{(T)*}} \hat{\Phi}_{f_1^{(T')}} |\Omega\rangle}_{\substack{\text{recall } \hat{\Phi}_{f_1^{(T')}}|\Omega\rangle \approx \hat{\Phi}_{f_1^{(T)}}|\Omega\rangle \\ \text{independent of } \Delta T\ !}} \langle \Omega | \hat{\Phi}_{f_2^{(T)*}} \hat{\Phi}_{f_2^{(T')}} |\Omega\rangle$$

It follows that

$$\| \, |\Psi^{(T')}\rangle - |\Psi^{(T)}\rangle \, \| \to 0 \quad \text{as } T \to \pm\infty$$

Assuming the vanishing is sufficiently fast,

[in fact, with non-overlapping and smooth $\tilde{f}_i(k)$, $\| \, |\Psi^{(T')}\rangle - |\Psi^{(T)}\rangle \, \|$ vanishes faster than $|T|^{-N}$ for any $N > 0$ as $T \to \pm\infty$ (Hepp 1965)]

$$\lim_{T \to \pm\infty} \hat{\phi}_{f_1^{(T)}} \hat{\phi}_{f_2^{(T)}} |\Omega\rangle := |\phi_{f_1}, \phi_{f_2}\rangle^{out/in}$$

is a well-defined state in $\mathcal{H}$

Compare to:

$$\hat{\phi}_f |\Omega\rangle = \int d^{D-1}\vec{k} \; \tilde{f}(\omega_{\vec{k}}, \vec{k}) \left[\frac{Z}{(2\pi)^{D-1}} \frac{1}{2\omega_{\vec{k}}} \right]^{\frac{1}{2}} |\vec{k}\rangle$$

(assuming Supp $\tilde{f}$ is concentrated near the 1-particle mass shell, with no overlap with multi-particle momentum region)

We can write

$$|\phi_{f_1}, \phi_{f_2}\rangle^{out/in}$$

$$= \int \prod_{j=1}^{2} d^{D-1}\vec{k}_j \; \tilde{f}_j(\omega_{\vec{k}_j}, \vec{k}_j) \left[\frac{Z}{(2\pi)^{D-1}} \frac{1}{2\omega_{\vec{k}_j}} \right]^{\frac{1}{2}} |\vec{k}_1, \vec{k}_2\rangle^{out/in}$$

Using cluster property, one can verify that $|\vec{k}_1, \vec{k}_2\rangle^{out/in}$ obey the orthogonality

$$
{}^{out}\langle \vec{k}_1, \vec{k}_2 | \vec{k}_1', \vec{k}_2' \rangle^{out}
$$

$$
= \delta^{D-1}(\vec{k}_1 - \vec{k}_1')\, \delta^{D-1}(\vec{k}_2 - \vec{k}_2') + (k_1' \leftrightarrow k_2')
$$

and the same with

$$
{}^{in}\langle \vec{k}_1, \vec{k}_2 | \vec{k}_1', \vec{k}_2' \rangle^{in} .
$$

- Generalization to n-particle in/out-states $|\vec{k}_1, \cdots, \vec{k}_n\rangle^{in/out}$ is obvious.

Given $f_1(x), \ldots, f_n(x)$ whose Fourier transform $\tilde{f}_1(k), \ldots, \tilde{f}_n(k)$ are supported near the mass-shell $k^2+m^2=0$, $k^0>0$ and do not overlap

$$\lim_{T\to\pm\infty} \hat{\phi}_{f_1^{(T)}} \cdots \hat{\phi}_{f_n^{(T)}} |\Omega\rangle$$

$$:= |\phi_{f_1}, \ldots, \phi_{f_n}\rangle^{out/in}$$

$$= \int \prod_{j=1}^{n} \left(d^{D-1}\vec{k}_j \, \tilde{f}_j(\omega_{\vec{k}_j}, \vec{k}_j) \left[\frac{Z}{(2\pi)^{D-1}} \frac{1}{2\omega_{\vec{k}_j}} \right]^{\frac{1}{2}} \right.$$

$$\cdot |\vec{k}_1, \ldots, \vec{k}_n\rangle^{out/in} .$$

Normalization is such that

$${}^{out}\langle \vec{k}_1, \ldots, \vec{k}_n | \vec{k}'_1, \ldots \vec{k}'_m \rangle^{out} = \delta_{nm} \sum_{\sigma\in S_n} \prod_{j=1}^{n} \delta^{D-1}(\vec{k}_j - \vec{k}'_{\sigma(j)})$$

permutation

same for ${}^{in}\langle \ldots | \ldots \rangle^{in}$

The S-matrix elements are defined as

$$S(\vec{k}_1, \cdots, \vec{k}_n \,|\, \vec{k}_1', \cdots, \vec{k}_m')$$

$$= {}^{out}\langle \vec{k}_1, \cdots, \vec{k}_n \,|\, \vec{k}_1', \cdots, \vec{k}_m' \rangle^{in}$$

[Generalization to more than 1 species of particles is obvious.]

By construction, the S-matrix elements are determined by a suitable limit of integrated Wightman functions. Let us inspect this relation in more detail:

$${}^{out}\langle \phi_{f_1}, \cdots \,|\, \phi_{g_1}, \cdots \rangle^{in}$$

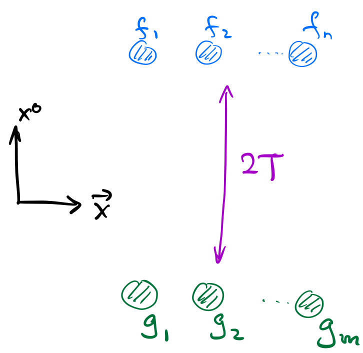
$$= \int d^{D-1}\vec{k}_1 \, \tilde{f}_1^*(\omega_{\vec{k}_1}, \vec{k}_1) \left[\frac{Z}{(2\pi)^{D-1}} \frac{1}{2\omega_{\vec{k}_1}} \right]^{\frac{1}{2}} \cdots\cdots$$

$$\times\, d^{D-1}\vec{k}_1' \, g_1(\omega_{\vec{k}_1'}, \vec{k}_1') \left[\frac{Z}{(2\pi)^{D-1}} \frac{1}{2\omega_{\vec{k}_1'}} \right]^{\frac{1}{2}} \cdots\cdots$$

$$\times\, S(\vec{k}_1, \cdots \,|\, \vec{k}_1', \cdots)$$

The LHS, by definition, is

$$= \lim_{T\to\infty} \langle \Omega | \underbrace{\hat{\phi}_{f_n^{(T)*}} \cdots \hat{\phi}_{f_1^{(T)*}}}_{\text{asymp. spacelike separated}} \underbrace{\hat{\phi}_{g_1^{(-T)}} \cdots \hat{\phi}_{g_m^{(-T)}}}_{\text{asymp. spacelike separated}} | \Omega \rangle$$



$$= \lim_{T\to\infty} \langle \Omega | T\, \hat{\phi}_{f_1^{(T)*}} \cdots \hat{\phi}_{g_1^{(-T)}} \cdots | \Omega \rangle$$

$$= \lim_{T\to\infty} \int d^D x_1\, f_1^{(T)*}(x_1) \cdots d^D y_1\, g_1^{(-T)}(y_1) \cdots$$

$$\times \underbrace{\langle \Omega | T\, \hat{\phi}(x_1) \cdots \hat{\phi}(y_1) \cdots | \Omega \rangle}_{\substack{\text{III} \\ G(x_1, \ldots, x_n, y_1, \ldots, y_m) \\ \text{III}}}$$

$$\int \frac{d^D k_1}{(2\pi)^D} e^{i k_1 \cdot x_1} \cdots \frac{d^D k_1'}{(2\pi)^D} e^{i k_1' \cdot y_1} \cdots$$

$$\times \tilde{G}(k_1, \ldots, k_1', \ldots)$$

$$= \lim_{T\to\infty} \int \frac{d^D k_1}{(2\pi)^D} \tilde{f}_1^{*}(k_1)\, e^{-i(k_1^0 - \omega_{\vec{k}_1})T} \cdots$$

$$\times \frac{d^D k_1'}{(2\pi)^D} \tilde{g}_1(-k_1')\, e^{i(-k_1'^0 - \omega_{\vec{k}_1'})(-T)} \cdots$$

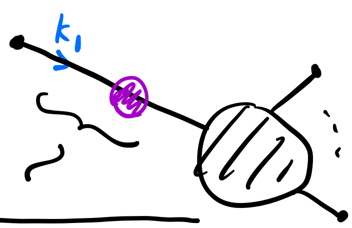
$$\times \tilde{G}(k_1, \ldots, k_n, k_1', \ldots, k_m') .$$

The $T\to\infty$ limit of the $k_i^0, k_j'^0$-integral vanishes unless $\tilde{G}$ is singular at

$$k_i^0 = \omega_{\vec{k}_i}$$

and $k_j'^0 = -\omega_{\vec{k}_j'}$.

Compare perturbative computation of $\tilde{G}$:



$$\underbrace{\frac{-i}{k_1^2 + m_0^2 - i\epsilon - \Sigma(k_1)}}_{\text{pole at } k_1^2 = -m_*^2 \text{ for physical mass } m_*}$$

m_0: bare mass

$$\lim_{T\to\infty}\int \frac{d^D k}{(2\pi)^D}\,\frac{\tilde{g}(-k)\,e^{i(-k^0-\omega_{\vec{k}})(-T)}}{k^2+m^2-i\epsilon}$$

is dominated by the contribution near $k^0 \simeq -\omega_{\vec{k}}$,

$$= \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}\,\tilde{g}(\omega_{\vec{k}},-\vec{k})\,\underbrace{\lim_{T\to\infty}\int\frac{dk^0}{2\pi}\,\frac{e^{i(-k^0-\omega_{\vec{k}})(-T)}}{k^2+m^2-i\epsilon}}_{\substack{\text{// residue at } k^0=-\omega_{\vec{k}}+i\epsilon \\ i\cdot\frac{1}{2\omega_{\vec{k}}}}}$$

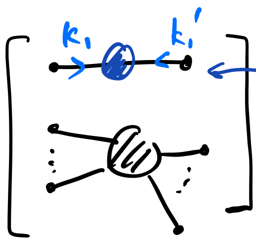
Similarly,

$$\lim_{T\to\infty}\int\frac{d^D k}{(2\pi)^D}\,\frac{\tilde{f}^*(k)\,e^{-i(k^0-\omega_{\vec{k}})T}}{k^2+m^2-i\epsilon}$$

$$= \int\frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}\,\tilde{f}^*(\omega_{\vec{k}},\vec{k})\,\frac{i}{2\omega_{\vec{k}}}.$$

- The time-ordered Green function $G(x_1,\dots,x_n)$ takes the form $\sum\prod G^{conn}$, where $G^{conn}(x_1,\dots,x_n)$ decays at large spacelike separation (cluster property). After Fourier transform, each $\tilde{G}^{conn}(\{\text{subset of } k_i\text{'s}\})$ contains a momentum-δ-function.

- In taking the $T \to \infty$ limit above, the momentum δ-functions in $\tilde{G}$ come along with the ride, with one exceptional case:



$$(2\pi)^D \delta^D(k_1 + k_1') \cdot \left[\frac{-iZ}{k_1^2 + m^2 - i\epsilon} + (\text{non-singular at } k_1^2 = -m^2) \right]$$

$$\lim_{T\to\infty} \int \frac{d^D k_1}{(2\pi)^D} \tilde{f}_1^*(k_1)\, e^{-i(k_1^0 - \omega_{\vec{k}_1})T}$$

$$\cdot \frac{d^D k_1'}{(2\pi)^D} \tilde{g}_1(-k_1')\, e^{i(-k_1'^0 - \omega_{\vec{k}_1'})(-T)} \cdot (2\pi)^D \delta^D(k_1 + k_1') \cdot \frac{-iZ}{k_1^2 + m^2 - i\epsilon}$$

$$= \lim_{T\to\infty} \int \frac{d^D k}{(2\pi)^D} \tilde{f}_1^*(k)\, \tilde{g}_1(k)\, e^{-2i(k^0 - \omega_{\vec{k}})T} \cdot \frac{-iZ}{k^2 + m^2 - i\epsilon}$$

$$= \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \tilde{f}_1^*(\omega_{\vec{k}}, \vec{k})\, \tilde{g}(\omega_{\vec{k}}, \vec{k}) \cdot \frac{Z}{2\omega_{\vec{k}}}$$

$$= \langle \phi_f | \phi_g \rangle \quad \text{as expected.}$$

Lehmann – Symanzik – Zimmermann (LSZ) reduction formula:

- Compare with the definition of the S-matrix elements, we conclude that

$$S(\vec{k}_1,\dots,\vec{k}_n \,|\, -\vec{k}_1',\dots,-\vec{k}_m')$$

$$= \prod_{i=1}^{n} \left[Z\cdot(2\pi)^{D-1} 2\omega_{\vec{k}_i} \right]^{-\frac{1}{2}} \prod_{j=1}^{m} \left[Z\cdot(2\pi)^{D-1} 2\omega_{\vec{k}_j'} \right]^{-\frac{1}{2}}$$

$$\times \lim_{\substack{k_i^0 \to \omega_{k_i} \\ k_j'^0 \to -\omega_{k_j'}}} \prod_{i=1}^{n} i(k_i^2+m^2) \prod_{j=1}^{m} i(k_j'^2+m^2)$$

$$\times \tilde{G}(k_1,\dots,k_n,k_1',\dots,k_m')$$

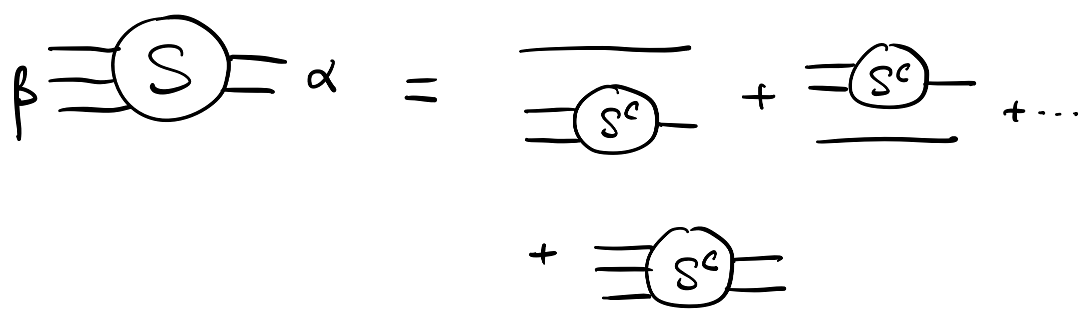
[with the exception of disconnected terms in $\tilde{G}$ that contains $\delta^D(k_i+k_j')$, which gives rise to disconnected factor $\delta^{D-1}(\vec{k}_i-\vec{k}_j')$ in S as discussed above.]

It follows from the cluster property of Green functions and LSZ that the S-matrix elements also obey a cluster property:

$$S(\beta|\alpha) \equiv {}^{out}\langle \beta | \alpha \rangle^{in}$$

$$= \sum_{\substack{\alpha = \amalg \alpha_I \\ \beta = \amalg \beta_I}} \prod_I S^{conn}(\beta_I | \alpha_I)$$

Pictorially :



Each connected S-matrix element comes with a momentum δ-function :

$$S^{conn}(\vec{k}_1, \dots, \vec{k}_n | \vec{k}_1', \dots, \vec{k}_m')$$

$$= i\,(2\pi)^D \delta^D(k_1 + \dots + k_n - k_1' - \dots - k_m')$$

$$\cdot\, \mathcal{M}\underbrace{(\vec{k}_1, \dots, \vec{k}_n | \vec{k}_1', \dots, \vec{k}_m')}_{\substack{\text{subject to } k_i^2 = k_j'^2 = -m^2 \\ k_1 + \dots + k_n = k_1' + \dots + k_m'}}$$

Convention

More precise version of LSZ:

$$S^{\text{conn.}}(\vec{k}_1, \cdots, \vec{k}_n \,|\, -\vec{k}_1', \cdots, -\vec{k}_m')$$

$$= \prod_{i=1}^{n} \left[Z \cdot (2\pi)^{D-1} 2\omega_{\vec{k}_i} \right]^{-\frac{1}{2}} \prod_{j=1}^{m} \left[Z \cdot (2\pi)^{D-1} 2\omega_{\vec{k}_j'} \right]^{-\frac{1}{2}}$$

$$\times \lim_{\substack{k_i^0 \to \omega_{k_i} \\ k_j'^0 \to -\omega_{k_j'}}} \prod_{i=1}^{n} i(k_i^2 + m^2) \prod_{j=1}^{m} i(k_j'^2 + m^2)$$

$$\times \tilde{G}^{\text{conn.}}(k_1, \cdots, k_n, k_1', \cdots, k_m')$$

for $n, m \geq 2$.

Due to our normalization convention of 1-particle states, it is convenient to further define

$$\mathcal{M}(\vec{k}_1, \dots, \vec{k}_n | \vec{k}_1, \dots, \vec{k}_m')$$

$$\equiv \prod_{i=1}^{n} \frac{1}{(2\pi)^{\frac{D-1}{2}} \sqrt{2\omega_{\vec{k}_i}}} \prod_{j=1}^{m} \frac{1}{(2\pi)^{\frac{D-1}{2}} \sqrt{2\omega_{\vec{k}_j'}}}$$

$$\times \mathcal{A}(k_1, \dots, k_n | k_1', \dots, k_m')$$

$\mathcal{A}$ is invariant under Lorentz transf $k \mapsto \Lambda k$.

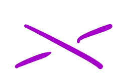
Example: $2 \to 2$ scattering

$\alpha = \{\vec{k}_1', \vec{k}_2'\}$, $\beta = \{\vec{k}_1, \vec{k}_2\}$

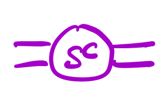
$$S(\vec{k}_1, \vec{k}_2 | \vec{k}_1', \vec{k}_2') \equiv {}^{out}\langle \vec{k}_1, \vec{k}_2 | \vec{k}_1', \vec{k}_2' \rangle^{in}$$

$$= \delta^{D-1}(\vec{k}_1 - \vec{k}_1')\, \delta^{D-1}(\vec{k}_2 - \vec{k}_2')$$

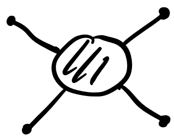
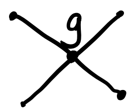
$$+ \delta^{D-1}(\vec{k}_1 - \vec{k}_2')\, \delta^{D-1}(\vec{k}_2 - \vec{k}_1')$$



$$+ i (2\pi)^D \delta^D(k_1 + k_2 - k_1' - k_2')\, \mathcal{M}(\vec{k}_1, \vec{k}_2 ; \vec{k}_1', \vec{k}_2')$$



ϕ^4 theory

 $=$  $+ \; \mathcal{O}(g^2)$

$$\tilde{G}^{\text{conn}}(k_1, k_2, k_3, k_4) = (2\pi)^D \delta^D(k_1 + \cdots + k_4) \times \left[(-ig) \cdot \prod_{j=1}^{4} \frac{-i}{k_j^2 + m^2 - i\epsilon} + \mathcal{O}(g^2) \right]$$

$$Z = 1 + \mathcal{O}(g^2)$$

LSZ:

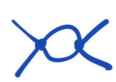
$$S^{\text{conn}}(\vec{k}_1, \vec{k}_2 \,|\, -\vec{k}_1', -\vec{k}_2')$$

$$= \prod_{i=1}^{2} \left[Z (2\pi)^{D-1} 2\omega_{\vec{k}_i} \right]^{-\frac{1}{2}} \cdot \prod_{j=1}^{2} \left[Z (2\pi)^{D-1} 2\omega_{\vec{k}_j'} \right]^{-\frac{1}{2}}$$

$$\times (2\pi)^D \delta^{D-1}(\vec{k}_1 + \vec{k}_2 + \vec{k}_1' + \vec{k}_2') \, \delta(\omega_{\vec{k}_1} + \omega_{\vec{k}_2} - \omega_{\vec{k}_1'} - \omega_{\vec{k}_2'})$$

$$\times \left[-ig + \mathcal{O}(g^2) \right]$$

$$\Rightarrow i\mathcal{A}(k_1, k_2 \,|\, k_1', k_2') = -ig + \mathcal{O}(g^2)$$

includes  $+ \cdots$

as well as correction

to the factor Z^{-2}.

- S-matrix as a unitary operator

$$S_{\beta\alpha} \equiv {}^{out}\langle\beta|\alpha\rangle^{in}$$

Define $\hat{S}$ via

$$\hat{S}\,|\alpha\rangle^{out} = |\alpha\rangle^{in}$$

$$\Rightarrow \quad {}^{out}\langle\beta|\,\hat{S}\,|\alpha\rangle^{out} = S_{\beta\alpha}$$

$\hat{S}$ preserves inner product:

$$\begin{array}{c} {}^{in}\langle\alpha|\alpha'\rangle^{in} = {}^{out}\langle\alpha|\alpha'\rangle^{out} \\ \shortparallel \\ {}^{out}\langle\alpha|\,\hat{S}^\dagger\hat{S}\,|\alpha'\rangle^{out} \end{array}$$

$$\Rightarrow \quad \hat{S}^\dagger\hat{S} = \mathbb{1} \quad (= \hat{S}\hat{S}^\dagger)$$

i.e. $\hat{S}$ is a unitary operator on $\mathcal{H}$

Pick an orthonormal out-basis $|\alpha\rangle^{out}$ such that

$${}^{out}\langle\beta|\alpha\rangle^{out} = \delta_{\beta\alpha}, \quad \int d\alpha\, |\alpha\rangle^{out}\,{}^{out}\langle\alpha| = \mathbb{1}$$

$$\Rightarrow \int d\gamma \, (S_{\gamma\beta})^* S_{\gamma\alpha} = \delta_{\beta\alpha}.$$

- Interpretation of S-matrix elements via transition amplitudes / probabilities

$|\alpha\rangle^{in}$ is the state that "looks like far-separated (incoming) particles with momenta $\alpha = \{\vec{k}_1, \dots, \vec{k}_n\}$ in the far past".

The probability of finding the state consisting of outgoing particles with momenta distributed in the range

$$\beta = \{\vec{k}_1', \dots, \vec{k}_m'\} \in \mathcal{D}$$

is
$$P_{\alpha\to\mathcal{D}} = \int_{\mathcal{D}} d\beta \, |S_{\beta\alpha}|^2$$

We have defined the scattering amplitude $\mathcal{M}$ via
$$S_{\beta\alpha} \equiv \delta_{\beta\alpha} + i(2\pi)^D \delta^D(P_\beta - P_\alpha)\, \mathcal{M}_{\beta\alpha}.$$

Assuming $\alpha \notin \mathcal{D}$,

$$P_{\alpha \to \mathcal{D}} = \int_{\mathcal{D}} d\beta \, \left| (2\pi)^D \delta^D(P_\beta - P_\alpha) \mathcal{M}_{\beta\alpha} \right|^2$$

$$= \underbrace{(2\pi)^D \delta^D(0)}_{\substack{\shortparallel \\ V \cdot T}} \int_{\mathcal{D}} d\beta \, (2\pi)^D \delta^D(P_\beta - P_\alpha) \left| \mathcal{M}_{\beta\alpha} \right|^2$$

$V \cdot T$, volume of spacetime

This spacetime volume divergence is no contradiction since we assumed that $|\alpha\rangle^{in}$ has definite energy-momentum P_α, and is hence not a normalizable state.

e.g. 2-particle in-state

$$|\alpha\rangle^{in} = |\vec{k}_1, \vec{k}_2\rangle^{in} \qquad (\vec{k}_1 \neq \vec{k}_2)$$

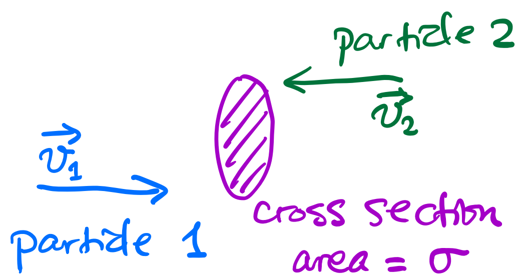
$$^{in}\langle \vec{k}_1, \vec{k}_2 | \vec{k}_1, \vec{k}_2 \rangle^{in} = \delta^{D-1}(0) \cdot \delta^{D-1}(0)$$

$$= \left(\frac{V}{(2\pi)^{D-1}} \right)^2 .$$

Transition rate per unit volume

$$\Gamma_{\alpha\to\mathscr{D}} \equiv \frac{P_{\alpha\to\mathscr{D}}}{VT}$$

$$= \int_{\mathscr{D}} d\beta \, (2\pi)^D \delta^D(P_\beta - P_\alpha) \, |\mathcal{M}_{\beta\alpha}|^2$$



relative velocity
(in lab frame)

$v \equiv |\vec{v}_1 - \vec{v}_2|$

The scattering cross section $\sigma_{\mathscr{D}}$ is defined such that the probability ratio of scattering into $\beta \in \mathscr{D}$ is

$$\frac{\sigma_{\mathscr{D}} v T}{V}$$

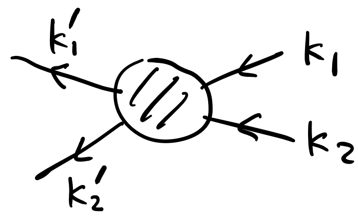
effective volume that passes through σ

total volume of space

$$\Gamma_{\alpha\to\mathscr{D}} = \frac{{}^{in}\langle\alpha|\alpha\rangle^{in} \cdot \frac{\sigma_{\mathscr{D}} v \cancel{T}}{V}}{V\cancel{T}} = \frac{\sigma_{\mathscr{D}} v}{(2\pi)^{2D-2}} .$$

$$\Rightarrow \quad \sigma_{\mathcal{D}} = \frac{(2\pi)^{2D-2}}{v} \int_{\mathcal{D}} d\beta \; (2\pi)^D \delta^D(P_\beta - P_\alpha) \times |\mathcal{M}_{\beta\alpha}|^2 .$$

For instance, $2 \to 2$ scattering in $D=4$



total cross section of 2-particle final state

identical particles

$$\sigma_{2\to 2} = \frac{(2\pi)^6}{v} \cdot \frac{1}{2} \int d^3\vec{k}_1{}' \, d^3\vec{k}_2{}' \times (2\pi)^4 \delta^4(k_1' + k_2' - k_1 - k_2) \cdot |\mathcal{M}(\vec{k}_1{}', \vec{k}_2{}'; \vec{k}_1, \vec{k}_2)|^2$$

$$= \frac{(2\pi)^6}{\frac{|\vec{k}_1|}{\omega_{k_1}} + \frac{|\vec{k}_2|}{\omega_{k_2}}} \frac{1}{2} \int d^3\vec{k}_1{}' \, (2\pi)^4 \delta(\omega_{k_1'} + \omega_{k_2'} - E) \times \frac{1}{(2\pi)^{12} \cdot 2^4 \omega_{k_1'} \omega_{k_2'} \omega_{k_1} \omega_{k_2}} |\mathcal{A}(k_1', k_2'; k_1, k_2)|^2$$

here $\vec{k}_2{}' = \vec{k}_1 + \vec{k}_2 - \vec{k}_1{}'$

In center-of-mass frame, $\vec{k}_1 + \vec{k}_2 = 0 = \vec{k}_1' + \vec{k}_2'$.

$$v = \frac{|\vec{k}_1|}{\omega_{k_1}} + \frac{|\vec{k}_2|}{\omega_{k_2}} = \frac{|\vec{k}_1| \cdot E}{\omega_{k_1}\omega_{k_2}} \quad (E = \text{total energy})$$

$$\int |\vec{k}_1'|^2 d|\vec{k}_1'| \, \delta(\omega_{k_1'} + \omega_{k_2'} - E)$$

$$= \frac{|\vec{k}_1'|^2}{\frac{\partial \omega_{k_1'}}{\partial |\vec{k}_1'|} + \frac{\partial \omega_{k_2'}}{\partial |\vec{k}_1'|}} = \frac{|\vec{k}_1'|^2}{\frac{|\vec{k}_1'|}{\omega_{k_1'}} + \frac{|\vec{k}_2'|}{\omega_{k_2'}}} = \frac{|\vec{k}_1'| \omega_{k_1'} \omega_{k_2'}}{E}$$

$$\sigma_{2\to 2} = \frac{(2\pi)^6}{\frac{|\vec{k}_1| \cdot E}{\cancel{\omega_{k_1}}\cancel{\omega_{k_2}}}} \cdot \frac{1}{2} \cdot (2\pi)^4 \int d\hat{k}_1' \cdot \frac{|\vec{k}_1'| \cancel{\omega_{k_1'}} \cancel{\omega_{k_2'}}}{E}$$

$$\times \frac{1}{(2\pi)^{12} \cdot 2^4 \cancel{\omega_{k_1'}} \cancel{\omega_{k_2'}} \cancel{\omega_{k_1}} \cancel{\omega_{k_2}}} |\mathcal{A}|^2$$

$$= \frac{1}{2^7 \pi^2 E^2} \int d\hat{k}_1' \, |\mathcal{A}|^2$$

For perturbative ϕ^4 theory,

$$\mathcal{A} = -g + \mathcal{O}(g^2)$$

$$\sigma_{2\to 2} = \frac{g^2}{32\pi E^2} + \mathcal{O}(g^3)$$

in COM frame.

• Partial wave basis for 2-particle states (D=4)

$$|E, \vec{P}, \ell, m\rangle$$

$\vec{J}^2 = \ell(\ell+1)$, $J_3 = m$.

$\ell = 0, 1, 2, 3, 4, \ldots$

only even ℓ for identical spinless particles.

$m = -\ell, -\ell+1, \cdots, \ell$

Relation to plane-wave basis (in COM frame i.e. $\vec{P}=0$)

$$\langle \vec{k}_1, \vec{k}_2 | E, \vec{P}=0, \ell, m\rangle$$

$$= \delta^3(\vec{k}_1+\vec{k}_2)\, \delta(\omega_{k_1}+\omega_{k_2}-E)\, \mathcal{N}(E)\, Y_{\ell,m}(\hat{k}_1)$$

normalization factor

spherical harmonics

Normalization convention:

$$\langle E, \vec{P}, \ell, m | E', \vec{P}', \ell', m'\rangle$$

$$= \delta(E-E')\, \delta^3(\vec{P}-\vec{P}')\, \delta_{\ell\ell'}\, \delta_{mm'}$$

Compare: $\langle E, \vec{P}, \ell, m | E', \vec{P}'=0, \ell', m'\rangle$

$$= \frac{1}{2}\int d^3\vec{k}_1 d^3\vec{k}_2 \langle E, \vec{P}, \ell, m | \vec{k}_1, \vec{k}_2\rangle \langle \vec{k}_1, \vec{k}_2 | E', \vec{0}, \ell', m'\rangle$$

identical particles

$$= \frac{1}{2}\,\delta(E-E')\,\delta^3(\vec{P})\cdot(\mathcal{N}(E))^2$$

$$\times \int \underbrace{d^3\vec{k}_1\ \delta(\omega_{k_1}+\omega_{k_2}-E)}_{d\hat{k}_1 \cdot \frac{|\vec{k}_1|\omega_{k_1}\omega_{k_2}}{E}} \cdot Y^*_{\ell m}(\hat{k}_1)\, Y_{\ell' m'}(\hat{k}_1)$$

$$\int d\hat{k}_1\ Y^*_{\ell m}(\hat{k}_1)\, Y_{\ell' m'}(\hat{k}_1) = \delta_{\ell\ell'}\,\delta_{mm'} .$$

identical particle

$$\Rightarrow \quad \mathcal{N}(E) = \sqrt{\frac{2E}{|\vec{k}_1|\omega_{k_1}\omega_{k_2}}}$$

[formula without the factor 2 applicable to non-identical 2-particle state of possibly different masses as well]

$2\to 2$ S-matrix elements in partial wave basis:

$$S(E',\vec{P}',\ell',m' \,|\, E,\vec{P},\ell,m)$$
$$\equiv {}^{out}\langle E',\vec{P}',\ell',m' \,|\, E,\vec{P},\ell,m\rangle^{in}$$
$$= \delta(E'-E)\,\delta^3(\vec{P}'-\vec{P})\ S_{\ell'm'|\ell m}(E,\vec{P})$$

In COM frame $\vec{P}=0$,

$$S_{\ell'm'|\ell m}(E,\vec{P}=0) = \underbrace{\delta_{\ell\ell'}\,\delta_{m'm}}_{\vec{J}\text{-conservation}}\ S_\ell(E)$$

$$|E,\vec{P}=0,\ell,m\rangle^{in} = S_\ell(E)\,|E,\vec{P}=0,\ell,m\rangle^{out}$$

$$+\ (\text{non-elastic scattering product})$$

unitarity $\Rightarrow$ $|S_\ell(E)| \le 1$

$|S_\ell(E)| = 1$ for elastic scattering

e.g. for $2m \le E < 3m$

in the case of 1 species of particle.

Using

$$|\vec{k}_1,\vec{k}_2\rangle = \mathcal{N}(E)\sum_{\ell,m} Y^*_{\ell m}(\hat{k}_1)\,|E,\vec{P}=0,\ell,m\rangle,$$

$$S(\vec{k}_1{}',\vec{k}_2{}'|\vec{k}_1,\vec{k}_2) = \delta^4(k_1'+k_2'-k_1-k_2)\cdot(\mathcal{N}(E))^2 \times \underbrace{\sum_{\substack{\ell=0\\ \ell\ \text{even}}}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\hat{k}_1{}')\,Y^*_{\ell,m}(\hat{k}_1)}_{=\frac{1}{4\pi}(2\ell+1)P_\ell(\hat{k}_1{}'\cdot\hat{k}_1)} S_\ell(E)$$

$$\| $$

$${}^{out}\langle\vec{k}_1{}',\vec{k}_2{}'|\vec{k}_1,\vec{k}_2\rangle^{out} + i(2\pi)^4\delta^4(k_1'+k_2'-k_1-k_2) \times \mathcal{M}(\vec{k}_1{}',\vec{k}_2{}'|\vec{k}_1,\vec{k}_2)$$

$P_\ell(x)$ = Legendre polynomial.

$$\Rightarrow\ i(2\pi)^4\mathcal{M}(\vec{k}_1{}',\vec{k}_2{}'|\vec{k}_1,\vec{k}_2)$$

$$= (\mathcal{N}(E))^2\cdot\frac{1}{4\pi}\sum_{\substack{\ell=0\\ \ell\ \text{even}}}^{\infty}(2\ell+1)\,P_\ell(\hat{k}_1{}'\cdot\hat{k}_1)\,(S_\ell(E)-1).$$

Total $2 \to 2$ cross section

$$\sigma_{2\to 2} = \frac{(2\pi)^6}{\frac{|\vec{k}_1|\cdot E}{\omega_{k_1}\omega_{k_2}}} \cdot \frac{1}{2} \cdot (2\pi)^4 \int d\hat{k}_1' \cdot \frac{|\vec{k}_1'|\,\omega_{k_1'}\omega_{k_2'}}{E}$$

$$\times \left| (2\pi)^{-4} \frac{2\,E}{|\vec{k}_1|\,\omega_{k_1}\omega_{k_2}} \frac{1}{4\pi} \sum_\ell (2\ell+1) P_\ell(\hat{k}_1'\cdot\hat{k}_1)(S_\ell(E)-1) \right|^2$$

$$= \frac{2}{4\,|\vec{k}_1|^2} \sum_{\ell,\ell'=0,\,\text{even}}^{\infty} (2\ell+1)(2\ell'+1) \underbrace{\int d\hat{k}_1'\, P_\ell(\hat{k}_1'\cdot\hat{k}_1) P_{\ell'}(\hat{k}_1'\cdot\hat{k}_1)}_{=\,\frac{4\pi}{2\ell+1}\delta_{\ell\ell'}}$$

$$\times \left(S_\ell(E)-1\right)\left(S_{\ell'}(E)-1\right)^*$$

$$= \frac{2\pi}{|\vec{k}_1|^2} \sum_{\ell=0,\,\text{even}}^{\infty} (2\ell+1)\,\left|S_\ell(E)-1\right|^2 .$$

Particle with internal (spin) degrees of freedom.

- restrict to $D=4$, $m>0$ for now

As before, pick a reference momentum $\vec{k}_R$

$$k_R = (\omega_{\vec{k}_R}, \vec{k}_R), \qquad k_R^2 + m^2 = 0$$

e.g. $\vec{k}_R = \vec{0}$, $k_R = (m, \vec{0})$

and choose "standard boost" $L^\mu{}_\nu(\vec{k})$ for any on-shell k^μ ($k^2+m^2=0$, $k^0>0$)

such that $k^\mu = L^\mu{}_\nu(\vec{k})\, k_R^\nu$.

1-particle states with momentum k_R are labeled

$$|\vec{k}_R, \sigma\rangle. \qquad \sigma = 1, \dots, N.$$

Define

$$|\vec{k}, \sigma\rangle = N(\vec{k})\, U(L(\vec{k}))\, |\vec{k}_R, \sigma\rangle$$

Normalization $\langle \vec{k}, \sigma | \vec{k}', \sigma'\rangle = \delta^3(\vec{k}-\vec{k}')\,\delta_{\sigma\sigma'}$

$$\Rightarrow \quad N(\vec{k}) = \sqrt{\frac{k_R^0}{k^0}} \quad \text{as before.}$$

For any Lorentz transf. Λ,

$$U(\Lambda)|\vec{k},\sigma\rangle = N(\vec{k})\, U(L(\Lambda \vec{k}))$$
$$\cdot\, U(\underbrace{L^{-1}(\Lambda\vec{k})\,\Lambda\, L(\vec{k})}_{W})\,|\vec{k}_R,\sigma\rangle$$

W fixed k_R,

$$\Rightarrow \quad U(W)\,|\vec{k}_R,\sigma\rangle = \sum_{\sigma'} D_{\sigma\sigma'}(W)\,|\vec{k}_R,\sigma'\rangle$$

Thus a general Lorentz transf. acts by

$$U(\Lambda)|\vec{k},\sigma\rangle = \frac{N(\vec{k})}{N(\vec{\Lambda k})} \sum_{\sigma'} D_{\sigma\sigma'}(W)\,|\vec{\Lambda k},\sigma'\rangle .$$

For $\vec{k}_R = 0$, the set of W's that leave $k_R = (m, \vec{0})$ invariant is the group of spatial rotations, $G = SO(3)$ "little group"

Expect:

$$U(W_1 W_2) = U(W_1)\, U(W_2)$$

$$\Rightarrow \quad D_{\sigma\sigma'}(W_1 W_2) = \sum_{\sigma''} D_{\sigma\sigma''}(W_2)\, D_{\sigma''\sigma'}(W_1)$$

i.e. as matrices, $D(w) = \left(D_{\sigma\sigma'}(w)\right)_{1\le\sigma,\sigma'\le N}$

obey

$$D(w_1 w_2) = D(w_2) \cdot D(w_1).$$

[opposite order of multiplication compared to that of $U(w)$'s]

Actually, we have made too strong an assumption

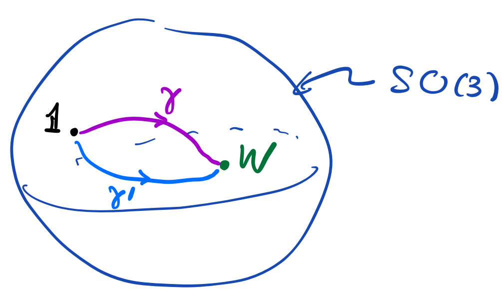
- it is possible that the rotation symmetry is represented projectively:

$$U(w_1)\, U(w_2) = e^{i\phi(w_1, w_2)}\, U(w_1 \cdot w_2)$$

a constant phase

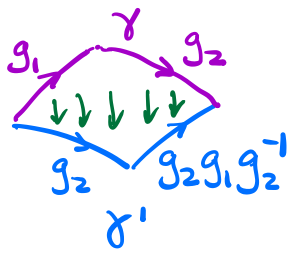
We can specify $U(w)$ using the generators $\vec{J}$, by composing infinitesimal rotations along a path $\gamma \subset SO(3)$, and denote the resulting unitary operator U_γ.

Suppose γ, γ' both start at $\mathbb{1}$ and end at w:



The commutation relation among J_i's is such that $U_\gamma = U_{\gamma'}$, provided that γ can be deformed into γ' continuously

e.g. for rotation group elements g_1, g_2 close to 1:

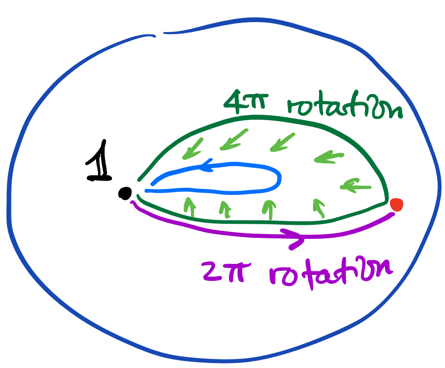


$$U_\gamma = U(g_2)\,U(g_1)$$
$$\parallel$$
$$U_{\gamma'} = U(g_2 g_1 g_2^{-1})\,U(g_2)$$

The only way for $U_{\gamma'}$ to be different from U_γ would be if γ' cannot be continuously deformed into γ (fixing end pt W.)

In particular, for $W = \mathbb{1}$,

$U_\gamma = 1$ if γ is a contractible path in $SO(3)$



$\widetilde{SO(3)}$ (covering group)

Fact: 2π rotation is not contractible, while 4π rotation is.

Consequence: $U(W)$ or $D(W)$ is defined not on $SO(3)$ but its 2-fold covering space $\widetilde{SO(3)}$ (the points of $\widetilde{SO(3)}$ are equivalent classes of paths $\gamma: \mathbb{1} \to W$ that are related by continuous deformations.)

$$D: \widetilde{SO(3)} \longrightarrow \text{unitary } N \times N \text{ matrices}$$
$$(D_{\sigma\sigma'}(W))$$

is a unitary representation of $\widetilde{SO(3)}$.

Theorem: can find a basis $|\sigma\rangle$ such that

$$D(W) = \begin{pmatrix} D^{(\ell_1)}(W) & & 0 \\ & D^{(\ell_2)}(W) & \\ 0 & & \ddots \end{pmatrix}$$

where $D^{(j)}(W)$ is a $(2j+1) \times (2j+1)$ matrix

$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ "spin - j representation".

We have already seen $j=0$ - trivial rep.

Focus now on $j=\frac{1}{2}$ case.

Infinitesimally, $W_{ij} = \delta_{ij} + \omega_{ij}$

Finite rotation:

$$U(W) = e^{\frac{i}{2}\omega_{ij}\hat{J}^{ij}} \qquad \omega_{ij} = -\omega_{ji}$$

$$i,j = 1,2,3.$$

$$\Downarrow$$

$$D(W) = e^{-\frac{i}{2}\omega_{ij}S^{ij}}$$

$$S^{ij} = \epsilon^{ijk}\frac{\sigma_k}{2}$$

Pauli matrices

obey same commutation rel^{n} as $\hat{J}^{ij}$'s.

e.g. $\omega_{12} = \theta$, $\hat{J}^{12} = \hat{J}_3$

$e^{i\theta\hat{J}_3}$ is the rotation around x_3-axis by angle θ.

$$e^{-i\theta S^{12}} = e^{-i\theta\frac{\sigma_3}{2}} = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$$

$(= -\mathbb{1}$ for $\theta = 2\pi)$.

A spin $j=\frac{1}{2}$ massive 1-particle state is labeled $|\vec{k},\sigma\rangle$, $\sigma=1,2$.

with

$$U(\Lambda)\,|\vec{k},\sigma\rangle = \sqrt{\frac{(\Lambda k)^0}{k^0}}\sum_{\sigma'} D_{\sigma\sigma'}(W)\,|\vec{\Lambda k},\sigma'\rangle$$

$$W = (L(\vec{\Lambda k}))^{-1}\Lambda L(\vec{k})$$

$D_{\sigma\sigma'}(W)$ constructed as above.

Q: What kind of local field operator can create a spin-$\frac{1}{2}$ 1-particle state?

Consider $\hat{\Phi}_\alpha(x)$ that obeys Lorentz transf.

$$U(\Lambda)\,\hat{\Phi}_\alpha(x)(U(\Lambda))^{-1} = (R(\Lambda))_\alpha{}^\beta\,\hat{\Phi}_\beta(\Lambda x)$$

Note: unlike $D(W)$, the matrix $R(\Lambda)$ need not be unitary.

$$U(\Lambda_1\Lambda_2) = U(\Lambda_1)\,U(\Lambda_2)$$

$$\Rightarrow\quad R(\Lambda_1\Lambda_2) = R(\Lambda_2)\cdot R(\Lambda_1)$$

just like $D(W)$, we will allow $R(\Lambda)$ to be defined on the covering group $\widetilde{SO(1,3)}$.

Consider the overlap of $\hat{\Phi}_\alpha(x)|\Omega\rangle$ with the 1-particle state,

$$\langle\vec{k},\sigma|\hat{\Phi}_\alpha(x)|\Omega\rangle = e^{-ik\cdot x}\langle\vec{k},\sigma|\hat{\Phi}_\alpha(0)|\Omega\rangle$$

Define

"(spinorial) polarization"

$$\langle\vec{k},\sigma|\hat{\Phi}_\alpha(0)|\Omega\rangle = C\cdot\mathcal{U}^\sigma_\alpha(\vec{k}).$$

a normalization constant, to be chosen later.

$$\|$$

$$\langle\vec{k},\sigma|(U(\Lambda))^\dagger\, U(\Lambda)\,\hat{\Phi}_\alpha(0)\,(U(\Lambda))^{-1}|\Omega\rangle$$

$$= \sqrt{\frac{(\Lambda k)^0}{k^0}}\sum_{\sigma'} D^*_{\sigma\sigma'}(W)\, R_\alpha{}^\beta(\Lambda)\,\langle\vec{\Lambda k},\sigma'|\hat{\Phi}_\beta(0)|\Omega\rangle$$

$$\Rightarrow\quad \mathcal{U}^\sigma_\alpha(\vec{k}) = \sqrt{\frac{(\Lambda k)^0}{k^0}}\, D^*_{\sigma\sigma'}(W)\, R_\alpha{}^\beta(\Lambda)\,\mathcal{U}^{\sigma'}_\beta(\vec{\Lambda k}).$$

In particular, we have

(1) For $\Lambda = (L(\vec{k}))^{-1}$, $\Lambda k = k_R$, $W = \mathbb{1}$.

$$\mathcal{U}^\sigma_\alpha(\vec{k}) = \sqrt{\frac{k_R^0}{k^0}}\, R_\alpha{}^\beta((L(\vec{k}))^{-1})\,\mathcal{U}^\sigma{}_\beta(\vec{k}_R).$$

(2) For $\vec{k} = \vec{k}_R$, $\Lambda = W$,

$$\mathcal{U}^{\sigma}_{\alpha}(\vec{k}_R) = D^{*}_{\sigma\sigma'}(W)\, R_{\alpha}{}^{\beta}(W)\, \mathcal{U}^{\sigma'}{}_{\beta}(\vec{k}_R).$$

Assuming $\mathcal{U}^{\sigma}_{\alpha}(\vec{k}_R)$ is non-degenerate, Schur's lemma implies that $R_{\alpha}{}^{\beta}(W)$ must also be (direct sum of) spinor representations of $\underset{\substack{\uplus \\ W}}{\widetilde{SO(3)}} \subset \underset{\substack{\uplus \\ \Lambda}}{\widetilde{SO(1,3)}}$.

To construct $R_{\alpha}{}^{\beta}(\Lambda)$ explicity, begin with infinitesimal Lorentz transf.

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu},$$

$\rightsquigarrow$ finite form

$$U(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}}.$$

$$\Big\Downarrow$$

$$R(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}.$$

$S^{\mu\nu}$ is a matrix $\left((S^{\mu\nu})_{\alpha}{}^{\beta}\right)$ that obey the same commutation relation as $\hat{J}^{\mu\nu}$'s, i.e.

$$[S^{\mu\nu}, S^{\rho\sigma}] = -i\left(\eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} - (\rho\leftrightarrow\sigma)\right).$$

If we have a set of matrices γ^μ that obey $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, then

$$S^{\mu\nu} \equiv -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

satisfy the desired property.

[A possible choice of γ^μ's is

$$\gamma^0 = -i\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^j = -i\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix},$$

$j = 1, 2, 3$.

but $\tilde{\gamma}^\mu = A\gamma^\mu A^{-1}$ would also do]

Having specified $R(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$, we have also determined the spinorial polarization $\mathcal{U}^\sigma_\alpha(\vec{k})$ in terms of $\mathcal{U}^\sigma_\beta(\vec{k}_R)$.

$$\mathcal{U}^\sigma_\alpha(\vec{k}) = \sqrt{\frac{k_R^0}{k^0}}\, R_\alpha{}^\beta\left((L(\vec{k}))^{-1}\right)\mathcal{U}^\sigma{}_\beta(\vec{k}_R).$$

Useful property:

$$[S^{\mu\nu}, \gamma^\rho] = -i\gamma^\mu \eta^{\nu\rho} + i\gamma^\nu \eta^{\mu\rho}.$$

$$\Rightarrow \quad R(\Lambda)\gamma^\mu (R(\Lambda))^{-1} = \Lambda^\mu{}_\nu \gamma^\nu .$$

In particular,

$$R(L(\vec{k}))\, k_\mu \gamma^\mu\, R((L(\vec{k}))^{-1})$$

$$= k_\mu (L(\vec{k}))^\mu{}_\nu \gamma^\nu$$

$$= ((L(\vec{k}))^{-1}\cdot k)_\nu \gamma^\nu = (k_R)_\mu \gamma^\mu .$$

Notation: $\quad \not{k} \equiv k_\mu \gamma^\mu .$

$$\not{k}\, \mathcal{U}^\sigma(\vec{k}) = \sqrt{\frac{k_R^0}{k^0}}\, \not{k}\, R((L(\vec{k}))^{-1})\, \mathcal{U}^\sigma(\vec{k}_R)$$

$$= \sqrt{\frac{k_R^0}{k^0}}\, R((L(\vec{k}))^{-1})\, \not{k}_R\, \mathcal{U}^\sigma(\vec{k}_R)$$

The matrix $\not{k}_R$ $(= -m\gamma^0$ for $k_R = (m, \vec{0}))$

obeys $\not{k}_R^2 = \frac{1}{2} k_{R\mu} k_{R\nu} \{\gamma^\mu, \gamma^\nu\}$

$$= k_R^2 = -m^2 .$$

$\not{k}_R$ has eigenvalues $\pm im$.

possible basis for $u^\sigma_\alpha(\vec{k}_R)$:

$$\not{k}_R \, u^\sigma_R = im \, u^\sigma_R ,$$

$$\not{k}_R \, v^\sigma_R = -im \, v^\sigma_R , \qquad \sigma = 1,2.$$

It follows that the corresponding spinor polarizations $u^\sigma_\alpha(\vec{k})$ and $v^\sigma_\alpha(\vec{k})$ obey

$$(\not{k} - im)_\alpha{}^\beta \, u^\sigma_\beta(\vec{k}) = 0$$

$$(\not{k} + im)_\alpha{}^\beta \, v^\sigma_\beta(\vec{k}) = 0$$

"Dirac equation".

· Some useful properties of $u^\sigma_\alpha(\vec{k})$:

- setting $\vec{k}_R = \vec{0}$ from now.

$$u^\sigma_\alpha(\vec{k}) = \sqrt{\frac{m}{k^0}} \, \underbrace{R_\alpha{}^\beta((L(\vec{k}))^{-1})}_{\text{NOT unitary}} \, u^\sigma_\beta(\vec{0})$$

$R_\alpha{}^\beta(\Lambda) = \left(e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\right)_\alpha{}^\beta$ is generally NOT unitary,

as $S^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ is not Hermitian :

with the choice $\gamma^0 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $\vec{\gamma} = -i\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$,

$$(\gamma^\mu)^\dagger = \gamma_\mu = \gamma^0 \gamma^\mu \gamma^0 .$$

<u>Define</u> $\beta \equiv i\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. $\quad (\beta^2 = \mathbb{1})$

$$(\gamma^\mu)^\dagger = -\beta\gamma^\mu\beta .$$

$$\Rightarrow \quad (S^{\mu\nu})^\dagger = \beta S^{\mu\nu}\beta$$

$$(R(\Lambda))^\dagger = \beta R(\Lambda^{-1})\beta .$$

$$(u^\sigma(\vec{k}))^\dagger = \sqrt{\frac{m}{k^0}}\,(u^\sigma(\vec{0}))^\dagger (R((L(\vec{k}))^{-1}))^\dagger$$

$$= \sqrt{\frac{m}{k^0}}\,(u^\sigma(\vec{0}))^\dagger \beta R(L(\vec{k}))\beta .$$

In particular,

$$(u^\sigma(\vec{k}))^\dagger \beta \cdot u^{\sigma'}(\vec{k})$$

$$= \frac{m}{k^0}(u^\sigma(\vec{0}))^\dagger \beta R(L(\vec{k}))\beta \cdot \beta R((L(\vec{k}))^{-1}) u^{\sigma'}(\vec{0})$$

$$= \frac{m}{k^0}(u^\sigma(\vec{0}))^\dagger \beta u^\sigma(\vec{0}) .$$

$$\not{k}_R = -m\gamma^0 = im\beta,$$

$$\beta u^\sigma_R = u^\sigma_R, \quad \beta v^\sigma_R = -v^\sigma_R.$$

Normalize:

$$(u^\sigma_R)^\dagger u^{\sigma'}_R = \delta_{\sigma\sigma'}, \qquad \sigma, \sigma' = 1, 2$$

$$(v^\sigma_R)^\dagger v^{\sigma'}_R = \delta_{\sigma\sigma'}$$

$$(u^\sigma_R)^\dagger v^{\sigma'}_R = 0.$$

$$\Rightarrow \quad (u^\sigma(\vec{k}))^\dagger \beta\, u^{\sigma'}(\vec{k}) = \frac{m}{k^0}\delta_{\sigma\sigma'}$$

$$(v^\sigma(\vec{k}))^\dagger \beta\, v^{\sigma'}(\vec{k}) = -\frac{m}{k^0}\delta_{\sigma\sigma'},$$

$$(u^\sigma(\vec{k}))^\dagger \beta\, v^{\sigma'}(\vec{k}) = 0.$$

Completeness:

$$\sum_\sigma u^\sigma_R (u^\sigma_R)^\dagger = \frac{1+\beta}{2}$$

$$\sum_\sigma v^\sigma_R (v^\sigma_R)^\dagger = \frac{1-\beta}{2}$$

It follows that

$$\sum_\sigma u^\sigma(\vec{k})\,(u^\sigma(\vec{k}))^\dagger$$

$$= \frac{m}{k^0} \sum_\sigma R(L^{-1})\, \mathcal{U}^\sigma_R\, (\mathcal{U}^\sigma_R)^\dagger \underbrace{(R(L^{-1}))^\dagger}_{=\beta R(L) \beta}$$

$$= \frac{m}{k^0}\, R(L^{-1})\, \frac{1+\beta}{2}\, \beta\, R(L)\, \beta$$

$$= \frac{1}{2k^0}\left(m - i \underbrace{R(L^{-1})\, \not{k}_R\, R(L)}_{=\not{k}}\right) \cdot \beta$$

$$= \frac{m - i\not{k}}{2k^0}\, \beta .$$

Similary, $\sum_\sigma v^\sigma(\vec{k})\, (v^\sigma(\vec{k}))^\dagger = \frac{-m - i\not{k}}{2k^0}\, \beta .$

- Now we inspect the relation

$$\mathcal{U}^\sigma_\alpha(\vec{0}) = D^*_{\sigma\sigma'}(W)\, R_\alpha{}^\beta(W)\, \mathcal{U}^{\sigma'}_\beta(\vec{0})$$

for $W = e^{i\theta J_{12}}$.

$$R(W) = e^{-i\theta\left(-\frac{i}{4}[\gamma^1, \gamma^2]\right)} = \begin{pmatrix} e^{-i\frac{\theta}{2}} & & & 0 \\ & e^{i\frac{\theta}{2}} & & \\ & & e^{-i\frac{\theta}{2}} & \\ 0 & & & e^{i\frac{\theta}{2}} \end{pmatrix}$$

$$D(W) = e^{-i\theta \frac{\sigma_3}{2}} = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$$

We have chosen the basis $|\vec{k}_R = \vec{0}, \sigma\rangle$ to be such that

$$\hat{J}_3 |\vec{0}, 1\rangle = -\tfrac{1}{2} |\vec{0}, 1\rangle$$
$$\hat{J}_3 |\vec{0}, 2\rangle = +\tfrac{1}{2} |\vec{0}, 2\rangle$$

$\Rightarrow$ up to overall phase,

$$u_R^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_R^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v_R^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \qquad v_R^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

Define "chirality matrix"

$$\gamma_5 \equiv -i \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$\gamma_5^2 = 1, \qquad \{\gamma_5, \gamma^\mu\} = 0 . \quad \mu = 0,1,2,3.$$

In our basis. $$\gamma_5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

$$v_R^\sigma = \gamma_5 u_R^\sigma .$$

Q: What is the field $\hat{\Phi}_\alpha(x)$ that has the property

$$\langle \vec{k}, \sigma | \hat{\Phi}_\alpha(x) | \Omega \rangle = e^{-ik\cdot x} \cdot C \cdot v^{\sigma}_{\alpha}(\vec{k}) \quad ?$$

Try constructing a free field

$$\hat{\Phi}_\alpha(x) = \int \frac{d^3\vec{k}}{(2\pi)^{\frac{3}{2}}}\Big[e^{-ik\cdot x} v^\sigma_\alpha(\vec{k})\, a^\dagger_\sigma(\vec{k})$$

$$+ \lambda\, e^{ik\cdot x} u^\sigma_\alpha(\vec{k})\, \tilde{a}_\sigma(\vec{k})\Big]_{k^0=\omega_{\vec{k}}}$$

$$(\not{k} - im)\, u^\sigma(\vec{k}) = (\not{k} + im)\, v^\sigma(\vec{k}) = 0$$

$$\Rightarrow \quad (\not{\partial} + m)\,\hat{\Phi}(x) = 0 \qquad \not{\partial} \equiv \gamma^\mu \partial_\mu$$

$$(\hat{\Phi}^\dagger(x)\beta)^\alpha = \int \frac{d^3\vec{k}}{(2\pi)^{\frac{3}{2}}}\Big[e^{ik\cdot x}(v^\sigma(\vec{k})^\dagger\beta)^\alpha a_\sigma(\vec{k})$$

$$+ \lambda^* e^{-ik\cdot x}(u^\sigma(\vec{k})^\dagger \beta)^\alpha\, \tilde{a}^\dagger_\sigma(\vec{k})\Big]_{k^0=\omega_{\vec{k}}}$$

$$\left[\hat{\Phi}_\alpha(x), (\hat{\Phi}^\dagger(y)\beta)^{\alpha'}\right]$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\Big[- e^{-ik\cdot(x-y)}\left(v^\sigma(\vec{k})\,(v^\sigma(\vec{k}))^\dagger \beta\right)_\alpha{}^{\alpha'}$$

$$+ |\lambda|^2 e^{ik\cdot(x-y)}\left(u^\sigma(\vec{k})\,(u^\sigma(\vec{k}))^\dagger \beta\right)_\alpha{}^{\alpha'}\Big]$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\left[- e^{-ik\cdot(x-y)}\frac{-m-i\not{k}}{2\omega_k} + |\lambda|^2 e^{ik\cdot(x-y)}\frac{m-i\not{k}}{2\omega_k}\right]_\alpha{}^{\alpha'}$$

For spacelike separated x, y,

e.g. $(x-y)^0 = 0$, $(\vec{x}-\vec{y}) \neq 0$,

$$[\hat{\Phi}_\alpha(x), (\hat{\Phi}^\dagger(y)\beta)^{\alpha'}]$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \Big[\underbrace{- \frac{-m+i\not{k}}{2\omega_k} + |\lambda|^2 \frac{m-i\not{k}}{2\omega_k}}_{\text{\& } 1+|\lambda|^2 \text{, do not cancel!}} \Big]_\alpha^{\ \alpha'}$$

$\neq 0$.

What if the $-$ sign is not there?

Try:

$$\hat{\psi}_\alpha(x) = \int \frac{d^3\vec{k}}{(2\pi)^{\frac{3}{2}}} \Big[e^{-ik\cdot x} v_\alpha^\sigma(\vec{k})\, b_\sigma^\dagger(\vec{k})$$

$$+ \underbrace{\cancel{\lambda}}_{\text{can set to } 1.} e^{ik\cdot x} u_\alpha^\sigma(\vec{k})\, \tilde{b}_\sigma(\vec{k}) \Big]_{k^0=\omega_{\vec{k}}}$$

$$\{ b_\sigma(\vec{k}), b^\dagger_{\sigma'}(\vec{k}') \} = \delta_{\sigma\sigma'} \delta^{D-1}(\vec{k}-\vec{k}')$$

$$= \{ \tilde{b}_\sigma(\vec{k}), \tilde{b}^\dagger_{\sigma'}(\vec{k}') \}$$

anti-commutator

Now

$$\left\{ \hat{\psi}_\alpha(x), \left(\hat{\psi}^\dagger(y)\beta\right)^{\alpha'} \right\}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \left[+ e^{-ik\cdot(x-y)} \frac{-m-i\not{k}}{2\omega_k} + e^{ik\cdot(x-y)} \frac{+m-i\not{k}}{2\omega_k} \right]_\alpha^{\alpha'}$$

$$= (-\not{\partial}_x + m)_\alpha{}^{\alpha'} \Big[\underbrace{\Delta_+(x-y; m^2) - \Delta_+(y-x; m^2)}_{\overset{\shortparallel}{0} \text{ for } (x-y)^2 > 0.} \Big]$$

✓

Conclusion: the causal field operator that creates a spin-$\frac{1}{2}$ particle should be fermionic !

Q: how do we formulate a quantum theory of fermionic fields ?

Option 1 : canonical quantization

- what is the classical theory to start with?

Option 2 : path integral

- what is the field space to integrate over?

Dirac vs Weyl vs Majorana

(D=4)

A Dirac spinor field operator $\hat{\psi}_\alpha(x)$ transforms under Lorentz symmetry according to

$$U(\Lambda)\,\hat{\psi}_\alpha(x)\,(U(\Lambda))^{-1} = (R(\Lambda))_\alpha{}^\beta\,\hat{\psi}_\beta(\Lambda\cdot x)$$

For $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, $\omega^\mu{}_\nu$ infinitesimal,

$$R(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$

$$S^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu].$$ - 4×4 matrices

- A priori, there are no linear relations between $(\hat{\psi}^\dagger)^\alpha(x)$ and $\hat{\psi}_\alpha(x)$.

  "4 complex components"

- Since γ_5 commutes with $S^{\mu\nu}$, $(\gamma_5\hat{\psi})_\alpha(x)$ transforms under Poincaré symmetry in the same way as $\hat{\psi}_\alpha(x)$ does.

We can define a new operator $\hat{\chi}^{\pm}_{\alpha}(x)$ as the chiral/anti-chiral projection

$$\hat{\chi}^{\pm}_{\alpha}(x) \equiv \left(\frac{1 \pm \gamma_5}{2} \hat{\psi}\right)_{\alpha}(x).$$

In our standard basis, $\gamma_5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$.

$$\frac{1+\gamma_5}{2} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Either $\hat{\chi}^{+}$ or $\hat{\chi}^{-}$ retains only 2 components of $\hat{\psi}$.

- A Weyl spinor field $\hat{\chi}_{\alpha}(x)$ is one that obeys $\gamma_5 \hat{\chi} = \hat{\chi}$. (chiral)

  or $\gamma_5 \hat{\chi} = -\hat{\chi}$. (anti-chiral)

- We have seen that a Dirac spinor field $\hat{\psi}$ contain a chiral Weyl spinor field $\hat{\chi}^{+}$ and an anti-chiral Weyl spinor field $\hat{\chi}^{-}$.

• In our convention,

$$\gamma^0 = -i\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \vec{\gamma} = -i\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix},$$

we have

complex conjugate without transpose

Note $\sigma_2^* = -\sigma_2$

$$(\gamma^\mu)^* = B\gamma^\mu B^{-1}$$

where B may be chosen as $B = \gamma_2$.

and so

$$(S^{\mu\nu})^* = -B S^{\mu\nu} B^{-1}.$$

$$(R(\Lambda))^* = B\, R(\Lambda)\, B^{-1}.$$

Hermitian conjugate component-wise

It follows that $(B^{-1}\hat{\psi}^*)_\alpha(x)$

has the same Poincaré transf. property as $\hat{\psi}_\alpha(x)$

$$U(\Lambda)\, B^{-1}\hat{\psi}^*(x)\, (U(\Lambda))^{-1} = B^{-1}\left(R(\Lambda)\hat{\psi}(\Lambda x)\right)^*$$

$$= B^{-1}\left(B\, R(\Lambda)\, B^{-1}\right)\hat{\psi}^*(\Lambda x)$$

$$= R(\Lambda)\cdot B^{-1}\hat{\psi}^*(\Lambda x).$$

Also, applying such operation twice gives back $\hat{\psi}$:

$$B^{-1}(B^{-1}\hat{\psi}^*)^* = \hat{\psi}. \qquad (\text{since } B^{-1}(B^{-1})^* = I)$$

- A Majorana spinor field $\hat{\zeta}_\alpha(x)$ is one that obeys

$$\hat{\zeta}^*_\alpha(x) = (B\zeta)_\alpha(x).$$

- Can associate a Weyl spinor $\hat{\chi}^+$ with a Majorana spinor via

$$\hat{\zeta} = \hat{\chi}^+ + B^*(\hat{\chi}^+)^*.$$

and vice versa:

$$\hat{\chi}^+ = \frac{1+\gamma_5}{2}\hat{\zeta}$$

Check: $$\frac{1+\gamma_5}{2} B^*(\hat{\chi}^+)^*$$

$$= B^* \frac{1-\gamma_5}{2}(\hat{\chi}^+)^*$$

$$= B^*\left(\frac{1-\gamma_5}{2}\hat{\chi}^+\right)^* = 0, \quad \checkmark$$

- Also note: given a chiral Weyl spinor $\hat{\chi}^+$, $B^*(\hat{\chi}^+)^*$ is anti-chiral:

$$\gamma_5 B^*(\hat{\chi}^+)^* = -B^*\gamma_5(\hat{\chi}^+)^* = -B^*(\hat{\chi}^+)^*.$$

In summary, Weyl and Majorana spinor fields can be obtained by projections on a Dirac spinor field. These notions of spinor fields are useful in constructing operators and Lagrangians. There is NO a priori logical connection between the existence of these different type of field operators, and the particle content of the QFT.

- Classical mechanics of Grassmann variables and its quantization

- Begin with "Grassmann coordinates" η_α, $\alpha=1,\cdots,N$

$$\eta_\alpha \eta_\beta = -\eta_\beta \eta_\alpha, \qquad \eta_\alpha^2 = 0.$$

In contrast to ordinary coordinates that parameterize the configuration space of a system, the "config. space" of Grassmann mechanics can only be characterized through relations among functions on the space $f(\eta_\alpha)$.

Lagrangian $\quad L = \sum_{\alpha,\beta=1}^{N} M_{\alpha\beta}\, \eta_\alpha \dot{\eta}_\beta$

$$S = \int dt\, M_{\alpha\beta}\, \eta_\alpha \dot{\eta}_\beta$$

symmetric in $(\alpha\beta)$

$$\overset{\int \text{by part}}{=\!=} -\int dt\, M_{\alpha\beta}\, \dot{\eta}_\alpha \eta_\beta = \int dt\, M_{\alpha\beta}\, \eta_\beta \dot{\eta}_\alpha$$

canonical momenta to η_α:

$$\pi_\alpha = L \frac{\overleftarrow{\partial}}{\partial \dot{\eta}_\alpha} = M_{\alpha\beta}\, \eta_\beta$$

Attempt of defining anti-commuting Poisson bracket:

$$\{\eta_\alpha, \eta_\beta\}_P = 0 = \{\pi_\alpha, \pi_\beta\}_P$$

$$\{\eta_\alpha, \pi_\beta\}_P = \delta_{\alpha\beta} = \{\pi_\alpha, \eta_\beta\}_P$$

But, $\pi_\alpha = M_{\alpha\beta}\eta_\beta$, contradiction ??

Dirac, "lectures on QM" (1964):

view $\chi_\alpha(\eta, \pi) \equiv \pi_\alpha - M_{\alpha\beta}\eta_\beta = 0$

as "second-class primary constraint"

meaning non-degenerate Poisson bracket among themselves, in contrast to "1^{st}-class" which has to do with "gauge redundancy".

in contrast to "secondary" constraints which follow from EOM.

$$K_{\alpha\beta} \equiv \{\chi_\alpha, \chi_\beta\}_P$$

$$= \frac{\partial \chi_\alpha}{\partial \pi_\gamma}\frac{\partial \chi_\beta}{\partial \eta_\gamma} + \frac{\partial \chi_\alpha}{\partial \eta_\gamma}\frac{\partial \chi_\beta}{\partial \pi_\gamma}$$

$$= -2M_{\alpha\beta}$$ (assume non-degenerate)

Define (classical) Dirac bracket

$$\{f, g\}_D := \{f, g\}_P - \{f, \chi_\alpha\}_P (K^{-1})^{\alpha\beta} \{\chi_\beta, g\}_P$$

to replace Poisson bracket.

Consistency with constraint $\chi_\alpha(\eta, \pi)$ "$= 0$"

is ensured via

$$\{f, \chi_\alpha\}_D = 0 \quad \text{for any } f(\eta, \pi).$$

Now,

$$\{\eta_\alpha, \eta_\beta\}_D = \cancelto{0}{\{\eta_\alpha, \eta_\beta\}_P} - \{\eta_\alpha, \chi_\gamma\}_P (K^{-1})^{\gamma\delta} \{\chi_\delta, \eta_\beta\}_P$$

$$= -(K^{-1})_{\alpha\beta} = \frac{1}{2}(M^{-1})_{\alpha\beta}$$

similarly, $\{\eta_\alpha, \pi_\beta\}_D = \frac{1}{2}\delta_{\alpha\beta}$.

Note the following $N=2$ example:

$$S = \int dt \; i(\eta_1 \dot{\eta}_1 + \eta_2 \dot{\eta}_2) = \int dt \; i \chi \dot{\eta}$$

where $\eta \equiv \eta_1 + i\eta_2$, $\chi \equiv \eta_1 - i\eta_2$.

Dirac bracket

$$\{\eta, \chi\}_D = -i \,, \quad \{\eta, \eta\}_D = 0 = \{\chi, \chi\}_D .$$

consistent with $\{\eta_a, \eta_b\}_D = -\frac{i}{2}\delta_{ab}$, $a,b = 1,2$.

Quantization: promote η_α to an (ordinary) linear operator $\hat{\eta}_\alpha$ that acts on a Hilbert space $\mathcal{H}$, such that

$$\{\hat{\eta}_\alpha, \hat{\eta}_\beta\} = i\hbar \cdot \frac{1}{2}(M^{-1})_{\alpha\beta}$$

Example: $N=2$

$$L = i\sum_{\alpha=1}^{2} \eta_\alpha \dot{\eta}_\alpha - i\omega \eta_1 \eta_2$$

$$\updownarrow$$

$$H = \sum_{\alpha=1}^{2} \pi_\alpha \dot{\eta}_\alpha - L = i\omega \eta_1 \eta_2$$

quantization: $\eta_\alpha \rightsquigarrow \hat{\eta}_\alpha$

$$\hat{\eta}_1^2 = \hat{\eta}_2^2 = \frac{1}{4}, \quad \{\hat{\eta}_1, \hat{\eta}_2\} = 0.$$

minimal choice: $\dim \mathcal{H} = 2$.

$$\hat{\eta}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\eta}_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

$$\hat{H} = i\omega\, \hat{\eta}_1 \hat{\eta}_2 = \begin{pmatrix} -\frac{\omega}{4} & 0 \\ 0 & \frac{\omega}{4} \end{pmatrix}.$$

– a 2-state system.

- **Grassmann path integral**

Consider $L = i\chi\dot{\eta} - H(\chi, \eta)$.

$\rightsquigarrow$ $\hat{\eta}$, $\hat{\chi}$ obey

$$\{\hat{\eta}, \hat{\chi}\} = 1, \qquad \hat{\eta}^2 = \hat{\chi}^2 = 0.$$

fermion annihilation ($\hat{\eta}$)

fermion creation ($\hat{\chi}$)

$$\begin{tikzcd} 0 & {|0\rangle} \arrow[l, "\hat{\eta}"'] \arrow[r, bend left, "\hat{\chi}"] & {|1\rangle} \arrow[l, bend left, "\hat{\eta}"] \arrow[r, "\hat{\chi}"] & 0 \end{tikzcd}$$

Define "position eigenstate"

$$|\eta\rangle = |1\rangle\, \eta + |0\rangle$$

a formal Grassmann parameter

$$\hat{\eta}\,|\eta\rangle = |0\rangle\, \eta = |\eta\rangle\, \eta .$$

Define the bra-state

$$\langle\langle \chi | = -\langle 1| + \chi \langle 0|$$

which obeys the property

$$\langle\langle \chi | \eta \rangle = \chi - \eta .$$

and

$$\langle\langle \chi | \hat{\eta} = \chi \langle 1| = -\chi \langle\langle \chi | .$$

Define Berezin integral $\int d\eta \, f(\eta)$

by $$\int d\eta \cdot \eta = 1$$

$$\int d\eta . 1 = 0 .$$

also, by convention $\eta . d\eta = - d\eta . \eta$.

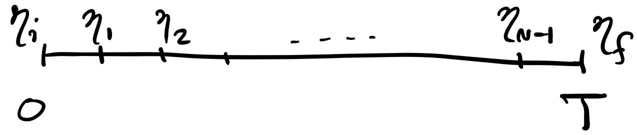
It follows that

$$\int |\eta\rangle \, d\eta \, \langle\langle \eta | = \int (|1\rangle \eta + |0\rangle) \, d\eta \, (-\langle 1| + \eta \langle 0|)$$

$$= |1\rangle\langle 1| + |0\rangle\langle 0| = \mathbb{1} .$$

– a completeness relation analogous to $\int dx \, |x\rangle\langle x| = 1$.

Now consider time evolution



$$\langle\langle \eta_f | U(T) | \eta_i \rangle$$

$$= \int \langle\langle \eta_f | e^{-i\hat{H}\frac{T}{N}} | \eta_{N-1} \rangle d\eta_{N-1} \langle\langle \eta_{N-1} | \cdots$$

$$\cdots | \eta_1 \rangle d\eta_1 \langle\langle \eta_1 | e^{-i\hat{H}\frac{T}{N}} | \eta_i \rangle .$$

using $\langle\langle \eta_{n+1} | \eta_n \rangle = \eta_{n+1} - \eta_n$

$$= -\int d\chi_n \, e^{-\chi_n(\eta_{n+1}-\eta_n)}$$

and $\langle\langle \eta_{n+1} | \hat{\chi} | \eta_n \rangle = \left(-\langle 1| + \eta_{n+1}\langle 0|\right) |1\rangle = -1$

$$= -\int d\chi_n \cdot \chi_n \, e^{-\chi_n(\eta_{n+1}-\eta_n)}$$

The factor $\langle\langle \eta_{n+1} | e^{-i\hat{H}\frac{T}{N}} | \eta_n \rangle$

is approximated by

$$-\int d\chi_n \, e^{-iH(\chi_n,\eta_n)\frac{T}{N} - \chi_n(\eta_{n+1}-\eta_n)}$$

We end up with

$$\langle\langle \overset{\eta_N}{\eta_f} \,|\, U(T) \,|\, \overset{\eta_0}{\eta_i} \rangle$$

$$= -\int d\chi_0 \prod_{n=1}^{N-1} d\eta_n d\chi_n \cdot e^{\sum_{n=0}^{N-1}\left[-iH(\chi_n,\eta_n)\frac{T}{N} - \chi_n(\eta_{n+1}-\eta_n)\right]}$$

$$\rightsquigarrow \int D\eta D\chi \cdot \underbrace{e^{\int_0^T dt\left[-\chi\dot{\eta} - iH(\chi,\eta)\right]}}_{= e^{iS}}$$

$$S = \int dt\left[\underbrace{i\chi\dot{\eta} - H(\chi,\eta)}_{= L}\right]$$

- Grassmann (Berezin) path integral.

fermionic spinor field variable

$$\psi_\alpha(x), \qquad (\psi^\dagger)^\alpha(x)$$

$$\bar{\psi}^\alpha \equiv (\psi^\dagger)^{\alpha'} \beta_{\alpha'}{}^{\alpha}$$

recall $\beta = i\gamma^0$,

$$(\gamma^\mu)^\dagger = -\beta\gamma^\mu\beta.$$

$\big\downarrow$ quantization

$$\hat{\psi}_\alpha(x), \quad (\hat{\psi}^\dagger)^\alpha(x)$$

Expect equal time commutator

$$\{\hat{\psi}_\alpha(x), \hat{\bar{\psi}}^{\alpha'}(y)\}\Big|_{y^0=x^0}$$

$$= (-\not{\partial}_x + m)_\alpha{}^{\alpha'} \left[\Delta_+(x-y; m^2) - \Delta_+(y-x; m^2)\right]\Big|_{y^0=x^0}$$

$$= -(\gamma^0)_\alpha{}^{\alpha'} \underbrace{\frac{\partial}{\partial x^0}\left[\Delta_+(x-y; m^2) - \Delta_+(y-x; m^2)\right]\Big|_{y^0=x^0}}_{= -i\delta^3(\vec{x}-\vec{y})}$$

$$= i(\gamma^0)_\alpha{}^{\alpha'}.$$

and

$$\{\hat{\psi}(x), \hat{\psi}(y)\}\Big|_{y^0=x^0} = 0$$

$$\{\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(y)\}\Big|_{y^0=x^0} = 0.$$

want $\Pi_\psi(x) = -\bar{\psi}(x)\gamma^0$

and equation of motion $(\not{\partial}+m)\psi(x)=0$.

Lagrangian density

$$\mathcal{L} = -\bar{\psi}(x)(\not{\partial}+m)\psi(x).$$

$$= -\bar{\psi}^{\alpha}(x)\left((\gamma^\mu)_\alpha{}^{\alpha'}\partial_\mu + m\delta_\alpha^{\alpha'}\right)\psi_{\alpha'}(x).$$

Path integral (Lorentzian)

$$Z = \int D\psi D\bar{\psi}\; e^{i\int d^4x\,\mathcal{L}}$$

Time-ordered 2-point function

$$\langle\Omega|T\hat{\psi}_\alpha(x)\hat{\psi}_\beta(y)|\Omega\rangle = \langle\Omega|T\hat{\bar{\psi}}^{\alpha'}(x)\hat{\bar{\psi}}^{\beta'}(y)|\Omega\rangle = 0.$$

$$\langle\Omega|T\hat{\psi}_\alpha(x)\hat{\bar{\psi}}^\beta(y)|\Omega\rangle$$

$$= (-\not{\partial}_x+m)_\alpha{}^\beta\,\Delta_F(x-y;m^2)$$

$$= \int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-y)}\frac{-i(-i\not{k}_\alpha{}^\beta+m\delta_\alpha^\beta)}{k^2+m^2-i\epsilon}.$$

* Compare with "Gaussian" Berezin integral

$$Z = \int \prod_a d\bar{\eta}_a d\eta_a \, e^{-\bar{\eta}^T A \cdot \eta} = \det A$$

$$\langle \eta_a \bar{\eta}^b \rangle = \frac{1}{Z} \int \prod d\bar{\eta} d\eta \cdot e^{-\bar{\eta}^T A \eta} \cdot \eta_a \bar{\eta}^b$$

$$= \frac{1}{Z} \int \prod d\bar{\eta} d\eta \cdot \eta_a (A^{-1})_c{}^b \frac{\partial}{\partial \eta_c} e^{-\bar{\eta}^T A \eta}$$

$$\overset{\int \text{by part}}{=\!=} \frac{1}{Z} \int \prod d\bar{\eta} d\eta \cdot (A^{-1})_a{}^b e^{-\bar{\eta}^T A \eta}$$

$$= (A^{-1})_a{}^b .$$

General fermion correlators w.r.t. Gaussian functional Berezin integral computed by Wick contractions, provided that appropriate signs are taken into account:

$$\langle \cdots \eta_a \underbrace{\cdots\cdots}_{N \ \eta \text{ or } \bar{\eta}\text{'s}} \bar{\eta}^b \cdots \rangle$$

$$= (-)^N \langle \eta_a \bar{\eta}^b \rangle \langle \cdots \rangle$$

More generally, in any QFT,

a (fermionic) spinor field operator $\hat{\psi}_\alpha(x)$

obeys

$$U(\Lambda, a)\, \hat{\psi}_\alpha(x)\, (U(\Lambda, a))^{-1}$$
$$= (R(\Lambda))_\alpha{}^\beta\, \hat{\psi}_\beta(\Lambda x + a)$$

For infinitesimal Lorentz transf.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu,$$
$$R(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}}.$$
$$S^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

Further assume a $U(1)$ (global) symmetry

$$U_\theta = e^{i\theta \hat{Q}}$$

$\hat{Q}$ is a Hermitian "charge" operator

$$U_\theta\, \hat{\psi}_\alpha(x)\, U_\theta^{-1} = e^{i\theta}\, \hat{\psi}_\alpha(x)$$
$$U_\theta\, \hat{\bar{\psi}}^\alpha(x)\, U_\theta^{-1} = e^{-i\theta}\, \hat{\bar{\psi}}^\alpha(x)$$

i.e. $$[\hat{Q}, \hat{\psi}_\alpha(x)] = \hat{\psi}_\alpha(x)$$
$$[\hat{Q}, \hat{\bar{\psi}}^\alpha(x)] = -\hat{\bar{\psi}}^\alpha(x).$$

Assume $\hat{Q}|\Omega\rangle = 0$,

spin-$\frac{1}{2}$ 1-particle states $|\vec{k}, \sigma, q = \pm 1\rangle$

$$\hat{Q}|\vec{k}, \sigma, q\rangle = q|\vec{k}, \sigma, q\rangle.$$

Lorentz sym implies

$$\langle \vec{k}, \sigma, +|\hat{\psi}_\alpha(x)|\Omega\rangle = \frac{e^{-ik\cdot x}}{(2\pi)^{\frac{3}{2}}}\left[Z_v^{\frac{1}{2}} v_\alpha^\sigma(\vec{k}) + Z_u^{\frac{1}{2}} u_\alpha^\sigma(\vec{k})\right]$$

Note: $(e^{i\zeta\gamma_5}\hat{\psi})_\alpha(x)$ has the same Lorentz transformation property as $\hat{\psi}_\alpha(x)$.

$$u^\sigma \xleftrightarrow{\gamma_5} v^\sigma$$

can always redefine $\hat{\psi}(x) \rightsquigarrow e^{i\zeta\gamma_5}\hat{\psi}(x)$

to set

$$\langle \vec{k}, \sigma, +|\hat{\psi}_\alpha(x)|\Omega\rangle = \frac{Z_\psi^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} e^{-ik\cdot x} v_\alpha^\sigma(\vec{k})$$

[$Z_\psi = 1$ agrees with our free fermion field convention]

It follows from microcausality

$$\langle\Omega|\{\hat{\psi}_\alpha(x),\ \hat{\bar{\psi}}^\beta(y)\}|\Omega\rangle = 0$$

for $(x-y)^2>0$, that

$$\langle\vec{k},\sigma,-|\hat{\bar{\psi}}^\alpha(x)|\Omega\rangle = \frac{Z_{\bar{\psi}}^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}}\, e^{-ik\cdot x}\,(\bar{\mathcal{U}}^\sigma)^\alpha(\vec{k}).$$

where $|Z_{\bar{\psi}}| = |Z_\psi|$.

can choose phase convention of 1-particle states to be such that $Z_\psi = Z_{\bar{\psi}} \geq 0$.

A similar derivation to that of Källén – Lehmann spectral rep. for scalar field operators gives

$$\langle\Omega|\,T\,\hat{\psi}_\alpha(x)\,\hat{\bar{\psi}}^\beta(y)\,|\Omega\rangle$$

$$= Z_\psi\,(-\not{\partial}_x + m)_\alpha{}^\beta\,\Delta_F(x-y\,;\,m^2)$$

physical mass of spin-$\frac{1}{2}$ particle

$$+\ (\text{multi-particle contribution})$$

do not affect LSZ pole at $k^2=-m^2$.

To relate a time-order Green function involving $\hat{\psi}_\alpha(x)$ or $\hat{\bar{\psi}}^\beta(x)$ to S-matrix elements involving in/out particle/anti-particle via LSZ:

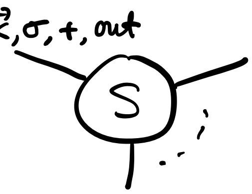
S-matrix element

momentum space Green function

$$\tilde{G}(k,\dots) = \int d^4x \, e^{-ik\cdot x} \langle \hat{\bar{\psi}}(x) \dots \rangle$$

out

+ particle



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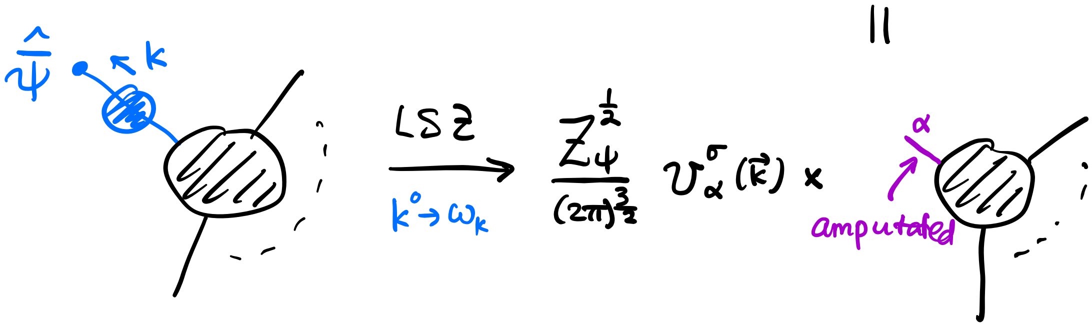
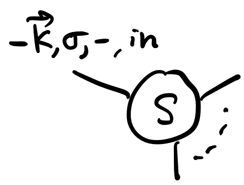


in

− particle



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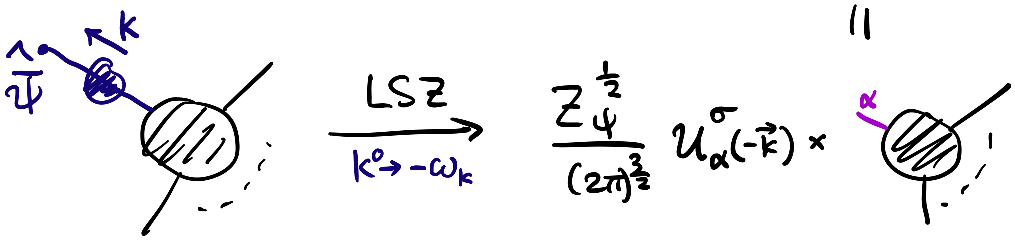
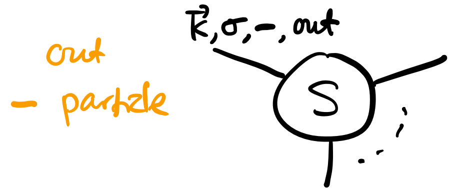


Similarly,



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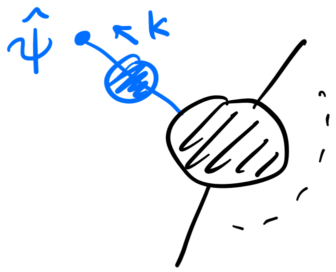
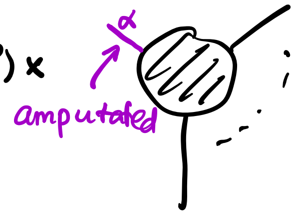
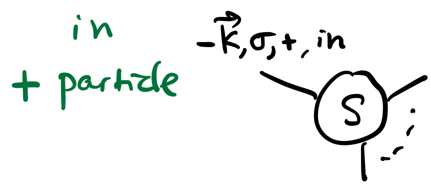
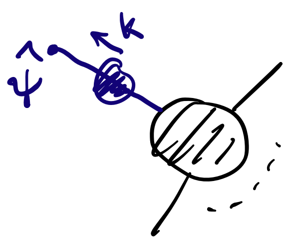


$$\xrightarrow[k^0 \to \omega_k]{\text{LSZ}} \quad \frac{Z_\psi^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} (\overline{u^\sigma})^\alpha (\vec{k}) \times$$

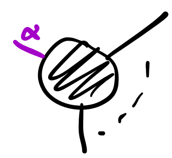




$\parallel$



$$\xrightarrow[k^0 \to -\omega_k]{\text{LSZ}} \quad \frac{Z_\psi^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} (\overline{v^\sigma})^\alpha (-\vec{k}) \times$$



Example:

$$\mathcal{L} = -\bar{\psi}(\not{\partial} + m)\psi - \frac{g}{2}(\bar{\psi}\psi)^2 .$$

At order g, $\langle \psi_\alpha(x_1)\, \psi_\beta(x_2)\, \bar{\psi}^\gamma(x_3)\, \bar{\psi}^\delta(x_4) \rangle^{\text{conn.}}$

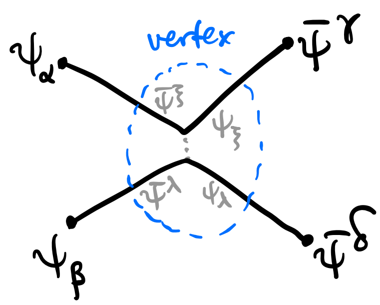
is computed by

$$\psi_\alpha(x_1)\,\psi_\beta(x_2)\,\bar\psi^\gamma(x_3)\,\bar\psi^\delta(x_4)\,\left(-i\frac{g}{2}\right)\int d^4y\,\bar\psi^\xi(y)\psi_\xi(y)\,\bar\psi^\lambda(y)\psi_\lambda(y)$$

and 3 other possible contractions

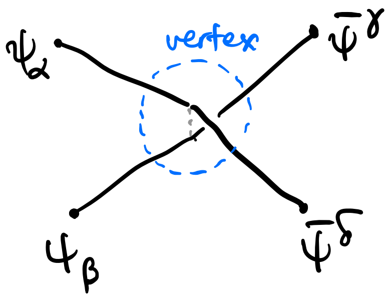
They can be represented by two diagrams

change sign due to ordering of ψ's



$$= +\,ig\int d^4y\,\langle\psi_\alpha\bar\psi^\xi\rangle\langle\psi_\xi\bar\psi^\gamma\rangle \cdot\langle\psi_\beta\bar\psi^\lambda\rangle\langle\psi_\lambda\bar\psi^\delta\rangle$$

and



$$= -ig\int d^4y\,\langle\psi_\alpha\bar\psi^\xi\rangle\langle\psi_\xi\bar\psi^\delta\rangle\langle\psi_\beta\bar\psi^\lambda\rangle\langle\psi_\lambda\bar\psi^\gamma\rangle$$

After LSZ reduction, the S-matrix element

${}_{out}\langle \vec k_1,\sigma_1,-;\ \vec k_2,\sigma_2,-\ |\ -\vec k_3,\sigma_3,-;\ -\vec k_4,\sigma_4,-\rangle_{in}$, for instance,

is computed at tree-level by stripping off the external propagators and attaching the

Spinor polarization factors.

e.g. $\langle \psi_\alpha \bar{\psi}^\xi \rangle \rightsquigarrow \frac{1}{(2\pi)^{\frac{3}{2}}} (\bar{u}^{\sigma_1})^\xi (\vec{k}_1)$

$\langle \psi_\xi \bar{\psi}^\gamma \rangle \rightsquigarrow \frac{1}{(2\pi)^{\frac{3}{2}}} u^{\sigma_3}_\xi (-\vec{k}_3)$, etc.

[Complete the computation of amplitude in Pset]

Consider a massless particle in 4D

choose reference momentum

$$k_R^\mu = (E, 0, 0, E)$$

little group ? $W \in SO(1,3)$ such that

$W \cdot k_R = k_R$?

Recall: a general Lorentz transformation $\Lambda^\mu{}_\nu$

can be written as $\Lambda = e^{\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}}$

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$(M^{\mu\nu})^\rho{}_\sigma = -i(\eta^{\mu\rho}\delta^\nu{}_\sigma - \eta^{\nu\rho}\delta^\mu{}_\sigma)$$

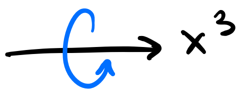
e.g. for infinitesimal $\omega_{\mu\nu}$, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\,(\eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - (\rho \leftrightarrow \sigma))$$

$\updownarrow$

$$[\hat{J}^{\mu\nu}, \hat{J}^{\rho\sigma}] = -i\,(\eta^{\nu\rho}\hat{J}^{\mu\sigma} - \eta^{\mu\rho}\hat{J}^{\nu\sigma} - (\rho \leftrightarrow \sigma))$$

$$U(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}}$$



take $(M_{12})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix}$

$$W = e^{i\theta M_{12}} = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}.$$

can also take

$$A = M_{01} + M_{31} = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$B = M_{02} + M_{32} = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$W = e^{i\alpha A + i\beta B} = \begin{pmatrix} 1+\zeta & \alpha & \beta & -\zeta \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \zeta & \alpha & \beta & 1-\zeta \end{pmatrix}.$$

$\zeta = \frac{\alpha^2+\beta^2}{2}.$

A general massless little group element takes the form

$$W(\alpha, \beta, \theta) = e^{i\alpha A + i\beta B} e^{i\theta M_{12}}.$$

$$A,\ B \longrightarrow \quad \hat{A} = \hat{J}_{01} + \hat{J}_{31}$$

$$\hat{B} = \hat{J}_{02} + \hat{J}_{32}$$

$$[A,B]=0 \longleftrightarrow [\hat{A},\hat{B}]=0$$

Within the subspace of states of 1-particle carrying momentum $\vec{k}_R$, we can simultaneously diagonalize $\hat{A}$, $\hat{B}$, with the basis states

$$|\vec{k}_R, \underset{\uparrow}{a}, \underset{\uparrow}{b}\rangle$$

eigenvalues w.r.t. $\hat{A}$, $\hat{B}$

Furthermore, using $[M_{12}, A] = iB$,

$$[M_{12}, B] = -iA,$$

$$\Rightarrow \quad [\hat{J}_{12}, \hat{A}] = i\hat{B}$$

$$[\hat{J}_{12}, \hat{B}] = -i\hat{A},$$

rotates (a,b)

$\hat{J}_{12}$ acts on $|\vec{k}_R, a, b\rangle$ as $ib\partial_a - ia\partial_b$ + (non-derivative)

$$\Rightarrow \quad e^{i\theta \hat{J}_{12}} |\vec{k}_R, a, b\rangle$$

$$\propto |\vec{k}_R, a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta\rangle$$

We would have a continuously family of massless particles at a given momentum, unless $a = b = 0.$

will assume this from now.

i.e. $\hat{A}, \hat{B}$ annihilate the 1-particle state of momentum $k_R = (E, 0, 0, E)$.

WLOG, we can then assume

$$\hat{J}_3 |\vec{k}_R, h\rangle = h |\vec{k}_R, h\rangle$$

"helicity"

Equivalently, under rotation in x_3-direction,

$$e^{i\theta \hat{J}_3} |\vec{k}_R, h\rangle = e^{ih\theta} |\vec{k}_R, h\rangle.$$

i.e.

$$U(W(\alpha, \beta, \theta)) |\vec{k}_R, h\rangle = e^{ih\theta} |\vec{k}_R, h\rangle.$$

completely determine Lorentz transf. of

$$|\vec{k}, h\rangle = \sqrt{\frac{k_R^0}{k^0}}\, U(L(\vec{k}))\, |\vec{k}_R, h\rangle.$$

Demand invariance under 4π rotation

$$\Rightarrow \quad h \in \tfrac{1}{2}\mathbb{Z}.$$

Focus on the case $h = \pm 1$ "photon".

Q: What kind of field operator can create a 1-photon state?

Try "vector field" $\hat{\Phi}_\mu(x)$

$$U(\Lambda)\,\hat{\Phi}_\mu(x)\,(U(\Lambda))^{-1} = \Lambda^\nu{}_\mu\,\hat{\Phi}_\nu(\Lambda\cdot x)$$

$$\langle\vec{k}, h|\,\hat{\Phi}_\mu(0)\,|\Omega\rangle = C\cdot\mathcal{U}^h_\mu(\vec{k}).$$

$$\parallel$$

$$\langle\vec{k}, h|\,(U(\Lambda))^\dagger\,U(\Lambda)\,\hat{\Phi}_\mu(0)\,(U(\Lambda))^{-1}|\Omega\rangle$$

$$= \sqrt{\frac{(\Lambda k)^0}{k^0}}\,(D(W))^*\,\Lambda^\nu{}_\mu\,\langle\vec{\Lambda k}, h|\,\hat{\Phi}_\nu(0)\,|\Omega\rangle$$

$$\Rightarrow \mathcal{U}^h_\mu(\vec{k}) = \sqrt{\frac{(\Lambda k)^0}{k^0}}\,(D(W))^*\,\Lambda^\nu{}_\mu\,\mathcal{U}^h_\nu(\vec{\Lambda k})$$

$$D(W(\alpha,\beta,\theta)) = e^{ih\theta}.$$

For $\vec{k} = \vec{k_R}$, $\Lambda = W(\alpha,\beta,\theta)$, we have

$$\mathcal{U}^h_\mu(\vec{k_R}) = e^{-ih\theta}(W(\alpha,\beta,\theta))^\nu{}_\mu\,\mathcal{U}^h_\nu(\vec{k_R}).$$

In the case $\alpha = \beta = 0$,

$$W(\alpha, \beta, \theta) = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}$$

$\Rightarrow$ $h = \pm 1$,

$$\mathcal{U}^{+1}(\vec{k}_R) \propto \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}$$

$$\mathcal{U}^{-1}(\vec{k}_R) \propto \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}.$$

"polarization vectors of right/left-handed photon".

But: consider $\theta = 0$, $\alpha, \beta \neq 0$.

$$\mathcal{U}^h_\mu(\vec{k}_R) = (W(\alpha, \beta, 0))^\nu{}_\mu \, \mathcal{U}^h_\nu(\vec{k}_R).$$

$$\begin{pmatrix} 1+\zeta & \alpha & \beta & -\zeta \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \zeta & \alpha & \beta & 1-\zeta \end{pmatrix}. \quad \zeta = \frac{\alpha^2+\beta^2}{2}.$$

$$\left((W(\alpha,\beta,0))^{\nu}{}_{\mu}\, \mathcal{U}^{\pm 1}_{\nu}(\vec{k}_R) \right) \propto \begin{pmatrix} \alpha \mp i\beta \\ 1 \\ \mp i \\ -\alpha \pm i\beta \end{pmatrix}$$

$$\not\propto$$

$$\left(\mathcal{U}^{\pm 1}_{\mu}(\vec{k}_R) \right) \quad ??$$

Conclusion: the operator $\hat{\Phi}_\mu(x)$ with the property that $\hat{\Phi}_\mu(x)|\Omega\rangle$ overlaps with 1-photon state DOES NOT EXIST (?!)

Note however that

$$(W(\alpha,\beta,0))^{\nu}{}_{\mu}\, \mathcal{U}^{h}_{\nu}(k_R) - \mathcal{U}^{h}_{\mu}(k_R) \propto (k_R)_\mu .$$

Possible interpretation:

While the operator $\hat{\Phi}_\mu(x)$ does not exist, $\partial_\mu \hat{\Phi}_\nu(x) - \partial_\nu \hat{\Phi}_\mu(x)$ does exist, and

$$\langle \vec{k}, h |\, (\partial_\mu \hat{\Phi}_\nu(x) - \partial_\nu \hat{\Phi}_\mu(x))\, |\Omega\rangle$$

$$= C \cdot e^{-ik\cdot x} (-i k_\mu \mathcal{U}^h_\nu + i k_\nu \mathcal{U}^h_\mu)$$ is well-defined.

Usually denote $\hat{\Phi}_\mu(x)$ by $\hat{A}_\mu(x)$.

"gauge redundancy"

$$\hat{A}_\mu(x) \sim \hat{A}_\mu(x) + \partial_\mu \zeta(x)$$

the "field strength" operator

$$\hat{F}_{\mu\nu}(x) = \partial_\mu \hat{A}_\nu(x) - \partial_\nu \hat{A}_\mu(x)$$

is a well-defined local field operator.

Classical Maxwell theory

"vector potential" $A_\mu(x)$ subject to gauge redundancy

$$A_\mu(x) \sim A_\mu(x) + \partial_\mu \zeta(x).$$

$\zeta(x)$ = arbitrary functio of spacetime coord.

Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ (gauge-invariant)

$$= \frac{1}{2} \sum_{i=1}^{3} (\partial_0 A_i - \partial_i A_0)^2 - \frac{1}{4} \sum_{i,j=1}^{3} (\partial_i A_j - \partial_j A_i)^2$$

Note: $\dot{A}_0$ does not appear.

A prior, view $A_\mu(t, \vec{x})$ at a given time t as generalized coordinates, whose corresponding conjugate canonical momentum densities are

$$\Pi^\mu(\vec{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}$$

$$\Rightarrow \quad \Pi_i = \partial_0 A_i - \partial_i A_0 = F_{0i}.$$

$$\boxed{\Pi^0 = 0.}$$

a "primary constraint"

Equation of motion includes

$$\underbrace{-\partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_0)} + \frac{\partial \mathcal{L}}{\partial A_0}}_{\parallel} = 0$$

$$-\partial_i(\partial_i A_0 - \partial_0 A_i)$$

$$\Rightarrow \quad \boxed{\partial_i \Pi_i(\vec{x}) = 0}$$

a "secondary constraint"

The constrained quantities $\Pi_0(\vec{x})$ and $\partial_i \Pi_i(\vec{x})$ have vanishing Poisson bracket among themselves (Dirac's "first-class constraints")

The 1^{st}-class constraints are closely tied to gauge redundancy. Consider

$$\delta A_\mu(\vec{x}) \equiv \int d^3\vec{y} \Big(\varepsilon(\vec{y}) \{\Pi^0(\vec{y}), A_\mu(\vec{x})\}_P + \tilde{\varepsilon}(\vec{y}) \{\partial_i \Pi_i(\vec{y}), A_\mu(\vec{x})\}_P \Big)$$

$$= \varepsilon(\vec{x})\, \delta^0_\mu - \partial_i \tilde{\varepsilon}(\vec{x})\, \delta_{i\mu}$$

$$= \partial_\mu \zeta(x) \Big|_{x^0 = t_0},$$

where $\zeta(t_0, \vec{x}) = -\tilde{\varepsilon}(\vec{x})$

$\partial_t \zeta(t_0, \vec{x}) = \varepsilon(\vec{x})$.

To define the phase space of a gauge theory, we need to remove 1^{st}-class constraints by "gauge fixing", i.e. impose constraints directly on generalized coordinates, so as to turn 1^{st}-class constraints into 2^{nd}-class.

For instance, consider "axial gauge"

$$A_3(x) = 0.$$

By choice, we have eliminated A_3 and Π_3 as canonical variables. Now

$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^{2}(\partial_0 A_i - \partial_i A_0)^2 - \frac{1}{2}(\partial_3 A_0)^2$$
$$- \frac{1}{2}(\partial_1 A_2 - \partial_2 A_1)^2 - \frac{1}{2}\sum_{i=1}^{2}(\partial_3 A_i)^2 .$$

Now the equation of motion obtained by varying A_0 is

$$\sum_{i=1}^{2} \partial_i(\partial_i A_0 - \partial_0 A_i) + \partial_3^2 A_0 = 0.$$

i.e.

$$\sum_{i=1}^{2} \partial_i \Pi_i + \partial_3^2 A_0 = 0.$$

Note:

$$\left\{ \Pi_0(\vec{x}), \sum_{i=1}^{2} \partial_i \Pi_i(\vec{y}) + \partial_3^2 A_0(\vec{y}) \right\}$$
$$= \partial_3^2 \delta^3(\vec{x}-\vec{y}) \neq 0.$$

The constraints

$$\Pi_0(\vec{x}) = 0$$

and $\displaystyle\sum_{i=1}^{2} \partial_i \Pi_i(\vec{x}) + \partial_3^2 A_0(\vec{x}) = 0$

are now "2nd - class" !

- Can proceed by replacing $\{\ ,\ \}_P$ with the Dirac bracket $\{\ ,\ \}_D$, or equivalently solve A_0 from the constraint

$$\partial_3^2 A_0 = -\sum_{i=1}^{2} \partial_i \Pi_i(\vec{x})$$

(at least for Fourier modes of non-zero wave number in x_3 direction)

and eliminate A_0, Π^0 altogether as canonical coordinates.

Conclusion: phase space of Maxwell theory is parameterized by $A_i(\vec{x})$, $\Pi_i(\vec{x})$, $i = 1, 2$ only, in axial gauge.

Can then proceed with canonical quantization.

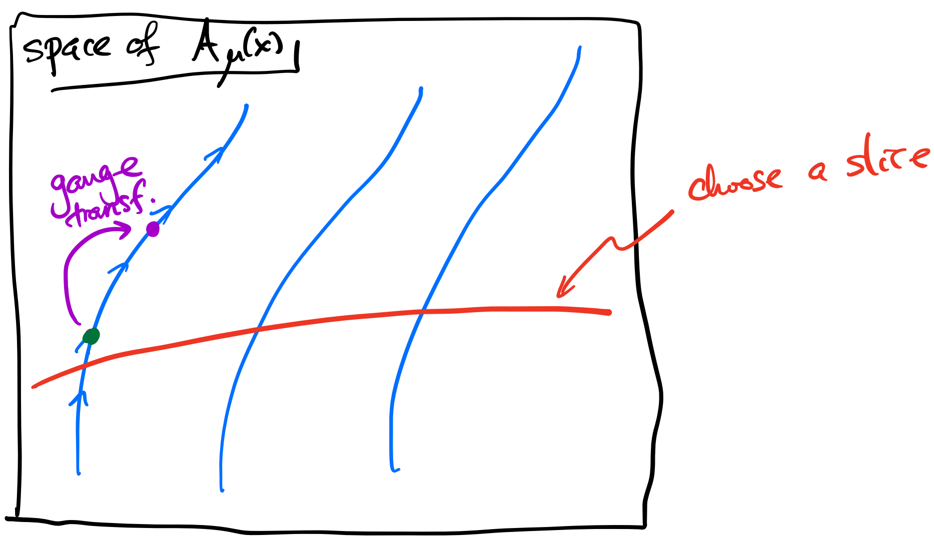
We will, however, pursue path integral quantization, for convenience of calculations.

Formally,

$$Z = \int [DA_\mu] \, e^{i \int d^4x \, \mathcal{L}}$$

$A_\mu \to A_\mu + \partial_\mu \zeta$, $\quad S = \int \mathcal{L}$ invariant

overcounting field configurations?



Toy model: $S(x,y) = (x-y)^2$

"path integral"

$$\int dx\, dy\, e^{-S(x,y)}$$

divergent due to shift symmetry

$$x \to x+\zeta, \quad y \to y+\zeta.$$

Impose gauge fixing condition

$$\ell(x,y) = 0$$

for now, linear function

gauge fixed integral

$$Z = \mathcal{N} \cdot \int dx\, e^{-S(x,y)}\Big|_{\ell(x,y)=0}$$

e.g. $\ell(x,y) = y + ax + b$.

$$Z = \mathcal{N} \cdot \int dx\, e^{-(x+ax+b)^2}$$

But, "physical observables" should be gauge-invariant !?

Consider $\mathcal{O}(x,y)$ (some function)

gauge invariance of $\mathcal{O}$:

$$\mathcal{O}(x+\xi,\, y+\xi) = \mathcal{O}(x,y).$$

$$\langle \mathcal{O}(x,y) \rangle = \frac{\int dx\, e^{-S(x,y)|_{\ell=0}}\, \mathcal{O}(x,y)|_{\ell=0}}{\int dx\, e^{-S(x,y)|_{\ell=0}}}$$

$$= \frac{\int dx\, e^{-(x+ax+b)^2}\, \overbrace{\mathcal{O}(x,-ax-b)}^{=\,\mathcal{O}(x+ax+b,0)}}{\int dx\, e^{-(x+ax+b)^2}}$$

$$\overset{\text{define } \tilde{x} = x+ax+b}{=\!=\!=\!=} \frac{\cancel{\frac{1}{1+a}} \int d\tilde{x}\, e^{-\tilde{x}^2}\, \mathcal{O}(\tilde{x},0)}{\cancel{\frac{1}{1+a}} \int d\tilde{x}\, e^{-\tilde{x}^2}}$$

Jacobian factors cancel

independent of a,b (!)

Can also use weighted average of gauge conditions, e.g.

$$Z = \int da\, db\, \underbrace{P(a,b)}_{\text{some positive function}} \int dx\, e^{-S(x,y)\big|_{y=-ax-b}}$$

expectation value of $\mathcal{O}(x,y)$ unaffected.

Maxwell theory path integral in axial gauge:

$$Z = \int [DA_\mu]\, \delta(A_3(x))\, e^{iS}$$

$$S = -\frac{1}{4}\int d^4x\, (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \frac{1}{2}\int d^4x\, A_\mu \underbrace{(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)}_{\equiv\, \mathbb{D}^{\mu\nu}} A_\nu$$

$\mathbb{D}^{\mu\nu}$ is not invertible unless we fix the gauge.

In axial gauge, set $A_3(x) = 0$.

$$S = \frac{1}{2}\int d^4x \, A_\mu(x) \, (\mathbb{D}^{\mu\nu} A_\nu)(x),$$

where $\mu, \nu = 0, 1, 2$.

$$\mathbb{D}^{\mu\nu} = \eta^{\mu\nu}\Box - \partial^\mu\partial^\nu$$

$\Box = \sum_{\rho=0}^{3} \partial^\rho \partial_\rho$ still.

In Fourier representation,

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \tilde{A}_\mu(x)$$

$$S = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(-k) \, \tilde{\mathbb{D}}^{\mu\nu}(k) \, A_\nu(k),$$

where $\tilde{\mathbb{D}}^{\mu\nu}(k) = -\eta^{\mu\nu}k^2 + k^\mu k^\nu$.

$\mu, \nu = 0, 1, 2$.

$$(\tilde{\mathbb{D}}(k))^{-1}_{\mu\nu} = -\frac{1}{k^2}\left(\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k_3^2}\right)$$

Check: $\sum_{\rho=0}^{2}(-\eta^{\mu\rho}k^2 + k^\mu k^\rho)\left(-\frac{1}{k^2}\right)\left(\eta_{\rho\nu} + \frac{k_\rho k_\nu}{k_3^2}\right)$

$$= \delta^\mu{}_\nu - \frac{k^\mu k_\nu}{k^2} + \frac{k^\mu k_\nu}{k_3^2} - \frac{k^\mu k_\nu \overbrace{\sum_{\rho=0}^{2} k^\rho k_\rho}^{= k^2 - k_3^2}}{k^2 \cdot k_3^2}$$

$$= \delta^\mu{}_\nu \qquad \checkmark$$

Taking into account the $i\epsilon$ prescription for Lorentzian time-ordered Green function, we conclude that in axial gauge,

$$\langle\Omega|\, T\, \hat{A}_\mu(x)\, \hat{A}_\nu(y)\, |\Omega\rangle$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-y)}\, \frac{-i}{k^2 - i\epsilon}\left(\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k_3^2}\right)$$

singular at $k_3 = 0$

having to do with the axial gauge condition is singular for $k_3 = 0$ modes

The lack of manifest Lorentz invariance is not a problem, as the physically meaningful expectation values are those of gauge-invariant operators,

e.g. $\langle \Omega | T \hat{F}_{\mu\nu}(x) \hat{F}_{\rho\sigma}(y) | \Omega \rangle$

not $\langle \Omega | T \hat{A}_{\mu}(x) \hat{A}_{\nu}(y) | \Omega \rangle$.

Indeed,

$$\langle \Omega | T \hat{F}_{\mu\nu}(x) \hat{F}_{\rho\sigma}(y) | \Omega \rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \frac{-i}{k^2 - i\epsilon}$$

$$\times \Big[\underbrace{(ik_\mu)(-ik_\rho)\left(\eta_{\nu\sigma} + \frac{k_\nu k_\sigma}{k_3^2}\right) - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma) + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma)}_{= \, k_\mu k_\rho \eta_{\nu\sigma} - k_\nu k_\rho \eta_{\mu\sigma} - k_\mu k_\sigma \eta_{\nu\rho} + k_\rho k_\sigma \eta_{\mu\nu}} \Big]$$

Lorentz invariant ✓

It would be useful to work with a gauge condition that is manifestly Lorentz invariant

Landau gauge $\partial_\mu A^\mu(x) = 0$

Starting with generic $A^\mu(x) = v^\mu(x)$,

a gauge transf. $A^\mu \to A^\mu + \partial^\mu \zeta$

choose ζ to be such that

$\Box \zeta = -\partial_\mu v^\mu$ to set $\partial_\mu A^\mu = 0$.

Generalization : consider the gauge condition

$$\partial_\mu A^\mu(x) = \varphi(x),$$

realized in the Euclidean path integral by inserting δ-functional:

$$\delta(\partial_\mu A^\mu - \varphi) = \int [D\lambda]\, e^{i\int d^4x_E\, \lambda(x)(\partial_\mu A^\mu(x) - \varphi(x))}$$

Further average over different $\varphi(x)$ with

$$\int [D\varphi]\, e^{-\frac{1}{2\xi}\int d^4x_E (\varphi(x))^2} \qquad (\xi > 0)$$

$$\int [D\varphi]\, e^{-\frac{1}{2\xi}\int d^4x^E (\varphi(x))^2}\, \delta(\partial_\mu A^\mu - \varphi)$$

$$= \int [D\varphi D\lambda]\, e^{-\frac{1}{2\xi}\int \varphi^2 + i\int \lambda(\partial_\mu A^\mu - \varphi)}$$

$$\underset{\text{Sort } \varphi}{=\!=} \int [D\lambda]\; e^{-\frac{\xi}{2}\int \lambda^2 + i\int \lambda \partial_\mu A^\mu}$$

$$\underset{\substack{\text{up to const}\\ \text{normalization}}}{=\!=} e^{-\frac{1}{2\xi}\int d^4x^E (\partial_\mu A^\mu)^2}$$

This amounts to adding to the Euclidean action

$$\Delta S^E = \frac{1}{2\xi}\int d^4x^E\, (\partial_\mu A^\mu)^2\ ,$$

or to the Lorentzian action

$$\Delta S = -\frac{1}{2\xi}\int d^4x\, (\partial_\mu A^\mu)^2\ .$$

"Gauge-fixed" form of Maxwell action

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \right]$$

$$= \frac{1}{2}\int d^4x \, A_\mu \Big(\underbrace{\eta^{\mu\nu}\Box - (1-\tfrac{1}{\xi})\partial^\mu\partial^\nu}_{\equiv \, \mathbb{D}_\xi^{\mu\nu}} \Big) A_\nu .$$

$\mathbb{D}_\xi^{\mu\nu}$ is invertible :

$$\widetilde{\mathbb{D}}_\xi^{\mu\nu}(k) = -\eta^{\mu\nu}k^2 + (1-\tfrac{1}{\xi})\, k^\mu k^\nu ,$$

$$\mu,\nu = 0,1,2,3.$$

$$\left(\widetilde{\mathbb{D}}_\xi(k)\right)^{-1}_{\mu\nu} = -\frac{\eta^{\mu\nu}}{k^2} + (1-\xi)\frac{k^\mu k^\nu}{(k^2)^2}$$

Check: $\left(-\eta^{\mu\rho}k^2 + (1-\tfrac{1}{\xi})k^\mu k^\rho\right)\left(-\frac{\eta_{\rho\nu}}{k^2} + (1-\xi)\frac{k_\rho k_\nu}{(k^2)^2}\right)$

$$= \delta^\mu{}_\nu - (1-\tfrac{1}{\xi})\frac{k^\mu k_\nu}{k^2} - (1-\xi)\frac{k^\mu k_\nu}{k^2} + (1-\tfrac{1}{\xi})(1-\xi)\frac{k^\mu k_\nu}{k^2}$$

$$= \delta^\mu{}_\nu . \quad \checkmark$$

Time-ordered Green function

$$\langle \Omega | T \hat{A}_\mu(x) \hat{A}_\nu(y) | \Omega \rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \frac{-i}{k^2 - i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2 - i\epsilon}\right)$$

$\xi \to 0$ Landau gauge

$\xi = 1$ "Feynman gauge", gives

the simplest form of photon propagator

$$\mu \overset{\xleftarrow{k}}{\sim\sim\sim} \nu \;=\; \frac{-i}{k^2 - i\epsilon}\,\eta^{\mu\nu}$$

The Maxwell theory is a free field theory.

$\hat{F}_{\mu\nu}(x)|\Omega\rangle$ is a linear combination of 1-photon states

$$|\vec{k}, h\rangle, \quad h = \pm 1$$

$$\langle \vec{k}, h|\, \hat{A}_\mu(x)\,|\Omega\rangle = \underbrace{\frac{e^{-ik\cdot x}}{\sqrt{(2\pi)^3 \cdot 2|\vec{k}|}}}_{\text{normalization factor similar to scalar case}} Z_A^{\frac{1}{2}}\, \underbrace{e^{h\,*}_\mu(\vec{k})}_{\text{polarization vector}}$$

convention

The polarization vector $e^h(\vec{k})$ is gauge-dependent: $e^h_\mu \sim e^h_\mu + \alpha k_\mu$.

Note: $\hat{A}_\mu(x)$ is NOT a well-defined local operator, unlike $\hat{F}_{\mu\nu}(x)$.

Indeed, the matrix element

$$\langle \vec{k}, h \,|\, \hat{F}_{\mu\nu}(x) \,|\Omega\rangle = \frac{e^{-ik\cdot x}}{\sqrt{(2\pi)^3 \cdot 2|\vec{k}|}} Z_A^{\frac{1}{2}} \left(-i k_\mu e_\nu^{h*}(\vec{k}) + i k_\nu e_\mu^{h*}(\vec{k})\right)$$

is well-defined, invariant under

$$e_\mu^h(\vec{k}) \to e_\mu^h(\vec{k}) + \alpha k_\mu .$$

We have already seen that at the reference momentum $k_R^\mu = (E, 0, 0, E)$, consideration of rotation symmetry around x_3-axis enforces (up to normalization)

$$e^{\pm 1}(\vec{k}_R) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \qquad \text{up to shift by } \alpha k_R$$

In particular, it demands $k_R^\mu e_\mu^h(\vec{k}_R) = 0$.

For general light-like momentum $k^\mu = (L(\vec{k}))^\mu{}_\nu k_R^\nu$,

$$e_\mu^h(\vec{k}) = (L(\vec{k}))_\mu{}^\nu e_\nu^h(\vec{k}_R) ,$$

still obeys $\boxed{k^\mu e_\mu^h(\vec{k}) = 0 .}$ transversality

$e^h_\mu(\vec{k})$ further obey

orthogonality: $\boxed{e^h_\mu(\vec{k})\, e^{h'*\mu}(\vec{k}) = \delta_{hh'}}$

and completeness relation

$$\sum_{h=\pm1} e^h_\mu(\vec{k})\, e^{h*}_\nu(\vec{k}) = \eta_{\mu\nu} + k_\mu c_\nu + k_\nu c_\mu$$

for some c^μ that obeys $k\cdot c = -1$, $c^2 = 0$.

e.g. we can take $c^\mu = \frac{1}{2|\vec{k}|}(k^0, -\vec{k})$.

Comparing

$$\langle\Omega|\, \hat{F}_{\mu\nu}(x)\, \hat{F}_{\rho\sigma}(y)\, |\Omega\rangle$$

$$= \sum_{h=\pm1} \int d^3\vec{k}\; \langle\Omega|\, \hat{F}_{\mu\nu}(x)\, |\vec{k}, h\rangle \langle \vec{k}, h|\, \hat{F}_{\rho\sigma}(y)\, |\Omega\rangle$$

$$= \sum_{h=\pm1} \int \frac{d^3\vec{k}}{(2\pi)^3 \cdot 2|\vec{k}|}\, e^{ik\cdot(x-y)}\, Z_A \Big(k_\mu k_\rho\, e^h_\nu(\vec{k})\, e^{h*}_\sigma(\vec{k}) - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + (\mu\leftrightarrow\nu, \rho\leftrightarrow\sigma)\Big)$$

to the Wightman function obtained by analytic continuation from time-ordered or Euclidean 2-pt function of $\hat{F}_{\mu\nu}$:

$$\langle \Omega | \hat{F}_{\mu\nu}(x) \; \hat{F}_{\rho\sigma}(y) | \Omega \rangle$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 \cdot 2|\vec{k}|} \, e^{ik\cdot(x-y)} \Big(k_\mu k_\rho \eta_{\nu\sigma} - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma) + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma) \Big)$$

In agreement, provided $Z_A = 1$. ✓

Remark: you may wonder what actually goes wrong if we define $\hat{A}_\mu(x)$ via the insertion of $A_\mu(x)$ in the gauge-fixed form of the path integral, and consider the state $\hat{A}_\mu(x)|\Omega\rangle$. It would have to contain more than just the 1-photon state in order to reproduce the 2-pt function $\langle \Omega | \hat{A}_\mu(x) \, \hat{A}_\nu(y) | \Omega \rangle$. In fact, $\hat{A}_\mu(x)|\Omega\rangle$ does contain "extra states" that are unphysical. A systematic treatment of the Hilbert Space based on the gauge-fixed path integral requires the BRST formalism, which will be discussed in 253b.

Quantum Electrodynamics

- Maxwell + charged fermion

$A_\mu(x)$ $\qquad\qquad$ $\psi_\alpha(x)$, $(\psi^\dagger)^\alpha(x)$

gauge redundancy

$$(A_\mu(x), \psi_\alpha(x))$$

$$\sim (A_\mu(x) + \partial_\mu \zeta(x), \quad e^{iq\zeta(x)}\psi(x))$$

"electric charge" q $(= -e$ for electron$)$

Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}(\gamma^\mu D_\mu + m)\psi$$

$$D_\mu \psi \equiv \partial_\mu \psi - iq A_\mu \psi$$

under gauge transf :

$$D_\mu \psi \to \partial_\mu (e^{iq\zeta}\psi) - iq(A_\mu + \partial_\mu \zeta) e^{iq\zeta}\psi$$
$$= e^{iq\zeta} D_\mu \psi .$$

Thus $\mathcal{L}$ is gauge-invariant, as needed.

Perturbation theory can be carried out via gauge-fixed path integral

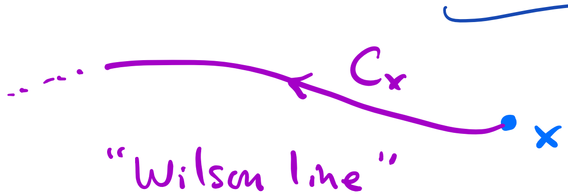
$$Z = \int [DA_\mu D\psi]\, e^{i\int d^4x\, (\mathcal{L} + \Delta\mathcal{L})}$$

$$\Delta\mathcal{L} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \quad \text{as in pure Maxwell}$$

Caution: $\hat{\psi}_\alpha(x)$ is not gauge-invariant, thus not a well-defined local field operator.

This may be fixed by considering

$$\underbrace{e^{iq\int_{C_x} \hat{A}_\mu(y)\, dy^\mu} \cdot \hat{\psi}_\alpha(x)}_{\text{gauge-invariant, but not local.}}$$



Consequently, correlation functions such as

$$\langle \Omega | T \, \hat{\psi}_\alpha(x) \, \hat{\bar{\psi}}^\beta(y) | \Omega \rangle$$

are not gauge-invariant, and generally involve unphysical states (in the sense of BRST), while

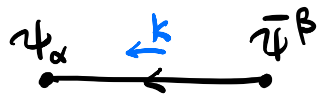
$$\langle \Omega | T \, \hat{\psi}_\alpha(x) \, e^{-iq \int_y^x \hat{A}_\mu(z) dz^\mu} \, \hat{\bar{\psi}}^\beta(y) | \Omega \rangle$$

is gauge-invariant and well-defined (modulo possible regularization needed to define the Wilson line operator itself.)

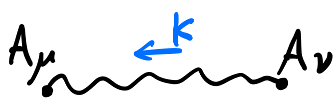
This issue is typically ignored in the formulation of scattering theory, where one anticipates that the LSZ limit of the Green function of gauge non-invariant operators nonetheless gives rise to physically meaningful S-matrix elements. One can in principle verify the consistency of the resulting S-matrix elmts such as unitarity and causality.

[This expectation is almost correct, modulo the issue of IR divergence.]

Feynman rules:

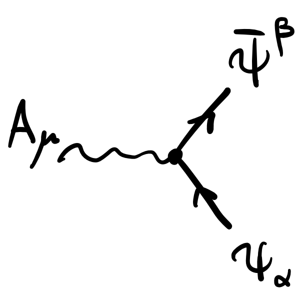


$$= \frac{-i(-i\not{k}+m)_\alpha{}^\beta}{k^2+m^2-i\epsilon}.$$



$$= \frac{-i}{k^2-i\epsilon}\left(\eta^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2-i\epsilon}\right)$$

[Feynman gauge $\xi=1$]

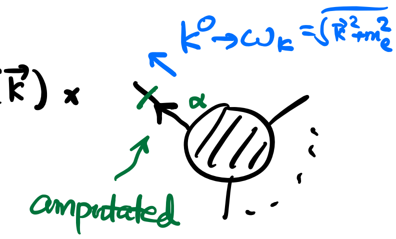


$$= -g(\gamma^\mu)_\beta{}^\alpha.$$

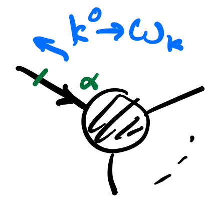
[$g=-e$]

In/out - particles:

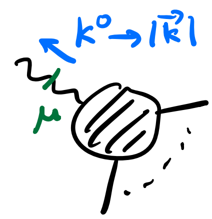
$${}_{\text{out}}\langle e^-, \vec{k}, \sigma ; \cdots | \cdots = \frac{Z_\psi^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} (\overline{u^\sigma})^\alpha(\vec{k}) \times$$



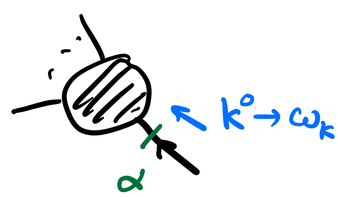
$${}_{\text{out}}\langle e^+, \vec{k}, \sigma ; \cdots | \cdots = \frac{Z_\psi^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} v_\alpha^\sigma(\vec{k}) \times$$



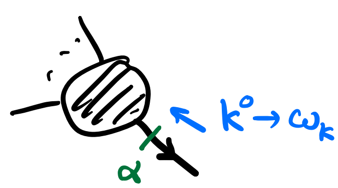
$${}_{\text{out}}\langle \gamma, \vec{k}, h ; \cdots | \cdots = \frac{Z_A^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}} \sqrt{2|\vec{k}|}} e_\mu^{h\,*}(k) \times$$



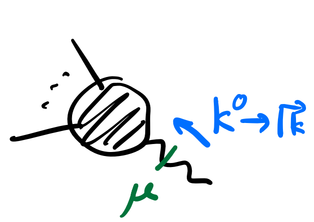
$$\cdots | \cdots, e^-, \vec{k}, \sigma \rangle^{\text{in}} = \frac{Z_\psi^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} u_\alpha^\sigma(\vec{k}) \times$$



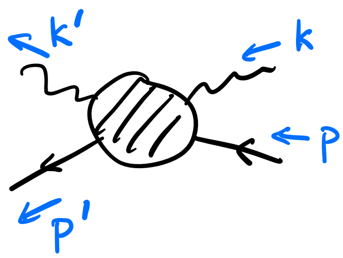
$$\cdots | \cdots, e^+, \vec{k}, \sigma \rangle^{\text{in}} = \frac{Z_\psi^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} (\overline{v^\sigma})^\alpha(\vec{k}) \times$$



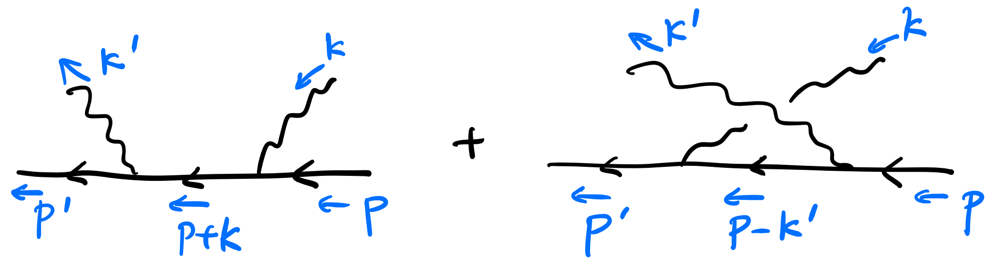
$$\cdots | \cdots ; \gamma, \vec{k}, h \rangle^{\text{in}} = \frac{Z_A^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}} \sqrt{2|\vec{k}|}} e_\mu^h(\vec{k}) \times$$



Example: Compton scattering



tree level



subject to $p'+k' = p+k$

scattering amplitude

stripping off $(2\pi)^4 \delta^4(p'+k'-p-k)$ in definition of $i\mathcal{M}$.

$$i\mathcal{M}(\vec{p}\,',\sigma';\vec{k}',h'\,|\,\vec{p},\sigma;\vec{k},h)$$

$$= \frac{(\overline{u}^{\sigma'})^{\beta}(\vec{p}\,')}{(2\pi)^{\frac{3}{2}}} \frac{e_{\nu}^{h'*}(\vec{k}')}{(2\pi)^{\frac{3}{2}}\sqrt{2|\vec{k}'|}} \frac{u_{\alpha}^{\sigma}(\vec{p})}{(2\pi)^{\frac{3}{2}}} \frac{e_{\mu}^{h}(\vec{k})}{(2\pi)^{\frac{3}{2}}\sqrt{2|\vec{k}|}}$$

$$\times \Big[e(\gamma^{\nu})_{\beta}{}^{\beta'} \frac{-i(-i(\not{p}+\not{k})+m)_{\beta'}{}^{\alpha'}}{(p+k)^2+m^2-i\epsilon} e(\gamma^{\mu})_{\alpha'}{}^{\alpha}$$

$$+ e(\gamma^{\mu})_{\beta}{}^{\beta'} \frac{-i(-i(\not{p}-\not{k}')+m)_{\beta'}{}^{\alpha'}}{(p-k')^2+m^2-i\epsilon} e(\gamma^{\nu})_{\alpha'}{}^{\alpha}\Big]$$

$$= \frac{-ie^2}{(2\pi)^6 \sqrt{2|\vec{k}'|}\sqrt{2|\vec{k}|}} \, \bar{u}^{\sigma'}(\vec{p}') \left[\not{e}^{h'*}(\vec{k}') \frac{-i(\not{p}+\not{k})+m}{(p+k)^2+m^2-i\epsilon} \not{e}^{h}(\vec{k}) \right.$$

$$\left. + \not{e}^{h}(\vec{k}) \frac{-i(\not{p}-\not{k}')+m}{(p-k')^2+m^2-i\epsilon} \not{e}^{h'*}(\vec{k}') \right] u^{\sigma}(\vec{p}) .$$

Note: $u^{\sigma}(\vec{p})$ obeys $(\not{p}-im)\,u^{\sigma}(\vec{p})=0$

and $u^{\sigma'}(\vec{p}')$ obeys $(\not{p}'-im)\,u^{\sigma'}(\vec{p}')=0$

$\iff \bar{u}^{\sigma'}(\not{p}'-im)=0.$

also, $k\cdot e^{h}(\vec{k})=0=k'\cdot e^{h'}(\vec{k}')$.

Under "gauge transformation" $e^{h}_{\mu} \to e^{h}_{\mu} + \alpha k_{\mu}$,

$\not{e}^{h}(\vec{k})\,u^{\sigma}(\vec{p})$ for instance shifts by

$$\alpha \not{k}\, u^{\sigma}(\vec{p}) = \alpha(\underbrace{\not{k}+\not{p}}-im)\,u^{\sigma}(\vec{p})$$

commutes w/ $\psi\bar{\psi}$ propagator

also, $\not{k}+\not{p}=\not{k}'+\not{p}'$

$\bar{u}^{\sigma'}(\vec{p}')\,\not{e}^{h'*}(\vec{k}') \dfrac{-i(\not{p}+\not{k})+m}{(p+k)^2+m^2-i\epsilon} \not{e}^{h}(\vec{k})\,u^{\sigma}(\vec{p})$ shifts by

$$-i\alpha\, \bar{u}^{\sigma'} \not{e}^{h'*}(\vec{k}')\, u^{\sigma}(\vec{p}),$$

cancels against a similar shift of the second term

$$\alpha\, \bar{u}^{\sigma'}(\vec{p}') \underbrace{(\not{k}-\not{p}'+im) \frac{-i(\not{p}-\not{k}')+m}{(p-k')^2+m^2-i\epsilon}}_{=\,i} \not{e}^{h'*}(\vec{k}')\,u^{\sigma}(\vec{p})$$

Amplitude is gauge-invariant ! ✓

total cross section

$$\sigma = \frac{(2\pi)^6}{v} \int d^3\vec{k}' d^3\vec{p}' \cdot (2\pi)^4 \delta^4(k'+p'-k-p)$$

relative velocity

$$\times \sum_{\substack{h'=\pm 1 \\ \sigma'=1,2}} \left| \mathcal{M}(\vec{p}', \sigma'; \vec{k}, h' \,|\, \vec{p}, \sigma; \vec{k}, h) \right|^2$$

Evaluation of polarization sum and momentum integral straightfoward but tedious.

[See Weinberg section 8.7]

So far, all fields and coupling/mass parameters are "bare", in that they appear in the bare Lagrangian

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^B - \partial_\nu A_\mu^B)(\partial^\mu A^{B\nu} - \partial^\nu A^{B\mu})$$

$$- \bar{\psi}^B(\gamma^\mu(\partial_\mu + ie^B A_\mu^B) + m^B)\psi^B$$

"renormalized fields"

$$A_\mu \equiv Z_A^{-\frac{1}{2}} A_\mu^B$$

$$\psi_\alpha \equiv Z_\psi^{-\frac{1}{2}} \psi_\alpha^B$$

are such that the 1-particle components of $\hat{F}_{\mu\nu}(x)|\Omega\rangle$ and $\hat{\psi}_\alpha(x)|\Omega\rangle$ are normalized as in free Maxwell and free fermion theories, respectively.

In terms of A_μ and ψ_α, we can write the bare Lagrangian as

$$\mathcal{L} = -\frac{1}{4} Z_A F_{\mu\nu}F^{\mu\nu} - Z_\psi \bar{\psi}(\gamma^\mu\partial_\mu + m^B)\psi$$

$$- i \underbrace{\boxed{e^B Z_A^{\frac{1}{2}}}}_{\parallel\; e} Z_\psi A_\mu \bar{\psi}\gamma^\mu\psi .$$

"renormalized charge"

gauge transf: $A_\mu \to A_\mu + \partial_\mu \zeta$. no room for modification!

$\psi \to \exp(-ie\zeta)\psi$

$$= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \bar{\psi}(\gamma^\mu\partial_\mu + m)\psi$$

"renormalized mass" (may choose to be physical mass)

$$- ie A_\mu \bar{\psi}\gamma^\mu\psi$$

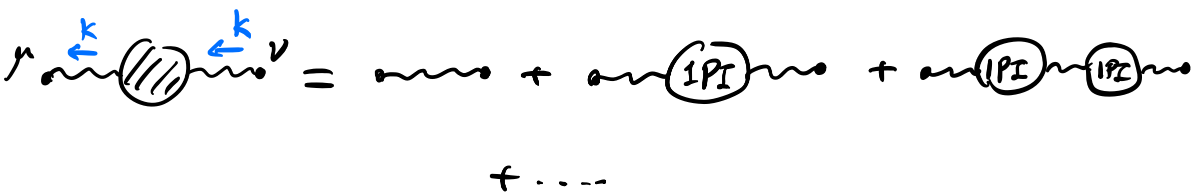
"counter terms"
$$- \frac{1}{4}(Z_A - 1)F^{\mu\nu}F_{\mu\nu} - (Z_\psi - 1)\bar{\psi}(\gamma_\mu\partial^\mu + m)\psi$$

$$- Z_\psi \delta m \bar{\psi}\psi - ie(Z_\psi - 1)A_\mu\bar{\psi}\gamma^\mu\psi .$$

$m_B = m + \delta m$

We begin by studying

$$\langle\Omega| T \hat{A}_\mu(x)\hat{A}_\nu(0)|\Omega\rangle = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \tilde{G}_{\mu\nu}(k)$$



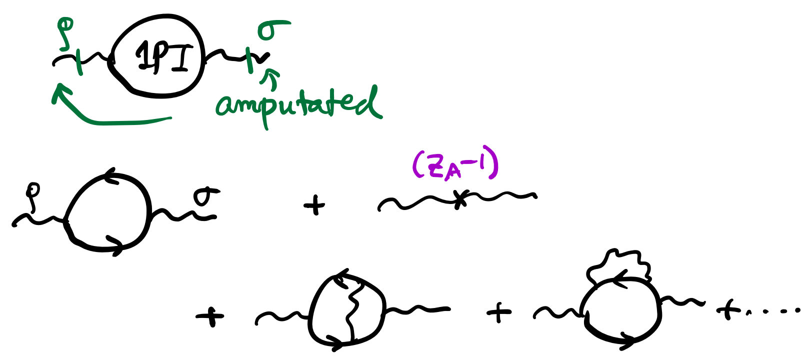
$$= -i\Delta_{\mu\nu}(k) = \frac{-i}{k^2 - i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - i\epsilon}\right)$$

$$\tilde{G}_{\mu\nu}(k) = -i\left[\Delta_{\mu\nu}(k) + \Delta_{\mu\rho}(k)\Pi^{\rho\sigma}(k)\Delta_{\sigma\nu}(k) + \cdots\right]$$

$$= -i\left(\left(\Delta^{-1}(k) - \Pi(k)\right)^{-1}\right)_{\mu\nu}$$

$$\equiv -i\left(\Delta_{\mu\nu}(k) + \Delta_{\mu\rho}(k)\, M^{\rho\sigma}(k)\,\Delta_{\sigma\nu}(k)\right)$$

$i\Pi^{\rho\sigma}(k) =$



Using $\quad 2k_\alpha k^{[\mu}\Delta^{\alpha]\rho} = \dfrac{-i}{k^2 - i\epsilon}\left(k^\mu k^\rho - k^2\eta^{\mu\rho}\right)$

We have

$$\langle\Omega|\, T\, \partial_\alpha \hat{F}^{\mu\alpha}(x)\, \partial_\beta \hat{F}^{\nu\beta}(y)\, |\Omega\rangle$$

contact term

$$(\star) \quad = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \Big[-i\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right)$$

$$- \frac{i}{(k^2 - i\epsilon)^2}\left(k^\mu k^\rho - k^2\eta^{\mu\rho}\right) M_{\rho\sigma}(k)\left(k^\sigma k^\nu - k^2\eta^{\sigma\nu}\right)\Big]$$

On the other hand, E.O.M. for A_μ implies the operator equation

$$\partial_\nu \hat{F}^{\mu\nu} = \hat{j}^\mu$$ E-M current

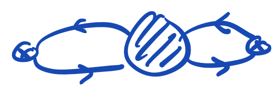
$$= -i Z_A^{-1} e Z_\psi \hat{\bar{\psi}} \gamma^\mu \hat{\psi}$$

$\hat{Q} \equiv \int d^3\vec{x}\, \hat{j}^0(x)$ is the electric charge.

In particular, $\partial_\mu \hat{j}^\mu = \partial_\mu \partial_\nu \hat{F}^{\mu\nu} \equiv 0$

$$(\star) \quad \overset{\text{up to contact term}}{=\!=} \langle \Omega | T\, \hat{j}^\mu(x)\, \hat{j}^\nu(y) | \Omega \rangle$$

$$\parallel$$



$$-i \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} M^{\mu\nu}(k).$$

$\partial_\mu \hat{j}^\mu = 0 \implies \boxed{k^\rho M_{\rho\sigma}(k) = 0}$ in agreement with $(\star)$ up to contact terms.

[also, no contact term in $\langle \partial_\mu j^\mu(x)\, j^\nu(y) \rangle$ as $j^\nu(y)$ is an uncharged operator]

It also follows that $k_\rho \Pi^{\rho\sigma}(k) = 0$,

and by Lorentz invariance, we must have

$$\Pi^{\rho\sigma}(k) = (\eta^{\rho\sigma} k^2 - k^\rho k^\sigma)\, \pi(k^2)$$

for some function $\pi(k^2)$.

using $\Delta_{\mu\rho}\Pi^{\rho\sigma} = \frac{1}{k^2-i\epsilon}\Pi_\mu{}^\sigma$

$$\tilde{G}_{\mu\nu}(k) = -i\Big[\Delta_{\mu\nu}(k) + \left(\frac{1}{k^2-i\epsilon}\right)^2 \Pi_{\mu\nu}(k) + \left(\frac{1}{k^2-i\epsilon}\right)^3 \Pi_{\mu\rho}(k)\Pi^\rho{}_\nu(k) + \cdots\Big]$$

$$= \frac{-i}{k^2-i\epsilon}\left[\frac{\eta_{\mu\nu}}{1-\pi(k^2)} - \frac{k_\mu k_\nu}{k^2-i\epsilon}\left(1-\xi+\frac{\pi(k^2)}{1-\pi(k^2)}\right)\right]$$

Expect: $\pi(k^2)$ regular at $k^2=0$,

$\longleftrightarrow$ photon massless.

Choice of Z_A (recall $A_\mu = Z_A^{-\frac{1}{2}} A_\mu^B$)

is such that $\pi(k^2=0) = 0$.

So that $\hat{F}_{\mu\nu}(x)|\Omega\rangle\Big|_{1\text{-particle}}$ is normalized

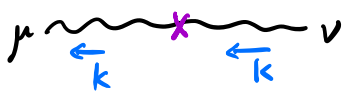
as in free Maxwell theory.

Now let us calculate $\pi(k^2)$ and Z_A

at 1-loop order.

The counter term vertex

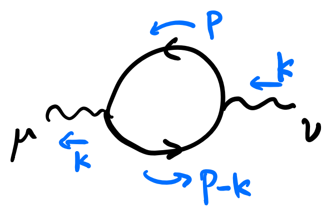
$-\frac{i}{4}(Z_A-1)F_{\rho\sigma}F^{\rho\sigma}$



$$= -i\,(Z_A-1)\,(k^2\eta_{\mu\nu} - k_\mu k_\nu)$$

contributes $1-Z_A$ to $\Pi(k^2)$.

The 1-loop diagram



$$= -\int \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\left[e\gamma_\mu \frac{-i(-i\not{p}+m)}{p^2+m^2-i\epsilon}\, e\gamma_\nu \frac{-i\left(-i(\not{p}-\not{k})+m\right)}{(p-k)^2+m^2-i\epsilon} \right]$$

fermion loop

$\bar\psi\,\psi\,\bar\psi\,\psi$

$$\overset{\text{Wick rotation}}{=\!=\!=} -i\int \frac{d^4p^E}{(2\pi)^4}\, \mathrm{Tr}\left[e\gamma_\mu \frac{-i(-i\not{p}+m)}{p^2+m^2}\, e\gamma_\nu \frac{-i\left(-i(\not{p}-\not{k})+m\right)}{(p-k)^2+m^2} \right]$$

(*)

divergent integral !

Dimensional Regularization :

$$\int \frac{d^4p}{(2\pi)^4} \longrightarrow \int \frac{d^Dp}{(2\pi)^D}. \qquad D = 4-\epsilon$$

maintain $\{\gamma_\mu, \gamma_\nu\} = \eta_{\mu\nu}$, but with $\eta^{\mu\nu}\eta_{\mu\nu} = D$

trace / Dirac spinor index is still $\mathrm{Tr}\, \mathbb{1} = 4$.

By Feynman trick,

$$\text{✪} = ie^2 \int \frac{d^Dp_E}{(2\pi)^D} \int_0^1 dx \frac{\mathrm{Tr}\left[\gamma_\mu(-i\not{p}+m)\gamma_\nu(-i(\not{p}-\not{k})+m)\right]}{(x(p-k)^2+(1-x)p^2+m^2)^2}$$

$$\underset{p\to p+xk}{=\!=} ie^2 \int_0^1 dx \int \frac{d^Dp}{(2\pi)^D} \frac{\mathrm{Tr}\left[\gamma_\mu(-i(\not{p}+x\not{k})+m)\gamma_\nu(-i(\not{p}-(1-x)\not{k})+m)\right]}{(p^2+x(1-x)k^2+m^2)^2}$$

using $\mathrm{Tr}[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta] = 4(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\beta}\eta_{\nu\alpha})$

$$\mathrm{Tr}[\cdots] = 4\Big[\eta_{\mu\nu}\big(+(p+xk)\cdot(p-(1-x)k)+m^2\big) - (p_\mu+xk_\mu)(p_\nu-(1-x)k_\nu) - (p_\nu+xk_\nu)(p_\mu-(1-x)k_\mu)\Big]$$

By SO(D) rotation symmetry, terms proportional to $p_\mu k_\nu$, $p_\nu k_\mu$, or $p\cdot k$ integrate to zero, whereas $p_\mu p_\nu$ can be replaced with $\frac{1}{D}\eta_{\mu\nu}p^2$.

We end up with

$$\circledast = ie^2 \int_0^1 dx \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + x(1-x)k^2 + m^2)^2}$$

$$\times 4 \left[\eta_{\mu\nu}(p^2 - x(1-x)k^2 + m^2) - \frac{2}{D}\eta_{\mu\nu} p^2 + 2x(1-x) k_\mu k_\nu \right]$$

using $\displaystyle\int \frac{d^D p}{(2\pi)^D} \cdots = \frac{\mathrm{vol}(S^{D-1})}{(2\pi)^D} \int d|p| \cdot |p|^{D-1} \cdots$

$$\mathrm{vol}(S^{D-1}) = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})},$$

the integral can be evaluated using

$$\int_0^\infty dp \cdot \frac{p^{D-1}}{(p^2+M^2)^2} = \frac{1}{2}(M^2)^{\frac{D}{2}-2} \Gamma(\tfrac{D}{2}) \Gamma(2-\tfrac{D}{2})$$

and similarly

$$\int_0^\infty dp \cdot \frac{p^{D+1}}{(p^2+M^2)^2} = \frac{1}{2}(M^2)^{\frac{D}{2}-1} \Gamma(\tfrac{D}{2}+1) \Gamma(1-\tfrac{D}{2})$$

defined by analytic continuation from $D<2$.

$$\circledast = 4ie^2 \cdot \frac{2}{(4\pi)^{\frac{D}{2}} \Gamma(\frac{D}{2})} \cdot \frac{1}{2} \int_0^1 dx$$

$$\times \Big[\left(1-\frac{2}{D}\right) \eta_{\mu\nu} \left(x(1-x)k^2+m^2 \right)^{\frac{D}{2}-1} \Gamma\left(\frac{D}{2}+1\right) \Gamma\left(1-\frac{D}{2}\right)$$

$$+ \Big(2x(1-x)\, k_\mu k_\nu + \eta_{\mu\nu}\left(-x(1-x)k^2+m^2\right) \Big)$$

$$\times \left(x(1-x)k^2+m^2 \right)^{\frac{D}{2}-2} \Gamma\left(\frac{D}{2}\right) \Gamma\left(2-\frac{D}{2}\right) \Big]$$

$$= - \frac{4ie^2}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)} \Gamma\left(\frac{D}{2}\right) \Gamma\left(2-\frac{D}{2}\right) \cdot 2\left(\eta_{\mu\nu}k^2 - k_{\mu\nu}\right) \checkmark$$

$$\times \int_0^1 dx \cdot x(1-x) \left(x(1-x)k^2+m^2 \right)^{\frac{D}{2}-2} .$$

Total contribution to $\pi(k^2)$ is

$$\pi(k^2) = 1 - Z_A$$

$$- \frac{8e^2}{(4\pi)^{\frac{D}{2}}} \Gamma\left(2-\frac{D}{2}\right) \int_0^1 dx \cdot x(1-x) \left(x(1-x)k^2+m^2 \right)^{\frac{D}{2}-2}$$

Setting $\pi(0)=0$ fixes Z_A to be

$$Z_A = 1 - \frac{8e^2}{(4\pi)^{\frac{D}{2}}} \Gamma\left(2-\frac{D}{2}\right) \cdot \frac{1}{6} m^{D-4}$$

$$\overset{D=4-\epsilon}{=\!=} 1 + e^2 \left(- \frac{1}{6\pi^2} \cdot \frac{1}{\epsilon} + \text{finite} \right)$$

In the $\epsilon \to 0$ limit, $\pi(k^2)$ is finite:

$$\pi(k^2) = \frac{e^2}{2\pi^2} \int_0^1 dx \cdot x(1-x) \ln \frac{x(1-x)k^2 + m^2}{m^2} .$$

1-electron state $|\vec{p},\sigma\rangle$

Let us investigate the matrix element

$$\langle \vec{p}\,',\sigma'|\,\hat{j}^\mu(x)\,|\vec{p},\sigma\rangle$$

$$= e^{i(p-p')\cdot x}\langle \vec{p}\,',\sigma'|\,\hat{j}^\mu(0)\,|\vec{p},\sigma\rangle .$$

"E-M form factor"

- Lorentz invariance implies

$$\langle \vec{p}\,',\sigma'|\,\hat{j}^\mu(0)\,|\vec{p},\sigma\rangle$$

$$= \langle \vec{p}\,',\sigma'|\,(U(\Lambda))^{-1}\cdot U(\Lambda)\,\hat{j}^\mu(0)(U(\Lambda))^{-1}\cdot U(\Lambda)\,|\vec{p},\sigma\rangle$$

$$= \sqrt{\frac{(\Lambda p')^0}{p'^0}}\sum_{\lambda'} D^*_{\sigma'\lambda'}(W)\cdot \Lambda_\nu{}^\mu \sum_\lambda D_{\sigma\lambda}\,\langle \overrightarrow{\Lambda p'},\lambda'|\,\hat{j}^\nu(0)\,|\Lambda\vec{p},\lambda\rangle$$

Most general solution can be written as

$$\langle \vec{p}\,',\sigma'|\,\hat{j}^\mu(0)\,|\vec{p},\sigma\rangle$$

$$= \frac{iq}{(2\pi)^3}\,\bar{u}^{\sigma'}(\vec{p}\,')\,M^\mu(p',p)\,u^\sigma(\vec{p}) .$$

where $M^\mu(p',p)$ is a Lorentz-covariant combination of p'^ν, p^ν, and γ^ν's.

Using $(\not{p} - im)\, u^{\sigma}(\vec{p}) = 0$

and $\bar{u}^{\sigma'}(\vec{p}')\,(\not{p}' - im) = 0$,

the only independent Lorentz - covariant expressions are :

$$M^{\mu} = p^{\mu},\; p'^{\mu},\; \gamma^{\mu}$$

e.g. $M^{\mu} = \gamma_5 \gamma^{\mu}$ is not compatible with parity, while $M^{\mu} = \gamma_5 \gamma_{\nu} \epsilon^{\mu\nu\rho\sigma} p_{\rho} p'_{\sigma}$ which is compatible with parity does not give a linearly independent result.

- Conservation law $\partial_{\mu} \hat{j}^{\mu}(x) = 0$ implies

$$(p - p')_{\mu} \langle \vec{p}', \sigma' | \hat{j}^{\mu}(0) | \vec{p}, \sigma \rangle = 0$$

$$\Rightarrow \quad (p - p')_{\mu}\, \bar{u}^{\sigma'}(\vec{p}')\, M^{\mu}(p', p)\, u^{\sigma}(\vec{p}) = 0$$

$$\Rightarrow \quad M^{\mu}(p', p) = \gamma^{\mu} F(k^2) - \frac{i}{2m} (p + p')^{\mu} G(k^2)$$

where $k^2 \equiv (p - p')^2$,

for some real-valued functions $F(k^2)$, $G(k^2)$.

Note: in free fermion theory,

we would have $F(k^2) = 1$, $G(k^2) = 0$.

Generally, take $p' \to p$ limit

$$\langle \vec{p}, \sigma' | \hat{j}^\mu(x) | \vec{p}, \sigma \rangle$$

$$= \frac{iq}{(2\pi)^3} \bar{u}^{\sigma'}(\vec{p}) \left(\gamma^\mu F(0) - \frac{i}{m} p^\mu G(0) \right) u^\sigma(\vec{p})$$

using $$\bar{u}^{\sigma'}(\vec{p}) \{ \gamma^\mu, \not{p} \} u^\sigma(\vec{p}) = 2p^\mu \bar{u}^{\sigma'}(\vec{p}) u^\sigma(\vec{p})$$

$$\parallel$$

$$2im \, \bar{u}^{\sigma'}(\vec{p}) \gamma^\mu u^\sigma(\vec{p}),$$

and $$\bar{u}^{\sigma'}(\vec{p}) u^\sigma(\vec{p}) = \frac{m}{p^0} \delta_{\sigma\sigma'},$$

we have

$$\langle \vec{p}, \sigma' | \hat{j}^\mu(x) | \vec{p}, \sigma \rangle = \frac{q}{(2\pi)^3} \frac{p^\mu}{p^0} \delta_{\sigma\sigma'} \left(F(0) + G(0) \right),$$

Compare to $$\hat{Q} = \int d^3\vec{x} \, \hat{j}^0(x)$$

$$\hat{Q} | \vec{p}, \sigma \rangle = q | \vec{p}, \sigma \rangle$$

$$\Rightarrow \langle \vec{p},\sigma' | \hat{Q} | \vec{p},\sigma \rangle = q\, \delta_{\sigma\sigma'}\, \delta^3(\vec{0})$$

$$= \frac{q}{(2\pi)^3}\, \delta_{\sigma\sigma'} \int d^3\vec{x} \cdot 1$$

$$\Rightarrow \quad F(0) + G(0) = 1 .$$

- The interpretation of $F(0)$:

if we turn on a weak, slow-varying background EM field $A_\mu^{cl}(x)$, the Hamiltonian receives the contribution

$$\Delta H = - \int d^3\vec{x}\; A_\mu^{cl}(x)\, \hat{j}^\mu(x)$$

The energy of an electron propagating in such a bkgrd EM field is given by $p' \to p$ limit of the matrix element

$$\langle \vec{p}',\sigma' | \Delta H | \vec{p},\sigma \rangle$$

$$= - \int d^3\vec{x}\; A_\mu^{cl}(x)\, \langle \vec{p}',\sigma' | \hat{j}^\mu(x) | \vec{p},\sigma \rangle$$

$$= -\int d^3\vec{x}\, A^{cl}_\mu(x)\, e^{i(p-p')\cdot x}$$

$$\times \frac{iq}{(2\pi)^3} \boxed{\bar{u}^{\sigma'}(\vec{p}')\left[\gamma^\mu F(k^2) - \frac{i}{2m}(p+p')^\mu G(k^2)\right] u^\sigma(\vec{p})}$$

$(k^2 \equiv (p-p')^2)$

rewrite, using

$$\bar{u}^{\sigma'}(\vec{p}')[\gamma^\mu,\gamma^\nu](p'-p)_\nu u^\sigma(\vec{p})$$

$$= \bar{u}^{\sigma'}(\vec{p}')[\gamma^\mu, \not{p}'-\not{p}]\, u^\sigma(\vec{p})$$

$$= \bar{u}^{\sigma'}(\vec{p}')\big(-2\not{p}'\gamma^\mu + \{\gamma^\mu,\not{p}'\} + \{\gamma^\mu,\not{p}\} - 2\gamma^\mu\not{p}\big)\, u^\sigma(\vec{p})$$

$$= -4im\, \bar{u}^{\sigma'}(\vec{p}')\gamma^\mu u^\sigma(\vec{p}) + 2(p^\mu + p'^\mu)\, \bar{u}^{\sigma'}(\vec{p}')u^\sigma(\vec{p}),$$

and $-\frac{i}{4}[\gamma^\mu,\gamma^\nu] = S^{\mu\nu}$,

$$\langle \vec{p}',\sigma'|\Delta H|\vec{p},\sigma\rangle = -\int d^3\vec{x}\, A^{cl}_\mu(x)\, e^{i(p-p')\cdot x}$$

(*) $$\times \frac{iq}{(2\pi)^3}\left[-\frac{i}{2m}(p+p')^\mu\big(F(k^2)+G(k^2)\big)\,\bar{u}^{\sigma'}(\vec{p}')u^\sigma(\vec{p}) - \frac{1}{m}(p'-p)_\nu F(k^2)\,\bar{u}^{\sigma'}(\vec{p}')S^{\mu\nu}u^\sigma(\vec{p})\right]$$

In the case of a bkgrd magnetic field, we can set $A_0^{cl}=0$, $\vec{A}^{cl}$ to be such that

$$\nabla \times \vec{A}^{cl} = \vec{B}.$$

The first term in ⊛ represents the effect of Lorentz force; it is spin-independent in the non-relativistic limit ($|\vec{p}|, |\vec{p}'| \ll m$).

The second term in ⊛ represents spin-magnetic field coupling:

$$\langle \vec{p}', \sigma' | \Delta H^{spin} | \vec{p}, \sigma \rangle$$

$$= \frac{i q}{(2\pi)^3 m} \int d^3\vec{x}\, A_i^{cl}(x)\, e^{i(p-p')\cdot x} (p'-p)_\nu$$

$$\times F(k^2)\, \bar{u}^{\sigma'}(\vec{p}')\, S^{i\nu} u^{\sigma}(\vec{p})$$

$$\xrightarrow{|\vec{p}|, |\vec{p}'| \ll m} \frac{i q}{(2\pi)^3 m} \int d^3\vec{x}\, \underbrace{A_i^{cl}(x)\, e^{i(p-p')\cdot x} (p'-p)_j}_{\int \text{by part} \;\to\; i F_{ij}^{cl}(x)\, e^{i(p-p')\cdot x}}$$

$$\times F(0)\, \underbrace{\bar{u}^{\sigma'}(\vec{0})\, S^{ij} u^{\sigma}(\vec{0})}_{\epsilon^{ijk}\left(\frac{\sigma^k}{2}\right)_{\sigma'\sigma}}$$

$$= -\frac{q\,F(0)}{(2\pi)^3 m}\int d^3\vec{x}\; e^{i(p-p')\cdot x}\,\vec{B}\cdot\left(\frac{\vec{\sigma}}{2}\right)_{\sigma'\sigma}$$

i.e. acting on electron state at small spatial momentum,

$$\Delta H^{spin} = -\underbrace{\boxed{\frac{q\,F(0)}{2m}}}_{\mu}\,\vec{B}\cdot\vec{\sigma}$$

magnetic moment

$$= -g\cdot\frac{q}{2m}\,\vec{B}\cdot\frac{\vec{\sigma}}{2}.$$

$$g = 2F(0) \qquad \text{"}g\text{-factor"}$$

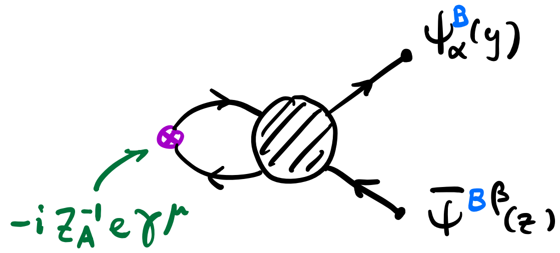
Perturbative Computation of EM form factor

$$j^\mu = -i\,Z_A^{-1}e\,\underbrace{Z_\psi\,\bar{\Psi}\gamma^\mu\Psi}_{=\,\bar{\Psi}^B\gamma^\mu\Psi^B}$$

A Green function such as

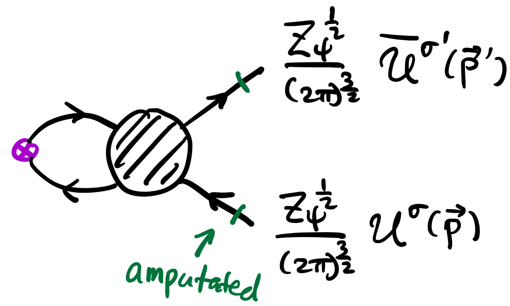
$$\langle\Omega|\,T\,\Psi^B_\alpha(y)\,j^\mu(x)\,\bar{\Psi}^{B\beta}(z)\,|\Omega\rangle$$

is computed in perturbation theory via



After taking LSZ limit on the $\psi^B_\alpha(y)$ and $\bar{\psi}^{B\beta}(z)$, the form factor is computed by

$$\langle \vec{p}', \sigma' | j^\mu(0) | \vec{p}, \sigma \rangle =$$



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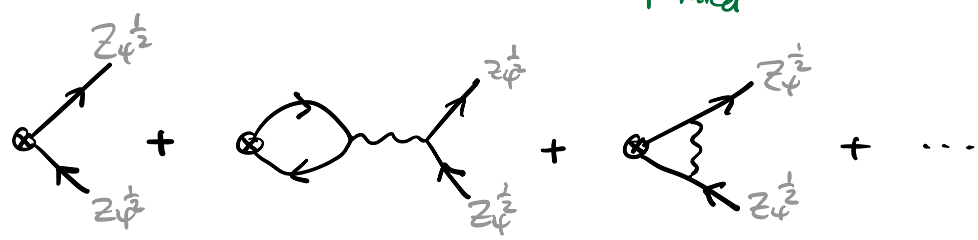
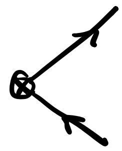


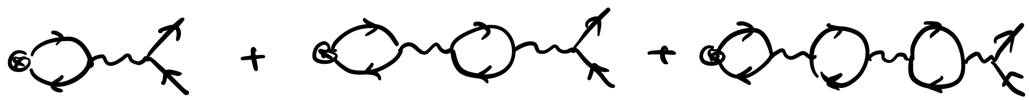
(diagrams here are constructed via bare vertices and bare field propagators)

Note: if we work with ψ instead of ψ^B, the $Z_\psi^{\frac{1}{2}}$ factors on external lines would be absent, but we would need to include additional vertices corresponding to counter terms, and the final result for the form factor would be identical.



$$= -i Z_A^{-1} e \cdot \frac{Z_\psi}{(2\pi)^3} \bar{u}^{\sigma'}(\vec{p}') \gamma^\mu u^\sigma(\vec{p})$$

combines with



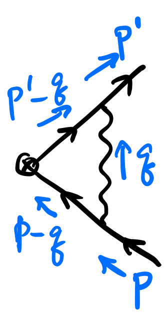
to give

$$-i \cancel{Z_A^{-1}} e \cdot \frac{Z_\psi}{(2\pi)^3} \frac{\cancel{Z_A}}{1-\pi(k^2)} \bar{u}^{\sigma'}(\vec{p}') \gamma^\mu u^\sigma(\vec{p})$$

It contributes $\frac{Z_\psi}{1-\pi(k^2)}$ to $F(k)$ and does not contribute to $G(k)$.

At order e^2, the only contribution to $G(k)$ comes from the diagram

Feynman gauge



$$= -i Z_A^{-1} e \frac{Z_\psi}{(2\pi)^3} \int \frac{d^4 q}{(2\pi)^4} \frac{-i}{q^2 - i\epsilon}$$

$$\times \bar{u}^{\sigma'}(\vec{p}') e \gamma^\rho \frac{-i(-i(\not{p}' - \not{q}) + m)}{(p'-q)^2 + m^2 - i\epsilon} \gamma^\mu \frac{-i(-i(\not{p} - \not{q}) + m)}{(p-q)^2 + m^2 - i\epsilon} e \gamma_\rho u^\sigma(\vec{p})$$

To proceed, we can

1. Wick rotate $q^0 \to i q^4$
2. dimensional regularization
3. use Feynman trick to combine the denominators in the q-integrand

After taking into account the Dirac eqn

$$(i\not{p} + m)\, u^\sigma(\vec{p}) = 0 = \bar{u}^{\sigma'}(\vec{p}\,')(i\not{p}' + m),$$

one can simplify the q-integral into

$$\bar{u}^{\sigma'}(\vec{p}\,')\,\gamma^\mu\, u^\sigma(\vec{p}) \cdot A(k^2)$$

$$+ (p+p')^\mu\, \bar{u}^{\sigma'}(\vec{p}\,')\, u^\sigma(\vec{p}) \cdot B(k^2)$$

$A(k^2)$ contributes to $F(k^2)$.

contains both UV and IR divergences, cancel against those coming from Z_4

$B(k^2)$ contributes to $G(k^2)$.

turns out to be finite on its own.

Result (See Weinberg section 11.3)

$$G(k^2) = -\frac{e^2 m^2}{4\pi^2} \int_0^1 dx \int_0^x dy \, \frac{x(1-x)}{m^2 x^2 + k^2 y(x-y)}$$

$$G(0) = -\frac{e^2}{8\pi^2} = -\frac{\alpha}{2\pi}.$$

$\Rightarrow$ electron g-factor

$$\begin{aligned} g &= 2F(0) \\ &= 2(1 - G(0)) \\ &= 2 + \frac{\alpha}{\pi} \quad + \mathcal{O}(\alpha^2) \end{aligned}$$