Classical $\longrightarrow$ Quantum

phase space

(q_i, p_i)

Hilbert Space $\mathcal{H}$

state vector $|\psi\rangle$

$\updownarrow$

wave function $\psi(q_i)$

Inner product

$$\langle \psi_1, \psi_2 \rangle = \int d^N q \; \psi_1^*(q)\, \psi_2(q)$$

functions on phase space (observables)

$f(q_i, p_i)$ $\rightsquigarrow$

linear operators on $\mathcal{H}$

$A \quad (\sim f(\hat{q}_i, \hat{p}_i))$

Poisson bracket

$\{f, g\}_P$ $\rightsquigarrow$

commutator

$[\hat{f}, \hat{g}]$

e.g. $[\hat{q}_i, \hat{p}_j] = i\hbar\, \delta_{ij}$

Action

$$S = \int dt \; L(q_i, \dot{q}_i)$$ $\rightsquigarrow$

path integral

$$Z = \int [D q_i(t)]\, e^{\frac{i}{\hbar} S}$$

$$\langle q^{out} |\, e^{iHT} \,| q^{in} \rangle = \int [D q_i(t)] \Big|_{q_i(0) = q_i^{in}}^{q_i(T) = q_i^{out}} e^{\frac{i}{\hbar} \int_0^T dt\, L}$$

- What is the measure $[Dq_i(t)]$?

- useful to work with Hamiltonian version of path integral

$$\int Dq_i(t)\, Dp_i(t)\; e^{\frac{i}{\hbar}\int dt\left[p_i \dot{q}_i - H(p_i, q_i)\right]}$$

think of as continuum "limit" of $\prod_t \underbrace{\prod_i \frac{dq_i(t)\, dp_i(t)}{2\pi\hbar}}$

1 per $(2\pi\hbar)^N$ volume of phase space

unambiguous normalization

To convert to Lagrangian form of path integral, need to assume 2-deriv. kinetic term, i.e. $H(p, q)$ quadratic in p_i

$\longrightarrow$ perform $p_i(t)$ integration as Gaussian.

- Caution: kinetic term can have coeff. that depend on q_i, e.g.

$$L = \frac{1}{2} g_{ij}(q)\, \dot{q}_i \dot{q}_j - V(q)$$

$$H = \frac{1}{2} (g^{-1})^{ij}(q)\, p_i p_j + V(q).$$

Integrating out $Dp_i(t)$ in the H-form of the path integral results in the measure

$$[Dq_i(t)] \propto \prod_t \underbrace{d^N q_i(t) \sqrt{\det g_{ij}(q(t))}}$$

volume form
on the config. space
as a Riemannian manifold
with metric $g_{ij}(q)$.
In particular, invt under redef
of var. $q_i \to \tilde{q}_i$.

- Wick rotation $t = -i\tau$

$$L(q_i, \dot{q}_i) \equiv -L^E\left(q_i, i\frac{\partial q_i}{\partial \tau}\right)$$

$$Z = \int Dq_i(\tau)\, e^{-\frac{1}{\hbar} S^E}, \quad S^E = \int d\tau\, L^E$$

$$\langle q^{out} | e^{-HT} | q^{in} \rangle = \int Dq_i(\tau) \Big|_{q_i(0) = q_i^{in}}^{q_i(T) = q_i^{out}} e^{-\frac{1}{\hbar}\int_0^T d\tau L^E}$$

better defined (L^E typically bounded from below)

- Periodic Euclidean time

$$\tau \sim \tau + \beta$$

$$\mathrm{Tr}\, e^{-\beta H} = \int d^N q \, \langle q | e^{-HT} | q \rangle$$

$$= \int \mathcal{D} q_i(\tau) \Big|_{q_i(\beta) = q_i(0)} e^{-\frac{1}{\hbar} S^E}$$

thermal partition fn

$Z(\beta)$ temperature $T = 1/\beta$.

- can expand $q_i(\tau)$ in Fourier modes

$$q_i(\tau) = \sum_{k \in \mathbb{Z}} q_{i,k} \, e^{i \frac{2\pi k \tau}{\beta}}$$

$$(q_{i,-k} = q_{i,k}^*)$$

and write the Euclidean path integral as the limit

$$\lim_{\Lambda \to \infty} \mathcal{N}_\Lambda \int \Big[\prod_{|k| < \Lambda} d q_{i,k} \Big] e^{-\frac{1}{\hbar} S^E}$$

provided that the very high frequency modes are "sufficiently weakly coupled".

Example: particle on a circle

$$L = \frac{1}{2}\dot{x}^2, \qquad x \sim x + 2\pi R$$

$$L^E = \frac{1}{2}(\partial_\tau x)^2,$$

$$\tau \sim \tau + \beta, \qquad x(\tau) = \sum_k x_k \, e^{2\pi i \frac{k\tau}{\beta}} + 2\pi R w \frac{\tau}{\beta}.$$

$$x_{-k} = x_k^*, \quad x_k \in \mathbb{C}, \ k \neq 0.$$

$$x_0 \in [0, 2\pi R].$$

$$w \in \mathbb{Z}$$

$$[Dx(\tau)] \underset{?}{\rightsquigarrow} \sum_w \prod_k dx_k$$

Caution: possible β-dependent normalization! To obtain the correct normalization, go back to H-form path integral (set $\hbar = 1$)

$$\int^{\beta}$$

$$Z(\beta) = \int Dx(\tau)\, Dp(\tau)\; e^{\int_0^1 d\tau\,(i p \partial_\tau x - H)}$$

$$= \sum_{w\in\mathbb{Z}} \int \prod_k \frac{dx_k\, dp_k}{2\pi}$$

$$\times \exp\Big[\sum_{k\neq 0} \Big(-2\pi k\, p_{-k} x_k - \frac{\beta}{2} p_{-k} p_k\Big)$$

$$+ 2\pi i R w\, p_0 - \frac{\beta}{2} p_0^2 \Big]$$

$$= \sum_{w\in\mathbb{Z}} \int_0^{2\pi R} \frac{dx_0}{\sqrt{2\pi\beta}} \int \prod_{k\neq 0} \frac{dx_k}{\sqrt{2\pi\beta}}$$

$$\exp\Big[\underbrace{-\sum_{k\neq 0} \frac{(2\pi k)^2}{2\beta} x_{-k} x_k - \frac{(2\pi R w)^2}{2\beta}}_{-S^E} \Big]$$

$$[Dx(\tau)] = \sum_{w\in\mathbb{Z}} \prod_k \frac{dx_k}{\sqrt{2\pi\beta}} .$$

$$= \mathcal{N} \cdot \frac{2\pi R}{\sqrt{2\pi\beta}} \sum_{w\in\mathbb{Z}} e^{-\frac{(2\pi R w)^2}{2\beta}}$$

β-ind. const

Poisson resummation:

$$\sum_{w\in\mathbb{Z}} \to \int dw \sum_{\ell\in\mathbb{Z}} \delta(w-\ell)$$

$$= \int dw \sum_{n\in\mathbb{Z}} e^{2\pi i n w}$$

$$\sum_{w\in\mathbb{Z}} e^{-\frac{(2\pi R w)^2}{2\beta}} = \sum_{n\in\mathbb{Z}} \int dw\, e^{2\pi i n w - \frac{(2\pi R w)^2}{2\beta}}$$

$$= \sum_{n\in\mathbb{Z}} \frac{\sqrt{2\pi\beta}}{2\pi R}\, e^{-\frac{\beta}{2}\left(\frac{n}{R}\right)^2}$$

$$\Rightarrow \quad Z(\beta) = \underbrace{\mathcal{N}}_{=1} \cdot \sum_{n\in\mathbb{Z}} e^{-\frac{\beta}{2}\left(\frac{n}{R}\right)^2}$$

$E_n = \frac{1}{2}\left(\frac{n}{R}\right)^2$ ✓

A note on regularization of path integral measure.

In class, we rewrote $[Dx(t)\, Dp(t)]$ as $\prod_k \frac{dx_k\, dp_k}{2\pi\hbar}$, where x_k, p_k are Fourier coeff. of $x(t)$, $p(t)$.

To regularize the product over $k \in \mathbb{Z}$, we can impose a cut off $|k| \leq k_{max}$

Q: how is this related to discretization of time τ?

$\tau \sim \tau + \beta$, discretize into

$$\tau_n = \frac{n\beta}{M}, \qquad n = 0, 1, \cdots, M-1$$

$M \gg 1$.

$$X(\tau_n) = \sum_{|k| \leq k_{max}} x_k\, e^{2\pi i k \tau_n / \beta}$$

(ignore winding modes in this discussion)

and similarly for $p(\tau_n)$

To be sure that the change of var.

$$\{x(\tau_n)\}_{n=0,\dots,M-1} \longrightarrow \{x_k\}_{|k|\leq k_{max}}$$

is non-degenerate, let's take

$$M = 2k_{max} + 1.$$

There is a Jacobian factor

$$\det\left(\frac{\partial x(\tau_n)}{\partial x_k}\right) = \det\left(e^{2\pi i k\tau_n/\beta}\right)$$

$$\underset{\text{Vandermonde}}{=} \prod_{0\leq n<m\leq M-1} \left(e^{2\pi i \tau_n/\beta} - e^{2\pi i \tau_m/\beta}\right)$$

$$= \Bigg[\underbrace{\prod_{n=1}^{M-1}\left(2\sin\frac{2\pi n}{M}\right)}_{\substack{\parallel \text{ for odd } M \\ (-)^{\frac{M-1}{2}}\cdot M}}\Bigg]^{\frac{M}{2}}$$

Thus,

$$\prod_{n=0}^{M-1} dx(\tau_n) = \prod_{|k|\leq k_{max}} dx_k \cdot M^{\frac{M}{2}}$$

In class, when we evaluated the Gaussian integral over x_k and p_k, we dropped a factor $\prod_{k=1}^{k_{max}} \frac{1}{(2\pi k)^2}$

In total, the correction factor is

$$M^M \cdot \prod_{k=1}^{\frac{M-1}{2}} \frac{1}{(2\pi k)^2} = \frac{M^M}{(2\pi)^{M-1}\left(\Gamma\left(\frac{M+1}{2}\right)\right)^2}.$$

$$\xrightarrow{\text{Stirling approx}} \left(\frac{e}{\pi}\right)^M \left(1+\mathcal{O}\left(\tfrac{1}{M}\right)\right).$$

$M = \beta/\delta \qquad \delta = \text{time step}$

cancel by adding a constant counter term

$\Delta L^E = \frac{1}{\delta}(1-\log\pi)$ in the Lagrangian

Conclusion: this is the counter term required in the frequency-cutoff reg. scheme

In class we evaded it by forgetting the Jacobian factor above as well as dropping $\prod_k \frac{1}{(2\pi k)^2}$.

- Generalization to Grassmann (anti-commuting) variables

$\eta_\alpha(t)$ anti-commuting $\quad \alpha = 1, \cdots, N$

const matrix

Lagrangian $L = M_{\alpha\beta} \eta_\alpha \dot{\eta}_\beta$

Action $S = \int dt \, M_{\alpha\beta} \eta_\alpha \dot{\eta}_\beta$

anti-sym part of $M_{\alpha\beta}$ gives total deriv.

assume $M_{\alpha\beta} = M_{\beta\alpha}$. non-degenerate

$\delta S = 0 \iff$ EOM $\quad M_{\alpha\beta} \dot{\eta}_\beta = 0$

- What does it mean?
  - nothing in classical physics
  - quantization?

view η_α as generalized coord.

canonical momentum

$$\pi_\alpha = L \frac{\overleftarrow{\partial}}{\partial \dot{\eta}_\alpha} = M_{\alpha\beta}\, \eta_\beta$$

act from right

Attempt: $\{\eta_\alpha, \eta_\beta\} = 0, \quad \{\eta_\alpha, \pi_\beta\} = i\hbar\, \delta_{\alpha\beta}$

but $\pi_\alpha = M_{\alpha\beta}\eta_\beta$. contradiction ?!

Begin by treating η_α and π_α as independent Grassmann coords on "phase space"

View

$$\chi_\alpha(\eta, \pi) \equiv \pi_\alpha - M_{\alpha\beta}\, \eta_\beta \overset{?}{=} 0$$

as a constraint that we would like to impose.

Dirac: second-class primary constraint

meaning non-degenerate Poisson bracket among themselves,

in contrast to 1^{st}-class which has to do with gauge redundancy (not encountered here)

in contrast to "secondary" constraint which are ones that follow from EOM.

Check Poisson (anti-) bracket

$$K_{\alpha\beta} \equiv \{\chi_\alpha, \chi_\beta\}^P$$

$$= \frac{\partial \chi_\alpha}{\partial \pi_\gamma}\frac{\partial \chi_\beta}{\partial \eta_\gamma} + \frac{\partial \chi_\alpha}{\partial \eta_\gamma}\frac{\partial \chi_\beta}{\partial \pi_\gamma}$$

$$= -2 M_{\alpha\beta} .$$

(classical) Dirac bracket

$$\{f, g\}^D := \{f, g\}^P - \{f, \chi_\alpha\}^P (K^{-1})^{\alpha\beta} \{\chi_\beta, g\}^P$$

replacing Poisson bracket by Dirac bracket ensures consistency with contraint, e.g.

$$\{f, \chi_\alpha\}^D = 0 , \quad \forall f .$$

Now,

$$\{\eta_\alpha, \eta_\beta\}^D = \underbrace{\{\eta_\alpha, \eta_\beta\}^P}_{=0} - \{\eta_\alpha, \chi_\gamma\}^P (K^{-1})^{\gamma\delta} \{\chi_\delta, \eta_\beta\}^P$$

$$= -(K^{-1})_{\alpha\beta} = \frac{1}{2}(M^{-1})_{\alpha\beta}$$

Similarly,

$$\{\eta_\alpha, \pi_\beta\}^D = \{\eta_\alpha, \pi_\beta\}^P - \{\eta_\alpha, \chi\}^P K^{-1} \{\chi, \pi_\beta\}^P$$

$$= \delta_{\alpha\beta} - (K^{-1})_{\alpha\gamma}(-M_{\gamma\beta})$$

$$= \frac{1}{2}\delta_{\alpha\beta}.$$

(half the naive expectation)

Quantization :

There is a Hilbert space $\mathcal{H}$

on which $\eta_\alpha \rightsquigarrow \hat{\eta}_\alpha$ acts as linear op.

Such that

$$\{\hat{\eta}_\alpha, \hat{\eta}_\beta\} = i\,\frac{1}{2}(M^{-1})_{\alpha\beta}.$$

anti-Commutator

can change basis by writing

$$M^{-1} = A\,A^T$$

$$\hat{\eta}_\alpha \equiv \sqrt{\tfrac{i}{2}}\, A_{\alpha\beta}\,\theta_\beta.$$

$$\Rightarrow \{\theta_\alpha, \theta_\beta\} = \delta_{\alpha\beta}$$ Clifford algebra

Example: $N=2$.

$$L = \frac{i}{2}\sum_{\alpha=1}^{2} \eta_\alpha \dot{\eta}_\alpha - i\omega \eta_1 \eta_2$$

$$H = i\omega \eta_1 \eta_2 .$$

quantization: $\eta_\alpha \longrightarrow \hat{\eta}_\alpha$

$$\hat{\eta}_1^2 = \hat{\eta}_2^2 = \frac{1}{2}, \quad \{\hat{\eta}_1, \hat{\eta}_2\} = 0$$

minimal possible Hilbert space

$\mathcal{H}$ is 2-dimensional.

on which $\hat{\eta}_1, \hat{\eta}_2$ act as 2×2 matrices

$$\hat{\eta}_1 = \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_1}, \quad \hat{\eta}_2 = \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_2}$$

$$\hat{H} = i\omega \hat{\eta}_1 \hat{\eta}_2 = -\frac{\omega}{2}\sigma_3 .$$

$$Z(\beta) \equiv \mathrm{Tr}\, e^{-\beta \hat{H}} = 2\cosh\frac{\omega}{2} .$$

· Path integral formulation ?

$t \to -i\tau$.

$$L^E = -L = \frac{1}{2}\eta_\alpha \partial_\tau \eta_\alpha + i\omega \eta_1 \eta_2 .$$

$$H = i\omega \eta_1 \eta_2 ,$$

The constraint $\pi_\alpha = \frac{i}{2}\eta_\alpha$ means that the H-form of the path integral only involves (Grassmann) integration in $\eta_\alpha(\tau)$:

$$\tilde{Z}(\beta) = \int D\eta_\alpha(\tau)\; e^{\int_0^\beta d\tau \left(\underbrace{i\pi_\alpha \partial_\tau \eta_\alpha - H}_{= -L^E} \right)}$$

Expand $\eta_\alpha(\tau) = \sum_k \eta_{\alpha,k}\, e^{2\pi i k \tau/\beta}$

$$\tilde{Z}(\beta) = \int \prod_{k,\alpha} d\eta_{\alpha,k}$$

$$\times \exp\left[\sum_k \left(-\pi i k\, \eta_{\alpha,-k}\eta_{\alpha,k} - i\beta\omega \eta_{1,-k}\eta_{2,k} \right) \right]$$

For given $k>0$, $\int$ over $\eta_{\alpha,\pm k}$ gives

$$\int \prod_{\alpha=1}^{2} d\eta_{\alpha,k}\, d\eta_{\alpha,-k}$$

$$\times \exp\Big[\underbrace{-2\pi i k\, \eta_{\alpha,-k}\eta_{\alpha,k} - i\beta\omega\, \eta_{1,-k}\eta_{2,k} + i\beta\omega\, \eta_{2,-k}\eta_{1,k}}_{=\ \eta_{\alpha,-k} A_{\alpha\beta}\eta_{\beta,k}}\Big]$$

$$(A_{\alpha\beta}) = \begin{pmatrix} -2\pi i k & -i\beta\omega \\ i\beta\omega & -2\pi i k \end{pmatrix}$$

$$\det A = -(2\pi k)^2 - (\beta\omega)^2$$

$$\Rightarrow \tilde{Z}(\beta) \propto \prod_{k} \sqrt{(2\pi k)^2 + (\beta\omega)^2}$$

If we assume periodicity

$$\eta_\alpha(\beta) = \eta_\alpha(0),$$

i.e., $k \in \mathbb{Z}$

$$\tilde{Z}(\beta) \propto \beta\omega \prod_{k=1}^{\infty}\left[1 + \left(\frac{\beta\omega}{2\pi k}\right)^2\right]$$

$$= 2\sinh\frac{\beta\omega}{2}$$

$$= e^{\frac{\beta\omega}{2}} - e^{-\frac{\beta\omega}{2}}$$

NOT the partition function

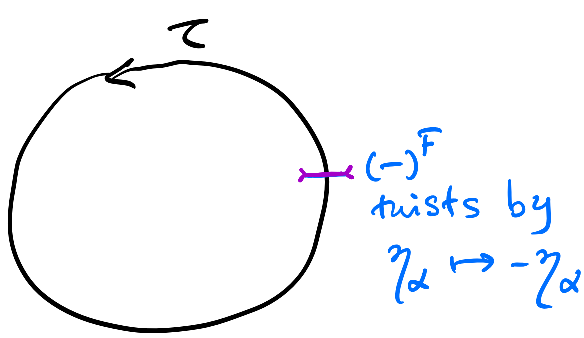
$$Z(\beta) = e^{\frac{\beta\omega}{2}} + e^{-\frac{\beta\omega}{2}}.$$

Rather, $\tilde{Z}(\beta) = \mathrm{Tr}\,(-)^F e^{-\beta H}$

$$(-)^F = \begin{cases} +1, & |0\rangle \\ -1, & |1\rangle \end{cases} \quad \eta_\alpha$$

"fermion number" operator.

To get $Z(\beta)$, need extra twist by $(-)^F$ before taking trace



⤳ anti-periodicify

$$\eta_\alpha(\beta) = -\eta_\alpha(0).$$

Indeed:

$$Z(\beta) = \int D\eta_\alpha \Big|_{\substack{\eta_\alpha(\beta) \\ = -\eta_\alpha(0)}} e^{-S^E}$$

$$= \underbrace{\mathcal{N}}_{=2} \cdot \underbrace{\prod_{k \in \mathbb{Z}+\frac{1}{2}} \sqrt{1 + \left(\frac{\beta\omega}{2\pi k}\right)^2}}_{= \cosh\frac{\beta\omega}{2}} \quad \checkmark$$

Conclusion: canonical quantization of Grassmann variables using Dirac bracket agrees with Grassmann path integral, provided that one assigns anti-periodicity for Grassmann var. around thermal circle.

A note on conventions in the Grassmann path integral.

Begin with Grassmann coord $\eta, \bar{\eta}$

and the Lagrangian $\boxed{L = i\bar{\eta}\dot{\eta}}$.

$$\pi_\eta = L \frac{\overleftarrow{\partial}}{\partial \dot{\eta}} = i\bar{\eta},$$

$$\pi_{\bar{\eta}} = 0.$$

After taking into account these constraints between $\pi_\eta, \pi_{\bar{\eta}}$ and $\eta, \bar{\eta}$,

the Dirac (anti-) bracket is

$$\{\eta, \bar{\eta}\}_D = -i. \qquad \{\eta,\eta\}_D = 0 = \{\bar{\eta},\bar{\eta}\}_D$$

Quantization:

$$\eta, \bar{\eta} \rightsquigarrow \hat{\eta}, \hat{\bar{\eta}}.$$

$$\hat{\eta}^2 = \hat{\bar{\eta}}^2 = 0, \quad \{\hat{\eta}, \hat{\bar{\eta}}\} = i\hbar(-i) = \hbar$$

will set $\hbar = 1$ from now.

Hamiltonian $\boxed{H = 0}$.

Hilbert space $\mathcal{H}$ spanned by 2 states

$$\begin{tikzcd} 0 & {|0\rangle} \arrow[l, "\hat{\bar{\eta}}"'] \arrow[r, bend left, "\hat{\eta}"] & {|1\rangle} \arrow[l, bend left, "\hat{\bar{\eta}}"] \arrow[r, "\hat{\eta}"] & 0 \end{tikzcd}$$

represented as 2×2 matrices

$$\hat{\eta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{\bar{\eta}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

* wave function $\phi(\eta)$

$$\hat{\eta}\cdot\phi(\eta) = \eta\phi(\eta)$$
$$\hat{\bar{\eta}}\cdot\phi(\eta) = \frac{\partial}{\partial\eta}\phi(\eta).$$

$$|0\rangle \leftrightarrow \phi_0(\eta) = 1$$
$$|1\rangle \leftrightarrow \phi_1(\eta) = \eta.$$

inner product

$$\langle\psi|\phi\rangle = \int d\eta\, d\eta^*\, e^{\eta^*\eta}\,(\psi(\eta))^*\,\phi(\eta).$$

check: $$\langle 0|0\rangle = \int d\eta\, d\eta^*\, e^{\eta^*\eta}\; 1 = 1.$$
$$\langle 1|1\rangle = \int d\eta\, d\eta^*\, e^{\eta^*\eta}\; \eta^*\eta = 1.$$
$$\langle 0|1\rangle = \int d\eta\, d\eta^*\, e^{\eta^*\eta}\; \eta = 0.$$
✓

So far while the GM model is completely

defined (two states, $\hat{H}=0$),
the Lagrangian has no meaning until
we define the path integral."

To proceed, we introduce:

* $\hat{\eta}$-eigenbasis with Grassmann-odd eigenvalues.

$$\boxed{|\eta\rangle := |0\rangle\eta + |1\rangle.}$$

$$\hat{\eta}|\eta\rangle = |1\rangle\eta = |\eta\rangle\eta. \quad \checkmark$$

- Define $\langle\langle\chi|$ by the property

$$\langle\langle\chi|\eta\rangle = \chi-\eta$$

$$= \delta(\chi-\eta)$$

Grassmann δ-fn is just equal to its argument.

$$\Rightarrow \quad \boxed{\langle\langle\chi| = -\langle 0| + \chi\langle 1|}.$$

Equivalently, we can say

$$\langle\langle 0| := \langle 1|, \quad \langle\langle 1| := -\langle 0|.$$

$$\langle\langle\chi| = \chi\langle\langle 0| + \langle\langle 1|.$$

We have:

$$\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$= \int d\eta \left(-|0\rangle\eta + |1\rangle\right)\left(-\langle 0| + \eta\langle 1|\right)$$

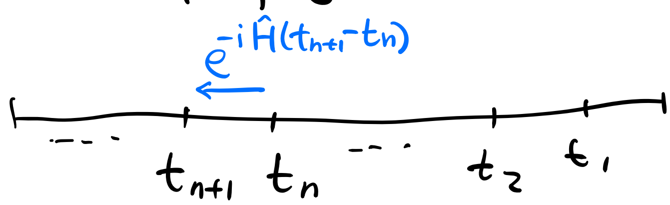
$$= \int d\eta\, |-\eta\rangle \langle\!\langle \eta| .$$

$$= \int |\eta\rangle\, d\eta\, \langle\!\langle \eta| .$$

The Grassmann path integral can be derived by inserting

$$\int |\eta(t_n)\rangle\, d\eta(t_n)\, \langle\!\langle \eta(t_n)|$$

into the propagator at time $t = t_n$.



In this very simple model, $\hat{H} = 0$,

we just have factors

$$\langle\!\langle \eta(t_{n+1}) | \eta(t_n) \rangle\!\rangle \, d\eta(t_n) = \left(\eta(t_{n+1}) - \eta(t_n) \right) d\eta(t_n)$$

Bring back $\bar{\eta}(t)$ variables by rewriting

$$\eta(t_{n+1}) - \eta(t_n)$$

$$= -\int d\bar{\eta}(t_n) \, e^{-\bar{\eta}(t_n)\left(\eta(t_{n+1}) - \eta(t_n)\right)}$$

we end up with integral of

$$\prod_n d\eta(t_n) \, d\bar{\eta}(t_n) \, e^{\underbrace{-\sum_n \bar{\eta}(t_n)\left(\eta(t_{n+1}) - \eta(t_n)\right)}_{i\int dt \, \boxed{i\bar{\eta}\dot{\eta}}}}$$

$$i\bar{\eta}\dot{\eta} = L$$

· Partition function

$$Z(\beta) = \mathrm{Tr} \, e^{-\beta \hat{H}}$$

$$= \int \mathrm{Tr}_{\mathcal{H}} \, |\eta\rangle d\eta \langle\!\langle \eta | \, e^{-\beta \hat{H}}$$

$$= \int d\eta \, \langle\langle \eta | \, e^{-\beta \hat{H}} \, | -\eta \rangle$$

this is the reason for the anti-periodicity of $\eta(\tau)$, i.e.

$\eta(\beta) = -\eta(0)$.

Consider a non-relativistic particle freely moving on a Riemannian manifold M

$$ds^2 = g_{ij}(x)\, dx^i dx^j$$

Lagrangian

$$L = \frac{1}{2} g_{ij}(x)\, \dot{x}^i \dot{x}^j$$

("non-linear sigma model")

$$P_i = \frac{\partial L}{\partial \dot{x}^i} = g_{ij} \dot{x}^j$$

$\Rightarrow$ Hamiltonian

$$H = \frac{1}{2} g^{ij}(x) P_i P_j$$

inverse matrix of g_{ij}

Quantum version:

states $\leftrightarrow$ wave function $\Psi(x)$

inner product

$$\langle \Psi_1 | \Psi_2 \rangle = \int \underbrace{d^N x \sqrt{\det g_{ij}}}_{\text{volume element}} \Psi_1^*(x) \Psi_2(x)$$

invariant under

Coord transf $x^i \to \tilde{x}^i$

$$g_{ij} \to \tilde{g}_{ij} = g_{mn} \frac{\partial x^m}{\partial \tilde{x}^i} \frac{\partial x^n}{\partial \tilde{x}^j}$$

[could also work with "naive measure" $d^N x$

and rescaled wave fn $\tilde{\psi}(x) \equiv (\det g)^{\frac{1}{4}} \psi(x)$.

$$\langle \psi_1 | \psi_2 \rangle = \int d^N x \; \tilde{\psi}_1^*(x) \, \tilde{\psi}_2(x) .$$

but $\tilde{\psi}(x)$ would not be inv't under coord-redefinition, not a well-defined function on M.]

canonical commutation relations

$$[\hat{x}^i, \hat{P}_j] = i\hbar \delta^i_j$$

$$[\hat{P}_i, \hat{P}_j] = 0$$

$$\Rightarrow \quad \hat{P}_j \cdot \psi(x) = -i\hbar f^{-1}(x) \, \partial_j [f(x) \psi(x)] .$$

will choose $f(x) \equiv 1$ in defining $\hat{P}_j$

so that

$$\hat{P}_j \cdot \psi(x) = -i\hbar \partial_j \psi(x) .$$

Caution: $\hat{P}_j$ defined in this way is $\underline{\text{NOT}}$ self-adjoint!

- Given a linear op. $A : |\Phi\rangle \mapsto |A\Phi\rangle$ the adjoint operator $A^\dagger$ is defined by $A^\dagger : |\Phi\rangle \mapsto |A^\dagger \Phi\rangle$ such that

$$\langle A^\dagger \Phi | \Psi \rangle = \langle \Phi | A \Psi \rangle$$

for any $|\Phi\rangle$, $|\Psi\rangle$.

- $$\langle \phi | \hat{P}_j \psi \rangle = \int d^N x \sqrt{\det g}\, \phi^*(x) (-i\hbar \partial_j) \psi(x)$$
$$= \int d^N x \; i\hbar \partial_j \left(\sqrt{\det g}\, \phi^*(x) \right) \psi(x)$$
$$= \langle \hat{P}_j^\dagger \phi | \psi \rangle$$

where $$\hat{P}_j^\dagger = \frac{1}{\sqrt{\det g}} (-i\hbar \partial_j) \sqrt{\det g}.$$

Q: What is the quantum Hamiltonian?

$$\hat{H} \sim \frac{1}{2} g^{ij}(\hat{x}) \hat{P}_i \hat{P}_j$$

ordering ?

will assume reparameterization invariance

$$x^i \to \tilde{x}^i = \tilde{x}^i(x)$$

$$g_{ij} \to \tilde{g}_{ij} = g_{mn} \frac{\partial x^m}{\partial \tilde{x}^i} \frac{\partial x^n}{\partial \tilde{x}^j}$$

$$\tilde{g}^{ij} = g^{mn} \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial \tilde{x}^j}{\partial x^n}.$$

$$\hat{P}_i = -i\hbar \partial_j \to \hat{\tilde{P}}_i = -i\hbar \frac{\partial}{\partial \tilde{x}^j} = -i\hbar \frac{\partial x^m}{\partial \tilde{x}^j} \partial_m.$$

$$\hat{H} = \frac{1}{2} \frac{1}{\sqrt{\det g}} \hat{P}_i \left(\sqrt{\det g}\, g^{ij} \hat{P}_j \right)$$

is invariant

$$= \frac{1}{2} \hat{P}_i^{\dagger} g^{ij} \hat{P}_j$$ self-adjoint ✓

In fact, $\hat{H}\psi(x) = -\frac{\hbar^2}{2} \underbrace{\nabla_i \nabla^i}_{\text{(Scalar) Laplacian}} \psi(x)$

QM NLσM:

$$\hat{H} = -\frac{\hbar^2}{2}\nabla^2 \quad \overset{?}{\longleftrightarrow}$$

path integral with

$$L = \frac{1}{2} g_{ij}(x)\,\dot{x}^i \dot{x}^j$$

invt under

(*) $\hbar \to \lambda \hbar$
$g_{ij} \to \lambda^2 g_{ij}$

regularization ambiguity in discretizing path integral.

Choose a reg. scheme that is invt under $x^i \to \tilde{x}^i$, $g \to \tilde{g}$.

$\to$ possible counter terms to add to L also covariant, e.g. scalar curvature $R(g)$.

Euclidean path integral

$$Z = \int [Dx^i]\, e^{-\int d\tau \left[\frac{1}{\hbar}\frac{1}{2} g_{ij}\partial_\tau x^i \partial_\tau x^j + \hbar c_1 R(g) + \hbar c_2 R_{ij}\partial_\tau x^i \partial_\tau x^j + \dots\right]}$$

$\hbar$-dependence of possible counter terms dictated by invariance w.r.t. (*)

$$\hbar \to \lambda\hbar, \quad g_{ij} \to \lambda^2 g_{ij}$$

$$\tau \to \tau' = \lambda\tau, \quad \partial_\tau x^i \to \partial_{\tau'} x^i = \lambda^{-1}\partial_\tau x^i$$

under which

$$R_{ij} \to R_{ij}, \quad R \to \lambda^{-2} R.$$

So

$$\hbar \int d\tau \, R, \quad \hbar \int d\tau \, R_{ij} \, \partial_\tau x^i \partial_\tau x^j,$$

are invariant.

Conclusion: counter terms may be needed, but they come with extra powers of $\hbar$, and become unimportant in $\hbar \to 0$ limit

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Since we have not specified the reg. scheme for the path integral at the moment, the coeff. c_1, c_2, etc. are also unspecified. Our working assumption is that with a given reg. scheme, there will be suitable choice of $c_1, c_2, \cdots$, such that the path integral reproduce the time evolution w.r.t. $\hat{H} = -\frac{\hbar^2}{2} \nabla^2$.

- NLσM with Grassmann variables ("fermions in 1D")

begin with config. manifold M

$$ds^2 = g_{ij}(x)\, dx^i dx^j$$

define local frame (tetrad)

$$g_{ij} = \sum_{a=1}^{N} e_i^a e_j^a$$

write $e^i{}_a$ as the inverse matrix of e_i^a

[i.e. $e^i{}_a e_i{}^b = \delta_a{}^b$, $e^i{}_a e_j{}^a = \delta^i{}_j$.
and $g^{ij} = \sum_a e^i{}_a e^j{}_b$.]

we can think of $v_a \equiv e^i{}_a \frac{\partial}{\partial x^i}$
as local trivialization of the tangent bundle.

- Now introduce Grassmann coordinates

η^a and $\bar{\eta}^a$, $a = 1, \cdots, N$.

anti-commuting w/ one another, with
Poisson (anti-)bracket

$$\{\eta^a, \eta^b\}_P = \{\bar{\eta}^a, \bar{\eta}^b\}_P = 0$$

$$\{\eta^a, \bar\eta^b\}_P = -i\,\delta^{ab}.$$

can view η^a as canonical coord,

$\bar\eta^a$ as their conjugate momenta.

- So far, $\eta^a, \bar\eta^b$ have nothing to do with x^i or the config. mfld M.

We now tie them together by demanding $\eta^a, \bar\eta^a$ to transform as coordinates on the tangent bundle TM with local trivilization $V_a = e^i{}_a \frac{\partial}{\partial x^i}$.

Namely, $\psi^i \equiv e^i{}_a(x)\,\eta^a$

and $\bar\Psi^i \equiv e^i{}_a(x)\,\eta^a$

are invariant under frame rotation

$$e^i{}_a \to \tilde e^i{}_a = \mathcal{O}_a{}^b e^i{}_b \;,\; \mathcal{O} \in SO(N)$$
$$\eta^a \to \tilde\eta^a = (\mathcal{O}^T)^a{}_b\,\eta^b.$$

and transform as vector fields on M:

under $x^i \to \tilde{x}^i$,

$$\psi^i \to \tilde{\psi}^i = \psi^j \frac{\partial \tilde{x}^i}{\partial x^j}.$$

Quantization

$\Rightarrow$ Hilbert $\mathcal{H}$ spanned by wave functions

$$\Phi(x^i, \eta^a) \equiv \sum_{p=0}^{N} \frac{1}{p!} \Phi_{a_1 \cdots a_p}(x)\, \eta^{a_1} \cdots \eta^{a_p}$$

$$= \sum_{p=0}^{N} \frac{1}{p!} \Phi_{i_1 \cdots i_p}(x)\, \psi^{i_1} \cdots \psi^{i_p}.$$

$$\updownarrow$$

differential p-form

$$\frac{1}{p!} \Phi_{i_1 \cdots i_p}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

- $\mathcal{H}$ is the space of differential forms on M !

- inner product

$\langle \Phi_1 | \Phi_2 \rangle$

$$= \int d^N x \sqrt{\det g} \sum_{p=0}^{N} \frac{1}{p!} \Phi_{1\,a_1 \dots a_p}^{*}(x)\, \Phi_{2\,a_1 \dots a_p}(x)$$

$$= \int d^N x\, d^N \eta\, d^N \eta^* \sqrt{\det g}\; e^{\sum_{a=1}^{N} \eta_a^* \eta_a}$$

$$\cdot \left(\Phi_1(x,\eta)\right)^* \Phi_2(x,\eta) .$$

$$(\eta_a \eta_b \cdots)^* \equiv \cdots \eta_b^* \eta_a^*$$

- sign convention for Berezin measure:

$$\int d^N \eta\, d^N \eta^* \prod_{a=1}^{N} \eta_a^* \eta_a = 1 .$$

- What is the adjoint $\eta_a^\dagger$?

$$\langle \Phi_1 | \eta_a \Phi_2 \rangle = \int \cdots e^{\sum_b \eta_b^* \eta_b}\, \Phi_1^* \eta_a \Phi_2$$

$$= \int \cdots \Phi_1^* \frac{\partial}{\partial \eta_a^*} e^{\sum_b \eta_b^* \eta_b}\, \Phi_2$$

$$= \int \cdots \Phi_1^* \frac{\partial}{\partial \eta_a^*} e^{\sum_b \eta_b^* \eta_b}\, \Phi_2$$

$$= \langle \frac{\partial}{\partial \eta_a} \Phi_1 | \Phi_2 \rangle$$

Conclusion: $\eta_a^\dagger = \bar{\eta}_a = \frac{\partial}{\partial \eta_a}$

canonical commutator $\{\eta_a, \bar{\eta}_b\} = \delta_{ab}$.

- Having defined the Hilbert space of our QM model, we now turn to the Hamiltonian $\hat{H}$

want: $\hat{H}$ invariant under coord. transf

$$x^i \to \tilde{x}^i, \qquad p_i \to \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} p_j$$
$$\psi^i \to \tilde{\psi}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \psi^j$$

Simplest choice, generalizing the scalar Laplacian, is the Laplacian (Laplace - de Rham operator) on forms.

Begin with exterior deriv.

$$d: \quad \Phi_{i_1 \cdots i_p}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

$$\mapsto \quad dx^j \wedge \partial_j \Phi_{i_1 \cdots i_p}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

takes $\quad \Phi_{a_1 \cdots a_p}\, \eta^{a_1} \cdots \eta^{a_p}$

to $\eta^b e^j_b\, \partial_j \left(e^{a_1}{}_{i_1} \cdots e^{a_p}{}_{i_p} \Phi_{a_1 \cdots a_p} \right) e^{i_1}_{b_1} \cdots e^{i_p}_{b_p}\, \eta^{b_1} \cdots \eta^{b_p}$

$$= \eta^b e_b{}^j \partial_j \Phi_{a_1 \cdots a_p}\, \eta^{a_1} \cdots \eta^{a_p}$$

$$+ \Big[\eta^b e_b{}^j\, (\partial_j e^{a_1}{}_{i_1})\, \Phi_{a_1 \cdots a_p}\, e^{i_1}_{b_1}\, \eta^{b_1} \eta^{a_2} \cdots \eta^{a_p}$$

$$+ (\partial_j e^{a_2}{}_{i_2})\, \Phi_{a_1 \cdots a_p}\, \eta^{a_1} e^{i_2}_{b_2}\, \eta^{b_2} \cdots \eta^{a_p}$$

$$+ \cdots \Big]$$

Can trade with $\nabla_j e^{a_1}{}_{i_1} = \partial_j e^{a_1}{}_{i_1} - \Gamma^m_{j i_1} e^{a_1}{}_m$, due to contraction w/ $\psi^j \psi^{i_1}$

In other words, d on the form is represented by the operator

$$\hat{d} \equiv \eta^a e_a{}^i \Big(\partial_i + \underbrace{e^j_b \nabla_i e^c_j}_{\equiv\, \omega_{ibc}}\, \eta^b \bar{\eta}^c \Big)$$

acts as $\dfrac{\partial}{\partial \eta^c}$.

spin connection

[defining property:

$$\nabla_i e^a{}_j + \omega_i{}^a{}_b e^b{}_j \equiv 0$$]

Conclusion:

$$\hat{d} = \eta^a e_a^i (\partial_i + \omega_{ibc} \eta^b \bar{\eta}^c)$$

Define $Q \equiv -i \hat{d}$. ("supercharge")

$$Q = \eta^a e^i_a (\hat{p}_i - i\omega_{ibc} \eta^b \bar{\eta}^c)$$

its adjoint operator is

$$Q^\dagger = (\hat{p}_i^\dagger + i\omega_{ibc} \eta^c \bar{\eta}^b) e^i_a \bar{\eta}^a$$

$$\underset{\text{verify!}}{=} \bar{\eta}^a e^i_a (\hat{p}_i - i\omega_{ibc} \bar{\eta}^b \eta^c)$$

Note: $Q^2 = 0 = (Q^\dagger)^2$.

We will take the Hamiltonian to be

$$H = \frac{1}{2} \{ Q, Q^\dagger \}$$

acts on forms as $dd^\dagger + d^\dagger d$
Laplace-de Rham op.

• What is the Lagrangian?

First, let us write the explicit expression of the classical Hamiltonian using the Poisson bracket between $Q, Q^\dagger$

$\Rightarrow$ [after some tedious calculation]

$$H^{cl} = \frac{1}{2} g^{ij} (p_i - i\omega_{iab}\eta^a\bar{\eta}^b)(p_j - i\omega_{jcd}\bar{\eta}^c\eta^d)$$

$$- \frac{1}{2} \underbrace{\boxed{(\partial_i \omega_{jab} - \partial_j \omega_{iab} + \omega_{iac}\omega_{jcb} - \omega_{jac}\omega_{icb})}}_{R_{ijab} = R_{ijkl}\, e^k_a\, e^l_b} \bar{\psi}^i \psi^j \bar{\eta}^a \eta^b$$

"$R_{ijab} = R_{ijkl}\, e^k{}_a\, e^l{}_b$

Riemann tensor

$\Updownarrow$ Legendre transf. $\qquad \dot{x}^i = g^{ij}(p_j - i\omega_{jab}\bar{\eta}^a\eta^b)$

Lagrangian

$$L = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j + \frac{i}{2}(\bar{\eta}^a\dot{\eta}^a + \eta^a\dot{\bar{\eta}}^a)$$

$$+ i\omega_{iab}\bar{\eta}^a\eta^b\dot{x}^i$$

$$+ \frac{1}{2} R_{abcd}\bar{\eta}^a\eta^b\bar{\eta}^c\eta^d.$$

Power counting:

$$e^{\frac{i}{\hbar}\int dt\, L} \quad \text{invariant under}$$

$$\hbar \to \lambda \hbar, \quad g_{ij} \to \lambda^2 g_{ij}, \quad t \to t' = \lambda t,$$

$$e^a_i \to \lambda e^a_i, \quad \omega_{iab} \to \omega_{iab}, \quad R_{abcd} \to \lambda^{-2} R_{abcd}.$$

$$\dot{x}^i \to \lambda^{-1} \dot{x}^i, \quad \bar{\eta}^a \eta^b \to \lambda \bar{\eta}^a \eta^b.$$

[Possible reg. scheme dependent counter terms are necessarily higher order in $\hbar$.]

Path integral representation of thermal partition function

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$$

$$= \int [Dx^i D\eta^a D\bar{\eta}^a]\Big|_{\substack{\eta(\beta) = -\eta(0) \\ \bar{\eta}(\beta) = -\bar{\eta}(0)}} e^{-\frac{1}{\hbar}\int_0^\beta d\tau\, L^E}.$$

On the other hand,

"Witten index"

$$\tilde{Z}(\beta) = \mathrm{Tr}\,(-)^F e^{-\beta H}$$

$$= \int [Dx\, D\eta\, D\bar{\eta}]\Big|_{\substack{\eta(\beta)=\eta(0)\\ \bar{\eta}(\beta)=\bar{\eta}(0)}} e^{-\frac{1}{\hbar}\int_0^\beta d\tau\, L^E}.$$

"______________"

- Special property of $\tilde{Z}(\beta)$ in supersymmetric QM

Define $Q_+ \equiv \frac{1}{\sqrt{2}}(Q + Q^\dagger)$

$H = Q_+^2$. $\quad\quad \{(-)^F, Q_+\} = 0.$

$$\frac{\partial}{\partial\beta}\tilde{Z}(\beta) = \mathrm{Tr}\left[(-)^F e^{-\beta H}(-Q_+^2)\right]$$

$$\underset{\text{move one } Q_+ \text{ to left}}{=} \mathrm{Tr}\left[Q_+ (-)^F e^{-\beta H} Q_+\right]$$

$$\underset{\text{cyclicity of trace}}{=} \mathrm{Tr}\left[(-)^F e^{-\beta H} Q_+^2\right]$$

$$\Rightarrow \frac{\partial}{\partial\beta}\tilde{Z}(\beta) = 0$$

$$\partial_\beta Z(\beta) = 0,$$

Z is independent of β !

Let's inspect this more closely :

consider H- eigen basis $|n\rangle$,

$$H|n\rangle = E_n |n\rangle.$$

$$\begin{aligned} H\left(Q_+|n\rangle\right) &= Q_+ H|n\rangle \\ &= E_n Q_+|n\rangle. \end{aligned}$$

$$\begin{aligned} \| Q_+|n\rangle \|^2 &= \langle n| Q_+^\dagger Q_+ |n\rangle \\ &= E_n \langle n|n\rangle. \end{aligned}$$

$$\Rightarrow \quad E_n \geq 0, \qquad E_n = 0 \iff Q_+|n\rangle = 0.$$

The non-zero energy eigenstates come in pairs

$$|n\rangle \underset{Q_+}{\overset{Q_+}{\rightleftarrows}} \frac{1}{\sqrt{E_n}} Q_+|n\rangle$$

$|n\rangle$ and $\frac{1}{\sqrt{E_n}} Q_+|n\rangle$ have same energy but opposite $(-)^F$, thereby cancel in their contribution to $\tilde{Z}(\beta)$.

On the other hand, $H=0$ (ground) states do not pair up in this way, thus

$$\tilde{Z}(\beta) = n_+ - n_-$$

# of ground states with $(-)^F = +1$ and -1 resp.

Caution: the above argument fails for SQM with a continuous spectrum, as the trace involved are not obviously well-defined.

e.g. $\mathcal{H} = L^2(\mathbb{R})$,

$$\mathrm{Tr}\,[p, f(\hat{x})] = -i\int dx\, \partial_x f(x) = -i\,[f(\infty) - f(-\infty)]$$

could be nonzero while $\mathrm{Tr}\, f(\hat{x})$ is ill-defined

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Now let's turn to the path $\int$ representation

$$\tilde{Z}(\beta) = \int [Dx\, D\eta\, D\bar{\eta}]\Big|_{\text{periodic}}$$

$$\cdot \exp\Big[-\frac{1}{\hbar}\int_0^\beta d\tau \Big(\frac{1}{2} g_{ij}\partial_\tau x^i \partial_\tau x^j$$

$$+ \frac{1}{2}(\bar{\eta}^a \partial_\tau \eta^a + \eta^a \partial_\tau \bar{\eta}^a) + \omega_{iab}\bar{\eta}^a \eta^b \partial_\tau x^i$$

$$+ \frac{1}{2} R_{abcd}\bar{\eta}^a \eta^b \bar{\eta}^c \eta^d \Big)$$

$$+ \mathcal{O}(\hbar) \text{ counter terms} \Big]$$

Change variable $\tau \equiv \beta s$

$$\partial_\tau x^i = \frac{1}{\beta}\partial_s x^i. \qquad \eta^a \equiv \beta^{-\frac{1}{2}}\tilde{\eta}^a$$

$$\bar{\eta}^a \equiv \beta^{-\frac{1}{2}}\bar{\tilde{\eta}}^a.$$

$$\tilde{Z}(\beta) = \int [Dx\, D\eta\, D\bar{\eta}]$$

$$\cdot \exp\Big[-\frac{1}{\beta\hbar}\int_0^1 ds \Big(\frac{1}{2} g_{ij}\partial_s x^i \partial_s x^j$$

$$+ \frac{1}{2}(\tilde{\eta}^a \partial_s \bar{\tilde{\eta}}^a + \bar{\tilde{\eta}}^a \partial_s \tilde{\eta}^a) + \omega_{iab}\bar{\tilde{\eta}}^a \tilde{\eta}^b \partial_s x^i$$

$$- \frac{1}{2} R_{abcd} \bar{\tilde{\eta}}^a \tilde{\eta}^b \bar{\tilde{\eta}}^c \tilde{\eta}^d \Big)$$

$$+ \mathcal{O}(\beta\hbar) \text{ counter terms} \Big]$$

$\beta \to 0$ same as $\hbar \to 0$ limit,

- classical limit, counter terms become unimportant.

calculate the above functional integral in this limit, suffices to expand around stationary point of $S^E[x, \eta, \bar{\eta}]$

$$X^i(s) = x_0^i + (\beta\hbar)^{\frac{1}{2}} e^i{}_a(x_0)\, \delta x^a(s)$$

$$\tilde{\eta}^a(s) = \tilde{\eta}_0^a + (\beta\hbar)^{\frac{1}{2}} \delta\eta^a(s)$$

and similarly for $\bar{\tilde{\eta}}^a$.

Here $\delta x^a(s)$ and $\delta\eta^a(s)$ contain only non-zero Fourier modes, i.e.

$$\delta x^a(s) = \sum_{k \in \mathbb{Z}} x_k^a e^{2\pi i s}, \quad \delta\eta^a(s) = \sum_{k \in \mathbb{Z}} \eta_k^a e^{2\pi i s}$$

To leading order in the expansion in β, the functional integral over $\delta x^a(s)$ and $\delta\eta^a(s)$, $\delta\bar{\eta}^a(s)$ reduces to Gaussian, and furthermore the bosonic and fermionic functional determinants cancel.

We are left with integration over the zero-Fourier modes x_0^i, η_0^a, $\bar{\eta}_0^a$.

To carefully fix the β-dependence of measure, we can restore integration in $p_{i,0}$, and write

$$\lim_{\beta\to 0}\tilde{Z}(\beta) = \int_{T^*M}\frac{d^N x_0\, d^N p_0}{(2\pi)^N}\, e^{-\frac{\beta}{2}g^{ij}(x_0)p_{i,0}p_{j,0}}$$

$$\cdot\int d^N\eta\, d^N\bar{\eta}\, e^{\underbrace{\frac{\beta}{2}R_{abcd}(x^0)\bar{\eta}^a\eta^b\bar{\eta}^c\eta^d}}$$

using Bianchi identity $R_{a[bcd]}=0$, same as

$$-\frac{\beta}{4}R_{acbd}(x^0)\,\bar{\eta}^a\bar{\eta}^c\eta^b\eta^d$$

$$= \int_M d^N x_0 \, \frac{\sqrt{\det g}}{(2\pi \cancel{\beta})^{N/2}} \, \frac{\left(\frac{\cancel{\beta}}{4}\right)^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!}$$

$$\cdot \, \epsilon^{a_1 \cdots a_N} \epsilon^{b_1 \cdots b_N} R_{a_1 a_2 b_1 b_2}(x_0) \cdots R_{a_{N-1} a_N b_{N-1} b_N}(x_0)$$

for N even,

($= 0$ for N odd)

Result:

Witten index

$$= \frac{1}{(2\pi)^{\frac{N}{2}} \cdot 2^N \left(\frac{N}{2}\right)!} \int_M d^N x \sqrt{\det g} \, \epsilon^{a_1 \cdots a_N} \epsilon^{b_1 \cdots b_N} R_{a_1 a_2 b_1 b_2} \cdots R_{a_{N-1} a_N b_{N-1} b_N}.$$

$= n_+ - n_-$

What is the geometric interpretation of $n_\pm$?

$$H = \frac{1}{2} \{ Q, Q^\dagger \}.$$

$$H|\psi\rangle = 0 \quad \Rightarrow \quad \underbrace{\langle\psi|\psi\rangle}_{\frac{1}{2}\left(\|Q|\psi\rangle\|^2 + \|Q^\dagger|\psi\rangle\|^2\right)} = 0$$

$$\Rightarrow \quad Q|\psi\rangle = 0 \quad \text{and} \quad Q^\dagger|\psi\rangle = 0.$$

Recall $\psi(x,\eta) = \sum \frac{1}{p!}\psi_{a_1\cdots a_p}(x)\,\eta^{a_1}\cdots\eta^{a_p}$

$\longleftrightarrow$ differential form

$$\Omega_\psi = \sum \frac{1}{p!}\psi_{a_1\cdots a_p}(x)\, e^{a_1}_{i_1}\cdots e^{a_p}_{i_p} \cdot dx^{i_1}\wedge\cdots\wedge dx^{i_p}.$$

Q acts as $-id$.

i.e. $\Omega_{Q\psi} = -i\, d\, \Omega_\psi$

likewise, can write

$$\Omega_{Q^\dagger\psi} = i\, d^* \Omega_\psi,$$

where d^* takes a p-form to a $(p-1)$-form.
(it is called the codifferential)

Can identify $(-)^F$ with $(-)^p$

$$\text{Witten index} = \sum_{p=0}^{N} (-)^p\, b_p \quad = \chi$$ "Euler characteristic"

where $b_p =$ # of linearly independent p-forms that are annihilated by d and d^*

"Betti number"

such forms are called harmonic forms.

From the $\beta \to 0$ limit of the path integral, we have derived

$$\chi = \frac{1}{(2\pi)^{\frac{N}{2}} \, 2^N \left(\frac{N}{2}\right)!} \int_M d^N x \sqrt{\det g} \, \epsilon \, \epsilon \, \overbrace{R \cdots R}^{N/2}$$

Chern - Gauss - Bonnet theorem

[Chern, 1944]

Some variants:

- $\mathbb{Z}_2$-symmetry σ : $\sigma^{-1} \eta^a \sigma = \bar{\eta}^a$.

$\mathrm{Tr}\, \sigma (-)^F e^{-\beta H}$ = path integral with twist boundary condition

$$\eta^a(\beta) = \bar{\eta}^a(0)$$
$$\bar{\eta}^a(\beta) = \eta(0).$$

$\rightsquigarrow$ Hirzebruch signature theorem

- $\mathcal{N}=1$ SQM $\quad \{\eta^a, \eta^b\} = \delta^{ab}$

(no $\bar{\eta}^a$)

wave function $\longleftrightarrow$ spinor fields on M

$H = Q^2$, $\quad Q \longleftrightarrow i \not{D}$

Dirac operator

$\mathrm{Tr}(-)^F e^{-\beta H}$

$\rightsquigarrow$ Atiyah-Singer index thm.

- Deformation of SQM.

  family of supercharge Q_λ

  $H_\lambda = Q_\lambda^2$.

$$\tilde{Z}_\lambda(\beta) = \mathrm{Tr}\,(-)^F e^{-\beta H_\lambda}$$

$$\frac{\partial}{\partial\lambda}\tilde{Z}_\lambda(\beta) = \mathrm{Tr}\,(-)^F e^{-\beta H_\lambda}(-\beta)\{Q_\lambda, \partial_\lambda Q_\lambda\}$$

$$= 0 \quad \text{by cyclicity of trace}$$

and $[Q_\lambda, H_\lambda] = 0$

$\{Q_\lambda, (-)^F\} = 0$.

- Witten index invariant under deformation.

For instance, NLσM SQM

$Q, Q^\dagger$ depend on metric on M

deform g_{ij}. Witten index = Euler char,

remains invariant.

• A different family of deformation.

$$Q = \eta^a e^i{}_a (\hat{p}_i - i\omega_{ibc} \eta^b \bar{\eta}^c)$$

$$Q^\dagger = \bar{\eta}^a e^i{}_a (\hat{p}_i - i\omega_{ibc} \bar{\eta}^b \eta^c)$$

as before.

$$Q_\lambda \equiv e^{-\lambda h} Q e^{\lambda h} .$$

$$Q_\lambda^\dagger = e^{\lambda h} Q^\dagger e^{-\lambda h}$$

$h = h(x)$ a smooth function on M.

$$H_\lambda = \frac{1}{2} \{ Q_\lambda , Q_\lambda^\dagger \} .$$

calculate: $Q_\lambda = Q + \lambda [Q, h]$

$$= Q - i \lambda \psi^i \partial_i h .$$

$$H_\lambda = H_0 - \frac{i}{2} \lambda \{ \psi^i \partial_i h , Q^\dagger \}$$

$$+ \frac{i}{2} \lambda \{ \bar{\psi}^i \partial_i h , Q \}$$

$$+ \frac{\lambda^2}{2} \{ \psi^i \partial_i h , \bar{\psi}^j \partial_j h \}$$

$$= H_0 + \lambda \nabla_i \partial_j h [\psi^i \bar{\psi}^j]$$

$$+ \frac{\lambda^2}{2} g^{ij} \partial_i h \, \partial_j h .$$

- Witten index independent of λ.

- $\lambda \to \infty$ limit, ground state

$$\updownarrow ?$$

minima of $g^{ij}\partial_i h \partial_j h$.

i.e. critical pts of $h(x)$

$$\partial_i h = 0.$$

Near a critical point of $h(x)$, curvature of M unimportant, can approximate

$$H \approx \frac{P_i^2}{2} + \frac{\lambda}{2} \partial_i \partial_j h [\psi^i, \bar{\psi}^j] + \frac{\lambda^2}{2} (\partial_i h)^2$$

W.L.O.G, assume critical pt at $x^i = 0$,

$$h(x) = \frac{1}{2} h_{ij} x^i x^j + \mathcal{O}(x^3)$$

assume non-degenerate

$$x^i = \mathcal{O}^i{}_j \tilde{x}^j . \qquad \mathcal{O} \in SO(N)$$

can find $\mathcal{O}$ such that

$$h_{ij}\,\mathcal{O}^i{}_k\,\mathcal{O}^j{}_\ell = u_k\,\delta_{k\ell}.$$

i.e. $\displaystyle h = \frac{1}{2}\sum_k u_k (\tilde{x}^k)^2.$

$$H \approx \sum_{k=1}^{N}\Big(\underbrace{\frac{\tilde{p}_k^{\,2}}{2} + \frac{\lambda^2}{2}u_k^2\,\tilde{x}^{k\,2}}_{\substack{\text{harmonic osc.}\\ \text{grd state } E=\frac{1}{2}\lambda|u_k|}} + \underbrace{\frac{\lambda}{2}u_k[\tilde{\psi}^k,\bar{\tilde{\psi}}^k]}_{\substack{\text{eigenvalue}\\ \pm\frac{\lambda}{2}u_k}}\Big)$$

$$[\eta,\bar{\eta}] = 2\eta\frac{\partial}{\partial\eta} - 1$$

has eigenvalue $\begin{cases} +1 & \text{acting on } \Phi(\eta)=\eta \\ -1 & \text{acting on } \Phi(\eta)=1. \end{cases}$

Conclusion: lowest energy wave function

$\Phi(x,\eta)$ localized at $x=0$

has energy $E\approx 0$, with

$$\Phi(x,\eta) \approx \prod_{k=1}^{N}\psi_0(\tilde{x}^k;u_k)\cdot\tilde{\eta}^{k_1}\cdots\tilde{\eta}^{k_m}$$

where $k_1,\dots,k_m$ are such that

$$u_k < 0, \qquad k \in \{k_1, \dots, k_m\}$$

$$u_k > 0, \qquad k \notin \{k_1, \dots, k_m\}.$$

has $(-)^F = (-)^m$

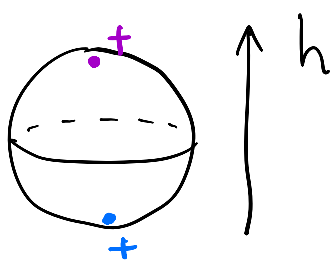
χ "

$$\text{Witten index} = \sum_{\text{grd states}} (-)^F$$

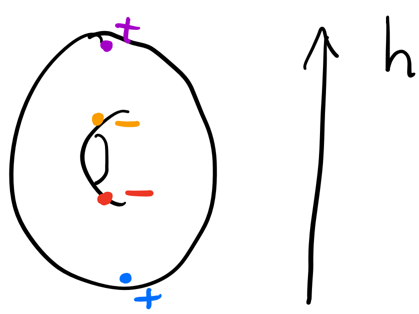
$$= \sum_{\substack{\text{critical pts} \\ \text{of } h}} (-)^{\#\,\text{negative eig. val. of } (\partial_i \partial_j h)}$$

"Morse index"

Examples:



$\chi = 2$



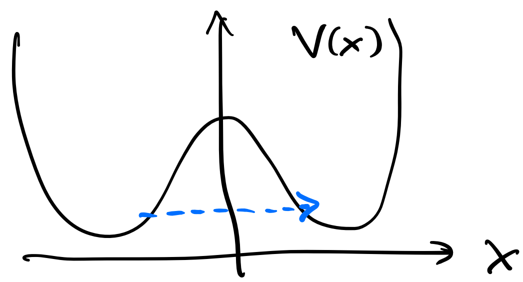
$\chi = 0$

✓

- But wait, are we sure that these are actual $(E=0)$ ground states?

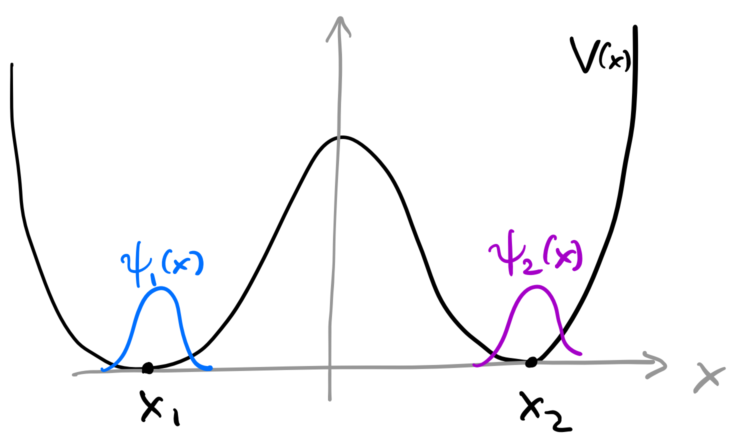
Not so fast!

Recall for $H = p^2 + V(x)$



tunnelling → exp. small split between 2 lowest energy states.

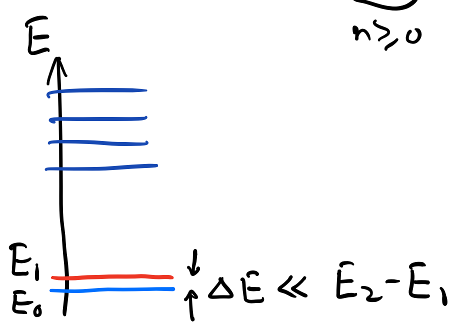
While in this case one can understand the split using WKB approx., we'll give a more general argument using the path integral.



ψ_1, ψ_2 "approx. ground" wave functions localized at $x \simeq x_1$ and $x \simeq x_2$ resp.

Study : $\langle \psi_2 | e^{-\hat{H}T} | \psi_1 \rangle$

$$= \sum_{n \geq 0} e^{-E_n T} \langle \psi_2 | n \rangle \langle n | \psi_1 \rangle$$



Isolate two lowest-E states with

$$\frac{1}{E_2 - E_1} \ll T \sim \mathcal{O}\left(\frac{1}{\Delta E}\right).$$

Path integral representation

$$\int dx_f \, \psi_2^*(x_f) \int dx_i \, \psi_1(x_i)$$

$$\times \int [Dx]_{x(0)=x_i}^{x(T)=x_f} e^{-\frac{1}{\hbar}\int_0^T d\tau \, L^E}$$

$$L^E = \frac{1}{2}(\partial_\tau x)^2 + V(x).$$

Approx. w/ saddle pt - solution to the Euclidean eqn of motion

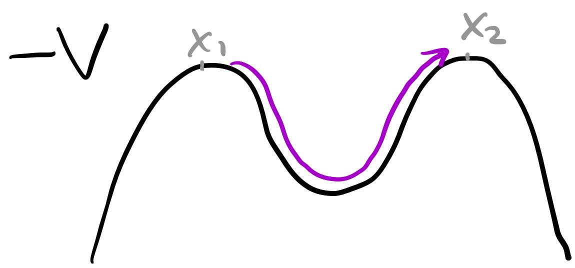
$$\delta S^E = 0 \Rightarrow \quad -\partial_\tau^2 x + V'(x) = 0$$

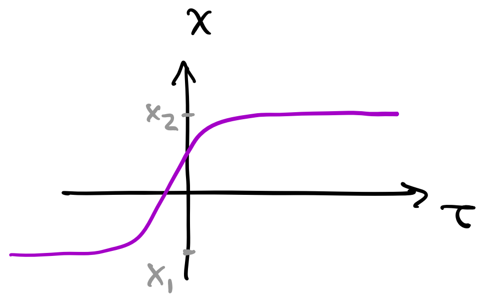
subject to boundary cond

subject to boundary cond.

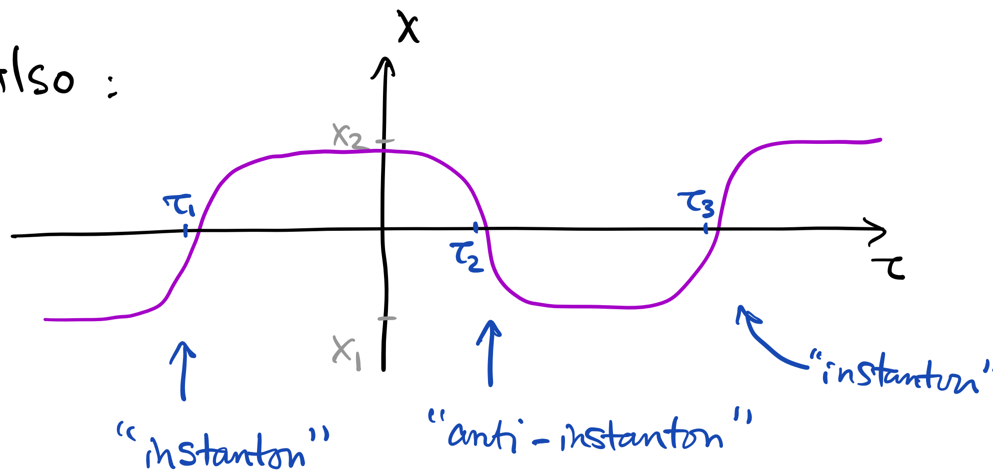
$$x(T) \approx x_2, \quad x(0) \approx x_i$$

"rolling in the inverted potential $-V$"





Also :



etc.

Integrate over (approx.) saddle pts corresponding to instantons/anti-instantons at

$$0 < \tau_1 < \tau_2 < \cdots < \tau_{2n+1} < T$$

Eucl. time separations $\gg \mathcal{O}\left(\frac{1}{E_2 - E_1}\right)$.

$\langle x_2 | e^{-HT} | x_1 \rangle$

action of single

$\langle \psi_2 | e \quad | \psi_1 \rangle$

instanton

$$\approx C \cdot \sum_{n=0}^{\infty} \int_{0<\tau_1<\cdots<\tau_{2n+1}<T} d\tau_1 \cdots d\tau_{2n+1} \left(e^{-\frac{1}{\hbar} S_0} \cdot K \right)^{2n+1}$$

result of Gaussian integral over fluctuations.

$$= C \cdot \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(e^{-\frac{1}{\hbar} S_0} K T \right)^{2n+1}$$

$$= \frac{C}{2} \left(e^{e^{-\frac{1}{\hbar} S_0} K T} - e^{-e^{-\frac{1}{\hbar} S_0} K T} \right)$$

- $$\partial_\tau^2 x(\tau) = V'(x(\tau)).$$

$$x(+\infty) = x_2, \qquad x(-\infty) = x_1$$

$$S_0 = \int_{-\infty}^{\infty} d\tau \left[\frac{1}{2} (\partial_\tau x)^2 + V(x(\tau)) \right]$$

(Assume $V(x_1) = V(x_2) = 0$)

Like wise, we have

$$\langle \psi_1 | e^{-HT} | \psi_1 \rangle$$

$$\approx C \cdot \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(e^{-\frac{1}{\hbar} S_0} K T \right)^{2n}$$

$$= \frac{C}{2}\left(\underbrace{e^{e^{-\frac{1}{\hbar}S_0} K T}}_{e^{-E_0 T}} + \underbrace{e^{-e^{-\frac{1}{\hbar}S_0} K T}}_{e^{-E_1 T}} \right)$$

Conclusion:

$$E_0 \approx -e^{-\frac{1}{\hbar}S_0} K$$

$$E_1 \approx e^{-\frac{1}{\hbar}S_0} K .$$

Split $\quad \Delta E \approx 2 e^{-\frac{1}{\hbar}S_0} K .$

Now return to h-deformed SQM

critical pts x_α, $\quad \partial_i h(x_\alpha) = 0$.

$\rightsquigarrow$ approx. grd. state $|\alpha\rangle$

whose wave fn $\Phi(x,\eta)$

is localized at $x \approx x_\alpha$.

Due to possible tunnelling, however, $\langle \alpha | H_\lambda | \alpha \rangle$ may not be exactly 0, i.e. $Q_\lambda |\alpha\rangle$ and $Q_\lambda^T |\alpha\rangle$

may be non-zero.

Expect:

$$Q_\lambda |\alpha\rangle \approx \sum_\beta e^{-\frac{1}{\hbar} S_{\alpha\beta}} K_{\alpha\beta} |\beta\rangle .$$

where $S_{\alpha\beta}$ is the action of an instanton

with $x(\tau = -\infty) = x_\alpha$
$x(\tau = +\infty) = x_\beta$.

$K_{\alpha\beta}$ is the Gaussian determinant of fluctuations around the instanton.

. What are the instantons?

bosonic part of Euclidean action:

$$S^E = \int d\tau \left[\frac{1}{2} g_{ij} \partial_\tau x^i \partial_\tau x^j + \frac{\lambda^2}{2} g^{ij} \partial_i h \partial_j h \right]$$

$$= \int d\tau \left[\underbrace{\frac{1}{2} g_{ij} (\partial_\tau x^i \pm \lambda g^{ik} \partial_k h)(\partial_\tau x^j \pm \lambda g^{jl} \partial_l h)}_{\geq 0} \mp \lambda \underbrace{\partial_\tau x^i \partial_i h}_{= \partial_\tau h(x(\tau))} \right]$$

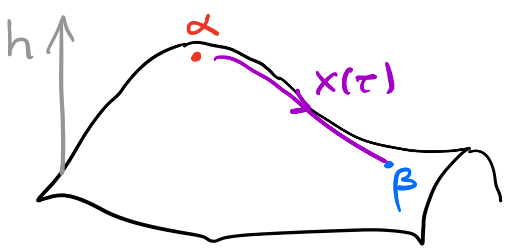
$\Rightarrow$ Instanton interpolating x_α and x_β has action

$$S_{\alpha\beta} = \lambda \,| h(x_\alpha) - h(x_\beta) | .$$

Two types:

- $h(x_\alpha) > h(x_\beta)$, $\quad \partial_\tau x^i = -\lambda g^{ik} \partial_k h$

steepest descent path



- $h(x_\alpha) < h(x_\beta)$, $\quad \partial_\tau x^i = \lambda g^{ik} \partial_k h$

steepest ascent path.

- What about $K_{\alpha\beta}$?

Lagrangian

$$L_\lambda = L_0 - \frac{\lambda^2}{2} g^{ij} \partial_i h \partial_j h + \lambda \nabla_i \partial_j h \, \bar{\psi}^i \psi^j$$

Consider 1D example

$$L_\lambda = \frac{1}{2}\dot{x}^2 + i \bar{\eta} \dot{\eta} + \lambda h''(x) \bar{\eta} \eta - \frac{\lambda^2}{2} (h'(x))^2 .$$

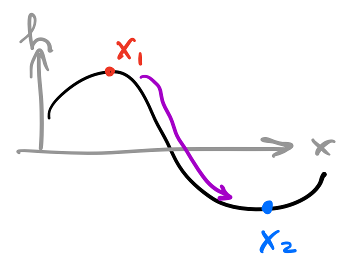
Euclidean:

$$L^E_\lambda = \frac{1}{2}(\partial_\tau x)^2 + \bar{\eta} \partial_\tau \eta - \lambda h''(x) \bar{\eta} \eta + \frac{\lambda^2}{2} (h'(x))^2 .$$

Steepest descent instanton:

$x = x_0(\tau)$ obeys

$$\partial_\tau x_0(\tau) = -\lambda h'(x_0(\tau)) .$$



Expand $\quad x(\tau) = x_0(\tau) + \delta x(\tau)$

$$S^E = \int d\tau \, L^E_\lambda$$

$$= S_0 + \int d\tau \left[\frac{1}{2} \left(\partial_\tau \delta x + \lambda h''(x_0) \delta x \right)^2 + \bar\eta \partial_\tau \eta - \lambda h''(x_0) \bar\eta \eta + \dots \right]$$

Decompose $\delta x(\tau)$, $\eta(\tau)$, $\bar\eta(\tau)$ into modes

$$\delta x(\tau) = \sum_k \delta x_k \cdot f_k(\tau),$$

$$(-\partial_\tau + \lambda h''(x_0))(\partial_\tau + \lambda h''(x_0)) f_k(\tau) = \omega_k^2 f_k(\tau)$$

$$\eta(\tau) = \sum_k \eta_k \cdot g_k(\tau), \qquad \bar\eta(\tau) = \sum_k \bar\eta_k f_k(\tau)$$

non-zero modes of $\eta, \bar\eta$ come in pairs.

$$(\partial_\tau + \lambda h''(x_0)) f_k = \omega_k g_k$$

$$(-\partial_\tau + \lambda h''(x_0)) g_k = \omega_k f_k.$$

Normalize f_k s.t. $\int d\tau\, f_k(\tau) f_{k'}(\tau) = \delta_{kk'}$

$$\Rightarrow S^E = S_0 + \sum_k \left[\frac{\omega_k^2}{2} (\delta x_k)^2 - \omega_k \bar\eta_k \eta_k \right] + \dots$$

$$\int \prod_k dx_k d\eta_k d\bar\eta_k \, e^{-\frac{1}{\hbar} S^E}$$

$$\approx e^{-S_0} \cdot \frac{\cancel{\prod_k \omega_k}}{\cancel{(\prod_k \omega_k^2)^{\frac{1}{2}}}}$$

bosonic and fermionic determinants over non-zero modes cancel !

Danger: zero-modes (i.e. $\omega_\kappa = 0$)
can lead to divergent or vanishing det.

We already know a zero mode of $\delta x(\tau)$,
namely $f_0(\tau) := \partial_\tau x_0(\tau)$.

Indeed, $\partial_\tau f_0(\tau) + \lambda h''(x_0(\tau)) f_0(\tau) = 0$

$$\Rightarrow \omega_0 = 0.$$

This amounts to time-translation of the instanton, and we have already seen its interpretation in giving rise to energy split.

· What about $\eta(\tau)$ and $\bar{\eta}(\tau)$?

$f_0(\tau) = \partial_\tau x_0(\tau) \rightarrow$ z.m. for $\bar{\eta}$.

no z.m. for η

Had we been considering a steepest ascent instanton instead,
there would be no z.m. for $\bar{\eta}$.

whereas $g_0(\tau) = \partial_\tau X_0(\tau)$

$\longrightarrow$ z.m. for η.

---

* Generalizes to any steepest descent instanton along which the Morse index p decreases by 1:

  there is exactly 1 zero mode for $\bar{\eta}$,

  namely $\bar{\eta}(\tau) = \bar{\eta}_0 \cdot \partial_\tau X_0(\tau)$,

  and no zero mode for η.

  For steepest ascent instanton,

  role of η and $\bar{\eta}$ exchanged.

Consequence:

- Integration over z.m. of either η or $\bar{\eta}$ would give 0 unless there are explicit insertions of η or $\bar{\eta}$ in the path integral.

- Insertion of Q_λ would introduce an η
  $Q_\lambda^\dagger$ - - - . $\bar{\eta}$.

Let us inspect

$$\langle \beta | Q_\lambda^\dagger | \alpha \rangle \approx \frac{\langle \beta | Q_\lambda^\dagger h - h Q_\lambda^\dagger | \alpha \rangle}{h(x_\alpha) - h(x_\beta)}$$

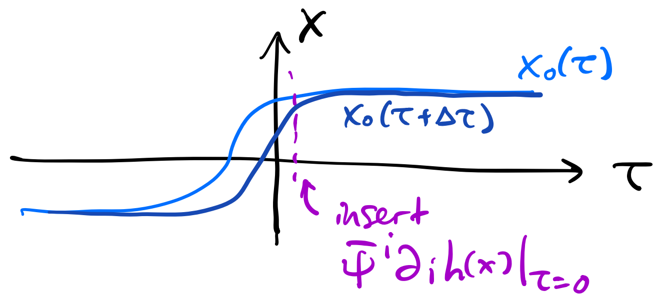
$$= \frac{1}{h(x_\alpha) - h(x_\beta)} \langle \beta | \underbrace{[Q_\lambda^\dagger, h]}_{-i\lambda \bar\psi^i \partial_i h} | \alpha \rangle$$

Consider

$$\langle \beta | e^{-HT/2} \bar\psi^i \partial_i h \, e^{-HT/2} | \alpha \rangle$$

T large.

1-instanton contribution:



fermion zero mode $\bar\psi^i(\tau) = \bar\eta_0 \cdot \partial_\tau x_0^i(\tau)$

Integration over bosonic and fermionic z.m. with the insertion of

$$\bar\psi^i(\tau=0)\, \partial_i h(x(\tau=0))$$

gives

$$\int_{-\infty}^{\infty} d\tau \int d\bar{\eta}_0 \cdot \bar{\eta}_0 \, \partial_\tau x_0^i(\tau) \cdot \partial_i h(x_0(\tau))$$

$$= h(x_\beta) - h(x_\alpha).$$

Remaining expansion of e^{-HT} in powers of T comes from additional pairs of instantons and anti-instantons.

The 1-instanton contribution gives

$$\langle \beta | [Q_\lambda^\dagger, h] | \alpha \rangle$$

$$\approx e^{-\frac{1}{\hbar} S_{\alpha\beta}} (-i\lambda)(h(x_\beta) - h(x_\alpha))$$

$$\Rightarrow \langle \beta | Q_\lambda^\dagger | \alpha \rangle \approx i\lambda \, e^{-\frac{1}{\hbar} S_{\alpha\beta}}$$

Caution: sign ambiguity in the measure of fermion z.m. integral !?

Note: if Morse indices of α and β differ by r, there is generically an $(r-1)$-parameter family of steepest descent paths (not counting time translation) and r fermion zero modes.

$r > 1$, no contribution to $\langle \beta | Q_\lambda^\dagger | \alpha \rangle$.

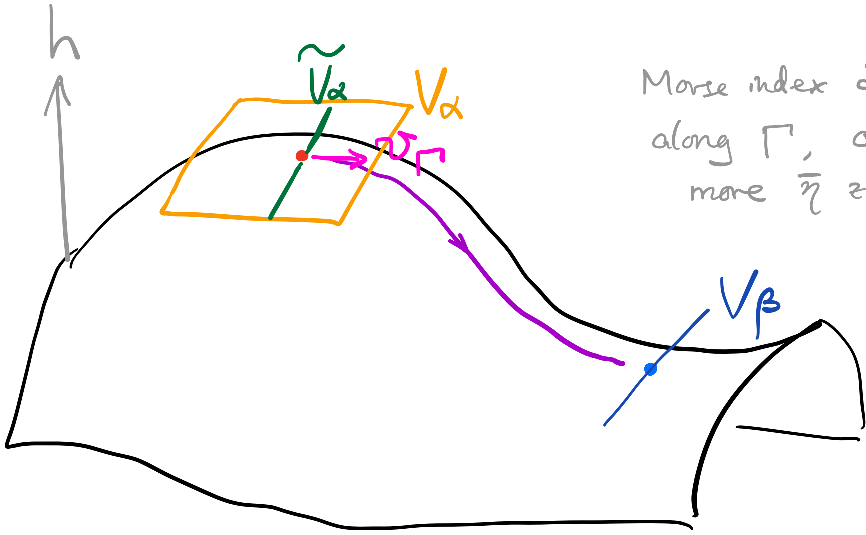
Conclusion:

$$Q_\lambda^\dagger \underbrace{|\alpha\rangle}_{p} \approx i\lambda \sum_\beta e^{-\frac{1}{\hbar} S_{\alpha\beta}} K_{\alpha\beta} |\beta\rangle_{p-1}$$

Morse index

$$K_{\alpha\beta} = \sum_{\substack{\text{descending paths} \\ \Gamma : \alpha \to \beta}} (\pm 1)$$

sign to be fixed.



Morse index decreases by 1 along Γ, otherwise more $\bar{\eta}$ zero modes can not be absorbed.

V_α = vector space spanned by tangent negative eigen-vectors of $\nabla_i \partial_j h$

v_Γ = tangent vector to steepest descent path Γ at x_α

$\widetilde{V}_\alpha = v_\Gamma^\perp$ in V_α.

Γ transports $\widetilde{V}_\alpha$ to V_β.

$\rightsquigarrow$ a linear map $\mathcal{O}_\Gamma : \widetilde{V}_\alpha \to V_\beta$.

with chosen orientation for each V_α, there is unambiguously defined orientation for $\widetilde{V}_\alpha$, and thus $\mathcal{O}_\Gamma$ either

$\underbrace{\text{preserves}}_{n_\Gamma := 1}$ or $\underbrace{\text{reverses}}_{n_\Gamma := -1}$ orientation.

Result: $$K_{\alpha\beta} = \sum_{\Gamma : \alpha \searrow \beta} n_\Gamma .$$

$$Q_\lambda^\dagger |\alpha\rangle_p \simeq i\lambda \sum_\beta e^{-\frac{\lambda}{\hbar}(h(x_\alpha) - h(x_\beta))} K_{\alpha\beta} |\beta\rangle_{p-1}.$$

rescaled basis $|\bar{\alpha}\rangle_p \equiv e^{\frac{\lambda}{\hbar} h(x_\alpha)} |\alpha\rangle_p$.

write $Q_\lambda^\dagger \equiv i\lambda\delta$, $\delta^2 = 0$.

$$\delta |\bar{\alpha}\rangle_p = \sum_\beta K_{\alpha\beta} |\bar{\beta}\rangle_{p-1}$$

Let W_p be the vector space spanned by the approx grd states $|\alpha\rangle_p$ of Morse index p.

$$W_N \xrightarrow{\delta} W_{N-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} W_1 \xrightarrow{\delta} W_0$$

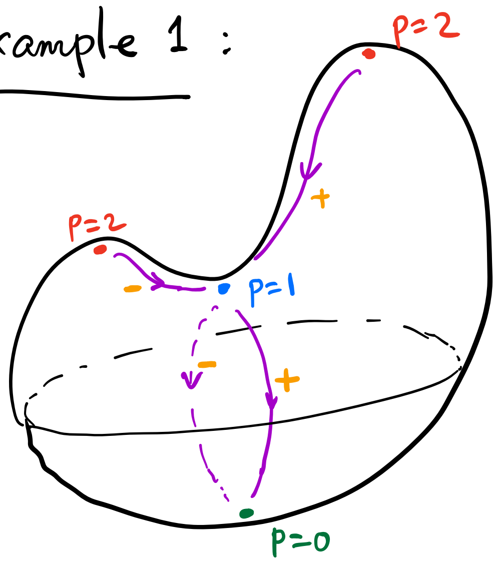
chain complex. $\delta^2 = 0$

Define cohomology

$$H_p := \frac{\mathrm{Ker}(\delta : W_p \to W_{p-1})}{\mathrm{Im}(\delta : W_{p+1} \to W_p)}$$

$\dim H_p = b_p$ Betti number.

Example 1 :



$$\begin{array}{ll} W_2 \simeq \mathbb{C}^2 & H_2 \simeq \mathbb{C} \\ \big\downarrow \delta \text{ onto} & \\ W_1 \simeq \mathbb{C} & H_1 \simeq 0 \\ \big\downarrow \delta = 1-1 = 0 & \\ W_0 \simeq \mathbb{C} & H_0 \simeq \mathbb{C} \end{array}$$