Physics 253a Problem set 1

Due Tuesday September 20, 2022

Problem 1. In the first lecture we introduced a scalar field operator in the system of free relativistic particles of mass m in D-dimensional Minkowskian spacetime, of the form

$$\hat{\phi}(x) = \int \frac{d^{D-1}\vec{k}}{\sqrt{(2\pi)^{D-1}2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}x^0} + a^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}x^0} \right), \tag{1}$$

where $x^\mu \equiv (x^0, \vec{x})$ is the space-time coodinate, $\vec{x}$ is the $(D-1)$-dimensional spatial position vector, $\vec{k}$ is the spatial momentum vector, and $\omega_{\vec{k}} \equiv \sqrt{\vec{k}^2 + m^2}$ is the energy of a particle according to the relativistic dispersion relation. $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ are annihilation and creation operators that obey the commutation relations

$$[a_{\vec{k}}, a_{\vec{k}'}] = 0 = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}], \qquad [a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta^{D-1}(\vec{k}-\vec{k}'). \tag{2}$$

(a) Verify the *equal-time* commutation relations

$$[\hat{\phi}(x^0,\vec{x}), \hat{\phi}(x^0,\vec{y})] = 0, \qquad [\hat{\phi}(x^0,\vec{x}), \frac{\partial}{\partial x^0}\hat{\phi}(x^0,\vec{y})] = i\,\delta^{D-1}(\vec{x}-\vec{y}). \tag{3}$$

(b) The stress-energy tensor operator $\hat{T}_{\mu\nu}(x)$ is given by

$$\hat{T}_{\mu\nu}(x) =: \partial_\mu\hat{\phi}(x)\partial_\nu\hat{\phi}(x) : -\eta_{\mu\nu}\left[\frac{1}{2} : \partial^\rho\hat{\phi}(x)\partial_\rho\hat{\phi}(x) : +\frac{1}{2}m^2 : (\hat{\phi}(x))^2 :\right], \tag{4}$$

where $: \cdots :$ stands for the normal ordering, defined simply by moving the $a^\dagger$'s to the left of the a's in a product of creation and annihilation operators. The derivation of this expression through the Noether procedure will be discussed in class later. For now you are asked to simply take the expression as given, and verify the conservation relation

$$\partial_\mu \hat{T}^\mu{}_\nu(x) = 0, \tag{5}$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$, and the Lorentz indices are raised or lowered by contracting with $\eta^{\mu\nu}$ or $\eta_{\mu\nu}$.[1] A consequence is that

$$\hat{P}^\mu = \int d^{D-1}\vec{x}\,\hat{T}^{0\mu}(x) \tag{6}$$

[1] In our convention, the Minkowskian metric $(\eta_{\mu\nu})$, as well as its inverse matrix $(\eta^{\mu\nu})$, is given by $\mathrm{diag}\{-1, 1, \cdots, 1\}$.

is independent of time x^0, i.e. conserved. Verify that $\hat{P}^0$ is the Hamiltonian and $\vec{\hat{P}}$ is the total spatial momentum operator as mentioned in the first lecture.

Also verify that the boost-angular-momentum operator

$$\hat{J}^{\mu\nu} = \int d^{D-1}\vec{x} \left[x^\mu \hat{T}^{0\nu}(x) - x^\nu \hat{T}^{0\mu}(x) \right] \tag{7}$$

is conserved.

(c) In class we asserted the Poincaré transformation property of $\hat{\phi}(x)$,

$$\hat{\phi}(\Lambda x + a) = U(\Lambda, a)\hat{\phi}(x)(U(\Lambda, a))^{-1}. \tag{8}$$

Verify this relation for the infinitesimal transformation and its corresponding unitary operator

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu, \qquad U(\Lambda, a) = 1 - ia^\mu \hat{P}_\mu + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}, \tag{9}$$

where $\omega_{\mu\nu}$ and a^μ are infinitesimal constants (i.e. we keep only first order terms in the Taylor series in ω and a). *Hint: you may simplify your computations making use of the result from step* **(a)**.

Problem 2. Consider a free non-relativistic particle moving on a circle of radius R, whose Lagrangian is

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^2, \tag{10}$$

where the coordinate q is periodically valued, namely

$$q \sim q + 2\pi R. \tag{11}$$

In this problem you will study the energy spectrum by analyzing the thermal partition function [2]

$$Z(\beta) = \mathrm{Tr} e^{-\beta \hat{H}} = \int dq \, \langle q | e^{-\beta \hat{H}} | q \rangle, \tag{12}$$

where the trace is taken over the Hilbert space of the system, using the Euclidean path integral representation

$$Z(\beta) = \int [Dq] \exp\left[- \int_0^\beta d\tau \frac{1}{2} (\partial_\tau q)^2 \right], \tag{13}$$

where $q(\tau)$ is subject to the boundary condition

$$q(\beta) = q(0) \bmod 2\pi R. \tag{14}$$

(a) A general real function $q(\tau)$ subject to (14) may be expanded in terms of Fourier modes as

$$q(\tau) = 2\pi R w \frac{\tau}{\beta} + \sum_{k=-\infty}^{\infty} q_k e^{2\pi i k \tau / \beta}, \tag{15}$$

where w is an integer, q_k's for $k \neq 0$ are complex coefficients that obey $q_{-k} = (q_k)^*$, and q_0 is a real periodically valued parameter, $q_0 \sim q_0 + 2\pi R$. One might attempt to replace the functional integration with integration over q_k's and sum over w:

$$\int [Dq] \overset{?}{\to} \sum_{w=-\infty}^{\infty} \int_0^{2\pi R} dq_0 \int \prod_{k=1}^{\infty} dq_k dq_k^*. \tag{16}$$

In fact, this is almost correct, up to certain normalization factors that depend on β. To determine this normalization, one can start with the Hamiltonian form of the path integral,

$$Z(\beta) = \int DpDq \, \exp\left[\int_0^\beta d\tau (ip\partial_\tau q - H(p, q)) \right], \tag{17}$$

[2] We have henceforth set $\hbar$ to 1.

where $q(\tau)$ obeys (14) while $p(\tau)$ obeys $p(\beta) = p(0)$, and express the functional integral as one over the Fourier modes of q and p in the form

$$\int DpDq \to \mathcal{N} \sum_{w} \int \prod_{k} \frac{dq_k dp_k}{2\pi}, \tag{18}$$

where $\mathcal{N}$ is assumed to be independent of β. Your task is to perform the integral over q_k, p_k, and express the result in the form

$$Z(\beta) = \mathcal{N} \sum_{w=-\infty}^{\infty} f(w), \tag{19}$$

for some function $f(w)$ (which also depends on β).

(b) Using *Poisson resummation formula*

$$\sum_{w=-\infty}^{\infty} f(w) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dw\, e^{2\pi i n w} f(w), \tag{20}$$

rewrite your result for (19) as a sum over n. Compare with what you expect from the eigenvalues of the quantum Hamiltonian of this system. What is the value of the normalization factor $\mathcal{N}$?

Remark: the RHS of (18), while reasonable looking, isn't quite the same as the integration over p, q at discrete time steps in the original derivation of the path integral from Hamiltonian formalism. The two different ways of doing the functional integral differ by a Jacobian factor. This difference can be absorbed by shifting the Lagrangian by a constant counter term that depends on the time step. You are not required to analyze this Jacobian factor, but you may have extra fun by doing so.