Physics 253a   Problem set 2

Due Thursday October 6, 2022

Problem 1. The quantum anharmonic oscillator is defined through the Hamiltonian

$$H = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{q}^2 + \frac{1}{4!}g\hat{q}^4. \tag{1}$$

In class we discussed the path integral representation of the system with the Lagrangian

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2 - \frac{1}{4!}gq^4, \tag{2}$$

or more conveniently, the Euclidean path integral with the Euclidean Lagrangian

$$L^E(q,\partial_\tau q) = \frac{1}{2}(\partial_\tau q)^2 + \frac{1}{2}q^2 + \frac{1}{4!}gq^4, \tag{3}$$

where the Euclidean time τ is related to the real time t by Wick rotation $t = -i\tau$. In this problem you will analyze the two-point Euclidean Green function $\mathbb{G}(\tau)$, defined as

$$\mathbb{G}(\tau) \equiv \begin{cases} \langle 0|\hat{q}(\tau)\hat{q}(0)|0\rangle, & \tau \geq 0 \\ \langle 0|\hat{q}(0)\hat{q}(\tau)|0\rangle, & \tau < 0 \end{cases} \tag{4}$$

where $|0\rangle$ stands for the ground state of the system, perturbatively through its path integral representation

$$\mathbb{G}(\tau) = \frac{\int [Dq] e^{-\int d\tau L^E} q(\tau)q(0)}{\int [Dq] e^{-\int d\tau L^E}}. \tag{5}$$

(a) Compute $\mathbb{G}(\tau)$, or its Fourier transform $\widetilde{\mathbb{G}}(k)$ related by

$$\mathbb{G}(\tau) = \int \frac{dk}{2\pi} \widetilde{\mathbb{G}}(k) e^{ik\tau}, \tag{6}$$

at order g^2, using Feynman diagrams as described in class.

(b) In class, we argued that $\widetilde{\mathbb{G}}(k)$ takes the form

$$\widetilde{\mathbb{G}}(k) = \frac{1}{k^2 + 1 - \Sigma(k)}, \tag{7}$$

where $\Sigma(k)$ is computed by 1PI diagrams, and that the poles of $\widetilde{\mathbb{G}}(k)$ on the positive imaginary k-axis correspond to possible excitation energies of the state $\hat{q}(0)|0\rangle$. What can you conclude about the energy of excited states using the expression for $\Sigma(k)$ up to order g^2?

Remark: You could have extra fun by checking your answer against Hamiltonian perturbation theory to second order, or even by solving Schrödinger's equation numerically, but you are not required to do so.

Problem 2. Consider a scalar field theory in D dimensions, defined through the path integral representation with the action

$$S[\phi] = \int d^D x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}m^2(\phi(x))^2 - \frac{1}{3!}g(\phi(x))^3 \right], \tag{8}$$

where $x^\mu = (x^0, x^1, \cdots, x^{D-1})$ and $(\eta^{\mu\nu}) = \mathrm{diag}\{-1, 1, \cdots, 1\}$, or more conveniently, the Euclidean path integral with the Euclidean action[1]

$$S^E[\phi] = \int d^D x \left[\frac{1}{2}(\partial_\mu\phi(x))^2 + \frac{1}{2}m^2(\phi(x))^2 + \frac{1}{3!}g(\phi(x))^3 \right], \tag{9}$$

where $x = (x^1, \cdots, x^D)$, x^D being the Euclidean time related to the real time x^0 by the Wick rotation $x^0 = -ix^D$.

(a) We begin by analyzing the vacuum expectation value

$$\langle\phi\rangle \equiv \langle 0|\hat{\phi}(x)|0\rangle = \frac{\int [D\phi] e^{-S^E[\phi]}\phi(x)}{\int [D\phi] e^{-S^E}}, \tag{10}$$

where $|0\rangle$ stands for the vacuum state, $\hat{\phi}(x)$ is a local field operator defined through the path integral quantization. Calculate $\langle\phi\rangle$ at order g. You should find a divergent result whenever $D \geq 2$. We will cure this problem by carefully define the path integral using the following regularization:

$$[D\phi] \to \prod_{|k|<\Lambda} d\widetilde{\phi}(k), \tag{11}$$

where $\widetilde{\phi}(k)$ are the Fourier modes of $\phi(x)$. This will effectively cut off the momentum integral in the diagram for $\langle\phi\rangle$ and result in an answer that depends on the cutoff Λ.

At this point we could proceed by redefining $\phi'(x) = \phi(x) - \langle\phi\rangle$, and work with $\phi'(x)$ which has vanishing vacuum expectation value. Alternatively, we may modify the action S^E by adding a counter term

$$\Delta S^E = \int d^D x\, c_\Lambda \phi(x), \tag{12}$$

such that $\langle\phi\rangle = 0$ after taking into account the contribution from this counter term. These two procedures are equivalent up to a redefinition of m^2. It will be convenient to adopt the second point of view, and define the theory with the regularization (11) and the counter term (12) included in the action, and take the limit $\Lambda \to \infty$ in the end.

[1] This Euclidean action is not bounded from below due to the cubic coupling, but we will ignore this issue while working only perturbatively in g.

(b) Now you will analyze the two-point Euclidean Green function $\mathbb{G}(x)$, defined as

$$\mathbb{G}(x) \equiv \begin{cases} \langle 0|\hat{\phi}(x)\hat{\phi}(0)|0\rangle, & x^D \geq 0 \\ \langle 0|\hat{\phi}(0)\hat{\phi}(x)|0\rangle, & x^D < 0 \end{cases} \tag{13}$$

through its path integral representation

$$\mathbb{G}(x) = \frac{\int [D\phi] e^{-S^E[\phi]}\phi(x)\phi(0)}{\int [D\phi] e^{-S^E}}. \tag{14}$$

Compute $\mathbb{G}(x)$, or its Fourier transform $\widetilde{\mathbb{G}}(k)$ related by

$$\mathbb{G}(x) = \int \frac{d^D k}{(2\pi)^D} \widetilde{\mathbb{G}}(k) e^{ik\cdot x}, \tag{15}$$

at order g^2. You should find a well-defined answer when $D = 2, 3$ (and a divergent result when $D = 4$).

(c) By similar diagrammatics as discussed in the anharmonic oscillator example, we expect $\widetilde{\mathbb{G}}(k)$ here to take the form

$$\widetilde{\mathbb{G}}(k) = \frac{1}{k^2 + m^2 - \Sigma(k)}, \tag{16}$$

and you have just determined $\Sigma(k)$ at order g^2 in step (b). Compare with the spectral decomposition

$$\mathbb{G}(x) = \int d\alpha \langle 0|\hat{\phi}(0)|\alpha\rangle e^{i\vec{P}_\alpha \cdot \vec{x} - P^0_\alpha x^D} \langle \alpha|\hat{\phi}(0)|0\rangle \qquad (x^D > 0) \tag{17}$$

where $|\alpha\rangle$ is a basis of energy-momentum eigenstates that obey $\hat{P}^\mu|\alpha\rangle = P^\mu_\alpha|\alpha\rangle$, and the measure $d\alpha$ is normalized such that $\int d\alpha |\alpha\rangle\langle\alpha| = 1$. What can you conclude about the invariant masses of the states $|\alpha\rangle$ that contribute to this Green function?

Remark: unlike in problem 1 where $\widetilde{\mathbb{G}}(k)$ has only poles on the upper half imagine k-plane, corresponding to discrete (normalizable) energy eigenstates, here $\widetilde{\mathbb{G}}(k)$ as a function of complex k^D (say at fixed spatial momentum $\vec{k}$) may have poles as well as branch cuts, the latter corresponding to a continuum of (delta-function-normalizable) energy eigenstates.