Physics 253a Problem set 3

Due Thursday October 20, 2022

Problem 1. Consider in D-dimensional spacetime a free scalar field theory with the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2. \tag{1}$$

In class we have derived the Fourier representation of the time-ordered 2-point function

$$G(x,y) \equiv \langle\Omega|\mathbf{T}\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle = -i\int \frac{d^Dk}{(2\pi)^D}\frac{e^{ik\cdot(x-y)}}{k^2+m^2-i\epsilon} \tag{2}$$

where $k\cdot(x-y) = \vec{k}\cdot(\vec{x}-\vec{y}) - k^0(x^0-y^0)$ and $k^2 = -(k^0)^2+\vec{k}^2$, as well as the Euclidean 2-point function

$$G_E(x_E,y_E) \equiv \langle\hat{\phi}_E(x_E)\hat{\phi}_E(y_E)\rangle = \int \frac{d^Dk_E}{(2\pi)^D}\frac{e^{ik_E\cdot(x_E-y_E)}}{k_E^2+m^2}, \tag{3}$$

where $k_E\cdot(x_E-y_E) = \sum_{\mu=1}^D k_E^\mu(x_E-y_E)^\mu$ and $k_E^2 = \sum_{\mu=1}^D (k_E^\mu)^2$, and $\hat{\phi}_E(x_E)$ is related to $\hat{\phi}(x)$ by the analytic continuation in x^0 that takes $x^0 \to -ix_E^D$.

For general D, the k-integral can be expressed in terms of Bessel functions. When D is an odd integer, the result is an elementary function. In this problem, you are asked to consider the $D=3$ case, and write an elementary expression for $G(x,y)$, $G_E(x_E,y_E)$, as well as the Wightman function

$$W(x,y) \equiv \langle\Omega|\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle. \tag{4}$$

Explain how $W(x,y)$ is related to $G_E(x_E,y_E)$ by analytic continuation, and how $W(x,y)$ differs from $W(y,x)$ for time-like separated x,y.

To be sure that you understand what's going on, make a plot for both the real and imaginary part of $W(x,0)$ as a function of (real time) x^0 in the case $m=0$ for some fixed spatial coordinate $\vec{x}\neq 0$.

Problem 2. Consider in D-dimensional spacetime a free *complex* scalar field theory with the Lagrangian density

$$\mathcal{L} = -\eta^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi - m^2\varphi^*\varphi, \tag{5}$$

where $\varphi = \phi_1 + i\phi_2$, $\varphi^* = \phi_1 - i\phi_2$. The $U(1)$ symmetry that rotates the phase of φ is associated with a Noether current operator, of the form[1]

$$\hat{j}_\mu(x) = -i : \hat{\varphi}^*(x)\overleftrightarrow{\partial}_\mu\hat{\varphi}(x) : \tag{7}$$

where $f\overleftrightarrow{\partial}_\mu g \equiv f\partial_\mu g - (\partial_\mu f)g$.

(a) Verify that $\hat{j}^\mu$ is conserved, namely $\partial_\mu\hat{j}^\mu = 0$ as an operator.

(b) Calculate the time-ordered 3-point correlation function

$$\begin{aligned} H^\mu(x_1, x_2, x_3) &\equiv \langle\Omega|\mathbf{T}\hat{\varphi}^*(x_1)\varphi(x_2)\hat{j}^\mu(x_3)|\Omega\rangle \\ &= \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\widetilde{H}^\mu(k_1, k_2)e^{ik_1\cdot x_1 + ik_2\cdot x_2 - i(k_1+k_2)\cdot x_3}. \end{aligned} \tag{8}$$

(c) Calculate

$$\frac{\partial}{\partial x_3^\mu}H^\mu(x_1, x_2, x_3) = \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}(-i(k_1+k_2)_\mu)\widetilde{H}^\mu(k_1, k_2)e^{ik_1\cdot x_1 + ik_2\cdot x_2 - i(k_1+k_2)\cdot x_3}. \tag{9}$$

Does the result vanish? Explain whether/how it is compatible with your answer to **(a)**.

[1] An equivalent expression for $\hat{j}_\mu$ in terms of $\hat{\phi}$ is

$$\hat{j}_\mu(x) = -i\lim_{y\to x}\left[\hat{\varphi}^*(y)\overleftrightarrow{\partial}_\mu\hat{\varphi}(x) - \left\langle\hat{\varphi}^*(y)\overleftrightarrow{\partial}_\mu\hat{\varphi}(x)\right\rangle\right]. \tag{6}$$

Problem 3. In class we introduced the 1-particle state $|\vec{k}\rangle$ for a massive particle with no internal degree of freedom, normalized according to $\langle\vec{k}|\vec{k}'\rangle = \delta^{D-1}(\vec{k}-\vec{k}')$, on which the Lorentz transformation acts by

$$U(\Lambda)|\vec{k}\rangle = \sqrt{\frac{(\Lambda k)^0}{k^0}}|\overrightarrow{\Lambda k}\rangle, \tag{10}$$

where $k^0 = \omega_{\vec{k}} \equiv \sqrt{\vec{k}^2+m^2}$. Recall that for infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$, $U(\Lambda) = 1 + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}$. Restricting to the case of $D=4$ spacetime dimensions, the usual angular momentum operator $\vec{J} = (J_1, J_2, J_3)$ is related by $\hat{J}_{ij} = \sum_{k=1}^3 \epsilon_{ijk} J_k$.

Now consider a two-particle state $|\vec{k}_1, \vec{k}_2\rangle$. For now you may assume that the two particles are not interacting, and that $U(\Lambda)$ acts by the obvious generalization of (10) [2]

$$U(\Lambda)|\vec{k}_1, k_2\rangle = \sqrt{\frac{(\Lambda k_1)^0(\Lambda k_2)^0}{k_1^0 k_2^0}}|\overrightarrow{\Lambda k_1}, \overrightarrow{\Lambda k_2}\rangle. \tag{11}$$

Assuming the two particles are identical bosons, it is natural to normalize the two-particle state according to

$$\langle\vec{k}_1, \vec{k}_2|\vec{k}_1', \vec{k}_2'\rangle = \delta^3(\vec{k}_1-\vec{k}_1')\delta^3(\vec{k}_2-\vec{k}_2') + \delta^3(\vec{k}_1-\vec{k}_2')\delta^3(\vec{k}_2-\vec{k}_1'). \tag{12}$$

We can consider an alternative basis $|E, \vec{P}, \ell, m\rangle$ for the two-particle state, that are eigenstates with respect to the total energy-momentum vector (of eigenvalue $(E, \vec{P})$), and with respect to the angular momentum squared $\vec{J}^2 = \ell(\ell+1)$ and the angular momentum component $J_3 = m$ defined in the center-of-mass frame. We will normalize these states according to the convention

$$\langle E, \vec{P}, \ell, m|E', \vec{P}', \ell', m'\rangle = \delta(E-E')\delta^3(\vec{P}-\vec{P}')\delta_{\ell\ell'}\delta_{mm'}. \tag{13}$$

It is easy to see that the overlap between the two kinds of basis states are of the form

$$\langle\vec{k}_1, \vec{k}_2|E, \vec{P}, \ell, m\rangle = \delta(\omega_{\vec{k}_1}+\omega_{\vec{k}_2}-E)\delta^3(\vec{k}_1+\vec{k}_2-\vec{P})f_{\ell m}(E, \vec{P}; \hat{k}_1) \tag{14}$$

where the function $f_{\ell m}$ depends only on $E, \vec{P}$, and the unit vector $\hat{k}_1 \equiv \vec{k}_1/|\vec{k}_1|$. Your task is to determine $f_{\ell m}$ in the special case $\vec{P}=0$. In particular, show that

$$f_{\ell m}(E, 0; \hat{k}_1) = \mathcal{N}(E) Y_{\ell m}(\hat{k}_1), \tag{15}$$

where $Y_{\ell m}$ are the spherical harmonics, and determine the function $\mathcal{N}(E)$.

[2] We will see in class soon that this is true for the so-called in- and out- asymptotic scattering states in an interacting quantum field theory as well.

Problem 4. A key fact concerning the Lorentz transformation property of the 1-particle state, as discussed in class, is that the action of $U(\Lambda)$ on the 1-particle state is determined through that of $U(W)$, where W is a Lorentz transformation that leaves the momentum of the particle invariant. The set of all such W's for a given particle momentum is called the "little group". In this problem you will study the little group of a *massless* particle in $D = 4$ spacetime dimensions.

(a) We begin with a light-like reference momentum k_R^μ,

$$(k_R^0, k_R^1, k_R^2, k_R^3) = (E, 0, 0, E). \tag{16}$$

For any spatial momentum 3-vector $\vec{k}$, and the corresponding light-like 4-momentum vector $(k^0 = |\vec{k}|, \vec{k})$, we can construct a "standard" Lorentz transformation $L(\vec{k})$ that takes k_R to k as

$$L(\vec{k}) = M(\vec{k}) \cdot R_1, \tag{17}$$

where

$$(M(\vec{k}))^\mu{}_\nu = \delta^\mu_\nu + \frac{(k - k_R)^\mu (k - k_R)_\nu}{k \cdot k_R}, \tag{18}$$

and R_1 is the reflection in the x^1-direction. Verify that (17) is indeed a Lorentz transformation of the desired property.

(b) $W = (L(\overrightarrow{\Lambda k_R}))^{-1}\Lambda$, which leaves k_R invariant (i.e. $W k_R = k_R$) for any generic Lorentz transformation Λ, is an element of the little group. Find an example of W that is not merely a rotation about the x^3 axis nor a reflection.

Remark: The structure of the massless little group you find here, as we will see in a few weeks, is closely related to why the photon is necessarily described by a gauge theory.