Physics 253a Problem set 4

Due Thursday November 3, 2022

Our convention for scattering amplitudes is such that the $2 \to 2$ S-matrix element (in the case of identical in- and out-particles) is written as

$$\begin{aligned} S(\vec{k}_1', \vec{k}_2' | \vec{k}_1, \vec{k}_2) &\equiv {}^{out}\langle \vec{k}_1', \vec{k}_2' | \vec{k}_1, \vec{k}_2 \rangle^{in} \\ &= \delta^{D-1}(\vec{k}_1' - \vec{k}_1)\delta^{D-1}(\vec{k}_2' - \vec{k}_2) + \delta^{D-1}(\vec{k}_1' - \vec{k}_2)\delta^{D-1}(\vec{k}_2' - \vec{k}_1) \\ &\quad + i(2\pi)^D \delta^D(k_1' + k_2' - k_1 - k_2)\mathcal{M}(\vec{k}_1', \vec{k}_2', \vec{k}_1, \vec{k}_2), \end{aligned} \tag{1}$$

where $\mathcal{M}$ is related to the manifestly Lorentz invariant form of the amplitude $\mathcal{A}$ by

$$\mathcal{M}(\vec{k}_1', \vec{k}_2', \vec{k}_1, \vec{k}_2) \equiv \prod_{j=1}^{2} \frac{1}{(2\pi)^{\frac{D-1}{2}}\sqrt{2\omega_{k_j'}}} \prod_{i=1}^{2} \frac{1}{(2\pi)^{\frac{D-1}{2}}\sqrt{2\omega_{k_i}}} \mathcal{A}(k_1', k_2' | k_1, k_2), \tag{2}$$

where $\omega_k \equiv \sqrt{\vec{k}^2 + m^2}$.

Also recall that in $D = 4$ dimensions, the partial wave amplitude $S_\ell(E)$, at total energy E and angular momentum $\vec{J}^2 = \ell(\ell+1)$ in the center-of-mass frame ($\vec{k}_1 + \vec{k}_2 = \vec{k}_1' + \vec{k}_2 = 0$) is related by

$$i(2\pi)^4 \mathcal{M}(\vec{k}_1', \vec{k}_2' | \vec{k}_1, \vec{k}_2) = (\mathcal{N}(E))^2 \frac{1}{4\pi} \sum_{\ell} (2\ell+1) P_\ell(\hat{k}_1 \cdot \hat{k}_1')(S_\ell(E) - 1), \tag{3}$$

where $P_\ell(x)$ is the ℓ-th Legendre polynomial. For identical particles, $\mathcal{N}(E) = \sqrt{\frac{8}{|\vec{k}_1| E}}$, and the sum is restricted to even non-negative integer ℓ.

Problem 1. Consider a field theory in $D = 4$ dimensions described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2} m_1^2 \phi_1^2 - \frac{1}{2} m_2^2 \phi_2^2 - \frac{g}{2} \phi_1^2 \phi_2, \tag{4}$$

where $\phi_1(x)$ and $\phi_2(x)$ are a pair of scalar fields, with $m_2 > m_1 > 0$.

(a) Calculate the $2 \to 2$ scattering amplitude $\mathcal{A}(k_1', k_2' | k_1, k_2)$ of the lightest particle in perturbation theory up to order g^2 (tree level).

(b) Calculate the $\ell = 0$ partial wave (i.e. "s-wave") amplitude $S_0(E)$ up to order g^2. Below the energy threshold for inelastic processes, unitarity implies that $S_0(E)$ is a phase. Is your answer consistent with this expectation?

Problem 2. Consider the scalar field theory described by the same Lagrangian as in problem 1, but now in $D = 4 - \delta$ dimensions ($\delta > 0$), and with additional linear counter terms so that $\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$.

(a) Calculate the mass of the lightest particle, as well as the Z factor of ϕ_1, in perturbation theory up to order g^2.

(b) Calculate the self-energy $\Sigma_2(k)$ of ϕ_2 up to order g^2, related to the time-ordered 2-point function via

$$\langle \Omega | \mathbf{T} \hat{\phi}_2(x) \hat{\phi}_2(0) | \Omega \rangle = -i \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\cdot x}}{k^2 + m_2^2 - i\epsilon - \Sigma_2(k)}. \tag{5}$$

Describe the spectral function $\rho_2(\mu^2)$ related via the Källén-Lehmann spectral representation

$$\frac{1}{k^2 + m_2^2 - i\epsilon - \Sigma_2(k)} = \int_0^\infty d\mu^2 \, \frac{\rho_2(\mu^2)}{k^2 + \mu^2 - i\epsilon}. \tag{6}$$

Note: there are two qualitatively distinct cases: (1) $m_2 < 2m_1$*, and (2)* $m_2 > 2m_1$*.*

(c) Interpret your results in the $D \to 4$ (i.e. $\delta \to 0$) limit.

Problem 3. Consider the ϕ^4 theory defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!}\phi^4 \tag{7}$$

in $D = 4 - \delta$ spacetime dimensions.

(a) Calculate the $2 \to 2$ amplitude $\mathcal{A}(k_1', k_2' | k_1, k_2)$ up to order g^2 (1-loop order). Express your result as a function of the Mandelstam invariants

$$s = -(k_1 + k_2)^2, \quad t = -(k_1 - k_1')^2, \quad u = -(k_1 - k_2')^2. \tag{8}$$

Note: you may choose to express the result in terms of the physical mass; there is no need to compute the mass counter term explicitly.

(b) Now take the $D \to 4$ ($\delta \to 0$) limit. You should find a divergence that can be canceled by a counter term of the form

$$\Delta\mathcal{L} = \frac{C}{\delta} g^2 \phi^4, \tag{9}$$

where C is a numerical constant. The term (9) can be absorbed into the definition of the bare coupling in the Lagrangian. Define a renormalized coupling g_R via

$$-g_R \equiv \mathcal{A}\Big|_{s=t=u=\frac{4m^2}{3}}\,, \tag{10}$$

and then express the amplitude $\mathcal{A}$ at generic momenta as a function of the renormalized coupling. *(Hopefully, the result will be manifestly finite in the $\delta \to 0$ limit.)*

Note: the condition on the RHS of (10) requires analytic continuation of the momenta outside of the regime of physical kinematics; nonetheless, the mass-shell condition and the momentum conservation is preserved, and such an analytic continuation is unambiguous.