Physics 253a Problem set 5

Due Friday November 18, 2022

Problem 1. In class we have discussed massive spin-$\frac{1}{2}$ particle and the polarization of a spinor field $\hat{\psi}_\alpha(x)$. In this problem you will consider the case where the particle is *massless*. Let $|\vec{k}, h\rangle$ label the 1-particle state of a massless particle (energy $\omega_{\vec{k}} = |\vec{k}|$) with helicity h, namely

$$\vec{J} \cdot \hat{k} |\vec{k}, h\rangle = h|\vec{k}, h\rangle. \tag{1}$$

"spin-$\frac{1}{2}$" for a massless particle means that $h = \frac{1}{2}$ or $-\frac{1}{2}$. Note that h is invariant under Lorentz symmetry, and so it is possible that only the $h = +\frac{1}{2}$ state exists, or only the $h = -\frac{1}{2}$ state exists. The spinor polarization $u^h_\alpha(\vec{k})$ appears in the relation

$$\langle \vec{k}, h|\hat{\psi}_\alpha(x)|\Omega\rangle = Z_\psi^{\frac{1}{2}} \frac{e^{-ik\cdot x}}{(2\pi)^{\frac{3}{2}}} u^h_\alpha(\vec{k}), \tag{2}$$

where Z_ψ is the field renormalization constant.

Now consider the reference momentum $k_R = (E, 0, 0, E)$, in which case the little group consists of Lorentz transformations of the form

$$W(a, b, \theta) = e^{iaA+ibB} e^{i\theta M_{12}}, \tag{3}$$

where

$$A = M_{01} + M_{31} = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = M_{02} + M_{32} = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4}$$

and $(M_{\mu\nu})^\rho{}_\sigma = -i(\delta^\rho_\mu \eta_{\nu\sigma} - \delta^\rho_\nu \eta_{\mu\sigma})$. As seen in class, it follows from Lorentz symmetry that $u^h_\alpha(k_R)$ obeys

$$u^h_\alpha(\vec{k}_R) = (D(W))^* (R(W))_\alpha{}^\beta u^h_\beta(\vec{k}_R), \tag{5}$$

where $D(W(a, b, \theta)) = e^{ih\theta}$.

(a) What is the matrix $R(W(a, b, \theta))$?

(b) By inspecting (5) with nonzero a or b, show that $\gamma_5 u^h = 2hu^h$. (*This means that if $\hat{\psi}_\alpha(x)$ obeys say $\gamma_5\hat{\psi} = \hat{\psi}$, it can only create a particle of helicity $+\frac{1}{2}$ and not $-\frac{1}{2}$.*)

Problem 2. Consider in $D = 4$ spacetime dimensions a theory of Dirac fermion field $\psi_\alpha(x)$ with self-interaction described by the Lagrangian density

$$\mathcal{L} = -\bar{\psi}(\partial\!\!\!/ + m)\psi - \frac{1}{2}g(\bar{\psi}\psi)(\bar{\psi}\psi), \tag{6}$$

where g is a coupling constant.

(a) Calculate the tree level (i.e. order g) $2 \to 2$ scattering amplitude of a pair fermions of the same type (i.e. both particles, as opposed to anti-particles). You can express the result in terms of the spinor polarizations. *(Be careful with the relative sign between different diagrams/Wick contractions!)*

(b) In the center-of-mass frame, calculate the total scattering cross section due to such an elastic process at tree level, for a pair of incoming particles of helicity $+\frac{1}{2}$,[1] as a function of the total energy E.

Remark: in computing the cross section by summing over out-states, you can simplify your computation using the formulae

$$\begin{aligned}\sum_\sigma u^\sigma(\vec{k})\overline{u^\sigma(\vec{k})} &= \frac{m - ik\!\!\!/}{2\omega_k}, \quad \text{or} \\ \sum_\sigma v^\sigma(\vec{k})\overline{v^\sigma(\vec{k})} &= \frac{-m - ik\!\!\!/}{2\omega_k}.\end{aligned} \tag{7}$$

In dealing with the in-state spinor polarizations, you may find it convenient to assume (without loss of generality) that the incoming particle momenta are in the x^3-direction, in which case $u^\sigma(k\hat{x}_3)$ can be obtained from $u^\sigma(\vec{0})$ by the Lorentz boost

$$u^\sigma(k\hat{x}_3) = \sqrt{\frac{m}{\omega_k}}\exp\left(-i\nu_k S^{03}\right)u^\sigma(\vec{0}), \tag{8}$$

where $S^{03} = -\frac{i}{4}[\gamma^0, \gamma^3]$, and $\nu_k = \operatorname{arctanh}\frac{k}{\omega_k}$ is the rapidity of the particle.

[1] Helicity of a massive particle is defined the same way as in (1). Unlike the massless case, massive particle states of different helicities can mix under Lorentz transformations.