Outline of QFT II

Part 1: the S-matrix

- bound states and resonances
- implications of causality and unitarity
- infrared divergence

Part 2: Renormalization Group

- BPHZ renormalizability
- the 1PI effective action
- improved perturbation theory
  (via Gell-Mann-Low equation)
- Wilson-Polchinski renormalization group
- ϵ-expansion

Part 3: QCD

- quantization of non-Abelian gauge theory
- BRST and BV formalisms
- asymptotic freedom

- OPE and deep inelastic scattering
- $1/N$ expansion
- chiral symmetry breaking and the effective theory of pions
- anomalies
- instantons

Recap of some basic aspects of QFT (from 253a)

- QM system with Hilbert space $\mathcal{H}$ and Poincaré symmetry represented by unitary operator $U(\Lambda, a)$

$$U(\Lambda', a')\, U(\Lambda, a) = U(\Lambda'\Lambda, \Lambda' a + a')$$

infinitesimal

$$U(\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \,,\; a^\mu = \epsilon^\mu) = 1 - i\epsilon^\mu \hat{P}_\mu + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}$$

energy-momentum

$$\hat{P}^\mu \equiv (P^0 = H, P^i)$$

$$= \int d^{D-1}\vec{x}\; T^{0\mu}(x)$$

angular-momentum/boost

$$\hat{J}^{\mu\nu} = \int d^{D-1}\vec{x}\, \left[x^\mu T^{0\nu}(x) - (\mu \leftrightarrow \nu) \right]$$

$T^{\mu\nu}(x)$ - stress-energy tensor

Assume vacuum state $|\Omega\rangle$ Poincaré-invariant

$$\hat{P}^\mu |\Omega\rangle = \hat{J}^{\mu\nu} |\Omega\rangle = 0.$$

(local) field operator $\hat{\Phi}_\alpha(x)$

↳ Lorentz index

$$U(\Lambda, a)\, \hat{\Phi}_\alpha(x) \left(U(\Lambda, a)\right)^{-1}$$

$$= (R(\Lambda))_\alpha{}^\beta\, \hat{\Phi}_\beta(\Lambda x + a)$$

↑ represation matrix of Lorentz group (or its covering group)

Assuming the Hilbert space is spanned by asymptotic states of particles, in particular the 1-particle state $|\vec{k}\rangle$

the obeys $\hat{P}^i|\vec{k}\rangle = k^i|\vec{k}\rangle$

$$H|\vec{k}\rangle = \sqrt{\vec{k}^2+m^2}\,|\vec{k}\rangle$$

$\omega_k \equiv \sqrt{\vec{k}^2+m^2}$

for some mass m.

A typical scalar field operator $\hat{\phi}(x)$ has

(assuming 1 species of scalar particle)

$$\hat{\phi}(x)|\Omega\rangle$$

$$= \int d^{D-1}\vec{k}\; e^{-i\vec{k}\cdot\vec{x}+i\omega_k x^0}\left[\frac{Z_\phi}{(2\pi)^{D-1}}\,\frac{1}{2\omega_k}\right]^{\frac{1}{2}}|\vec{k}\rangle$$

$$+ \int_{m.p.} d\alpha\; e^{-iP_\alpha\cdot x}\,\langle\alpha|\hat{\phi}(0)|\Omega\rangle\,|\alpha\rangle$$

Spectral representation of

2-point Wightman function

$$\langle\Omega|\hat{\phi}(x)\,\hat{\phi}(y)|\Omega\rangle = \int_0^\infty d\mu^2\,\rho(\mu^2)\,\Delta_+(x-y;\mu^2)$$

$$\Delta_+(x;\mu^2) = \int\frac{d^Dk}{(2\pi)^{D-1}}\,\theta(k^0)\,\delta(k^2+\mu^2)\,e^{ik\cdot x}$$

vs that of time-ordered Green function

$$\langle \Omega | T \, \hat{\Phi}(x) \, \hat{\Phi}(y) \, | \Omega \rangle = \int_0^\infty d\mu^2 \, \rho(\mu^2) \, \Delta_F(x-y;\mu^2)$$

$$\Delta_F(x;\mu^2) = -i \int \frac{d^D k}{(2\pi)^D} \, \frac{e^{ik\cdot x}}{k^2+\mu^2-i\epsilon}$$

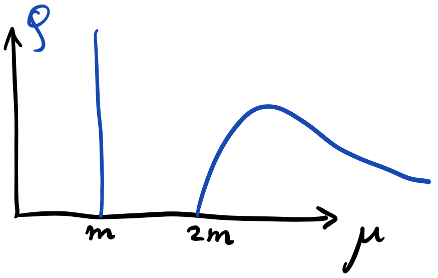
vs that of Euclidean Green function

$$\langle \phi(x^E) \, \phi(y^E) \rangle = \int_0^\infty d\mu^2 \, \rho(\mu^2) \, \Delta_E(x^E-y^E;\mu^2)$$

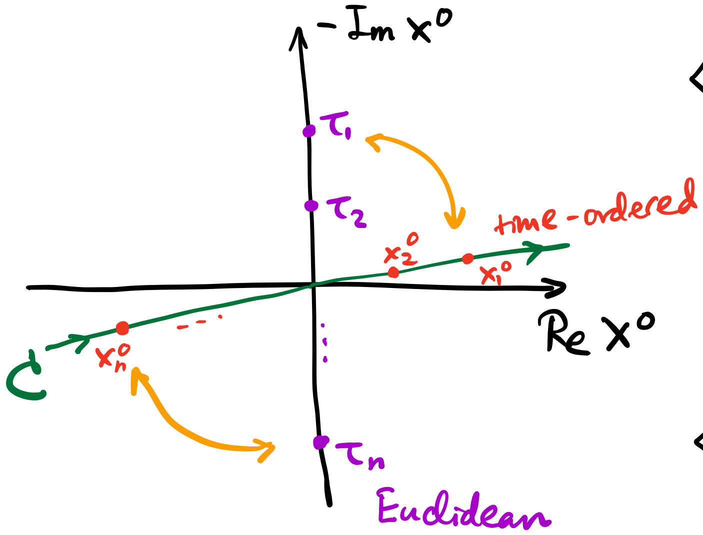
$$\Delta_E(x^E;\mu^2) = \int \frac{d^D k^E}{(2\pi)^D} \, \frac{e^{ik^E\cdot x^E}}{(k^E)^2+\mu^2} .$$

$$\rho(\mu^2) = Z_\phi \, \delta(\mu^2-m^2) + \underset{\uparrow}{\sigma(\mu^2)}$$

≥ 0, supported at m.p. invt masses



The time-ordered and Euclidean Green fns are related by Wick rotation:



$$\langle \Omega | T\, \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | \Omega \rangle$$

$$\Big\downarrow \quad x_a^0 = e^{-i\alpha}\tau_a \qquad \alpha : 0 \to \frac{\pi}{2}$$

$$\langle \phi(x_1^E) \cdots \phi(x_n^E) \rangle$$

$$x_a^D = \tau_a$$

Path integral representation:

action $\quad S = \int d^D x\, \mathcal{L}[\phi]$

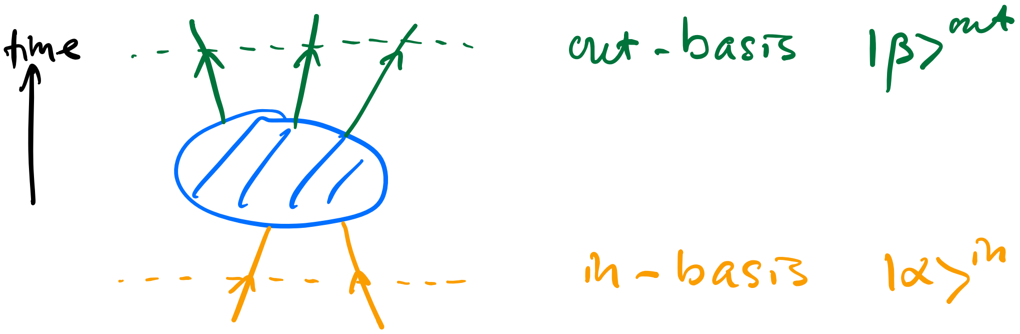
vs Euclidean action $\quad S^E = \int d^D x^E\, \mathcal{L}^E[\phi]$

$$\mathcal{L}^E[\phi] = -\mathcal{L}[\phi]\Big|_{\substack{x^0 \to -i(x^E)^D \\ \vec{x} = \vec{x}^E}} .$$

$$\langle \phi(x_1^E) \cdots \phi(x_n^E) \rangle = \frac{1}{Z} \int [D\phi]\, e^{-S^E[\phi]}\, \phi(x_1^E) \cdots \phi(x_n^E)$$

$$\langle \Omega | T\, \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | \Omega \rangle = \frac{1}{Z} \int [D\phi]\, e^{i\int_C dx^0 d^{D-1}\vec{x}\, \mathcal{L}}\, \phi(x_1) \cdots \phi(x_n)$$

at early or late times, a generic state admits particle interpretation:



S-matrix $\quad S_{\beta\alpha} \equiv {}^{out}\langle\beta|\alpha\rangle^{in}$

Equivalently,

$$|\alpha\rangle^{in} = \int d\beta \, S_{\beta\alpha} \, |\beta\rangle^{out}$$

measure is such that $\int d\beta \, |\beta\rangle^{out} \, {}^{out}\langle\beta| = 1$.

The asymptotic in/out-states are tied to local field operator via

$$|\phi_{f_1}, \cdots, \phi_{f_n}\rangle^{out/in} = \lim_{T\to\pm\infty} \hat{\phi}_{f_1}(T) \cdots \hat{\phi}_{f_n}(T) \, |\Omega\rangle \tag{$\star$}$$

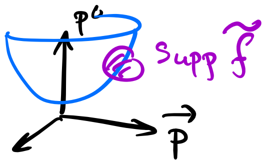
$$\|$$

$$\int \prod_{i=1}^{n} \Big(d^{D-1}\vec{k}_i \, \tilde{f}_i(\omega_i,\vec{k}_i) \left[\frac{Z_\phi}{(2\pi)^{D-1}} \frac{1}{2\omega_{k_i}} \right]^{\frac{1}{2}} |\vec{k}_1,\dots,\vec{k}_n\rangle^{out/in}$$

where $\hat{\phi}_f \equiv \int d^D x \; f(x)\, \hat{\phi}(x)$,

such that $f(x) \equiv \int \frac{d^D k}{(2\pi)^D} \tilde{f}(k)\, e^{ik\cdot x}$

$\tilde{f}(k)$ supported near 1-particle mass-shell



and $f^{(T)}(x)$ is the transform of $f(x)$ defined by

$$f^{(T)}(x) = \int \frac{d^D k}{(2\pi)^D} \tilde{f}(k)\, e^{i(k^0-\omega_k)T + ik\cdot x} .$$

In particular, $\hat{\phi}_{f^{(T)}} |\Omega\rangle = \hat{\phi}_f |\Omega\rangle$

(consisting of only 1-particle states)

LSZ reduction formula follows from ⊛ and the cluster property of Green fns.

Cluster property of S-matrix elmts:

$$S(\beta|\alpha) \equiv {}^{out}\langle\beta|\alpha\rangle^{in}$$

$$= \sum_{\substack{\alpha=\amalg\alpha_I \\ \beta=\amalg\beta_I}} \prod_I S^{conn}(\beta_I|\alpha_I)$$

LSZ: connected S-matrix elmts, or "scattering amplitudes", are related to Fourier transform of connected time-ordered Green function via

$$S^{conn}(\vec{k}_1,\dots,\vec{k}_n\,|-\vec{k}_1',\dots,-\vec{k}_m')$$

$$= \prod_{i=1}^{n}\left[Z_\phi(2\pi)^{D-1}2\omega_{k_i}\right]^{-\frac{1}{2}} \prod_{j=1}^{m}\left[Z_\phi(2\pi)^{D-1}2\omega_{k_j'}\right]^{-\frac{1}{2}}$$

$$\times \lim_{\substack{k_i^0\to\omega_{k_i}\\ k_j'^0\to-\omega_{k_j'}}} \prod_i i(k_i^2+m^2)\prod_j i(k_j'^2+m^2)$$

$$\times\ \tilde{G}^{conn}(k_1,\dots,k_n,k_1',\dots,k_m')$$

$$\tilde{G}^{conn} = \int d^Dx_1\cdots e^{-ik_1\cdot x_1-\cdots}\ \underset{\substack{\uparrow\\ \text{time-ordered}}}{G^{conn}}(x_1,\dots,x_1',\dots)$$

Convention:

$$S^{\text{conn}}(\vec{k}_1, \cdots | \vec{k}_1', \cdots)$$

$$\equiv i(2\pi)^D \delta^D(k_1 + \cdots + k_n - k_1' - \cdots - k_m')$$

$$\cdot \mathcal{M}(\vec{k}_1, \cdots, \vec{k}_n | \vec{k}_1', \cdots, \vec{k}_m')$$

$$\mathcal{M} \equiv \prod_i \frac{1}{(2\pi)^{\frac{D-1}{2}} \sqrt{2\omega_{k_i}}} \prod_j \frac{1}{(2\pi)^{\frac{D-1}{2}} \sqrt{2\omega_{k_j'}}}$$

$$\times \mathcal{A}(k_1, \cdots, k_n | k_1', \cdots, k_m')$$

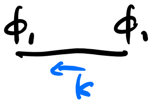
$\mathcal{A}$ is Lorentz-invariant.

e.g. ϕ_1, ϕ_2 scalar fields

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2} m_1^2 \phi_1^2$$

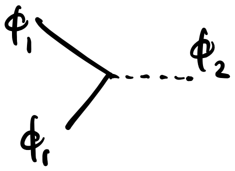
$$-\frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2} m_2^2 \phi_2^2 - \frac{1}{2} g \phi_1^2 \phi_2$$

Lorentzian Feynman rule



$$= \frac{-i}{k^2 + m_1^2 - i\epsilon}$$

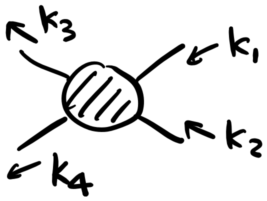
$$= \frac{-i}{k^2 + m_2^2 - i\epsilon}$$



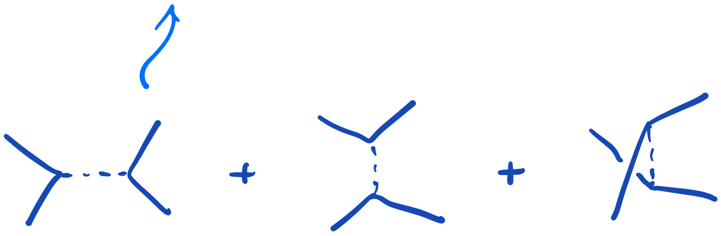
$$= -ig$$

Consider the 2→2 scattering amplitude of the particle created by ϕ_1

$$i\mathcal{A}(k_3,k_4;k_1,k_2) =$$



$$= i\mathcal{A}^{\text{tree}} + \mathcal{O}(g^4)$$



$$i\mathcal{A}^{\text{tree}} = (-ig)^2 \frac{-i}{(k_1+k_2)^2+m_2^2-i\epsilon} + (k_2 \to -k_3) + (k_2 \to -k_4)$$

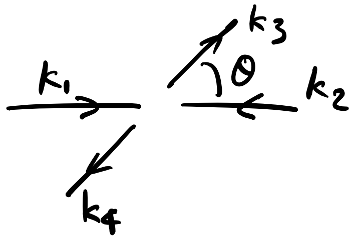
Define Mandelstam variables

$$s \equiv -(k_1+k_2)^2, \quad t = -(k_1-k_3)^2, \quad u = -(k_1-k_4)^2$$

$$i\mathcal{A}^{\text{tree}} = ig^2\left(\frac{1}{-s+m_2^2-i\epsilon} + \frac{1}{-t+m_2^2-i\epsilon} + \frac{1}{-u+m_2^2-i\epsilon}\right)$$

We will spend a little time interpreting this simple-looking result.

Note: in terms of center-of--mass total energy E and scattering angle θ,



$$s = E^2, \quad t = -\frac{E^2 - 4m_{1*}^2}{2}(1 - \cos\theta)$$

$$u = -\frac{E^2 - 4m_{1*}^2}{2}(1 + \cos\theta)$$

the physical mass m_{1*} of the particle may differ from m_1 at higher orders in g, but we will ignore this for now.

Case I : $m_2 < 2m_1$

For physically admissible kinematics,

$E \geq 2m_1 \;\Rightarrow\; s > m_2^2$ and $t, u < 0$

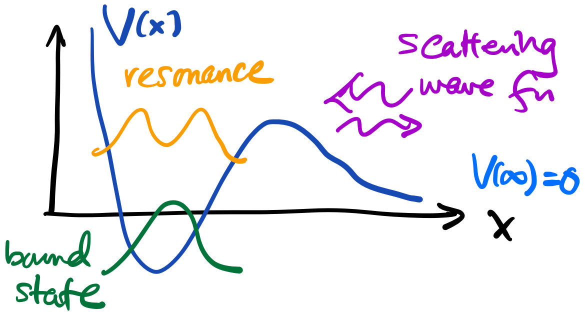
$\mathcal{A}^{\text{tree}}$ is always finite

If we analytically continue $\mathcal{A}^{\text{tree}}$ on the complex E-plane (or s-plane), we find a pole at $E = m_2$ $(< 2m_1)$

Interpretation: the particle of mass m_2 is a bound state of a pair of particles of mass m_1.

Comparison with a simple non-relativistic QM model: scattering in 1D

$$H = \frac{p^2}{2m} + V(x)$$



In the asymptotic region $x \to \infty$, a scattering wave function takes the form

“in-state”

$$\psi_k^{in}(x) \sim e^{-ikx} + \underbrace{S(k)}_{} e^{ikx}$$

scattering amplitude (analogous to partial wave amplitude in higher dim)

$$\updownarrow$$

$$\psi_k^{out}(x) = (S(k))^{-1} \psi_k^{in}(x)$$
$$\sim (S(k))^{-1} e^{-ikx} + e^{ikx} .$$

$$k = \sqrt{2mE}$$ (assume $V(\infty) = 0$)

- a bound state, on the other hand, has asymptotic wave function

$$\psi(x) \sim e^{-\alpha x}, \quad x \to \infty.$$

$$\alpha = \sqrt{-2mE} > 0 \quad (E<0)$$

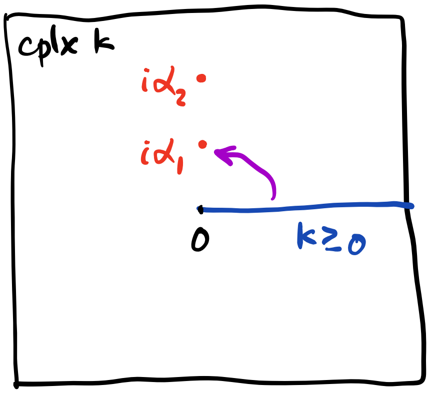
. What is the relation between $S(k)$ and the bound states ?

- If we do not impose the normalizability of the wave function in the asymptotic region $x \to \infty$, we can analytically continue $\psi_k^{out}(x)$ as a solution to the Schrödinger equation at complex k.

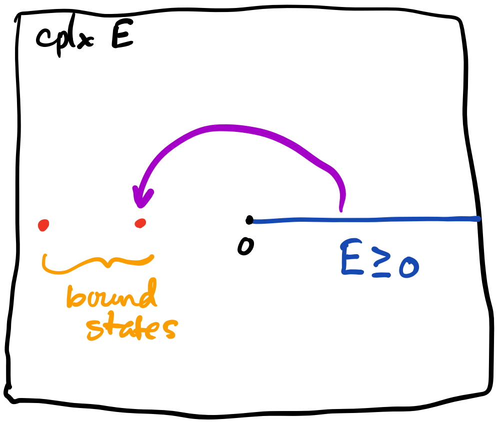
$k \to i\alpha$, $\psi_k^{out}(x)$ becomes

$$\psi_{i\alpha}^{out}(x) \sim \underbrace{(S(i\alpha))^{-1} e^{\alpha x}}_{\substack{\text{must vanish} \\ \text{for a bound state}}} + e^{-\alpha x}$$

bound state $\longleftrightarrow$ pole of $S(k)$ at $k=i\alpha$, $\alpha>0$.



$$E = \frac{k^2}{2m}$$



The precise analog of this in QFT is the bound state pole in a partial wave amplitude.

recall the relation between 2-(scalar-) particle plane-wave basis and partial-wave basis in C.O.M. frame (D=4 case):

$$\langle \vec{k}_1, \vec{k}_2 \,|\, E, \vec{P}=0, \ell, m \rangle$$

$$= \delta^3(\vec{k}_1 + \vec{k}_2)\, \delta(\omega_{k_1} + \omega_{k_2} - E)\, \mathcal{N}(E)\, Y_{\ell m}(\hat{k}_1)$$

$$\mathcal{N}(E) = \sqrt{\frac{2E}{|\vec{k}_1|\, \omega_{k_1} \omega_{k_2}}}$$

for identical particles

$$|E, \ell, m\rangle^{in} = S_\ell(E)\, |E, \ell, m\rangle^{out} + \text{inelastic}.$$

$$i(2\pi)^4\mathcal{M} = (\mathcal{N}(E))^2\cdot\frac{1}{4\pi}\sum_{\substack{\ell=0\\ \text{even}}}^{\infty}(2\ell+1)\,P_\ell(\cos\theta)\,(S_\ell(E)-1)$$

Equivalently,

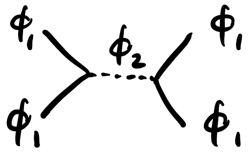
$$i\mathcal{A}(s,t) = i(2\pi)^6\cdot E^2\mathcal{M}$$

$$= \frac{8\pi E}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\,P_\ell(\cos\theta)\,(S_\ell(E)-1)$$

$\uparrow$ $E = 4(k^2+m^2)$.

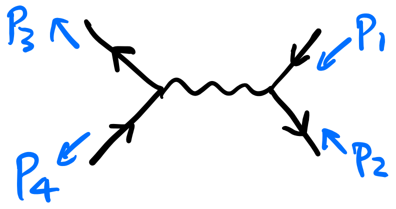
A spin-ℓ bound state particle of mass M would correspond to a pole of $S_\ell(E)$ at $E = M$ $(< 2m)$.

We have seen that



gives rise to a pole in the S-wave ($\ell=0$) amplitude

An example of spin-1 bound state: in scalar QED, particle - anti-particle scattering



$$\propto \frac{(P_1-P_2)^\mu(P_3-P_4)_\mu}{s}$$

$$= \frac{4k^2\cos\theta}{s}$$

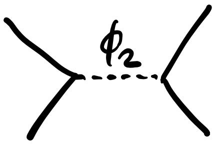
$P_1(\cos\theta) = \cos\theta$,

$\rightsquigarrow$ pole of $S_{\ell=1}$ at $s=0$.

- the photon can be viewed as a bound state!

Back to $\phi_1 \phi_1 \to \phi_1 \phi_2$ scattering

Case II: $m_2 > 2m_1$



has pole at $s = m_2^2$

now in physically admissible region

$\to \mathcal{A}^{\text{tree}}$, or $S^{\text{tree}}_{\ell=0}(E)$,

diverges at $E = m_2$.

But, unitarity demands $|S_{\ell=0}(E)| \leq 1$.

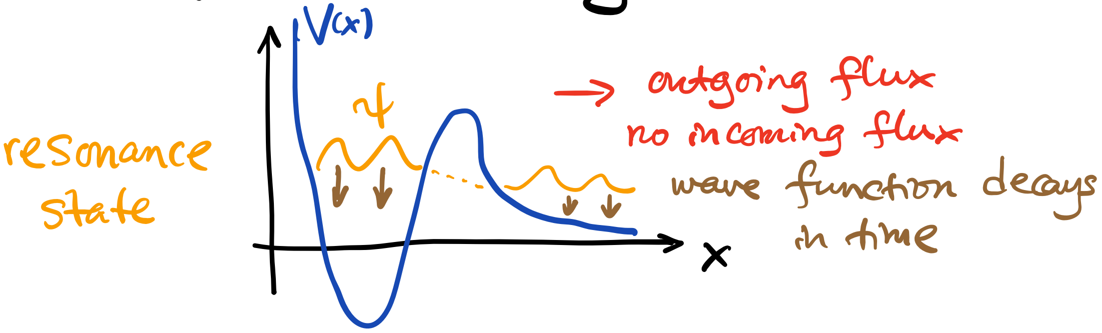
- Is there a contradiction?

Intuition: when $m_2 > 2m_1$, the "particle" created by ϕ_2 is unstable against decay into a pair of ϕ_1-particles; there is

NO stable particle of mass (near) m_2.
but rather, a resonance !

- What is the signature of resonance in $\mathcal{A}$ or $S_\ell(E)$?

. Let us revisit the simple NRQM model of 1D scattering.



The wave function of a resonance "state" that decay uniformly in time is necessarily non-normalizable (due to conservation of probability).

Consider once again the out-state wave function, analytically continued to complex $k = \sqrt{2mE}$.

$$\psi_k^{out}(x,t) \overset{x\to\infty}{\sim} \Big(\underset{\swarrow\,0}{\cancel{S(k)}}\Big)^{-1} e^{-ikx-iEt}$$

$$+\, e^{ikx-iEt}$$

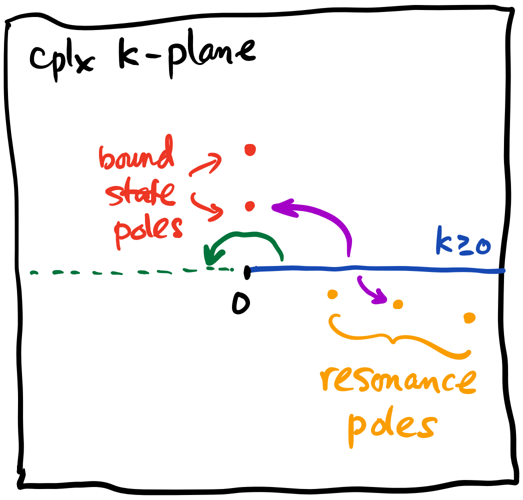
Suppose $S(k)$ has a pole at k_*

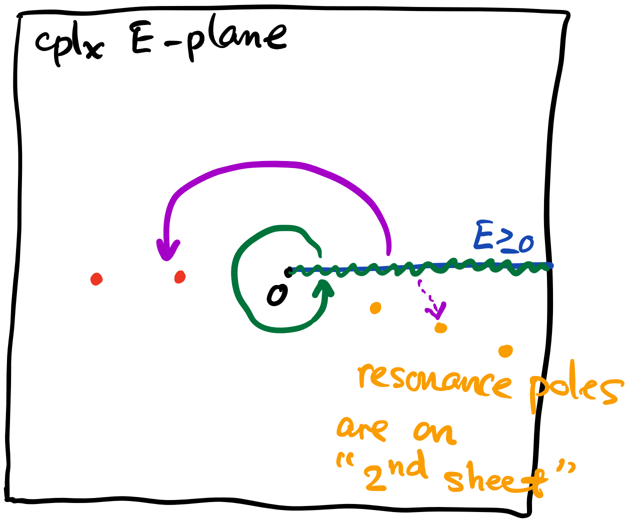
$$E = \frac{k_*^2}{2m} = \omega - i\gamma \text{ for some } \underset{\overset{\nearrow}{\text{decay width}}}{\gamma} > 0.$$

$$\psi_{k_*}^{out}(x,t) \sim e^{ik_* x - i\omega t - \gamma t}$$

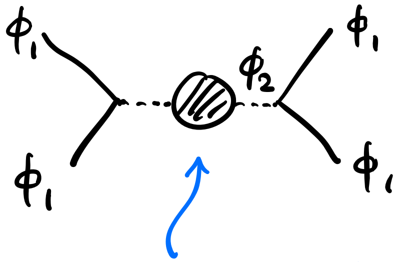
in the asymptotic region.

$$k_* = \sqrt{2m(\underset{\overset{\uparrow}{\omega>0}}{\omega} - i\gamma)}$$

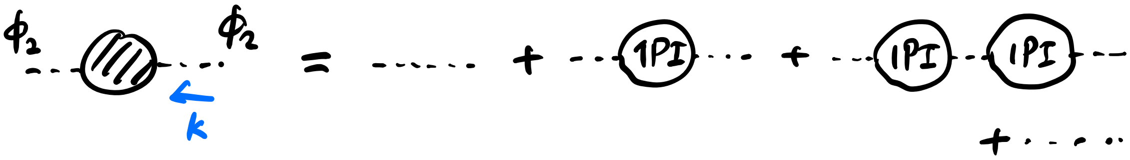




Back to $\phi_1 \phi_1 \to \phi_1 \phi_1$ scattering

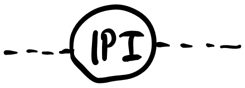


need to take into account the self-energy of ϕ_2 to account for the decay width of resonance



$$= \frac{-i}{k^2 + m_2^2 - i\epsilon - \Sigma(k)}$$

$i\Sigma(k) =$



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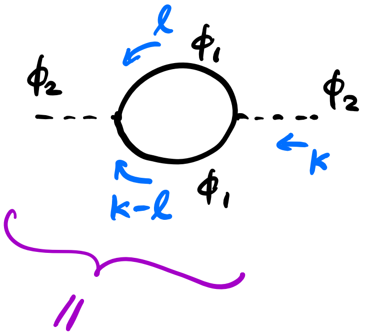


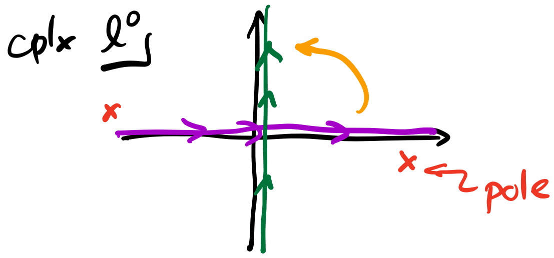
$+ \ \mathcal{O}(g^4)$

$$\frac{(-ig)^2}{2} \int \frac{d^D\ell}{(2\pi)^D} \frac{-i}{\ell^2 + m_1^2 - i\epsilon} \frac{-i}{(k-\ell)^2 + m_1^2 - i\epsilon}$$

$$= \frac{g^2}{2} \int \frac{d^D\ell}{(2\pi)^D} \int_0^1 dx \, \frac{1}{\left[\ell^2(1-x) + (k-\ell)^2 x + m_1^2 - i\epsilon\right]^2}$$

first shift $\ell^\mu \to \ell^\mu + x k^\mu$.

then Wick rotate $\ell^0 = i\ell^D$



$$= i \frac{g^2}{2} \int \frac{d^D\ell_E}{(2\pi)^D} \int_0^1 dx \, \frac{1}{\left(\ell_E^2 + k^2 x(1-x) + m_1^2 - i\epsilon\right)^2}$$

specialize to $D=4$ case, regularize with cutoff $|\ell_E| < \Lambda$ (a more systematic reg. scheme is dim reg, but the simple momentum cutoff scheme suffices for our discussion at 1-loop order)

$$= \frac{i g^2}{32\pi^2} \int_0^1 dx \left[\log \frac{\Lambda^2}{k^2 x(1-x) + m_1^2 - i\epsilon} - 1 + \mathcal{O}(\Lambda^{-2})\right]$$

The k-independent terms, that come with

UV divergence, can be absorbed into a redefinition of the mass parameter m_2. Equivalently, we can say

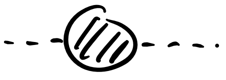
$$(m_2^{(\text{bare})})^2 = (m_2^R)^2 + \underbrace{\delta m_2^2}_{\text{counter term}}$$

where m_2^R is finite, and δm_2^2 serves to cancel the k-ind. UV divergence in $\Sigma(k)$.

We can write

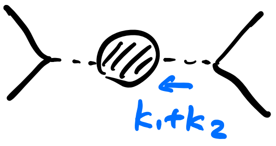
$$\Sigma(k) = \Sigma^R(k) + \delta m_2^2 \, ,$$

$$\Sigma^R(k) = \frac{-g^2}{32\pi^2} \int_0^1 dx \log \frac{k^2 x(1-x) + m_1^2 - i\epsilon}{m_1^2}$$



$$= \frac{-i}{k^2 + (m_2^R)^2 - i\epsilon - \Sigma^R(k)} \, .$$

Let us examine the analytic/singularity structure of



Its contribution to the scattering amplitude $\mathcal{A}$ is

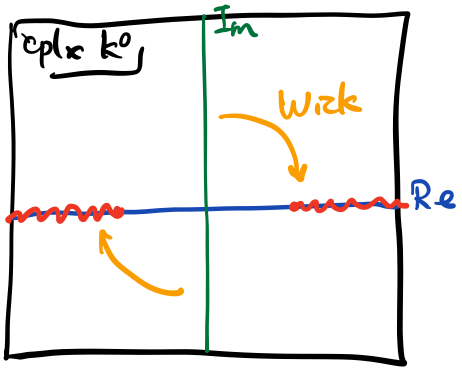
$$\frac{g^2}{-s+(m_2^R)^2-i\epsilon+\frac{g^2}{32\pi^2}\int_0^1 dx \log\left[1-\frac{s x(1-x)}{m_1^2}-i\epsilon\right]}$$

for $0 \leq x \leq 1$, $0 \leq x(1-x) \leq \frac{1}{4}$, argument of log is negative in the physical region

$s \in [4m_1^2, \infty)$

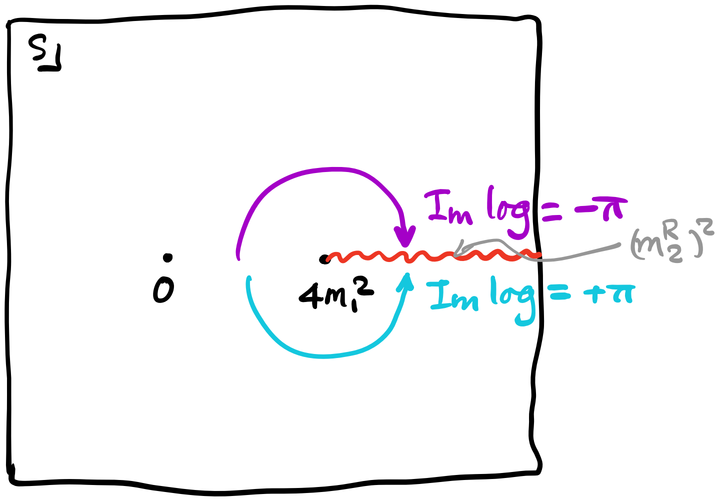
branch choice read off from the $i\epsilon$ prescription.

More fundamentally, the value of $\Sigma(k)$ at real (Lorentzian) k^0 is determined by the Wick rotation / analytic continuation relating the Euclidean Green function G_E to the time-ordered Green function G.



For real $k^0 > \sqrt{\vec{k}^2+M^2}$, $\Sigma(k)$ is given by analytic continuation from $k^0 \in i\mathbb{R}$ to positive real k^0-axis from above.

On the complex s-plane:

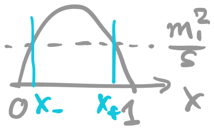


Thus, at real $s > 4m_1^2$,

$$\text{Im}\,\Sigma^R(k_1+k_2)\Big|_{\text{1-loop}} = -\frac{g^2}{32\pi^2}(-\pi)\,(x_+ - x_-)$$

$x_\pm$ are solutions to $x(1-x) = \frac{m_1^2}{s}$

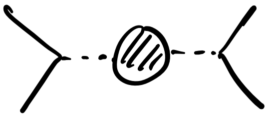
$$= \frac{g^2}{32\pi}\,\frac{\sqrt{s-4m_1^2}}{\sqrt{s}}\,.$$



The imaginary part of Σ^R renders the amplitude $\mathcal{A}$, or $S_{\ell=0}(E)$, finite at all physical energies.

At weak coupling, for s near $(m_2^R)^2$.

the amplitude $\mathcal{A}$ is dominated by

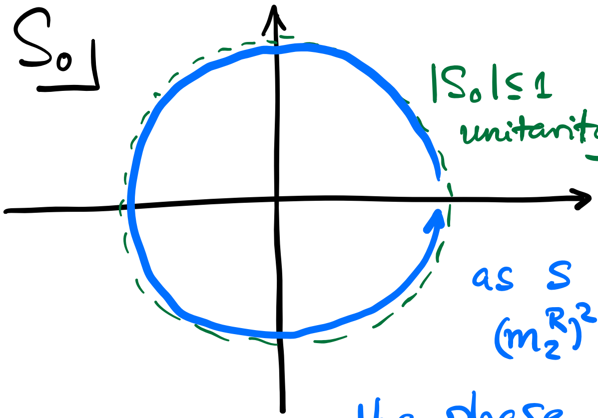


$$\mathcal{A} \approx \frac{g^2}{-s + (m_2^R)^2 - i \frac{g^2}{32\pi}\sqrt{\frac{s-4m_1^2}{s}}}$$

only $l=0$ partial wave

$$\approx -16\pi i \sqrt{\frac{s}{s-4m_1^2}}\,\left(S_0(s) - 1\right)$$

$$\Rightarrow \; S_0(s) \approx \frac{-s + (m_2^R)^2 + i \frac{g^2}{32\pi}\sqrt{\frac{s-4m_1^2}{s}}}{-s + (m_2^R)^2 - i \frac{g^2}{32\pi}\sqrt{\frac{s-4m_1^2}{s}}}$$



as s varies from below $(m_2^R)^2$ to above $(m_2^R)^2$

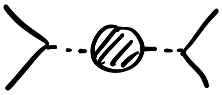
the phase of $S_0(s)$ changes by 2π.

Where is the singularity of $\mathcal{A}$, or S_0, associated with the ϕ_2 - resonance on the complex s - plane ?

As we analytically continue $\Sigma^R(k_1+k_2)$ from real $s < 4m_1^2$ to the upper-half complex s-plane ($\mathrm{Im}\, s > 0$), we have

$$\mathrm{Im} \log\left(1 - \frac{s\,x(1-x)}{m_1^2} - i\epsilon\right) < 0$$

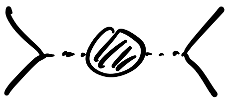
and $\mathrm{Im}(-s) < 0$.

It follows that  has no pole in s on the UHP.

Likewise, if we analytically continue in s to the LHP ($\mathrm{Im}\, s < 0$), we have

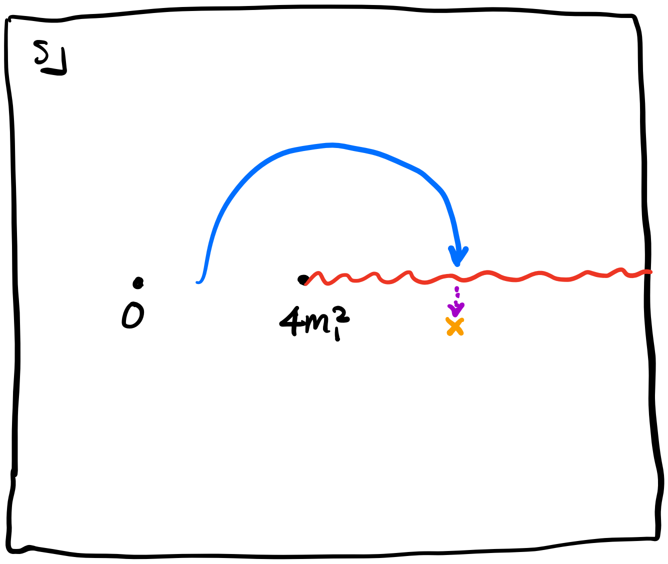
$$\mathrm{Im} \log\left(1 - \frac{s\,x(1-x)}{m_1^2} - i\epsilon\right) > 0$$

$$\mathrm{Im}(-s) > 0.$$

It once again follows that  has no pole in s on the LHP either!

Q: where is the resonance pole?

A: on the second sheet.



In agreement with earlier expectation from the NRQM model

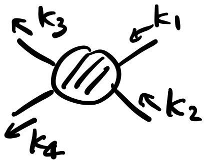
- Bound states as composite particles

Consider ϕ^4 theory in D spacetime dimensions

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}g\phi^4 .$$

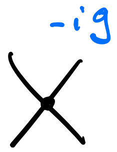
2 → 2 amplitude

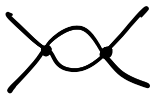
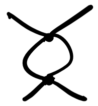
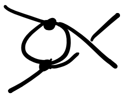
$i\mathcal{A} =$

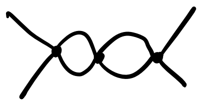
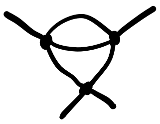


is a function of $s = -(k_1+k_2)^2$ and $t = -(k_1-k_3)^2$

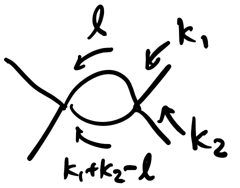
If a pair of ϕ-particles form bound states, the latter should show up as poles of $\mathcal{A}$ at $s < 4m^2$.

$i\mathcal{A} = \Big[$ 

$+$  $+$  $+$ 

$+$  $+$  $+ \cdots \Big] \times \left(Z_\phi^{\frac{1}{2}}\right)^4$

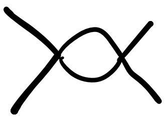
do not affect position of pole in s



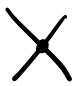
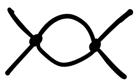
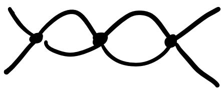
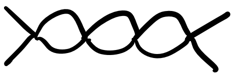
$$= \frac{(-ig)^2}{2}\int \frac{d^D l}{(2\pi)^D}\,\frac{-i}{l^2+m^2-i\epsilon}\,\frac{-i}{(k_1+k_2-l)^2+m^2-i\epsilon}$$

$$\underset{\text{Feynman trick + Wick rot}^n}{=} \frac{i g^2}{2}\int \frac{d^D l_E}{(2\pi)^D}\int_0^1 dx\,\frac{1}{\left(l_E^2 - s\,x(1-x)+m^2-i\epsilon\right)^2}$$

Specialize to D=2 case



$$= \frac{i g^2}{8\pi}\underbrace{\int_0^1 dx\,\frac{1}{-s x(1-x)+m^2-i\epsilon}}_{\equiv\, f(s)}$$

 $+$  $+$  $+$  $+ \cdots$

$$= -ig\sum_{L=0}^{\infty}\left(\frac{-g}{8\pi}f(s)\right)^L = \frac{-ig}{1+\frac{g}{8\pi}f(s)} \qquad \circledast$$

pole in s ?

For s approaching $4m^2$ from below,

$$s = 4m^2 - \delta \,, \qquad \delta \to 0^+$$

$$f(s) \approx \frac{\pi}{m\sqrt{\delta}} \,.$$

⊛ acquires a pole if $g<0$,

at $\delta \approx \left(\frac{g}{8m}\right)^2$.

Interpretation: for weak and negative coupling g, there is a bound state of

$$\text{mass} \approx 2m - \frac{g^2}{256m^3} \,.$$

You might worry that ϕ^4 theory with negative g is ill-defined (non-perturbatively) as the scalar potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4$ is not bounded from below, so that the perturbative vacuum is not the true vacuum state. Nonetheless, the same perturbative analysis applies to scalar theory with additional ϕ^6, ϕ^8 couplings etc.

i.e. $V(\phi) = \frac{m^2}{\beta}\cos(\beta\phi)$, $\beta = \frac{\sqrt{g}}{m}$. "sine-Gordon model"

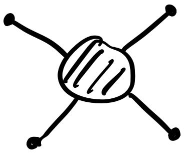
More generally, we can probe the state $|B\rangle$ of a composite particle using

$$\langle \Omega | T\, \hat{\phi}(x_1)\, \hat{\phi}(x_2)\, | B \rangle \equiv \Phi_B(x_1, x_2)$$

"Bethe - Salpeter amplitude"

Similarly to the $2 \to 2$ scattering amplitude, we can consider the time-ordered 4-point Green function

$$\tilde{G}(k_1, \dots, k_4) = \int d^D x_1 \cdots d^D x_4 \; e^{-i \sum_{j=1}^{4} k_j \cdot x_j} \langle \Omega | T\, \hat{\phi}(x_1) \cdots \hat{\phi}(x_4) | \Omega \rangle$$

$=$ 

The existence of a bound state $|B\rangle$ of (invariant) mass M gives rise to a pole of $\tilde{G}$ at $(k_1 + k_2)^2 = -M^2$.

[same derivation as Källén - Lehmann spectral rep of time-ordered 2-pt function]

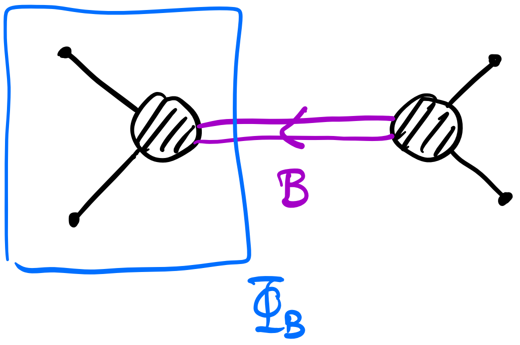
$$\tilde{G} \xrightarrow[k_B^0 = \sqrt{\vec{k}_B^2 + M^2}]{k_1+k_2 \to k_B} (2\pi)^D \delta(k_1 + \cdots + k_4)$$

$$\times \frac{-i}{(k_1+k_2)^2 + M^2 - i\epsilon} \, \tilde{\Phi}_B(k_1, k_2)\, \tilde{\Phi}_B^*(-k_3, -k_4)$$

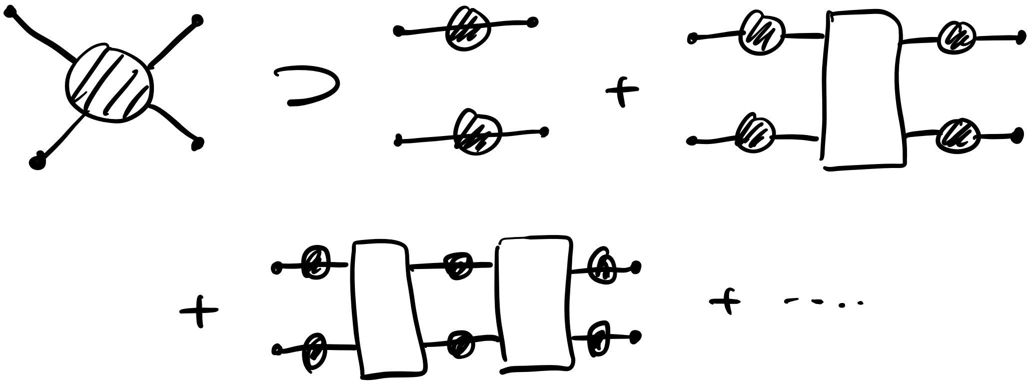
$$\Phi_B(x_1, x_2) \equiv \langle \Omega | T \hat{\phi}(x_1) \hat{\phi}(x_2) | B, \vec{k}_B \rangle$$

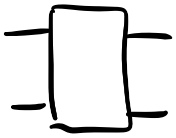
$$= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} e^{i k_1 \cdot x_1 + i k_2 \cdot x_2}$$

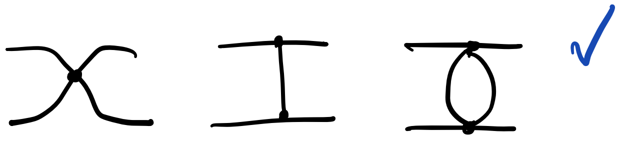
$$\times (2\pi)^D \delta^D(k_1 + k_2 - k_B) \frac{\tilde{\Phi}_B(k_1, k_2)}{(2\pi)^{\frac{D-1}{2}} \sqrt{2\omega_{k_B}}}$$

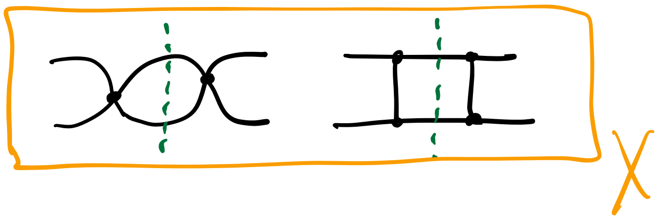


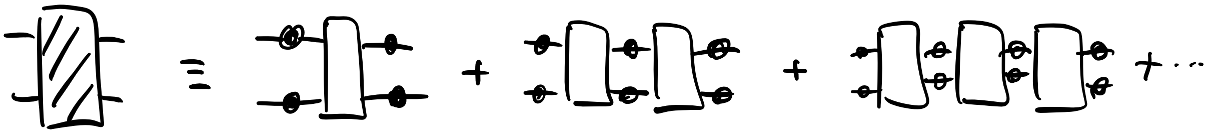
In the "2-particle approximation", the bound state pole comes from divergence in summing over diagrams of the type



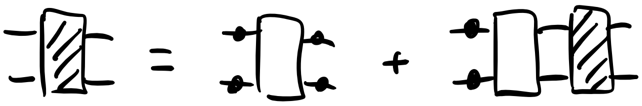
where  contains no "2-particle cuts"

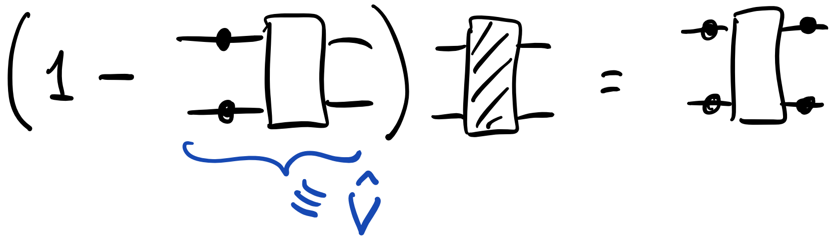
e.g. 

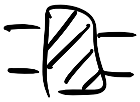
discard: 



obeys the recursive relation

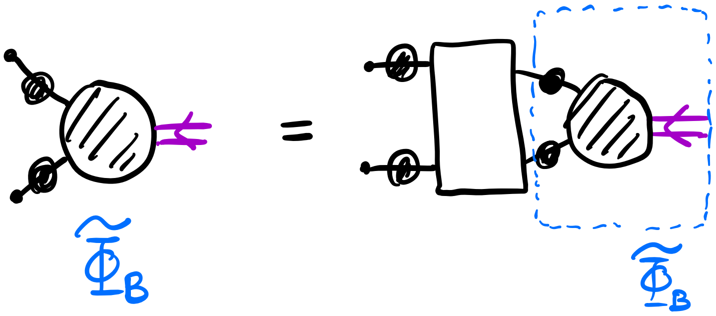


or 

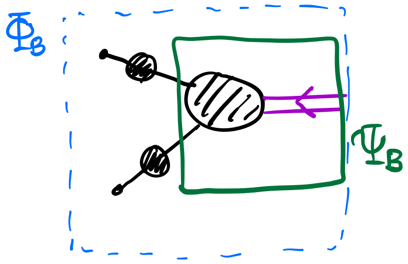
The residue of  at the bound state pole $P^2 = -M^2$ is porportional to $\tilde{\Phi}_B$, which obeys

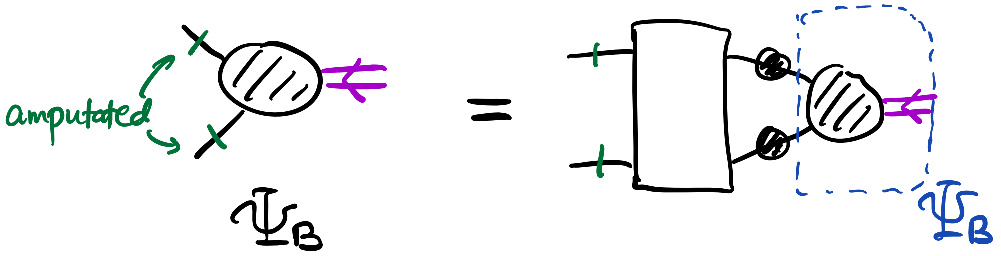
$$(1 - \hat{V})\tilde{\Phi}_B = 0,$$

or diagrammatically,



Equivalently, in terms of the amputated Bethe-Salpeter amplitude Ψ_B :

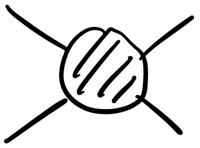


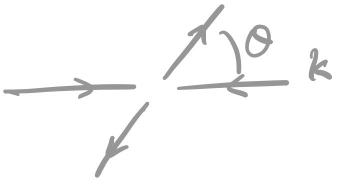


"Bethe-Salpeter equation"

General expectations on the analyticity properties of scattering amplitudes

2 → 2 amplitude (of identical, massive scalar particles)





$$s = E^2 = 4(k^2+m^2)$$

$$t = -2k^2(1-\cos\theta)$$

$\mathcal{A}(s,t)$

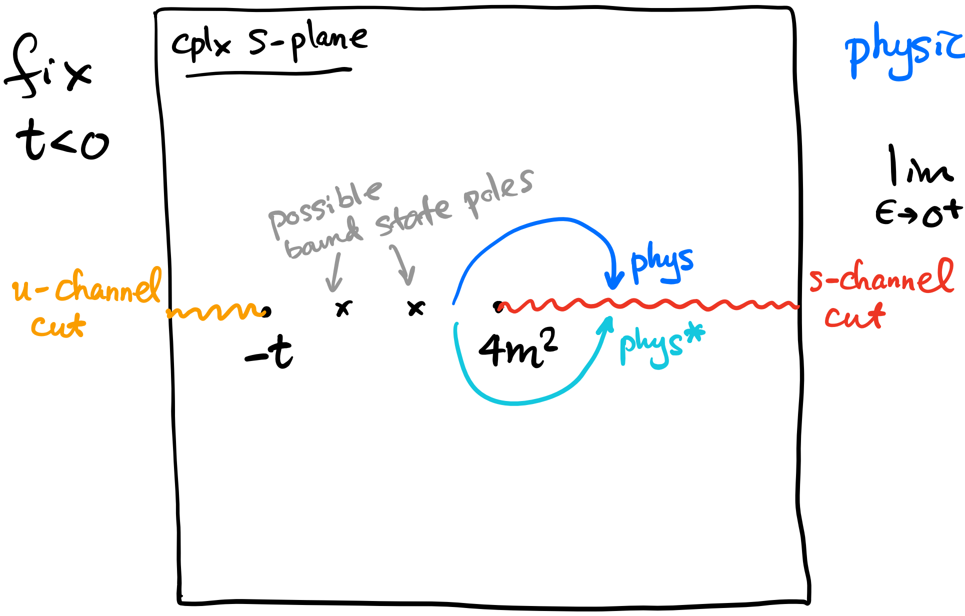
can be analytically continued away from the physical region $s, t \in \mathbb{R}$

$s \geq 4m^2 - t\,, \quad t \leq 0$

$(u = 4m^2 - s - t \leq 0)$

still denote by $\mathcal{A}(s,t)$ the result of analytic continuation

fix $t<0$



physical amplitude

$$\parallel$$

$$\lim_{\epsilon\to 0^+} \mathcal{A}(s+i\epsilon, t)$$

(1st sheet)

The analytically continued $\mathcal{A}(s,t)$ is expected to obey

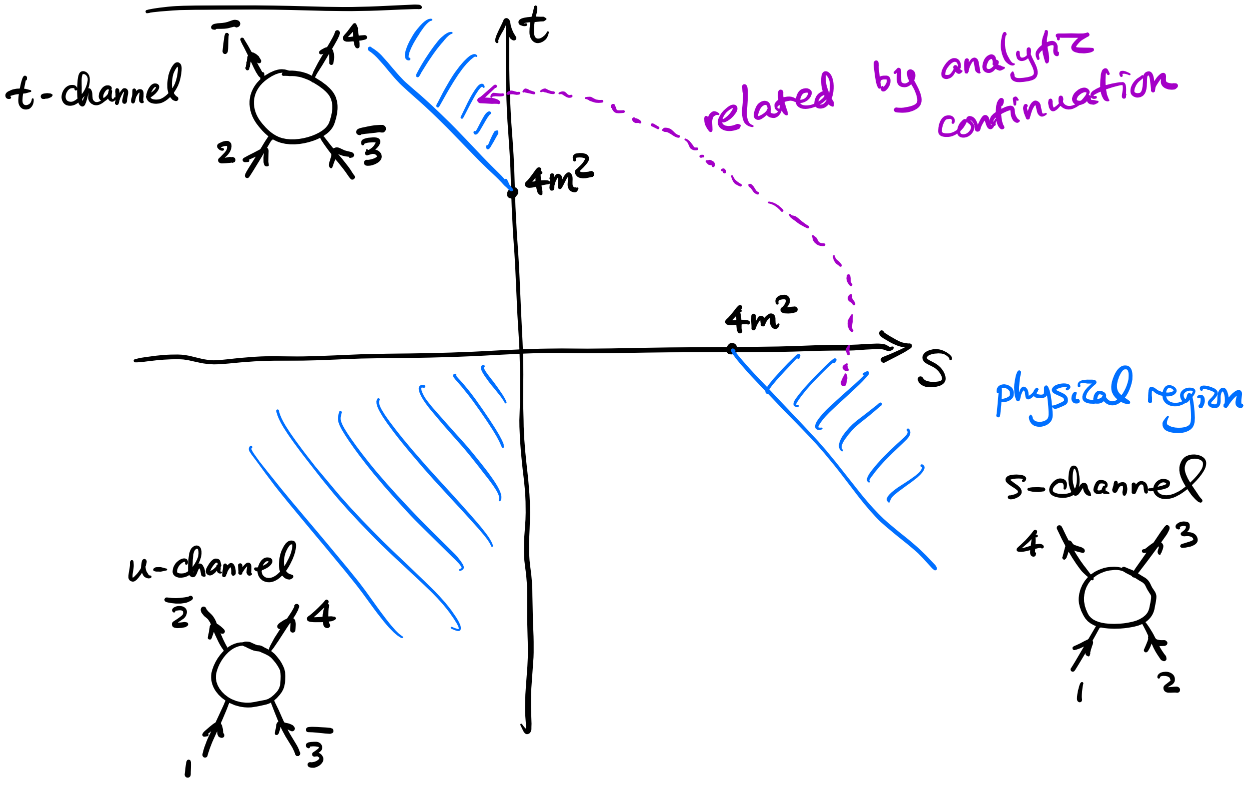
- real analyticity

$$\mathcal{A}(s^*, t^*) = (\mathcal{A}(s,t))^*$$

In perturbation theory, this follows from reality of coupling constants in the Lagrangian

- crossing [superficially, a consequence of LSZ and symmetry of Green fn]

real (s,t)-plane :



- unitarity

$$\mathcal{A}(s,t) = -16\pi i \sqrt{\frac{s}{s-4m^2}} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta) \left(S_\ell(s) - 1 \right)$$

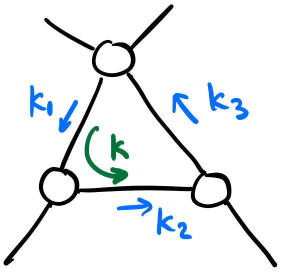
$$|S_\ell(s)| \leq 1 \quad \text{for} \quad s \geq 4m^2$$

Also: $$|S_\ell(s)| = 1 \quad \text{for} \quad 4m^2 \leq s \leq M^2_{inel}.$$

threshold of inelastic processes.

<u>Caution</u>: NOT all singularities of $\mathcal{A}$ (on 1st sheet) have the interpretation of an intermediate physical state (bound state or multi-particle scattering states).

"Anomalous threshold"



loop momentum k

$k_i = k +$ external momenta

polynomial in k_i's

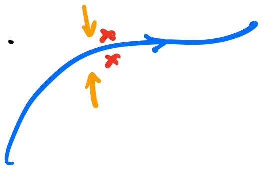
$$\int d^Dk \, \frac{(\cdots)}{\prod_{i=1}^{N} (k_i^2 + m_i^2 - i\epsilon)}$$

$$= \int d^D k \int_0^1 \prod_{i=1}^{N} d\alpha_i \, \delta\left(\sum \alpha_i - 1\right) \frac{(\cdots)}{\left[\sum_{i=1}^{N} \alpha_i (k_i^2 + m_i^2) - i\epsilon\right]^N}$$

As a function of external momenta, it can be analytically continued, so long as poles of the integrand stay away from the integration contour. Even if a pole hits the integration contour, we can still deform the (α_i or k^μ) contour to evade the pole, generically.

A singularity can occur only if the contour cannot be deformed away from the pole

e.g.



This happens when all of the derivatives of the denominator w.r.t. α_i and k^μ vanish

i.e.

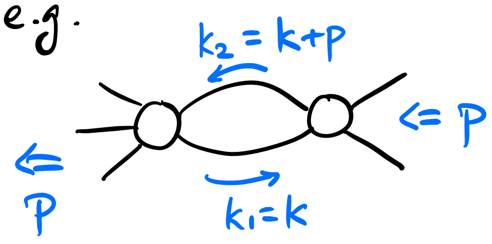
$$\begin{cases} k_i^2 + m_i^2 = 0, \quad i = 1, \ldots, N \\ \dfrac{\partial}{\partial k^\mu} \underbrace{\sum_{i=1}^{N} \alpha_i (k_i^2 + m_i^2)}_{\overset{\parallel}{2 \sum_i \alpha_i k_{i\mu}}} = 0 \end{cases}$$

"Landau equations"

On the 1st Riemann sheet (a.k.a "physical sheet", not to be confused with "physical region", the latter refers to real kinematics only) singularities of the amplitude occur at solutions to Landau equations with $\alpha_i > 0$.

[For more details, see Eden et al. " The Analytic S-Matrix" 1966 book]

e.g.



singularity ⓢ

$$\begin{cases} k^2+m^2 = 0 = (k+p)^2+m^2 \\ \alpha(k+p) + (1-\alpha)\,k = 0 \end{cases}$$

for some $0 \le \alpha \le 1$.

$$\Downarrow$$

$$\alpha = \frac{1}{2}, \quad k = -\frac{p}{2}, \quad p^2 = -4m^2$$

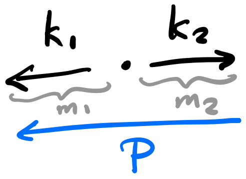
This is the 2-particle threshold.

A pictorial way to see the singularity:

think of $(k^0, i\vec{k})$ as a vector of length m

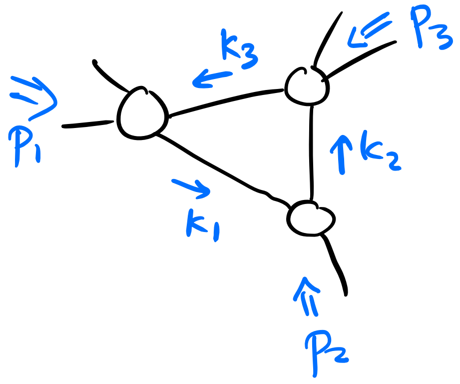
$$\alpha_1 k_1 + \alpha_2 k_2 = 0$$

$$0 < \alpha_1, \alpha_2 < 1$$

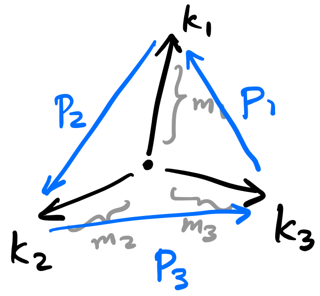


$$-p^2 = (m_1+m_2)^2$$

Now consider a triangle diagram

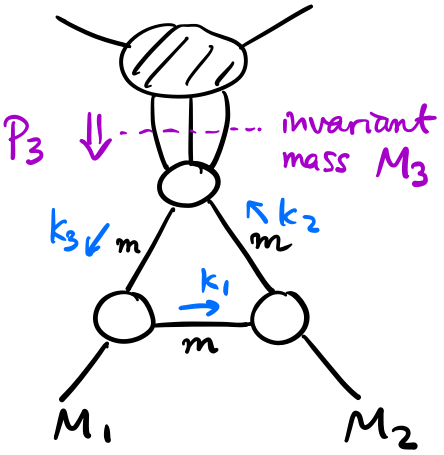


singularity occurs on the 1st sheet if



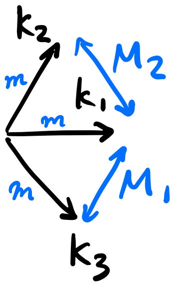
can be arranged.

Consider



$P_3^2 = -M_3^2$

If $M_1, M_2 < \sqrt{2}\, m$, can only arrange

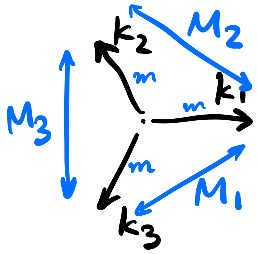


cannot satisfy Landau eqn.

⇒ amplitude non-singular.

On the other hand, if $M_1, M_2 > \sqrt{2}\, m$,

can arrange

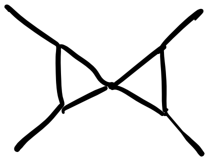


$\rightsquigarrow$ amplitude acquires Landau singularity at a specific value of $P_3^2 = -M_3^2$

(M_3 determined by M_1, M_2, m)

which does not admit the interpretation of a bound state.

e.g.



anomalous threshold singularity occurs in sine-Gordon model

[Coleman, Thun 1978]

Working hypothesis: the kind of analytic structure of scattering amplitudes found in perturbation theory also applies to the non-perturbative amplitude, provided that we interpret the on-shell internal propagators that give rise to Landau singularities as those of actual particles.

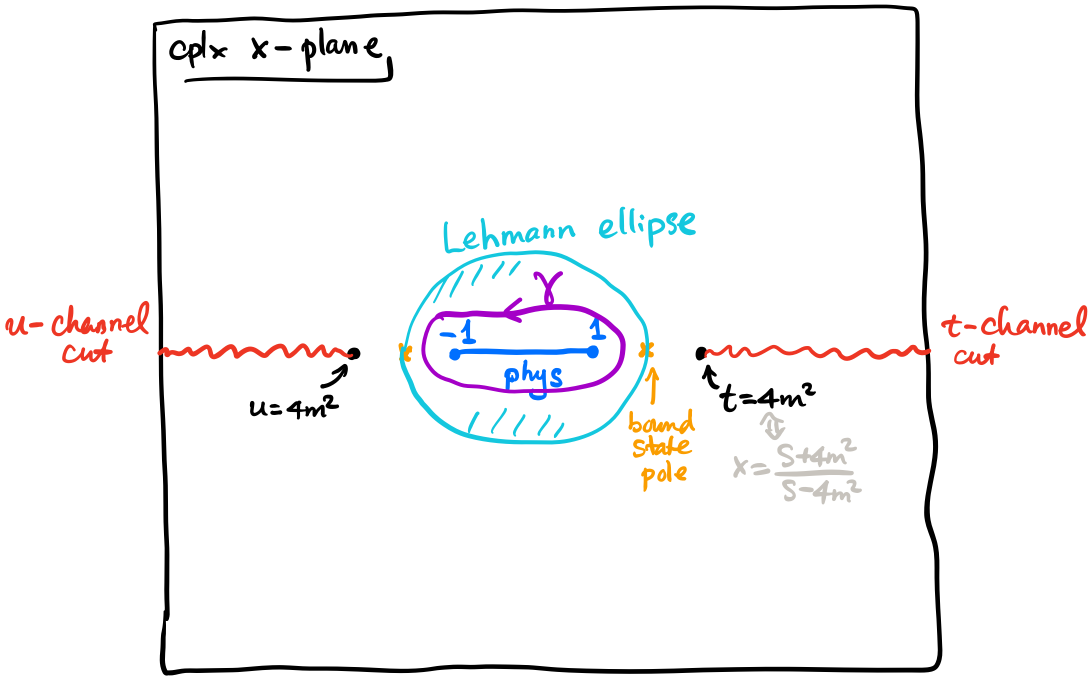
Only a small subset of such analyticity assumptions have been rigorously established based on robust axioms.

- Some consequences of analyticity, crossing, and partial wave unitarity.

For 2 → 2 scattering of lightest particles, write

$$x \equiv \cos\theta = 1 + \frac{2t}{s-4m^2}.$$

Fixing a physical value of s ($> 4m^2$), let us inspect $\mathcal{A}(s,t)$ viewed as an analytic function of x, denoted by $\mathcal{A}(x)$.



The physical value of x lies between -1 and 1 on the real axis. We expect $\mathcal{A}(x)$ to be analytic on the complex x- plane away from the t-channel branch cut $x \in \left[\frac{s+4m^2}{s-4m^2}, \infty\right)$

$\Leftrightarrow t \in [4m^2, \infty)$

the u-channel cut $x \in \left(-\infty, -\frac{s+4m^2}{s-4m^2}\right]$, and possible poles that admit interpretation as bound states in t and u- channels.

We can write:

$$\mathcal{A}(x) = \oint_\gamma \frac{dz}{2\pi i} \frac{\mathcal{A}(z)}{z-x}$$

Provided that γ encircles x and $\mathcal{A}(z)$ is analytic inside of γ.

Now use the identity

$$⊛ \quad \frac{1}{z-x} = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(x) \underbrace{Q_\ell(z)}_{\text{Legendre function of the } 2^{nd} \text{ kind}}$$

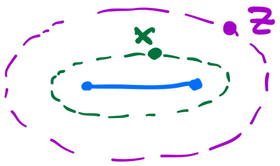
- holds when $|z+\sqrt{z^2-1}| > |x+\sqrt{x^2-1}|$

$$Q_\ell(z) = \frac{1}{2}\int_{-1}^{1} dx \, \frac{P_\ell(x)}{z-x} = \int_{z+\sqrt{z^2-1}}^{\infty} \frac{d\zeta}{\zeta^{\ell+1}\sqrt{1-2\zeta z+\zeta^2}}.$$

Indeed, at large ℓ,

$$P_\ell(x) \sim \left(x+\sqrt{x^2-1}\right)^{\ell}, \quad Q_\ell(z) \sim \left(z+\sqrt{z^2-1}\right)^{-\ell}.$$

RHS of ⊛ converges when $\left|z+\sqrt{z^2-1}\right| > \left|x+\sqrt{x^2-1}\right|$.



Note: $\left\{z : \left|z+\sqrt{z^2-1}\right| = \text{const}\right\}$ is an ellipse on the cplx plane with foci ±1.

We will take γ to be the largest ellipse with foci ±1 and encircles no singularity of $\mathcal{A}(z)$ (e.g. touching the first t-channel bound state pole) This largest ellipse is called the "Lehmann ellipse".

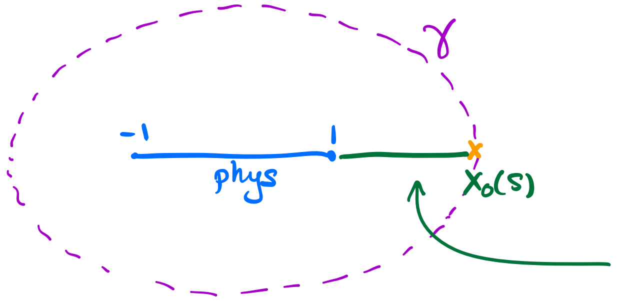
$$\mathcal{A}(x) = \oint_\gamma \frac{dz}{2\pi i} \underbrace{\sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(x)\, Q_\ell(z)\, \mathcal{A}(z)}_{\text{sum converges for } x \text{ inside Lehmann ellipse}}$$

can exchange $\oint_\gamma$ with $\sum_\ell$

$$= \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(x) \oint_\gamma \frac{dz}{2\pi i} Q_\ell(z)\, \mathcal{A}(z)$$

↳ partial wave expansion converges

for unphysical x, provided that x lies inside the Lehmann ellipse



$$1 < x < x_0(s)$$
$$\Updownarrow$$
$$0 < t < \underset{\min(4m^2,\; m_B^2)}{\underset{\shortparallel}{t_0}}$$

$$\mathcal{A}(x) = -16\pi i \sqrt{\frac{s}{s-4m^2}} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(x) \left(S_\ell(s) - 1\right)$$

$\Rightarrow$ For real x inside Lehmann ellipse,

$$\operatorname{Im} \mathcal{A}(x) = 16\pi \sqrt{\frac{s}{s-4m^2}} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(x) \Big(\underbrace{1 - \operatorname{Re} S_\ell(s)}_{\in [0,2]}\Big)$$

Also recall:

elastic cross section

$$\sigma_{2\to 2} = \frac{8\pi}{s-4m^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| S_\ell(s) - 1 \right|^2$$

inelastic cross section

$$\sigma_{\text{inel}} = \frac{8\pi}{s-4m^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left(1 - |S_\ell(s)|^2\right)$$

total cross section

$$\sigma_{tot} = \sigma_{2\to 2} + \sigma_{inel}$$

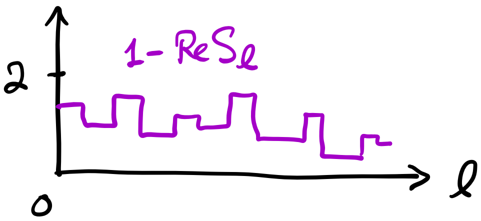
$$= \frac{16\pi}{s-4m^2}\sum_{\ell=0}^{\infty}(2\ell+1)\left(1-\text{Re}\,S_\ell(s)\right)$$

$$= \frac{1}{\sqrt{s(s-4m^2)}}\,\text{Im}\,\mathcal{A}(x=1)$$

"optical theorem"

Compare:

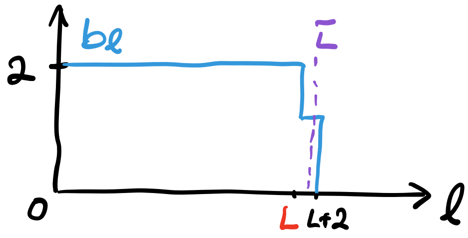
$$\frac{1}{16\pi}\sqrt{\frac{s-4m^2}{s}}\,\text{Im}\,\mathcal{A}(x) = \sum_{\ell=0}^{\infty}(2\ell+1)\,P_\ell(x)\,\left(1-\text{Re}\,S_\ell(s)\right)$$



$$\frac{s-4m^2}{16\pi}\,\sigma_{tot} \equiv \sum_{\substack{\ell=0\\ \text{even}}}^{\infty}(2\ell+1)\,b_\ell \equiv \bar{L}^2$$

$$\bar{L} = \sqrt{\frac{s-4m^2}{16\pi}\,\sigma_{tot}} \approx \underset{\nwarrow\,\text{integer}}{L}$$

(≫1 at high energies)



$$\text{Optical theorem} \Rightarrow \sum_{\ell=0}^{\infty}(2\ell+1)\left(1-\text{Re}\,S_\ell\right) = \sum_{\ell=0}^{\infty}(2\ell+1)\,b_\ell$$

- For $x>1$, $P_\ell(x)$ increases with ℓ.

$$\begin{array}{ccc}\Rightarrow \displaystyle\sum_{\ell=0}^{\infty}(2\ell+1)\,P_\ell(x)\left(1-\mathrm{Re}\,S_\ell\right) & \geq & \displaystyle\sum_{\ell=0}^{\infty}(2\ell+1)\,P_\ell(x)\,b_\ell \\ \| & & \rotatebox{90}{$\leq$} \\ \displaystyle\frac{1}{16\pi}\sqrt{\frac{s-4m^2}{s}}\;\mathrm{Im}\,A(x) & & \displaystyle\sum_{\ell=0,\,\text{even}}^{L}(2\ell+1)\,P_\ell(x)\cdot 2\end{array}$$

For $x>1$,

$$P_\ell(x) = \frac{1}{\pi}\int_0^{\pi}\left(x+\cos\alpha\sqrt{x^2-1}\right)^{\ell} d\alpha$$

$$> C_\delta\Big[\underbrace{1+(1-\delta)\sqrt{x^2-1}}_{\substack{\rotatebox{90}{$\equiv$}\\ y}}\Big]^{\ell}$$

for all ℓ, given arbitrarily small fixed $\delta>0$ and some constant $C_\delta>0$.

$$\Rightarrow \mathrm{Im}\,A(x) > \mathrm{Const}\times\sum_{\ell=0,\,\text{even}}^{L}(2\ell+1)\,y^{\ell}$$

$$\gtrsim \mathrm{Const}\times L\cdot\frac{y^{L}}{y-1}$$

For $s\gg m^2$, $L\approx\sqrt{\frac{s\,\sigma_{tot}}{16\pi}}$, $y-1\approx(1-\delta)\sqrt{\frac{4t}{s}}$.

$$y^{L}\approx e^{(1-\delta)\sqrt{\frac{t\,\sigma_{tot}}{4\pi}}}.$$

$$\Rightarrow \quad \mathrm{Im}\,A(x) > \text{Const} \times s\sqrt{\frac{\sigma_{tot}}{t}}\; e^{(1-\delta)\sqrt{\frac{t\,\sigma_{tot}}{4\pi}}}.$$

for $1 < x < x_0(s) \iff 0 < t < t_0$

For $x \lesssim x_0(s)$, if $A(x)$ grows no faster than polynomial in s at large s,

$$\mathrm{Im}\,A(x) < C \cdot s^N \quad \text{for some } N$$

it then follows that

$$\sigma_{tot} < \frac{(N-1)^2}{(1-\delta)^2}\,\frac{4\pi}{t_0}\left(\ln \frac{s}{M^2}\right)^2$$

$$\text{for } s \gg \underset{\substack{\uparrow \\ \text{some mass scale.}}}{M^2}$$

" Froissart – Martin bound "

Polynomial boundedness in (complex) energy

a toy model:

$$f_{\text{out}}(t) = \int dt'\, S(t-t')\, f_{\text{in}}(t')$$

"scattering amplitude"

$$\tilde{S}(\omega) = \int dt\, e^{i\omega t}\, S(t)$$

Causality: $S(t)$ supported at $t \geq 0$

⇙

$\tilde{S}(\omega)$ analytic for $\operatorname{Im}\omega > 0$

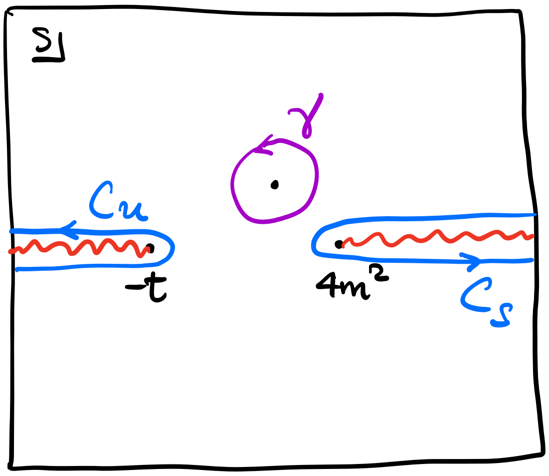
also: no exponential growth as $\omega \to \infty$ on the upper-half complex plane.

- Exponential growth as $\operatorname{Im}\omega \to \infty$ $\Longleftrightarrow$ "time advance".

Now let us investigate the behavior of $2 \to 2$ amplitude $\mathcal{A}(s,t)$ at large complex s.

For simplicity, we assume scattering of lightest scalar particle (no anomalous threshold) and no bound state (no self-coupling pole either).

Fix real $t<0$, and consider the analytic continuation of $\mathcal{A}(s,t)$ to the complex s-plane.



$$\mathcal{A}(s,t) = \oint_\gamma \frac{ds'}{2\pi i} \frac{\mathcal{A}(s',t)}{s'-s}$$

want to deform γ to $-(C_u+C_s)$

But $\mathcal{A}(s',t)$ may grow as $s'\to\infty$.

Assume $\mathcal{A}(s,t)$ is polynomial bounded at large complex s, for fixed $t<0$,

i.e. $$\lim_{s\to\infty}\left|\frac{\mathcal{A}(s,t)}{s^N}\right| = 0$$

for some integer N

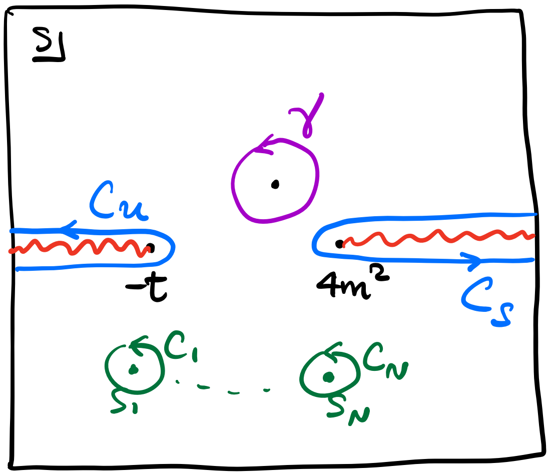
[the value of N may depend on t]

[rigorous result: Epstein, Glaser, Martin Comm. Math. Phys. 1969]

Then $\dfrac{A(s,t)}{\prod_{i=1}^{N}(s-s_i)} \to 0$ as $s \to \infty$

for any $s_1, \dots, s_N$. we can write

$$A(s,t) = \oint_\gamma \frac{ds'}{2\pi i} \frac{\prod_{i=1}^{N}(s-s_i)}{\prod_{j=1}^{N}(s'-s_j)} \frac{A(s',t)}{s'-s}$$



$\downarrow$ deform γ to $-(C_u + C_s + \sum C_i)$

$$= -\oint_{C_s+C_u} \frac{ds'}{2\pi i} \prod_{i=1}^{N} \frac{s-s_i}{s'-s_i} \frac{A(s',t)}{s'-s}$$

$$+ \sum_{i=1}^{N} \underbrace{\prod_{j\neq i} \frac{s-s_j}{s_i-s_j} A(s_i,t)}_{-\text{residue at } s'=s_i}$$

$$\parallel$$

$$P_{N-1}(s;t) = \sum_{n=0}^{N-1} c_n(t)\, s^n$$

some polynomial in s of deg $N-1$.

we can rewrite

$$-\oint_{C_s} \frac{ds'}{2\pi i} A(s',t)(\cdots) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \operatorname{Im} A(s',t)(\cdots)$$

Since for real t, s

$$2i \,\mathrm{Im}\, \mathcal{A}(s,t) = \mathcal{A}(s+i\epsilon, t) - \mathcal{A}(s-i\epsilon, t)$$
$$= \mathrm{disc}\, \mathcal{A}(s,t)$$

↑ discontinuity across the s-channel branch cut.

we arrive at (still for $t<0$)

$$\mathcal{A}(s,t) = P_{N-1}(s;t)$$
$$+ \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\mathrm{Im}\,\mathcal{A}(s',t)}{s'-s} \prod_{i=1}^{N} \frac{s-s_i}{s'-s_i}$$
$$+ \int_{4m^2}^{\infty} \frac{du'}{\pi} \frac{\mathrm{Im}\,\mathcal{A}(u',t)}{s'-s} \prod_{i=1}^{N} \frac{s-s_i}{s'-s_i}$$

"(N-time subtracted) dispersion relation"

In fact, to derive this expression for $N \geq 1$, we only need to assume

$$|\mathrm{Im}\,\mathcal{A}(s,t)| < C \cdot s^{N-\delta}$$ for large real s and some $\delta > 0$

Suppose not true, i.e. need N' subtractions for $N' \geq N+1$, then the N'-time subtract dispersion formula says

$$\mathcal{A}(s,t) \sim C(t) \cdot s^{\overbrace{N'-1}^{\geq N}}, \qquad s \to \infty$$

[If the $s^{N'-1}$ term in $P_{N'-1}$ happens to cancel aginst the order $s^{N'-1}$ term from $\int \mathrm{Im}\mathcal{A}(s',t)\cdots$, the leading large s behavior would be $s^{N'-2}$, contradicting the need for N' subtractions.]

On the other hand, it follows from unitarity

$$\mathrm{Im}\,\mathcal{A}(s,t=0) = \sqrt{s(s-4m^2)}\;\sigma_{tot}$$

$$\geq \sqrt{s(s-4m^2)}\;\sigma_{2\to2}$$

$$= 8\pi\sqrt{\frac{s}{s-4m^2}}\sum_{\ell}(2\ell+1)\,|S_\ell(s)-1|^2$$

$$= \frac{1}{64\pi}\sqrt{\frac{s-4m^2}{s}}\int_{-1}^{1}dx\,|\mathcal{A}(s,t=2k^2(x-1))|^2$$

$$= \frac{1}{32\pi\sqrt{s(s-4m^2)}}\int_{4m^2-s}^{0}dt\,|\mathcal{A}(s,t)|^2$$

$$\gtrsim (\text{positive const}) \times s^{2N-1}, \qquad s\to+\infty$$

Contradicting our assumption $\dfrac{\mathrm{Im}\mathcal{A}}{s^{N-\delta}} \to 0$.

Q: What is the minimal possible N ?

Froissart bound: for large real s,

$$\begin{array}{c} \mathrm{Im}\,\mathcal{A}(s,t=0) < C\cdot s\,(\log s)^2 \\ \rotatebox{-90}{$\geq$} \\ \mathrm{Im}\,\mathcal{A}(s,t<0) \end{array}$$

recall $\mathrm{Im}\,\mathcal{A}(s,t) = 16\pi\sqrt{\frac{s}{s-4m^2}}\sum_\ell (2\ell+1)P_\ell(x)\underbrace{(1-\mathrm{Re}S_\ell)}_{\geq 0}$

for $t<0$, $x=\cos\theta = 1+\frac{2t}{s-4m^2} < 1$

$$P_\ell(x) \leq P_\ell(1) = 1.$$

$\Rightarrow$ $N=2$ is good enough!

It follow that $\displaystyle\lim_{s\to\infty}\left|\frac{\mathcal{A}(s,t)}{s^2}\right| = 0$ for $t<0$.

- "$N=2$" can be extended to complex t in the disc $|t|<|t_0|$.

Idea: compare Taylor series in t of both sides of the dispersion relation, and use

$$\partial_t^n\,\mathrm{Im}\,\mathcal{A}(s,t)\Big|_{t=0} \geq 0$$

to show that the Taylor expansion in t commutes with dispersion integral, and that the same N-time subtraction extends to within the radius of convergence of the Taylor series in t.

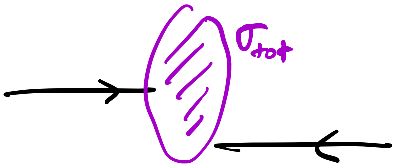
[For details see Jin, Martin Phys. Rev. 1964]

Thus "$N=2$" applies to Froissart - Martin bound:

$$\sigma_{tot} < \frac{1}{(1-\delta)^2}\,\frac{4\pi}{t_0}\left(\ln\frac{s}{M}\right)^2, \qquad s \gg M^2$$

- In D-dimensional spacetime, the analogous bound is

$$\sigma_{tot} < C\cdot(\ln s)^{D-2}$$



"size cannot grow faster than log(energy)"

Some caveats:

- we have assumed all particles are massive
- in a theory with massless particles, the total cross section may be infinite due to (collinear) divergence at zero scattering angle, as is the case in Coulomb / Rutherford scattering.

[a physically more meaningful observable would be the cross section excluding small scattering angles]

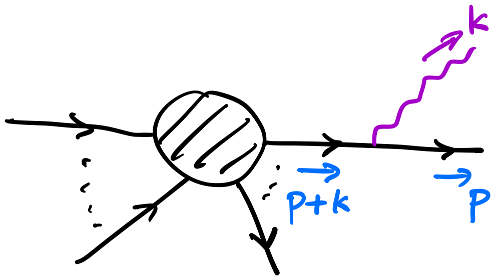
- the idea that "size cannot grow faster than log(E)" is violated in gravity: Schwarzschild black hole horizon size

$$r_H \sim (G_N \cdot E)^{\frac{1}{D-3}} .$$

Infrared Divergence

in QED (or any D=4 Abelian gauge theory with charged particles)

I. the effect of soft photon emission

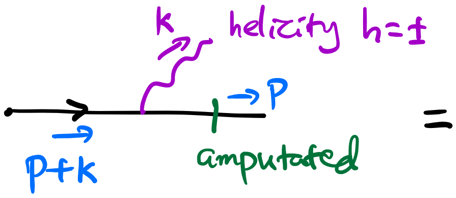


soft photon

$k^0 = |\vec{k}|$ small

(in lab frame)

In scalar QED,



$$= \frac{-i}{(p+k)^2+m^2-i\epsilon} \, i \overset{\text{charge}}{q} (2p+k)^\mu \overset{\text{polarization vector}}{e^{h*}_\mu(k)}$$

$$= q \, \frac{\cancel{2} p^\mu e^{h*}_\mu(k)}{\cancel{2} p\cdot k - i\epsilon}$$ (diverges as $k \to 0$)

Compare the amplitude $\mathcal{M}_{\beta|\alpha}$ to one with extra soft photon emission $\mathcal{M}_{\beta;k,h|\alpha'}$

$\alpha' \to \alpha$ when $k \to 0$

we find a simple relation

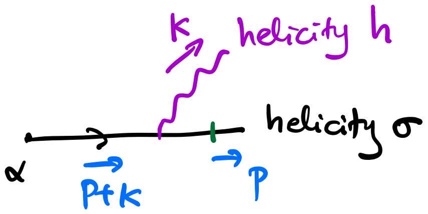
$$\mathcal{M}_{\beta;k,h|\alpha} \xrightarrow{k^0=|\vec{k}|\to 0} \mathcal{M}_{\beta\alpha} \sum_{\substack{\text{particle } n\\ \text{in } \alpha \text{ and } \beta}} \frac{q_n p_n^\mu e_\mu^{h*}}{\eta_n p_n\cdot k - i\epsilon}$$

✪

"soft theorem"

$$\eta_n = \begin{cases} + , & n\in\beta \quad \text{out} \\ - , & n\in\alpha \quad \text{in} \end{cases}$$

Similarly, in spinor QED



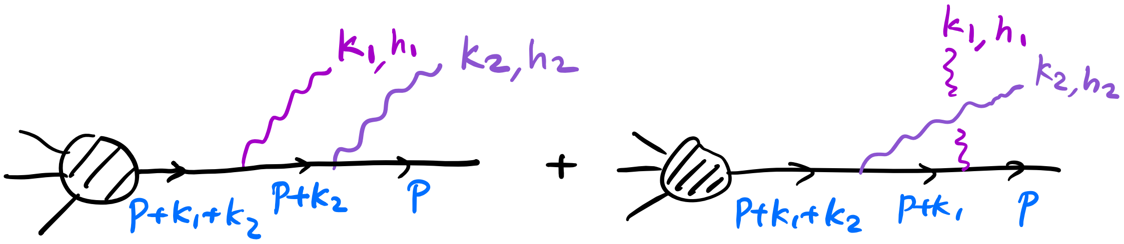
$$= \left[\bar{u}^\sigma(-q\gamma^\mu) \frac{-i(-i(\overbrace{\not p+\not k}^{\approx\, \not p \text{ as } k\to 0})+m)}{\underbrace{(p+k)^2+m^2}_{=\,2p\cdot k \;\rightsquigarrow\; \text{divergence as } k\to 0}-i\epsilon} \right]^\alpha e_\mu^{h*}(k)$$

$$\bar{u}^\sigma \gamma^\mu(-i\not p+m) = \underbrace{\bar{u}^\sigma(i\not p+m)}_{=\,0}\gamma^\mu - 2ip^\mu \bar{u}^\sigma$$

$$\approx q \frac{p^\mu e_\mu^{h*}(k)}{p\cdot k - i\epsilon} \bar{u}^{\sigma\alpha}$$

The same soft relation ✪ holds.

- Next, generalize to multiple soft photon emission



Scalar QED case : $k_1, k_2 \to 0$ limit

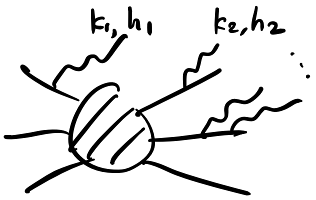
the diagram with 2 soft photon emission on the same external particle line acquires the factor

$$\frac{g\, p\cdot e^*_{h_2}(k_2)}{p\cdot k_2 - i\epsilon} \; \frac{g\, p\cdot e^*_{h_1}(k_1)}{p\cdot(k_1+k_2) - i\epsilon} \quad + \quad (1 \leftrightarrow 2)$$

$$= \frac{g\, p\cdot e^*_{h_1}(k_1)\; p\cdot e^*_{h_2}(k_2)}{(p\cdot k_1 - i\epsilon)\,(p\cdot k_2 - i\epsilon)}$$

soft factors simply multiply !

General number of soft photon emissions



enhances amplitude by the factor

$$\prod_{r=1}^{N} \sum_{n} \frac{g_n\, p_n\cdot e^*_{h_r}(k_r)}{\eta_n\, p_n\cdot k_r - i\epsilon}$$

$\eta_n = \pm 1$, $n \in$ out/in

Effect of soft photon emission on transition probability

$$\frac{\Gamma_{\alpha\to\beta+\text{soft photons}}}{\Gamma_{\alpha\to\beta}} = \sum_{N=0}^{\infty}\frac{1}{N!}\int\prod_{r=1}^{N}\frac{d^{D-1}\vec{k}_r}{(2\pi)^{D-1}2|\vec{k}_r|}$$

$$\times\sum_{\{h_r=\pm\}}\left|\sum_n\frac{q_n\, p_n\cdot e_{h_r}(k_r)}{\eta_n\, p_n\cdot k_r}\right|^2$$

using $\displaystyle\sum_{h_r=\pm} e_{h_r,\mu}(k_r)\, e^{*}_{h_r,\nu}(k_r) = \eta_{\mu\nu} + k_{r\mu}c_{r\nu} + k_{r\nu}c_{r\mu}$

$(k_r\cdot c_r = -1,\ c_r^2 = 0)$

$$\sum_{\{h_r=\pm\}}\left|\sum_n\frac{q_n\, p_n\cdot e_{h_r}(k_r)}{\eta_n\, p_n\cdot k_r}\right|^2 = \sum_{n,m}\frac{\eta_n\eta_m q_n q_m}{p_n\cdot k_r\, p_m\cdot k_r}$$

$$\times\Big(p_n\cdot p_m + \underbrace{p_n\cdot k_r\, p_m\cdot c_r + p_m\cdot k_r\, p_n\cdot c_r}_{\text{contribution}}\Big)$$

$\propto \sum_n \eta_n q_n = 0$

by charge conservation

$$= \sum_{n,m}\frac{\eta_n\eta_m q_n q_m\, p_n\cdot p_m}{p_n\cdot k_r\ p_m\cdot k_r}\ .$$

$\displaystyle\int\frac{d^{D-1}\vec{k}_r}{(2\pi)^{D-1}2|\vec{k}_r|}(\ \cdots\ )$ potential log divergence for $D=4$

In D=4, evaluate

$$\circledast \quad \underbrace{\int \frac{d^3\vec{k}}{(2\pi)^3\, 2|\vec{k}|^3}}_{\frac{1}{2(2\pi)^3}\int\frac{d\omega}{\omega}\int d^2\hat{k}} \sum_{n,m} \frac{\eta_n \eta_m g_n g_m\, p_n\cdot p_m}{(p_n^0 - \vec{p}_n\cdot\hat{k})(p_m^0 - \vec{p}_m\cdot\hat{k})}$$

$$\omega \equiv |\vec{k}|, \quad \hat{k} \equiv \frac{\vec{k}}{|\vec{k}|}.$$

Angular integral gives

$$\int d^2\hat{k}\ \frac{1}{(p_n^0 - \vec{p}_n\cdot\hat{k})(p_m^0 - \vec{p}_m\cdot\hat{k})}$$

$$= \int_0^1 dx \int d^2\hat{k}\ \frac{1}{\left[p_n^0 x + p_m^0(1-x) - (\vec{p}_n x + \vec{p}_m(1-x))\cdot\hat{k}\right]^2}$$

$$= \int_0^1 dx \int_0^{2\pi} d\theta\ \frac{2\pi \sin\theta}{\left[p_n^0 x + p_m^0(1-x) - |\vec{p}_n x + \vec{p}_m(1-x)|\cos\theta\right]^2}$$

$$= \cdots = -\frac{2\pi}{p_n\cdot p_m} \underbrace{\frac{1}{\beta_{nm}} \log\frac{1+\beta_{nm}}{1-\beta_{nm}}}_{\text{monotonic} \nearrow \text{ in } \beta_{nm}},$$

where $\beta_{nm} \equiv \sqrt{\underbrace{1 - \frac{m_n^2 m_m^2}{(p_n\cdot p_m)^2}}_{\geq 0}}\ .$

$$\circledast = \underbrace{-\frac{1}{8\pi^2}\sum_{n,m}\frac{\eta_n\eta_m q_n q_m}{\beta_{nm}}\log\frac{1+\beta_{nm}}{1-\beta_{nm}}}_{\substack{|||\\ A \,\geq\, 0.}} \cdot \int \frac{d\omega}{\omega}$$

e.g.

$$\sum_{n,m=1}^{2} \eta_n\eta_m q_n q_m \frac{1}{\beta_{nm}}\log\frac{1+\beta_{nm}}{1-\beta_{nm}}$$
$$= 2q^2\left(2 - \frac{1}{\beta_{12}}\log\frac{1+\beta_{12}}{1-\beta_{12}}\right) \leq 0$$

The soft photon emission enhances $T_{\alpha\to\beta}$ by the factor

$$\approx \sum_{N=0}^{\infty}\frac{1}{N!}A^N\int_{\mu}^{E_T}\prod_{r=1}^{N}\frac{d\omega_r}{\omega_r}$$

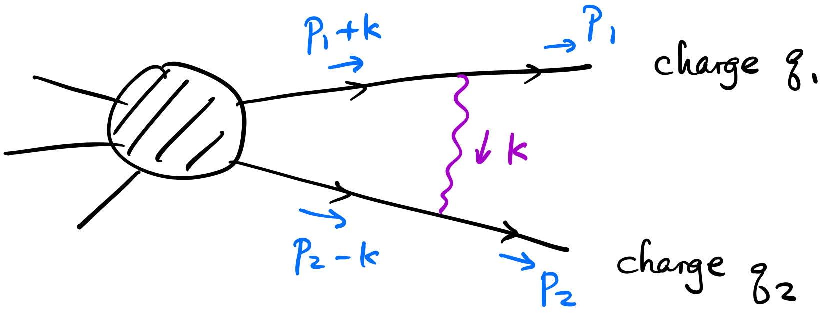
$$= e^{A\log(E_T/\mu)} = \left(\frac{E_T}{\mu}\right)^A$$

for sufficiently small energy E_T, below which the soft photon are not observed in experimental apparatus.

μ is an unphysical IR cutoff

Divergence as $\mu \to 0$?!

- The effect of soft photon loops.



adding a virtual photon loop with small $|\vec{k}|$ modifies the diagram by the factor

$$\int_{\mu<|\vec{k}|<M} \frac{d^4k}{(2\pi)^4} \frac{q_1 p_1^\mu}{p_1\cdot k - i\epsilon} \frac{q_2 p_2^\nu}{p_2\cdot(-k) - i\epsilon} \cdot \frac{-i\,\eta_{\mu\nu}}{k^2 - i\epsilon}$$

μ: unphysical IR cutoff

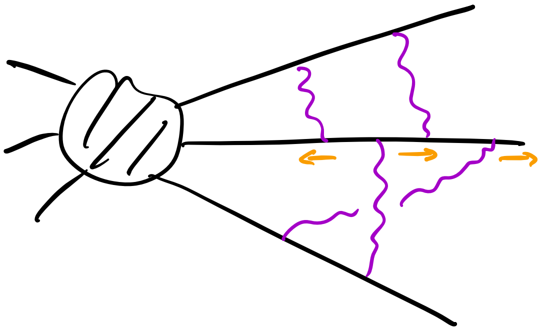
M: Suff. small mass for soft approximation

$\frac{-i\,\eta_{\mu\nu}}{k^2 - i\epsilon}$: Feynman gauge photon propagator

Similarly, a soft photon loop between two external lines of momenta p_n, p_m gives the factor $q_n q_m J_{nm}$, where

$$J_{nm} = -i\, p_n\cdot p_m \int_{\mu<|\vec{k}|<M} \frac{d^4k}{(2\pi)^4} \frac{1}{(\eta_n p_n\cdot k - i\epsilon)(-\eta_m p_m\cdot k - i\epsilon)(k^2 - i\epsilon)}$$

More generally, multiple soft photon loops



absorbing virtual soft photons in different orders add up to a simple result

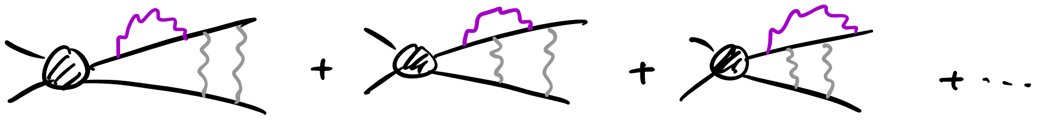
modifies amplitude by the factor

$$\sum_{N=0}^{\infty} \frac{1}{N!} \left[\frac{1}{2} \sum_{n \neq m} q_n q_m J_{nm} \right]^N$$

$$= \exp \left[\frac{1}{2} \sum_{n \neq m} q_n q_m J_{nm} \right]$$

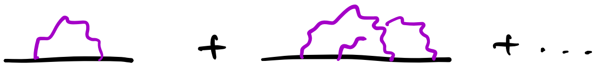
Still missing:

(1) soft photon propagators that begin and end on the same charged particle line;



(2) need to include $Z_n^{\frac{1}{2}}$ factors, which are also IR-divergent!

$\Sigma =$



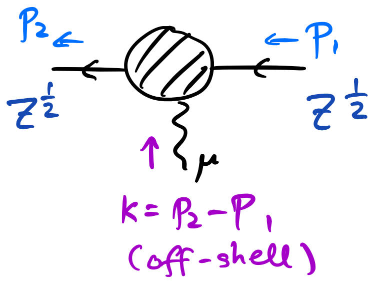
Recall : E-M current $\hat{j}^\mu(x)$

$$\parallel$$

$$\partial_\nu \hat{F}^{\mu\nu}(x)$$

E-M form factor

$$\langle \vec{p}_2 \,|\, \hat{j}^\mu(0) \,|\, \vec{p}_1 \rangle \;=$$



In scalar QED,

$$\langle \vec{p}_2 | \hat{j}^\mu(0) | \vec{p}_1 \rangle = \frac{q}{(2\pi)^3 \sqrt{2\omega_{p_1}}\sqrt{2\omega_{p_2}}} (p_1 + p_2)^\mu F(k^2)$$

$F(0) = 1$

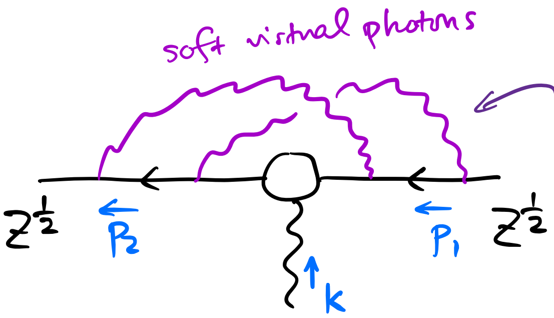
In spinor QED,

$$\langle \vec{p}_2, \sigma_2 | \hat{j}^\mu(0) | \vec{p}_1, \sigma_1 \rangle$$

$$= \frac{iq}{(2\pi)^3} \overline{u}^{\sigma_2}(p_2) \left(\gamma^\mu F(k^2) - \frac{i}{2m}(p_1+p_2)^\mu G(k^2) \right) u^{\sigma_1}(p_1).$$

$F(0) + G(0) = 1.$

The E-M factor in the $k^2 \to 0$ limit is free of IR divergence [recall that we have calculated $G(0) = -\frac{q^2}{8\pi^2} + \mathcal{O}(q^4)$; on the other hand, $\left.\frac{dF}{dk^2}\right|_{k^2=0}$ contains IR divergence, see Weinberg (11.3.27)]



soft factor

$$\sum_{N=0}^{\infty} \frac{1}{N!}(q^2 J_{12})^N = e^{q^2 J_{12}}$$

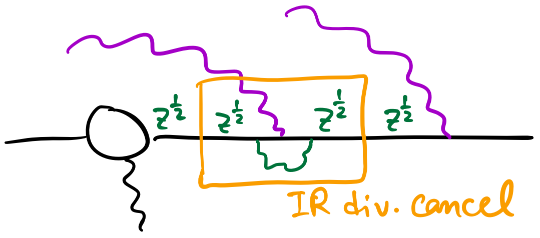
must cancel against the IR divergence in $(Z^{\frac{1}{2}})^2$ in $k \to 0$ limit!

$$\Rightarrow \quad Z_{IR} = e^{-q^2 J_{12}\big|_{\substack{p_2=p_1 \\ \eta_2=-\eta_1}}}$$

↑ IR divergent factor in Z.

[If so, no further IR divergence from further loop corrections to the soft vertices:

e.g.



]

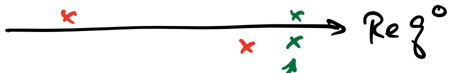
$$J_{12}\big|_{\substack{p_2=p_1 \\ \eta_2=+ \\ \eta_1=-}} = -i p_1^2 \int_{\mu<|\vec{q}|<M} \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2-i\epsilon)(-p_1\cdot q-i\epsilon)^2}$$

q^0-integral can be evaluated from residues on either UHP or LHP, giving a real result.

Compare to our earlier formula for J_{nm} :

Formally,

$$\text{``}J_{11}\text{''} = -i p_1^2 \int_{\mu<|\vec{q}|<M} \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2-i\epsilon)(p_1\cdot q-i\epsilon)(-p_1\cdot q-i\epsilon)}$$



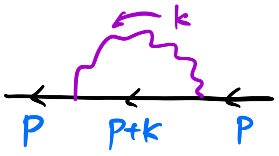
$$\operatorname{Re} J_{11} = -J_{12}\Big|_{\substack{p_2=p_1\\ \eta_2=-\eta_1}}$$

Contribute to Im J_{11}, divergent

So we must have

$$Z_{IR} = e^{g^2 \operatorname{Re} J_{11}}.$$

Let us check this at 1-loop order in scalar QED :



$$i\Sigma(p) = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{(p+k)^2+m^2-i\epsilon} \cdot \frac{-i}{k^2-i\epsilon}$$
$$\times (ig)^2 (2p+k)^2$$

the IR-divergent contribution (from small k) is

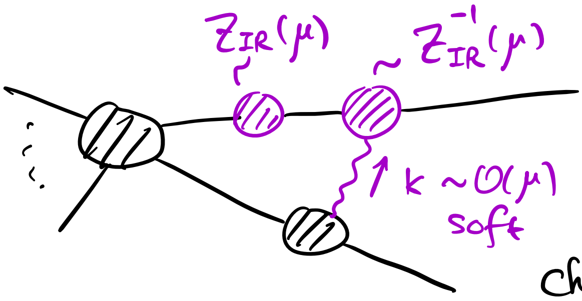
$$g^2\cdot 4p^2 \int_{|\vec{k}|<M} \frac{d^4k}{(2\pi)^4} \frac{1}{p^2+m^2+2p\cdot k-i\epsilon} \cdot \frac{1}{k^2-i\epsilon}$$

$$Z = \left(1 - \frac{\partial\Sigma}{\partial p^2}\Big|_{p^2=-m^2}\right)^{-1}$$

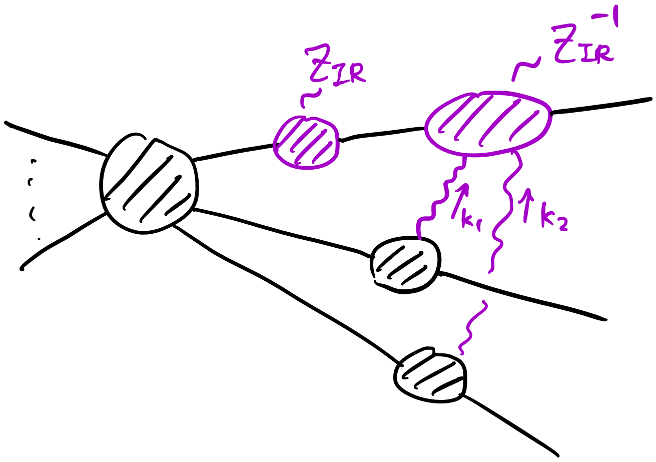
$$\ln Z = -\ln\left(1 - \frac{\partial \Sigma}{\partial p^2}\Big|_{p^2=-m^2}\right)$$

$$\overset{\text{IR},\ \mathcal{O}(q^2)}{\sim} \; i q^2 \cdot \cancel{4} p^2 \int_{|\vec{k}|<M} \frac{d^4k}{(2\pi)^4} \frac{1}{(\cancel{2}p\cdot k - i\epsilon)^2} \frac{1}{k^2 - i\epsilon}$$

$$= q^2 \,\mathrm{Re}\, J_{11} \qquad \checkmark$$



IR divergences due to soft photon loops beginning and ending on the same external charged particle line cancel!



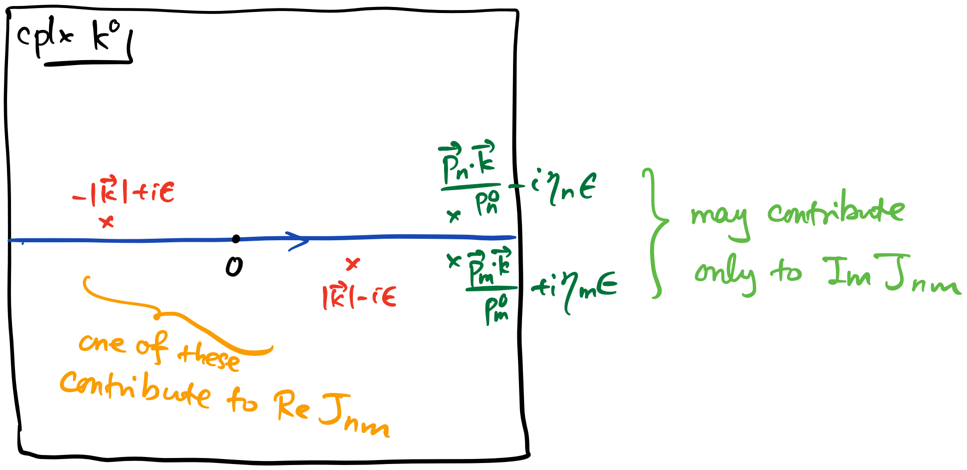
Final result:

soft photon loops modify the amplitude by the IR-divergent factor

$$e^{\frac{1}{2}\sum_{n\neq m} q_n q_m J_{nm}} \prod_n \left(Z_n^{IR}\right)^{\frac{1}{2}}$$

$$= (\text{phase}) \times e^{\frac{1}{2}\sum_{n,m} q_n q_m \operatorname{Re} J_{nm}} .$$

$$J_{nm} = -i\, p_n\cdot p_m \int_{\mu<|\vec{k}|<M} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-i\epsilon)(\eta_n p_n\cdot k - i\epsilon)(-\eta_m p_m\cdot k - i\epsilon)}$$



$$\operatorname{Re} J_{nm} = -\eta_n\eta_m \frac{p_n\cdot p_m}{2} \int_{\mu<|\vec{k}|<M} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{|\vec{k}|^3\cdot(p_n^0-\vec{p}_n\cdot\hat{k})(p_m^0-\vec{p}_m\cdot\hat{k})}$$

$$= \frac{\eta_n \eta_m}{8\pi^2 \beta_{nm}} \log\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right) \log \frac{M}{\mu}$$

$|\mathcal{A}_{\alpha\to\beta}|$ contains IR divergent factor

$$e^{\frac{1}{2}\sum_{n,m} q_n q_m \operatorname{Re} J_{nm}} = e^{-\frac{1}{2} A \log\frac{M}{\mu}} = \left(\frac{\mu}{M}\right)^{\frac{A}{2}}$$

$\longrightarrow 0$ as $\mu \to 0$ (recall $A > 0$ for generic kinematics)

- All S-matrix elements between in- and out-states involving charged particles and a definite number of photons vanish !?

- However, restricting to states with finitely many photons is unnatural: experimentally, cannot observe photons of energies below a certain threshold E_T:

  $\Gamma_{\alpha\to\beta+\text{soft}}$ acquires the factor

$$\underbrace{\left(\left(\frac{\mu}{M}\right)^{\frac{A}{2}}\right)^2}_{\text{soft photon loops}} \times \underbrace{\left(\frac{E_T}{\mu}\right)^{A}}_{\text{soft photon emission}} \longrightarrow \text{finite as } \mu \to 0.$$

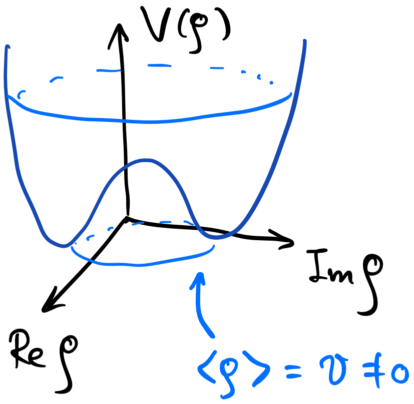
Remark: in the above analysis, the IR regulator μ was introduced in a rather ad hoc manner. One might worry about its consistency, i.e. why the same μ for virtual soft photons vs real soft photons emitted?

A more physical way to introduce the IR regulator would be to deform the QFT in a consistent manner, e.g. by giving the photon a small mass via an Abelian Higgs model:

- Introduce complex scalar field $\varphi(x)$ charged under $A_\mu(x)$, that acquires nonzero vacuum expection value $\langle\Omega|\varphi(x)|\Omega\rangle = v$.

$$\mathcal{L} \supset -|D_\mu \varphi|^2 - V(\varphi).$$

expand around $\varphi = v$, the term $-g^2|v|^2 A_\mu A^\mu$ gives a mass to the vector boson.



Consider a QFT defined through the path integral

bare action

$$Z = \int [D\phi]\, e^{iS[\phi]}$$

defined through some regularization
(e.g. dim reg)

$$\langle \mathcal{O}[\phi] \rangle = \frac{1}{Z} \int [D\phi]\, e^{iS[\phi]}\, \mathcal{O}[\phi].$$

Define <u>generating functional</u> $W[J]$ via

$$Z[J] \equiv \int [D\phi]\, e^{iS[\phi] + i\int d^Dx\, J(x)\,\phi(x)}$$

$$\equiv e^{iW[J]}.$$

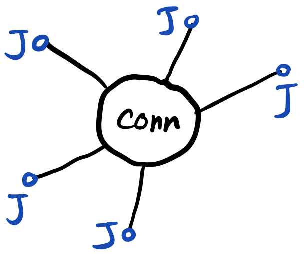
view as extra terms in action

new 1-point vertex

$= i\tilde{J}(k)$

$$\int d^Dx\, J(x)\,\phi(x) = \int \frac{d^Dk}{(2\pi)^D}\, \tilde{J}(k)\,\tilde{\phi}(-k)$$

$$iW[J] = \sum_{\substack{\text{all connected}\\ \text{diagrams}}}$$



$Z[J]$, or $W[J]$, capture all Green functions via

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{1}{Z}\frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta J(x_n)} Z[J]\Big|_{J=0}.$$

↑

in relating to Green fn $\langle\Omega|\hat{\phi}(x_1)\cdots|\Omega\rangle$, ordering determined by the choice of x^0-contour in $S[\phi]=\int_C dx^0\int d^{D-1}\vec{x}\,\mathcal{L}[\phi]$ in path integral.

Note in particular

$$\langle\phi(x)\rangle = \frac{\delta}{i\delta J(x)}\ln Z[J]\Big|_{J=0} = \frac{\delta W}{\delta J(x)}\Big|_{J=0}.$$

Now consider the Legendre transform

$$\underbrace{\Gamma[\phi_0]} \equiv W[J]-\int d^Dx\,\phi_0(x)J(x)\Bigg|_{J\text{ is such that }\underbrace{\frac{\delta W}{\delta J(x)}}_{\langle\phi\rangle_J}=\phi_0(x)}$$

"1PI effective action",
or "quantum eff. action"

expectation value of $\phi(x)$ "in the presence of source $J(x)$" → $\langle\phi\rangle_J$

Let us inspect the meaning of $\Gamma[\phi_0]$ in perturbation theory.

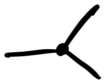
$$e^{i\Gamma[\phi_0]} = e^{iW[J] - i\int \phi_0 J}\Big|_{\frac{\delta W}{\delta J}=\phi_0}$$

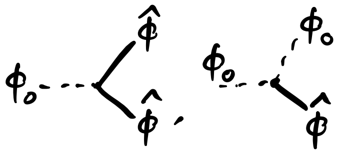
$$= \int [D\phi]\, e^{iS[\phi] + i\int (\phi-\phi_0) J}\Big|_{\langle\phi\rangle_J=\phi_0}$$

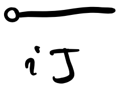
$$\underset{\phi\equiv\phi_0+\hat{\phi}}{=\!=} \int [D\hat{\phi}]\, e^{iS[\phi_0+\hat{\phi}] + i\int \hat{\phi} J}\Big|_{\langle\hat{\phi}\rangle_{J,\phi_0}=0}$$

how do we find such J ?

e.g. $S[\phi]$ contains the coupling ϕ^3

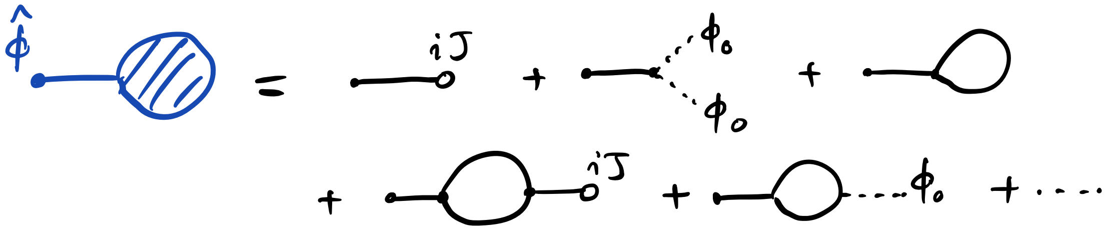
⇝ Feynman vertex 

$S[\phi_0+\hat{\phi}]$ ⇝ vertex 
, etc

$\int \hat{\phi} J$ ⇝ vertex 

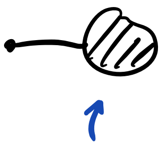
J is such that $\langle\hat{\phi}\rangle_{J,\phi_0}=0$

computed by

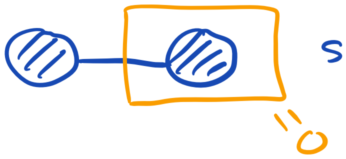


$= 0$ by choice of J !

$i\Gamma[\phi_0]$ is computed by summing over all connected diagrams with background field ϕ_0 and source J, the latter chosen so that

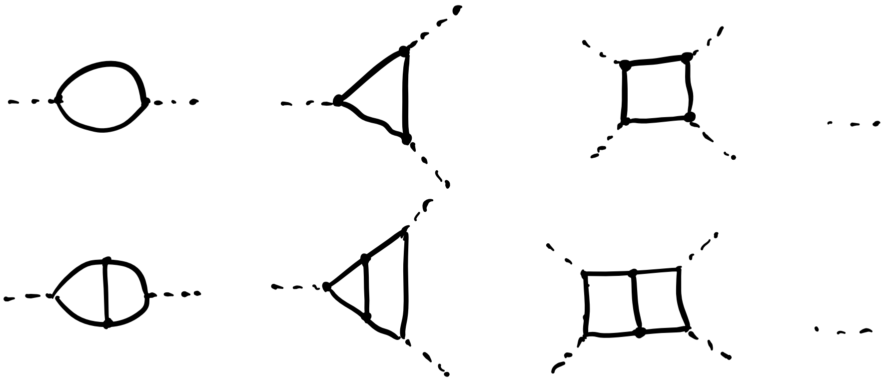
 $= 0$

the effect of J is to remove all possible "1-particle cuts": the subset of diagrams of the form



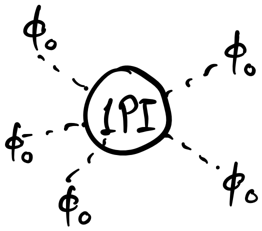
sum up to zero.

The only nontrivial contributions to $i\Gamma[\phi_0]$ come from 1PI diagrams, e.g.



In other words,

$$i\Gamma[\phi_0] = i\underset{\substack{\uparrow\\ \text{kinetic term}\\ \text{only}}}{S_{\text{kinetic}}[\phi_0]} + \sum_{\substack{\text{all 1PI graphs}\\ \text{with } n \text{ external}\\ \text{bkgrd field lines}}} \frac{1}{n!}$$



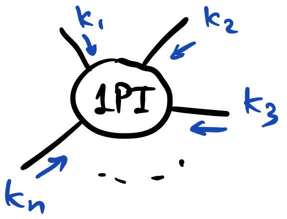
[Note: different convention for symmetry factor of graph when ϕ_0 is viewed as external line v.s. part of a vertex]

More precisely, Γ takes the form

$$\Gamma[\phi] = \sum_n \int \prod_{i=1}^{n} \frac{d^D k_i}{(2\pi)^D} \cdot (2\pi)^D \delta^D(\Sigma k_i)$$

$$\times \frac{1}{n!} \underbrace{\Gamma^{(n)}(k_1,\cdots,k_{n-1})} \tilde{\phi}(k_1)\cdots \underset{\substack{\uparrow\\ \phi(x) = \int \frac{d^Dk}{(2\pi)^D} e^{ik\cdot x}\tilde{\phi}(k)}}{\tilde{\phi}(k_n)}$$

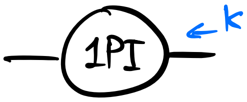
$i\Gamma^{(n)}(k_1,\cdots,k_{n-1})$ is computed by the sum of n-point amputated 1PI diagrams



Note: k_i's are off-shell

For instance,

$$i\Gamma^{(2)}(k) = -i(k^2+m^2) +$$

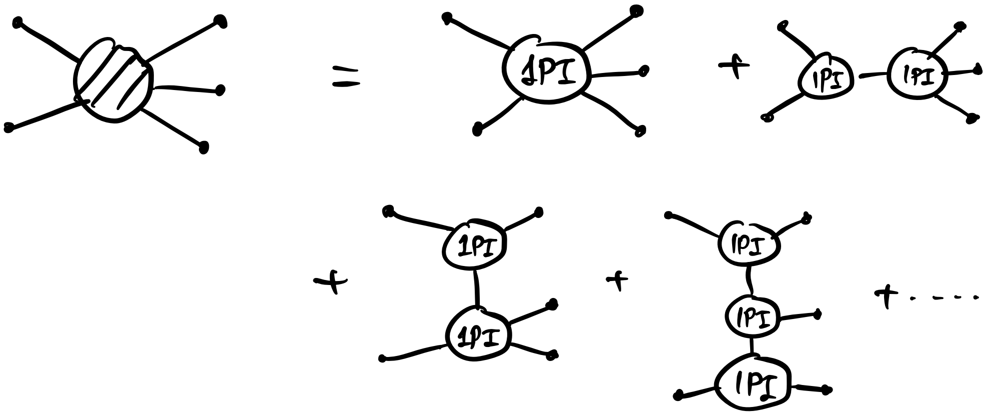


Note: derivative expansion of $\Gamma^{(n)}$ in k_i

$\Longleftrightarrow$ momentum expansion of "effective couplings"

[when there are loops of massless fields that contribute to $\Gamma^{(n)}$, the latter is typically not analytic in k_i's at $k_i = 0$.]

Interpretation of $\Gamma[\phi]$ as "quantum effective action" - the tree diagrams formed out of Feynman vertices of $\Gamma[\phi]$ give the exact Green functions:



More formally, we can go from $\Gamma[\phi]$ to $W[J]$ via the inverse Legendre transform

$$W[J] = \Gamma[\phi] + \int d^Dx \, \phi(x) J(x) \Big|_{\substack{\phi \text{ such that} \\ \frac{\delta \Gamma}{\delta \phi} = -J}}$$

check: $\Gamma[\phi] = W[J] - \int \phi J \Big|_{\frac{\delta W}{\delta J} = \phi}$.

$$\frac{\delta \Gamma}{\delta \phi(x)} = \int dy \underbrace{\frac{\delta W}{\delta J(y)}}_{\text{cancel}} \frac{\delta J(y)}{\delta \phi(x)} - J(x) - \int dy \underbrace{\phi(y)}_{\text{cancel}} \frac{\delta J(y)}{\delta \phi(x)}$$

$$= -J(x). \quad \checkmark$$

We can write equivalently

$$i W[J] = \lim_{\hbar \to 0} \hbar \log \int [D\phi]\, e^{\frac{i}{\hbar}\left(\Gamma[\phi] + \int \phi J\right)}$$

$=$ sum over tree diagrams built from $\Gamma[\phi]$.

Also note: in the absence of source J,

$$\phi \Big|_{\frac{\delta \Gamma}{\delta \phi} = 0} = \frac{\delta W}{\delta J}\Big|_{J=0} = \underbrace{\langle \phi \rangle}_{\text{vacuum expectation value}}$$

i.e. $\Gamma[\phi]$ is extremized at $\phi = \langle \phi \rangle$.

- In principle, $\Gamma[\phi]$ may be defined beyond perturbation theory - need $W[J]$ to be a concave functional for Legendre transform.

Consider the Euclidean version:

$$Z[J] \equiv e^{-W[J]} = \int [D\phi]\, e^{-S[\phi]-\int \phi J}$$

Assume bosonic fields

$\Rightarrow$ measure $[D\phi]$ positive (semi-) definite

For $0 \le t \le 1$,

$$e^{-W[tJ_1+(1-t)J_2]}$$

$$= \int [D\phi]\, e^{-S[\phi] - \int \phi (tJ_1 + (1-t) J_2)}$$

$$= \int [D\phi]\, e^{-t(S[\phi] + \int \phi J_1) - (1-t)(S[\phi] + \int \phi J_2)}$$

$$\le \left(\int [D\phi]\, e^{-(S[\phi] + \int \phi J_1)} \right)^{t} \left(\int [D\phi]\, e^{-(S[\phi] + \int \phi J_2)} \right)^{(1-t)}$$

Hölder's inequality

$$= e^{-tW[J_1] - (1-t)W[J_2]} .$$

$$\Rightarrow \quad W[tJ_1 + (1-t)J_2] \ge t\,W[J_1] + (1-t)\,W[J_2]$$

W concave,

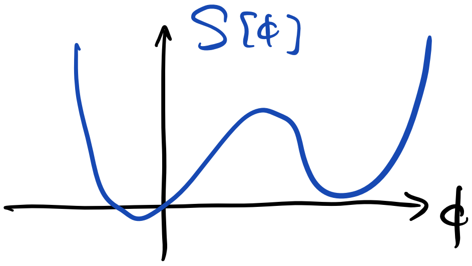
$$\Gamma[\phi] := W[J] - \int \phi J \Big|_{\frac{\delta W}{\delta J} = \phi}$$

is well-defined

An instructive toy model

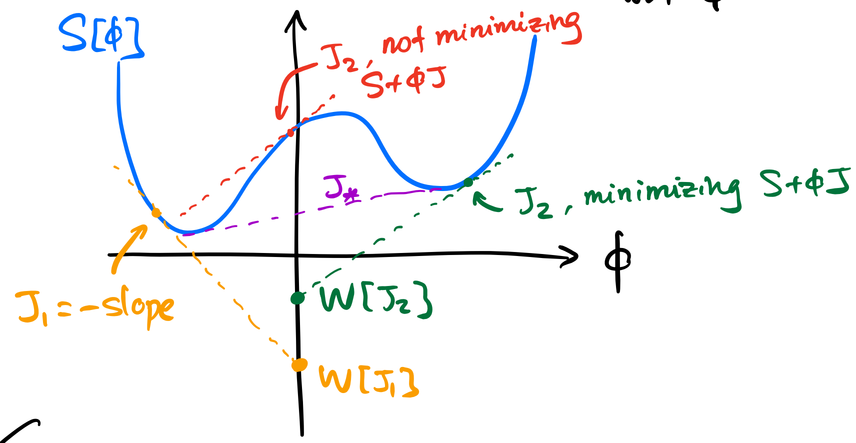
"D=0 QFT in $\hbar \to 0$ limit"

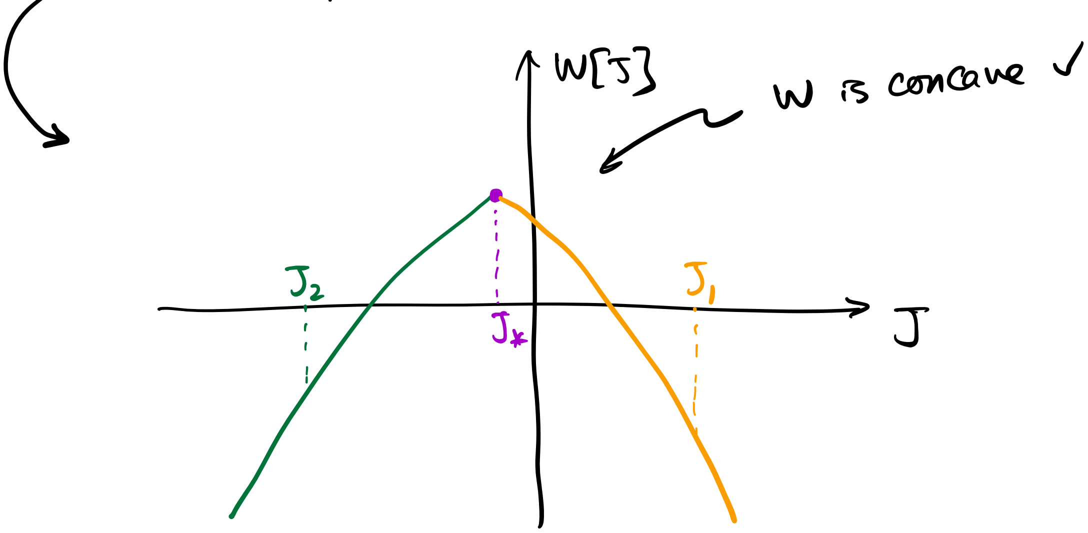
$\phi \in \mathbb{R}$,



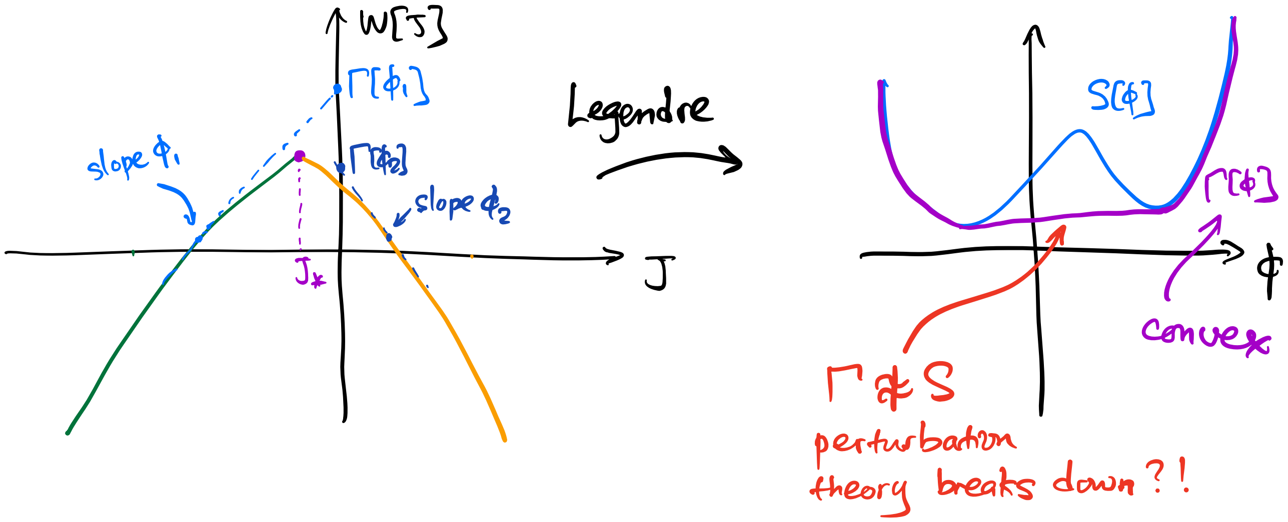
$$Z[J] = e^{-W[J]} = e^{-S[\phi]-\phi J}\Big|_{\substack{\text{maximize} \\ \text{wrt } \phi}}$$

$$W[J] = S[\phi] + \phi J\Big|_{\substack{\text{minimize} \\ \text{wrt } \phi}} \leftarrow \frac{\partial S}{\partial \phi} + J = 0$$





W is concave ✓



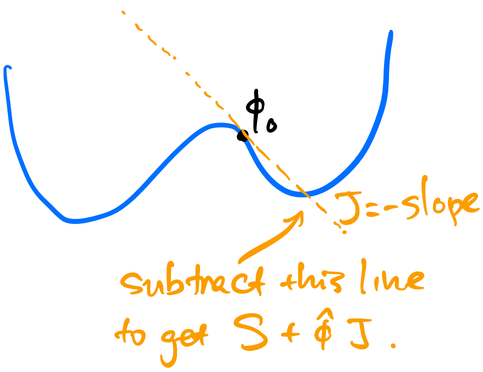
Γ ≠ S
perturbation theory breaks down ?!

Recall: (Euclidean) quantum eff. action

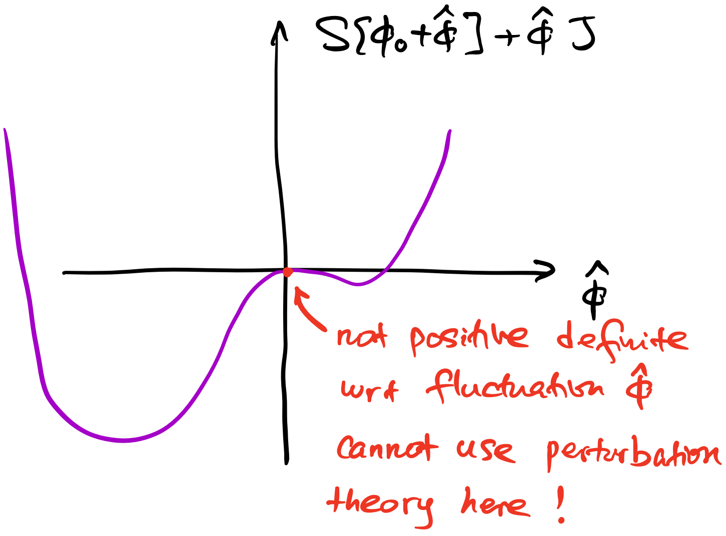
$$e^{-\Gamma[\phi_0]} = \int [D\hat{\phi}]\; e^{-S[\phi_0+\hat{\phi}] - \int \hat{\phi} J} \Bigg|_{\substack{J \text{ chosen} \\ \text{such that} \\ \langle\hat{\phi}\rangle_{J,\phi_0}=0.}}$$

can this be evaluated using perturbation theory ?

In our toy model,



subtract this line to get S + φ̂ J .



not positive definite wrt fluctuation $\hat{\phi}$

cannot use perturbation theory here !

Renormalizability

Consider a scalar field theory defined via the (Euclidean) path integral

$$Z = \int [D\phi] \, e^{-\int d^D x \, \mathcal{L}_E},$$

with bare Lagrangian

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \sum_I g_I \mathcal{O}_I$$

- each $\mathcal{O}_I$ is a function of $\phi(x)$ and its derivatives $\partial_\mu \phi(x)$, $\partial_\mu \partial_\nu \phi(x)$, etc.
- g_I are bare couplings (include all counter terms)
- the path integral measure is typically defined with a regularization scheme, e.g.
  - momentum cutoff: $\tilde{\phi}(k)$, $|k| < \Lambda$
  - dimensional regularization $D =$ integer$- \epsilon$
- Physical observables (e.g. mass spectrum of particles, scattering amplitudes) should be finite as we take $\Lambda \to \infty$ or $\epsilon \to 0$, while adjusting g_I's accordingly (as function of Λ or ϵ).

Q: Is this possible (for a Lagrangian with finitely many g_I's ?)

A sufficient criterion: Euclidean Green functions $\langle \phi_R(x_1) \cdots \phi_R(x_n) \rangle$ finite, where $\phi_R(x)$ is related to $\phi(x)$ by a rescaling

$$\phi_R(x) = Z_R^{-\frac{1}{2}} \phi(x),$$

where Z_R may be infinite in the limit $\Lambda \to \infty$ or $\epsilon \to 0$. This is equivalent to demanding that the 1PI effective action $\Gamma[\phi]$ to be such that

$$\Gamma[Z_R^{\frac{1}{2}} \phi_R]$$

is a finite functional of $\phi_R(x)$.

- We may equivalently work with ϕ_R as the field variable, $S[\phi = Z_R^{\frac{1}{2}} \phi_R] \equiv \tilde{S}[\phi_R]$,

  and demand $\Gamma[\phi = Z_R^{\frac{1}{2}} \phi_R] = \tilde{\Gamma}[\phi_R]$ to be finite.

e.g. $S[\phi] = \int \frac{1}{2} \phi (-\Box + \underset{\uparrow \text{ bare mass}}{m_0^2}) \phi + \cdots\cdot$

$$\tilde{S}[\phi_R] = \int \frac{1}{2} \phi_R (-\Box + m_R^2) \phi_R$$

$$+ \underbrace{(Z_R - 1) \frac{1}{2} \phi_R (-\Box + m_R^2) \phi_R + \frac{1}{2} \delta m^2 \phi_R^2}_{\text{view as counter terms}} + \cdots$$

Generally, $\tilde{S}[\phi_R] = S_{kin}[\phi_R] + \int \underset{\substack{\uparrow \\ \text{bare couplings, including all counter terms}}}{\sum_I g_I} \mathcal{O}_I[\phi_R]$

$$\tilde{\Gamma}[\phi_R] = S_{kin}[\phi_R] + \int \sum_I \underset{\substack{\uparrow \\ \text{finite "effective couplings",} \\ \text{linearly related to 1PI Green} \\ \text{functions of well-defined field} \\ \text{operator } \hat{\phi}_R \text{.}}}{g_I^{eff}} \mathcal{O}_I[\phi_R]$$

We would like g_I, and thereby g_I^{eff}, to be determined by a finite set of "physical couplings", λ_{phys}.

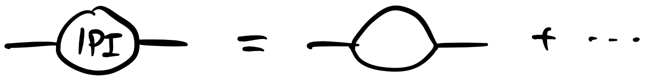
$$g_I = \underset{\substack{\uparrow \\ \text{may well diverge in the limit } \Lambda \to \infty}}{g_I(\lambda_{phys}, \Lambda)},$$

such that

$$\lim_{\Lambda \to \infty} g_I^{eff}(\lambda_{phys}, \Lambda) = \text{finite, all } I.$$

Let us inspect what kind of counter terms are needed in $g_I(\lambda_{phys}, \Lambda)$.

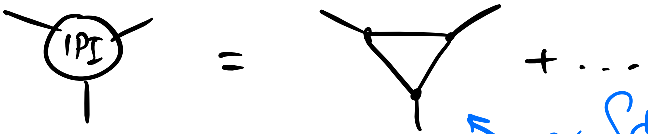
e.g. $D=4$, $S[\phi] \supset \int g_3 \phi^3$.



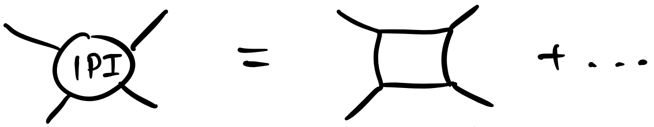
$\sim g_3^2 \int d^4k \frac{1}{(k^2+m^2)^2} \sim g_3^2 \log \Lambda$

divergent contribution to mass term in self-energy, canceled by mass counterterm

$\delta m^2 \sim g_3^2 \log \Lambda$.



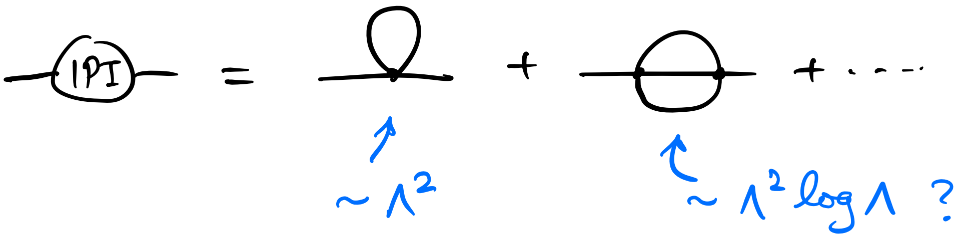
$\sim \int d^4k \frac{1}{(k^2+m^2)^3} = \text{finite}$

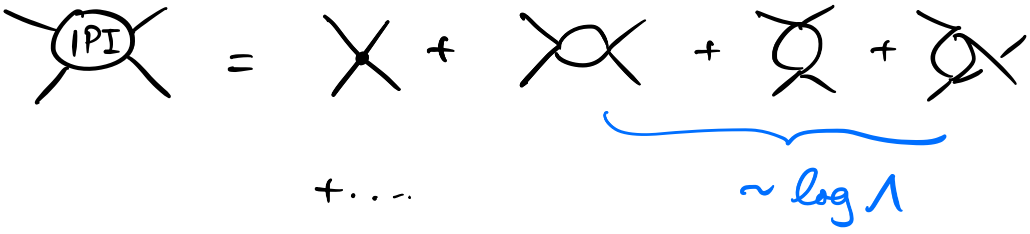


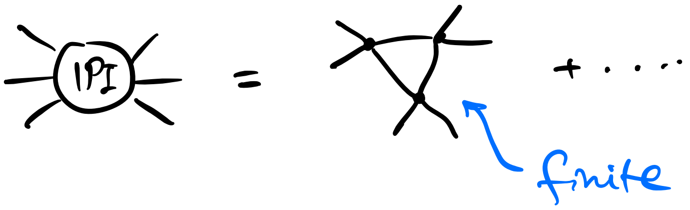
finite

No counter terms of the form $\delta g_4 \phi^4$, $\delta g_6 \phi^6$, etc. needed.

Now consider $D=4$, $S[\phi] \supset \int g_4 \phi^4$.







Need mass counter term $\delta m^2 \phi^2$

$$\delta m^2 \sim g_4 \Lambda^2 + \mathcal{O}(g_4^2)$$

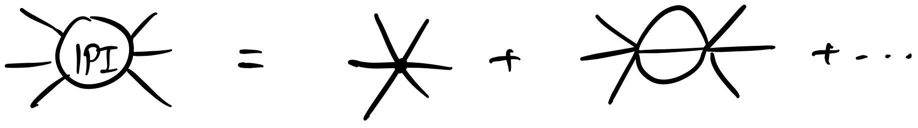
also $\delta Z (\partial_\mu \phi)^2$, $\delta Z \sim g_4^2 \log \Lambda$.

and $\delta g_4 \phi^4$, $\delta g_4 \sim g_4^2 \log \Lambda$.

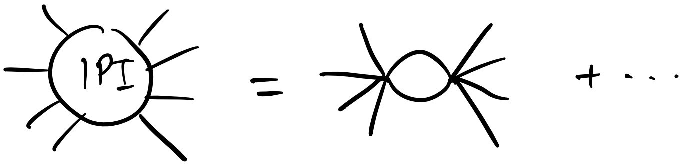
No counter terms of the form $\delta g_6 \phi^6$, etc. needed.

Next, consider still $D=4$,

$$S[\phi] \supset \int g_6 \phi^6 .$$

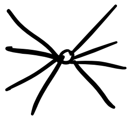


$\sim \Lambda^2 \log \Lambda$



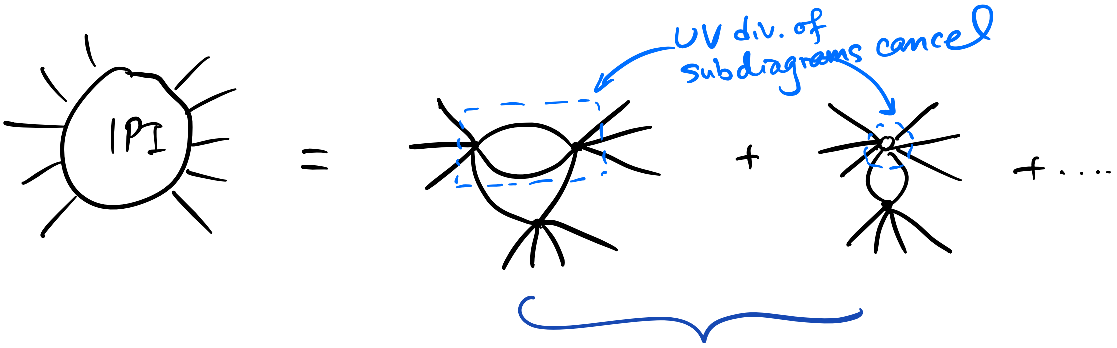
$\sim \log \Lambda$

- Need counter terms $\delta g_8 \cdot \phi^8$



$$\delta g_8 \sim g_6^2 \log \Lambda$$

- But there is also



$\sim \log \Lambda$

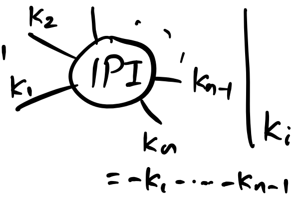
Need $\delta g_{10} \phi^{10}$, $\quad \delta g_{10} \sim g_6^3 \log \Lambda$,

and so forth, ad infinitum.

Superficially, the necessary counter terms are dictated by dimension analysis:

- action $S[\phi]$ is dimensionless
- ϕ has mass dimension $\frac{D-2}{2}$.
- $\mathcal{O}_I$ consisting n ϕ's and l derivatives has mass dimension $d_I = l + n\frac{D-2}{2}$.
- g_I has mass dimension $D-d_I$.

g_I^{eff} for $\mathcal{O}_I \sim \phi\,\partial^{l_1}\phi\ldots\partial^{l_{n-1}}\phi$

is computed through $\partial_{k_1}^{l_1}\cdots\partial_{k_{n-1}}^{l_{n-1}}$  $\Big|_{k_i=0}$

Assuming non-singular in $m\to 0$ limit, UV divergence in 1PI diagram contributing to g_I^{eff} takes the form

$$\prod_{J\in \text{1PI dgm}} g_J \cdot \Lambda^{(D-d_I)-\sum_J (D-d_J)}$$

$$\times\ (\log\Lambda)^{\le \#\text{loops}}.$$

Need (potentially divergent) counter term $\delta g_I \mathcal{O}_I$ in the action $\tilde{S}[\phi]$ if

$$D - d_I - \sum_J (D - d_J) \geq 0$$

- If $d_I \leq D$ for all couplings $g_I \mathcal{O}_I$ appearing in the action $\tilde{S}[\phi]$, potentially only the counter terms for such couplings are needed to ensure finite $\Gamma[\phi]$.

  - "renormalizable theory"

- If $\tilde{S}[\phi]$ contains some $g_I \mathcal{O}_I$ with $d_I > D$, there will be infinitely many counter terms $\delta g_I \mathcal{O}_I$ needed, with d_I arbitrarily large. Cannot produce finite $\Gamma[\phi]$ with only finitely many terms in $\tilde{S}[\phi]$.

  - "non-renormalizable theory"
  - cannot pin down theory entirely by measuring finitely many λ_{phys}

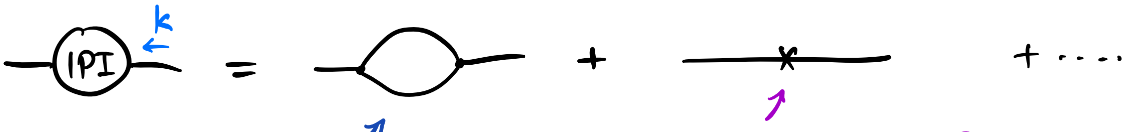
Let us understand in more detail the cancellation of divergences in a renormalizable theory.

e.g. $D=6$, ϕ^3 theory

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m_R^2\phi^2$$

$$+\delta Z\left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m_R^2\phi^2\right] + \frac{1}{2}\delta m^2\phi^2$$

$$+\frac{1}{3!}g_R\phi^3 + \frac{1}{3!}\delta g\cdot\phi^3$$



$$-\delta Z\,(k^2+m_R^2) - \delta m^2$$

$$\frac{1}{2}g_R^2\int\frac{d^6\ell}{(2\pi)^6}\frac{1}{(\ell^2+m_R^2)((k-\ell)^2+m_R^2)}$$

$$= \frac{1}{2}g_R^2\int\frac{d^6\ell}{(2\pi)^6}\int_0^1 dx\,\frac{1}{(\ell^2+k^2x(1-x)+m_R^2)^2}$$

- momentum cutoff scheme

$$\delta m^2 \sim g_R^2\Lambda^2 + g_R^2 m_R^2\log\Lambda$$

$$\delta Z \sim g_R^2\log\Lambda$$

- dim reg $D=6-\epsilon$,

$$\delta m^2 \sim g_R^2 m_R^2\cdot\frac{1}{\epsilon}, \quad \delta Z \sim g_R^2\cdot\frac{1}{\epsilon}.$$

It will be useful later to inspect the explicit k-dependence of the 1-loop contribution to $\Sigma(k)$

$$\frac{1}{2} g_R^2 \int \frac{d^D\ell}{(2\pi)^D} \int_0^1 dx \, \frac{1}{(\ell^2 + k^2 x(1-x) + m_R^2)^2}$$

$$= \frac{1}{2} g_R^2 \frac{\Gamma(2-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}} \int_0^1 dx \, (k^2 x(1-x) + m_R^2)^{\frac{D}{2}-2}$$

$$= \frac{g_R^2}{2\cdot(4\pi)^3} \Big[\frac{1}{\epsilon}\left(-\frac{8}{3}k^2 - 16 m_R^2\right)$$

some finite consts.

$$+ \left(\frac{4}{3}k^2 + 8m_R^2\right)\log(k^2) + \# \, k^2 + \# \, m_R^2$$

$$+ \text{(terms that vanish at large } k) \Big]$$

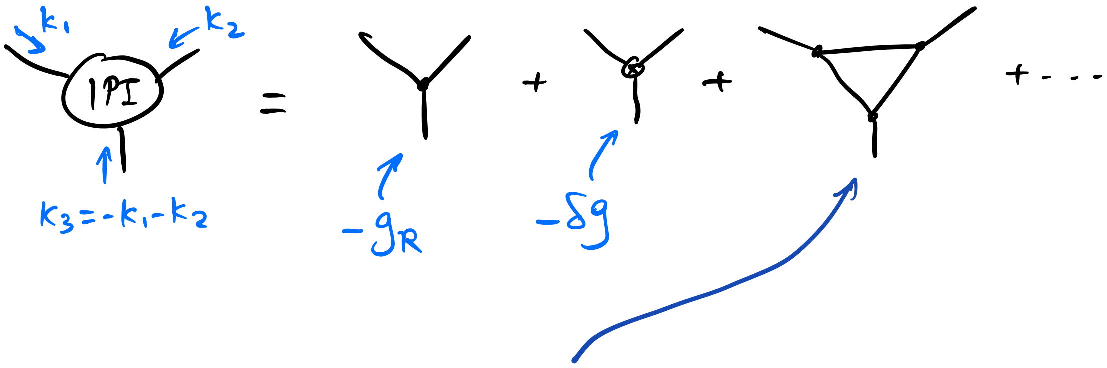
Choose $$\delta Z \Big|_{\mathcal{O}(g_R^2)} = \frac{g_R^2}{2\cdot(4\pi)^3}\left(-\frac{8}{3}\frac{1}{\epsilon} + \text{finite}\right)$$

choice of renorm. scheme

$$\delta m_R^2 \Big|_{\mathcal{O}(g_R^2)} = \frac{g_R^2 m_R^2}{2\cdot(4\pi)^3}\left(-\frac{40}{3}\frac{1}{\epsilon} + \text{finite}\right)$$

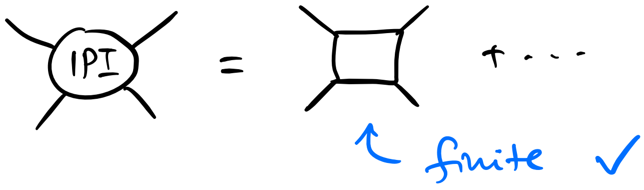
Cancel $\frac{1}{\epsilon}$ terms, resulting in a finite $\Sigma(k)$ that behaves at large k as

$$\Sigma(k)\Big|_{\mathcal{O}(g_R^2)} \sim (\alpha k^2 + \beta)\log(k^2). \qquad k^2 \gg m_R^2.$$

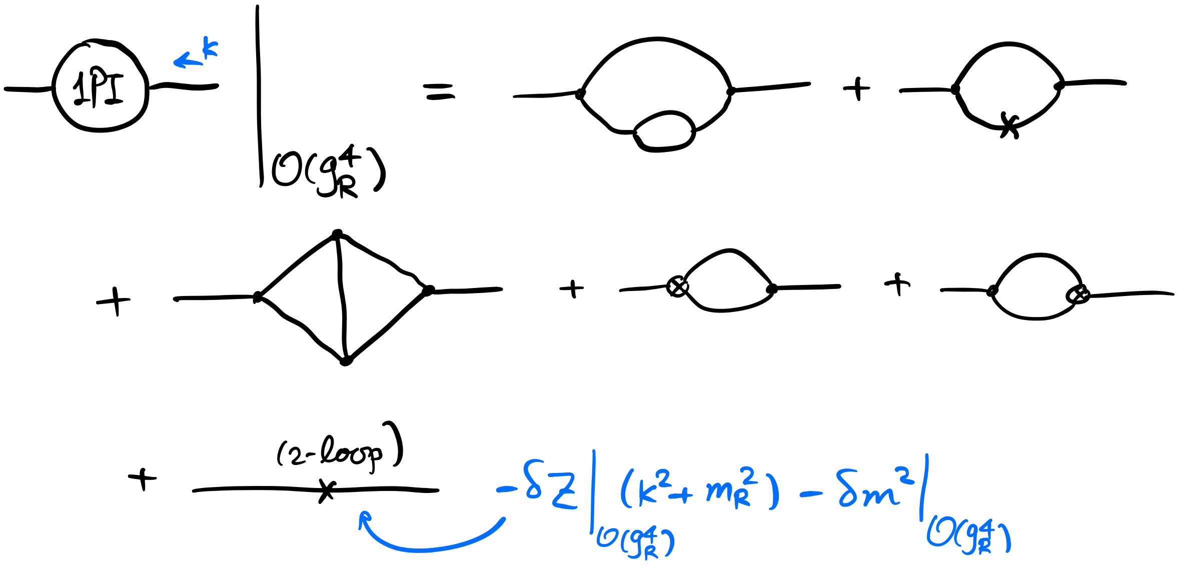


$$-g_R^3\int\frac{d^6\ell}{(2\pi)^6}\,\frac{1}{(\ell^2+m_R^2)((\ell+k_1)^2+m_R^2)((\ell+k_2)^2+m_R^2)}$$

Need $\delta g \sim g_R^3 \log\Lambda$ or $g_R^3\cdot\frac{1}{\epsilon}$



. What happens at the next order in perturbation theory?



part of $G(q) = \frac{1}{q^2 + m_R^2 - \Sigma(q)}$

$\Sigma(q)\big|_{q_R^2} \sim (\alpha q^2 + \beta) \log\left(\frac{q^2}{M^2}\right)$ at large q, for some finite α, β, M.

$$\sim \int \frac{d^6 q}{(2\pi)^6} \frac{\Sigma(q)}{(q^2+m_R^2)^2 \left((k-q)^2 + m_R^2\right)} \qquad \circledast$$

UV divergence has simple k-dependence:

$$\int \frac{d^6 q}{(2\pi)^6} \frac{\Sigma(q)}{(q^2+m_R^2)^2} \left[\frac{1}{(k-q)^2+m_R^2} - \frac{1}{q^2+m_R^2} - \frac{2k\cdot q - k^2}{(q^2+m_R^2)^2} - \frac{4(k\cdot q)^2}{(q^2+m_R^2)^3} \right]$$

$= \text{finite !}$

- Let's verify in dim reg:

  using Feynman trick, we can turn $\circledast$ into

$$\int \frac{d^D q}{(2\pi)^D} \int_0^1 dx \, \frac{2(1-x)\, \Sigma(q+kx)}{\left(q^2 + k^2 x(1-x) + m_R^2\right)^3}$$

The divergent part of the q-integral, involving $\Sigma(q) \sim (\alpha q^2 + \beta) \log \frac{q^2}{M^2}$, can be performed using

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+\Delta)^n} = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)} \Delta^{\frac{D}{2}-n}$$

$$\xrightarrow{D=6-\epsilon,\ n=2,3} \frac{\#}{\epsilon} \Delta^{3-n} + \text{finite}$$

and

$$\int \frac{d^Dq}{(2\pi)^D}\frac{\log(q^2+\Delta)}{(q^2+\Delta)^n} = -\frac{1}{(4\pi)^{\frac{D}{2}}}\frac{\partial}{\partial n}\left(\frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)}\Delta^{\frac{D}{2}-n}\right)$$

$$\xrightarrow{D=6-\epsilon,\ n=2,3} \quad \sim \frac{\partial}{\partial\epsilon}\left(\frac{\#}{\epsilon}+\#\log\Delta\right)\Delta^{3-n}+\frac{\#}{\epsilon}\Delta^{3-n}+\text{finite}$$

$$\sim \left(\frac{\#}{\epsilon^2}+\frac{\#}{\epsilon}\right)\Delta^{3-n}+\text{finite}$$

Importantly, no $\frac{1}{\epsilon}\log\Delta$ term!

This leads to

✪ $$\stackrel{D=6-\epsilon}{=\!=} A_\epsilon k^2 + B_\epsilon + \text{finite}$$

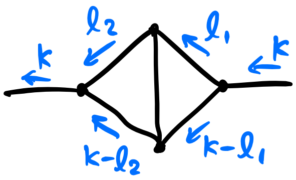
where $A_\epsilon = \frac{\alpha_1}{\epsilon}+\frac{\alpha_2}{\epsilon^2}$, $\quad B_\epsilon = \frac{\beta_1}{\epsilon}+\frac{\beta_2}{\epsilon^2}$

for some finite k-independent $\alpha_1, \alpha_2, \beta_1, \beta_2$.

These divergences can be canceled by including in δZ and δm^2 counterterms of order $g_R^4\frac{1}{\epsilon^2}$ and $g_R^4\frac{1}{\epsilon}$.

Note: If we had a divergence of the form $\sim\frac{1}{\epsilon}\log(k^2)$, we would be in trouble, as there is no counterterm (that corresponds to a local Lagrangian) that could cancel such a divergence. Luckily (?!), this did not occur.

- Next, let us inspect UV divergence of the diagram



$$= \frac{g_R^4}{2} \int \frac{d^6\ell_1}{(2\pi)^6} \frac{d^6\ell_2}{(2\pi)^6} P(\ell_1) P(k-\ell_1)$$
$$\times P(\ell_2) P(k-\ell_2)\, P(\ell_1-\ell_2)$$

We can write each propagator in the form

$$P(\ell) \equiv \frac{1}{k^2+m_R^2} = \int_0^\infty d\alpha\; e^{-\alpha(k^2+m_R^2)}$$

loop integral

$$\int_0^\infty d\alpha_1 \cdots d\alpha_5\; e^{-m_R^2 \sum_{i=1}^{5} \alpha_i} \int \frac{d^6\ell_1}{(2\pi)^6} \frac{d^6\ell_2}{(2\pi)^6}$$
$$\times \exp\left[-\alpha_1 \ell_1^2 - \alpha_2 (k-\ell_1)^2 - \alpha_3 \ell_2^2 - \alpha_4 (k-\ell_2)^2 - \alpha_5 (\ell_1-\ell_2)^2\right]$$

$$= \frac{1}{(4\pi)^6} \int_0^\infty d\alpha_1 \cdots d\alpha_5\, \frac{e^{-m_R^2 \sum \alpha_i - k^2 Q}}{(\det A)^3} \qquad ✪$$

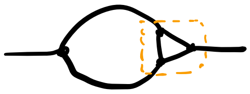
where $A = \begin{pmatrix} \alpha_1+\alpha_2+\alpha_5 & -\alpha_5 \\ -\alpha_5 & \alpha_3+\alpha_4+\alpha_5 \end{pmatrix}$,

$$Q = \alpha_2+\alpha_4 - (\alpha_2\ \alpha_4)\, A^{-1} \begin{pmatrix} \alpha_2 \\ \alpha_4 \end{pmatrix} \gtrsim 0 .$$

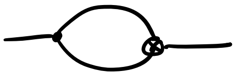
UV divergence comes from the region det $A \to 0$.

The generic UV divergence occurs in the limit

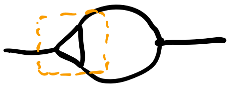
$\alpha_1, \alpha_2, \alpha_5 \to 0$



cancelling against ↕



or $\alpha_3, \alpha_4, \alpha_5 \to 0$



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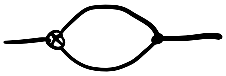
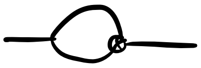
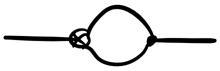


Writing ⊛ as

$$\frac{1}{(4\pi)^6}\int_0^\infty \left[\prod_{i=1}^{5} d\alpha_i\, e^{-m_R^2\alpha_i}\right]\cdot \mathcal{F}(k^2;\alpha_1,\cdots,\alpha_5),$$

with $$\mathcal{F} = \frac{e^{-k^2 Q}}{(\det A)^3}.$$

[obviously, $\rho^6\mathcal{F}(\frac{k^2}{\rho};\rho\alpha_1,\cdots,\rho\alpha_5) = \mathcal{F}(k^2;\alpha_1,\cdots,\alpha_5)$]

the effect of adding  and  amounts to replacing $\mathcal{F}$ with

$$\tilde{\mathcal{F}} := \mathcal{F} - \lim_{\rho\to\infty}\rho^3\mathcal{F}(\frac{k^2}{\rho};\rho\alpha_1,\rho\alpha_2,\alpha_3,\alpha_4,\alpha_5)$$
$$-\lim_{\rho\to\infty}\rho^3\mathcal{F}(\frac{k^2}{\rho};\alpha_1,\alpha_2,\rho\alpha_3,\rho\alpha_4,\alpha_5).$$

Remaining divergence in $\int \tilde{F}$ comes from the limit $\alpha_1, \dots, \alpha_5 \to 0$ simultaneously, and depends on k at most quadratically by power counting.

After cancelling divergence by adjusting δZ, δm, we are left with

$$\int \left[\prod_i d\alpha_i \, e^{-m_R^2 \alpha_i} \right] \cdot \Big(\underbrace{\tilde{F} - \tilde{F}\Big|_{k^2=0} - \frac{\partial \tilde{F}}{\partial k^2}\Big|_{k^2=0} \cdot k^2}_{\overset{|||}{F^R}} \Big)$$

can verify that

$$\int F^R \text{ is finite.}$$

This analysis can be extended to show that all divergences in 1PI diagrams can be cancelled by diagrams involving counter terms that correspond to nested subdiagrams, thereby proving perturbative renormalizability of naive-power-counting-renormalizable theories.

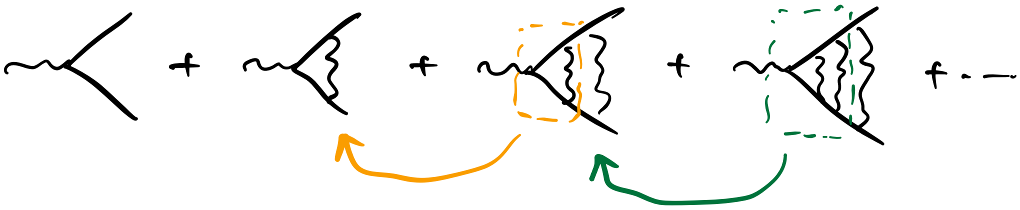
[Bogoliubov-Parasiuk-Hepp-Zimmermann theorem]

Renormalization Group

is a way of characterizing QFT by effective descriptions at different energy/momentum scales, and a computational scheme for relating such effective descriptions at different scales.

- "1PI RG" a.k.a. Gell-Mann-Low eqn
  Callan-Symanzik eqn

  is an improved version of perturbation theory in which we "recycle" computation of loop corrections in sub-diagrams, and perform a partial resummation e.g.



This is accomplished by defining energy-scale-dependent effective/"renormalized" coupling $g(\mu)$, and studying how $g(\mu)$ varies with μ.

- Wilsonian RG a.k.a. Wilson-Polchinski eqn.
  - define "Wilsonian effective action" $S_\Lambda[\phi]$ by performing the functional integration over modes $\tilde{\phi}(k)$ for $\underset{\substack{\nearrow \\ \text{finite}}}{\Lambda} < |k| < \underset{\substack{\uparrow \\ \text{UV cutoff}}}{\Lambda_0}$
  - study how $S_\Lambda[\phi]$ changes with Λ.

Let us begin with the example of $D=4$ ϕ^4 theory:

(bare) Euclidean Lagrangian

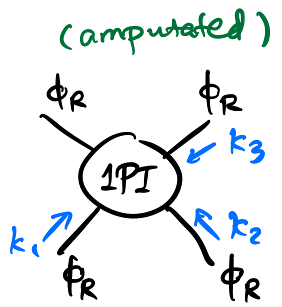
$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{1}{4!} g_0 \phi^4 .$$

or equivalently in term of $\phi_R = Z_R^{-\frac{1}{2}} \phi$,

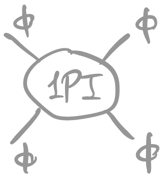
$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi_R)^2 + \frac{1}{2} m_R^2 \phi_R^2 + (Z_R - 1)\left(\frac{1}{2}(\partial_\mu \phi_R)^2 + \frac{1}{2} m_R^2 \phi_R^2\right) + \frac{1}{2}\delta m^2 \phi_R^2 + \frac{1}{4!} g_R \phi_R^4 + \frac{1}{4!} \delta g \cdot \phi_R^4 .$$

Consider

$-\Gamma^{(4)}(k_1, k_2, k_3) =$



[$= (Z_R^{\frac{1}{2}})^4 \times$



]

- We could define

$$g_R = -\Gamma^{(4)}(k_1 = k_2 = k_3 = 0),$$

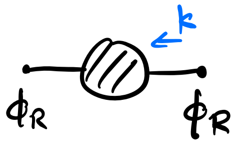
but this is just an arbitrary choice.

For mass scale μ, one migh expect

$$\Gamma^{(4)}\Big|_{\substack{k_i \sim \mathcal{O}(\mu) \\ i=1,2,3,4}}$$ to characterize the

effective coupling strength at scale μ.

This is not accurate, as the normalization of ϕ_R is still arbitrary.



$$= \frac{1}{k^2 + m_R^2 - \Sigma_R(k)}$$

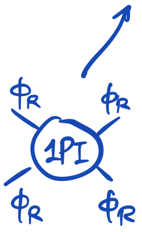
Rescale ϕ_R by

$$Z(\mu) \equiv \left(1 - \frac{\partial \Sigma_R}{\partial k^2}\right)^{-1}\Bigg|_{k^2=\mu^2} \quad \text{(Euclidean)}$$

so that the 1PI eff. action $\Gamma[\phi_R]$ viewed as a functional of $[\phi]_\mu \equiv (Z(\mu))^{-\frac{1}{2}}\phi_R$ has canonically normalized kinetic term at momentum $k \sim \mathcal{O}(\mu)$.

- Define a scale-dependent "running coupling"

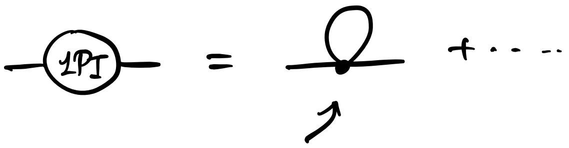
$$g(\mu) = -\left(Z(\mu)^{\frac{1}{2}}\right)^4 \underbrace{\Gamma^{(4)}(k_1,k_2,k_3)}\Bigg|_{\substack{k_i^2=\mu^2 \quad \text{(Euclidean)} \\ s=t=u=-\frac{4}{3}\mu^2}}$$



a specific renormalization scheme

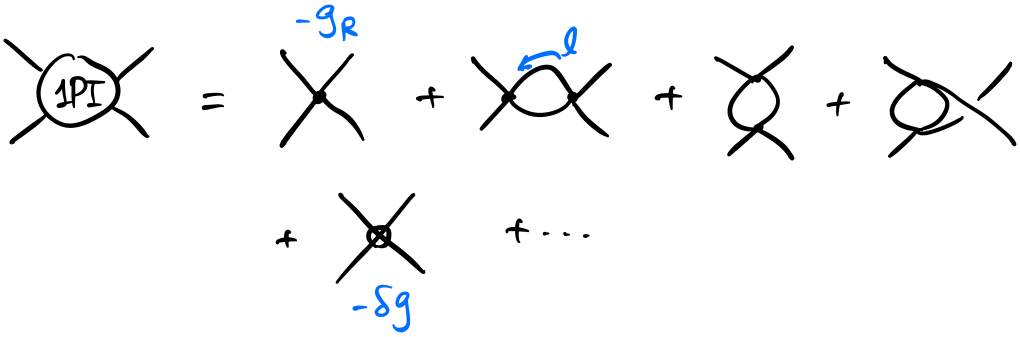
————— ''

1-loop example:



does not affect Z_R or $Z(\mu)$.

$Z_R = Z(\mu) = 1$ at 1-loop.



$$-g(\mu) = -g_R - \delta g$$
$$+\, 3\cdot\frac{g_R^2}{32\pi^2}\int_0^1 dx\left[\log\frac{\Lambda^2}{m_R^2 - s\,x(1-x)} - 1\right]\Bigg|_{s=-\frac{4}{3}\mu^2}$$
$$+\, \mathcal{O}(g_R^3).$$

By our convention, $g_R = g(\mu=0)$

$\rightarrow$ fix δg.

$$g(\mu) = g_R - \frac{3g_R^2}{32\pi^2}\int_0^1 dx\,\log\frac{m_R^2}{m_R^2 + \frac{4}{3}\mu^2 x(1-x)}$$
$$+\, \mathcal{O}(g_R^3)$$

- we have obtained unambiguous relations among physical couplings ✓
- However, 1-loop correction becomes large when $\mu \gg m_R$.

One could of course compute more terms in perturbation theory, but they will also grow as $\mu \gg m_R$.

Instead, let us compare $g(\mu)$ to $g(\mu')$ for μ' close to μ :

$$g(\mu') = g(\mu) - \frac{3}{32\pi^2} g(\mu)^2 \int_0^1 dx \, \underbrace{\log \frac{m_R^2 + \frac{4}{3}\mu^2 x(1-x)}{m_R^2 + \frac{4}{3}\mu'^2 x(1-x)}}_{\text{small for } \mu' \sim \mu}$$

$$+ \cancel{\mathcal{O}(g_R^3)} \quad \mathcal{O}((g(\mu))^3)$$

perturbative expansion in $g(\mu)$ more reliable when μ' is close to μ !

- To get most out of this relation, we take $\mu' = \mu + d\mu$,

$$\Rightarrow \quad \frac{d g(\mu)}{d \log \mu} = \frac{3}{16\pi^2} (g(\mu))^2 \cdot \int_0^1 dx \, \frac{\frac{4}{3}\mu^2 x(1-x)}{m^2 + \frac{4}{3}\mu^2 x(1-x)}$$

$$+ \mathcal{O}((g(\mu))^3) .$$

For $\mu \gg m$, we can write

$$\frac{d\,g(\mu)}{d\log\mu} = \frac{3\,(g(\mu))^2}{16\pi^2} + \mathcal{O}((g(\mu))^3)\,.$$

"renormalization group equation"

"β-function"

Solve the 1-loop RGE:

$$\frac{1}{g(\mu)} - \frac{1}{g(\mu')} = -\frac{3}{16\pi^2}\log\frac{\mu}{\mu'}$$

$$\Rightarrow \quad g(\mu) = \frac{g(\mu')}{1 - \frac{3}{16\pi^2}\,g(\mu')\log\frac{\mu}{\mu'}}\,. \qquad \circledast$$

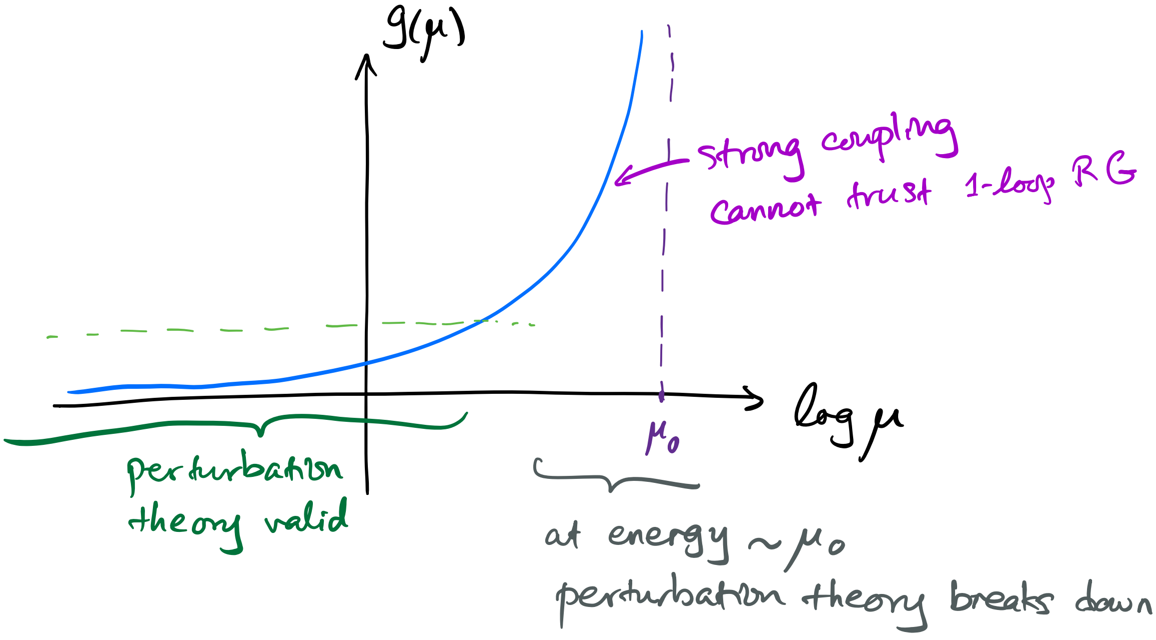
- a good approximation provided that $g(\mu'')$ is small for all $\mu < \mu'' < \mu'$.
  even if $g(\mu')\log\frac{\mu}{\mu'}$ is large!

- $\circledast$ contains all order contributions in $g(\mu')$, or g_R, even though we only perform a 1-loop computation!

At a scale

$$\mu_0 = \mu' e^{\frac{16\pi^2}{3} \frac{1}{g(\mu')}} ,$$

the 1-loop RGE predicts $g(\mu_0) = \infty$.



We can equivalently express the solution to 1-loop RGE as

$$g(\mu) = \frac{1}{\frac{3}{16\pi^2} \log \frac{\mu_0}{\mu}} .$$

Instead of a coupling constant, the D=4 ϕ^4 theory is characterized by a mass scale μ_0.   "dimensional transmutation"

- The $D=4$ ϕ^4 theory is weakly coupled at energy $E \ll \mu_0$, but strongly coupled at $E \sim \mu_0$. Perturbation theory is really an expansion in powers of $(\log \frac{\mu_0}{E})^{-1}$.

[As we loose perturbative control for $E \gtrsim \mu_0$, it is unclear (based on our currently available definition of ϕ^4 theory) that the $D=4$ ϕ^4 theory exists as a QFT that obeys locality/microcausality at arbitrarily short distance scales, or is even a well-defined QM system that admits exact Poincaré symmetry.]

Formulation of 1PI RG

- QFT defined via (Euclidean) path integral

$$Z = \int [D\phi]\, e^{-\int d^Dx\, \mathcal{L}_E}$$

↖ defined with UV reg. scheme

$$\mathcal{L}_E = \mathcal{L}_{kin} + \sum g_I \mathcal{O}_I .$$

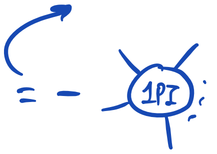
↖ bare coupling

contains counter terms including those due to field renorm. factor Z_R.

$\Big\downarrow$

1PI effective action

$$\Gamma[\phi] = S_{kin} + \sum_n \frac{1}{n!} \int \Gamma^{(n)}_{(k_1,\cdots)}\, \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_n)$$



For $\mathcal{O}_I$ of the schematic form $\underbrace{\phi\, \partial_x^{l_1} \phi \cdots \partial_x^{l_{n-1}} \phi}_{}$,

n ϕ's. indices on ∂_x contracted in some way

an appropriate definition of the renormalized coupling $g_I(\mu)$ is through expanding $\Gamma^{(n)}(k_1, \cdots, k_{n-1})$ in k_i's, around some $k_i \sim \mathcal{O}(\mu)$:

$$\Gamma^{(n)}(k_1 + \delta k_1, \cdots, k_{n-1} + \delta k_{n-1}) = \sum \Gamma^{(n; l_1, \cdots l_{n-1})}(k_1, \cdots, k_{n-1}) \times \underbrace{\delta k_1^{l_1} \cdots \delta k_{n-1}^{l_{n-1}}}_{}$$

indices contracted in some way

$$g_I(\mu) := -\left[Z(\mu)\right]^{\frac{n}{2}} \Gamma^{(n;\ell_1,\dots,\ell_{n-1})}(k_1,\dots,k_{n-1})\Big|_{k_i\sim O(\mu)}$$

choice of renorm. scheme

$$= g_I + \text{corrections}$$

in "naive/unimproved" pert. theory

where $$Z(\mu) = \left(1 - \frac{\partial\Sigma}{\partial k^2}\right)^{-1}\Bigg|_{k^2=\mu^2}.$$

[Note: in the special case of $g_{2,2}\,\phi\Box\phi$,

$$\Gamma^{(2;2)}\Big|_{k^2=\mu^2} = \frac{\partial\Sigma}{\partial k^2}\Big|_{k^2=\mu^2} = 1-(Z(\mu))^{-1}.$$

the renormalized coupling is

$$g_{2,2}(\mu) = -Z(\mu)\cdot\left(1-(Z(\mu))^{-1}\right) = 1-Z(\mu)$$

in agreement with our definition.]

- Now express $g_I(\mu')$ in terms of $g_I(\mu)$ for $\mu' = \mu + \delta\mu$
  - well-defined in the limit of removing UV regulator ($\Lambda\to\infty$ or $\epsilon\to 0$) i.e. free of UV divergences
  - small log corrections for small $\delta\mu$.

$\Rightarrow$ (1PI) RGE of the form

$$\frac{d\,g_I(\mu)}{d\log\mu} = \beta_I(\{g_J(\mu)\};\mu)$$

- Define dimensionless renormalized running couplings

$$\lambda_I(\mu) \equiv \mu^{d_I - D} g_I(\mu) .$$

e.g. mass term $m_0^2 \equiv g_2^{\text{bare}}$

$$g_2(\mu) = \mu^{-2} \cdot Z(\mu) \cdot (g_2^{\text{bare}} - \Sigma|_{k^2=\mu^2})$$

$$\frac{d\lambda_I(\mu)}{d\log\mu} = \beta_I(\{\lambda_J(\mu)\})$$

no further explicit μ-dependence

- The job of 1PI RG is to improve perturbation theory: in comparing phys. observables at widely separated energy scales, the naive perturbation theory may break down (due to "large logs") while the RG-improved perturbation theory i.e. expansion in $g(\mu)$, the latter solved from RGE, may still be valid.

Remark 1: in formulating the RGE we have assumed renormalizability in relating $g(\mu')$ to $g(\mu)$. Compare to "naive" perturbation theory in terms of bare coupling g_0, we may have

$$g(\mu) = g_0 - b_1 g_0^2 \log\frac{\Lambda}{\mu}$$
$$- b_2 g_0^3 \log\frac{\Lambda}{\mu} - c_1 g_0^3 \left(\log\frac{\Lambda}{\mu}\right)^2 + \mathcal{O}(g_0^4)$$

$$\Rightarrow \frac{dg(\mu)}{d\log\mu} = b_1 g_0^2 + b_2 g_0^3 + 2c_1 g_0^3 \log\frac{\Lambda}{\mu} + \mathcal{O}(g_0^4)$$

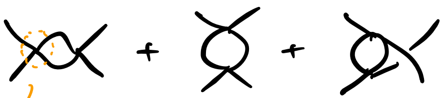
$$\overset{\text{re-express in } g(\mu)}{=\!=\!=} b_1 g(\mu)^2 + b_2 g(\mu)^3 + 2(c_1+b_1^2)\, g(\mu)^3 \log\frac{\Lambda}{\mu}$$

conflicting renormalizability

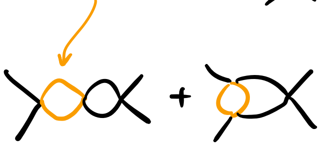
$$+\mathcal{O}(g(\mu)^4)$$

Consistency with renormalizability requires $c_1 = -b_1^2$.

e.g. $D=4$ ϕ^4 theory



$\sim b_1 g_0^2 \log\frac{\Lambda}{\mu}$



$+$ perms $\sim c_1 g_0^3 \left(\log\frac{\Lambda}{\mu}\right)^2$

$+$ subleading single log.

$c_1 = -b_1^2$ ✓

Note that the 2-loop self-energy  only contributes single log to δZ in this example.

Remark 2: in a different renorm. scheme, coupling $\tilde{\lambda}(\mu)$ may be related to $\lambda(\mu)$ by a finite redefinition $\tilde{\lambda}(\mu) = f(\lambda(\mu))$,

$f(\lambda) = \lambda + \alpha\lambda^2 + \cdots$

$$\frac{d\lambda(\mu)}{d\log\mu} = \beta(\lambda(\mu)) \iff \frac{d\tilde{\lambda}(\mu)}{d\log\mu} = \tilde{\beta}(\tilde{\lambda}(\mu)).$$

$$\beta(\lambda) = \frac{\tilde{\beta}(f(\lambda))}{f'(\lambda)} = (1-2\alpha\lambda+\cdots)\left(1+\alpha\lambda^2\frac{\partial}{\partial\lambda}+\cdots\right)\tilde{\beta}(\lambda)$$

$$= \tilde{\beta}(\lambda) + \alpha\lambda\left(\lambda\frac{\partial}{\partial\lambda}-2\right)\tilde{\beta}(\lambda) + \cdots$$

For a classically marginal coupling, typically

$$\begin{array}{ccccccccc}
\beta(\lambda) = & b_1\lambda^2 & + & b_2\lambda^3 & + & b_3\lambda^4 & + \cdots & & \\
 & \| & & \| & & \neq & & \tilde{b}_3 = b_3 - \alpha b_2 & \\
\tilde{\beta}(\lambda) = & \tilde{b}_1\lambda^2 & + & \tilde{b}_2\lambda^3 & + & \tilde{b}_3\lambda^4 & + \cdots & &
\end{array}$$

- The notion of "renormalized operators" in 1PI RG.

  - consider an infinitesimal deformation of the action

$$S \to S + \delta S.$$

$$\delta S = \int d^Dx \sum_I \delta g_I \cdot \mathcal{O}_I \qquad \circledast$$

corresponding, the renormalized running couplings $g_I(\mu)$ are deformed by

$$g_I(\mu) \to g_I(\mu) + \delta g_I(\mu).$$

δg_I and $\delta g_I(\mu)$ are linearly related:

$$\delta g_I = \sum_J \delta g_J(\mu)\, N_{JI}(\mu).$$

The <u>renormalized operator</u> $[\mathcal{O}_I]_\mu$ is defined by rewriting $\circledast$ as

$$\delta S = \int d^D x \sum_I \delta g_I(\mu)\, [\mathcal{O}_I]_\mu .$$

i.e. $$[\mathcal{O}_I]_\mu = \sum_J N_{IJ}(\mu)\, \mathcal{O}_J .$$

In fact, $N_{IJ}(\mu)$ are determined by the β-functions:

$$0 \equiv \frac{d\,\delta g_I}{d\log\mu} = \sum_J \frac{d}{d\log\mu}\left(\delta g_J(\mu)\, N_{JI}(\mu)\right)$$

$$= \sum_J \underbrace{\delta\beta_J(g(\mu))}_{\begin{array}{c}\| \\ \sum_K \frac{\partial\beta_J}{\partial g_K(\mu)}\,\delta g_K(\mu)\end{array}} N_{JI} + \delta g_J(\mu)\cdot \frac{dN_{JI}(\mu)}{d\log\mu}$$

$$\Rightarrow \qquad \frac{dN_{IJ}(\mu)}{d\log\mu} = -\sum_K \gamma_{IK}(g(\mu))\, N_{KJ}(\mu)$$

$\circledast$

$$\uparrow\ \gamma_{IK} \equiv \frac{\partial\beta_K(g(\mu))}{\partial g_I(\mu)} .$$

If we can find an operator basis in which γ_{IK} is diagonal, i.e. $\gamma_{IK} = \delta_{IK}\gamma_I$, can integrate $\circledast$ to obtain

$$[\mathcal{O}_I]_\mu = e^{-\int_{\mu'}^{\mu} \frac{d\tilde{\mu}}{\tilde{\mu}} \gamma_I(g(\tilde{\mu}))} \cdot [\mathcal{O}_I]_{\mu'}$$

We will see later that in a scale-invariant QFT, $\gamma_I = \text{constant}$,

$$[\mathcal{O}_I]_\mu = \left(\frac{\mu}{\mu'}\right)^{-\gamma_I} [\mathcal{O}_I]_{\mu'}$$

In this case, γ_I is called the "anomalous dimension" of $\mathcal{O}_I$.

- A special case
  - consider the operator ϕ itself

$$\delta S = \int d^D x \; \delta g_1 \cdot \phi(x)$$

amputated 1PI diagram

$$\textcircled{\scriptsize 1PI}\!\!-\!\!\!-\!\!\!- \;=\; \underset{-\delta g_1}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-} \qquad \text{no further corrections}$$

"renormalized running coupling"

$$\delta g_1(\mu) = -(Z(\mu))^{\frac{1}{2}}\, \textcircled{\scriptsize 1PI}\!\!-\!\!\!- \;=\; (Z(\mu))^{\frac{1}{2}}\, \delta g_1 .$$

therefore,

$$[\phi]_\mu = (Z(\mu))^{-\frac{1}{2}}\phi .$$

anomalous dimension of ϕ

$$\gamma_\phi = -\frac{d\log\left((Z(\mu))^{-\frac{1}{2}}\right)}{d\log\mu}$$

$$= \frac{1}{2}\frac{d\log Z(\mu)}{d\log\mu} .$$

In a scaling invariant theory,

$$\gamma_\phi = \text{const} \iff Z(\mu) \propto \mu^{2\gamma_\phi} .$$

- 1PI RG in minimal subtraction scheme

Begin with path integral formulation of QFT defined via dimensional regularization,

$$S = S_{kin} + \int d^{d-\epsilon}x \; g_I^\epsilon \, \mathcal{O}_I(x)$$

e.g. $\frac{1}{2}\int d^{d-\epsilon}x\,(\partial_\mu\phi)^2$

bare couplings depend on ϵ

"bare operators" built out of ϕ, $\partial_\mu\phi$, ...

Dimension analysis:

	mass dim
ϕ	$\frac{d-\epsilon-2}{2}$
g_I^ϵ	$-\delta_I(\epsilon)$
$\mathcal{O}_I$	$d-\epsilon+\delta_I(\epsilon)$

$\delta_I(\epsilon) = \delta_I^{(0)} + \epsilon\,\delta_I^{(1)}$

by construction of $\mathcal{O}_I$.

- The minimal subtraction (MS) is a renormalization scheme in which the dimensionless renormalized running couplings $\lambda_I(\mu)$ are defined such that the bare coupling g_I^ϵ, when expressed as a function of $\lambda_J(\mu)$ and ϵ, only involves negative powers of ϵ, i.e.

$$g_I^\epsilon = \mu^{-\delta(\epsilon)}\left[\lambda_I(\mu) + \underbrace{\sum_{n=1}^{\infty}\frac{1}{\epsilon^n}K_I^{(n)}(\{\lambda_J(\mu)\})}_{\text{``counter term'', contains no non-neg. powers of } \epsilon}\right]$$

This definition removes the freedom/ambiguity in finite redefinition of $\lambda_I(\mu)$ (at least in perturbation theory)

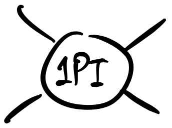
To see how MS scheme works in practice, let us revisit $d=4$ ϕ^4 theory.

with dim reg $D=4-\epsilon$, the coupling is

$$\mathcal{L}_E \supset \frac{1}{4!}\, \underset{\nwarrow \text{ mass dim } D-4\cdot\frac{D-2}{2}=\epsilon}{g^\epsilon}\, \phi^4$$

In a generic renorm scheme based on evaluating the 1PI diagram at momenta $\sim \mu$, we would define the renormalized dim-less coupling $\tilde{\lambda}(\mu)$ as

$$\mu^\epsilon \tilde{\lambda}(\mu) \equiv -\left(Z(\mu)\right)^2$$



$$\Bigg|_{\substack{k_i^2 \sim \mathcal{O}(\mu^2) \\ s=-\mu^2 a_s \\ t=-\mu^2 a_t \\ u=-\mu^2 a_u}}$$

for some constants a_s, a_t, a_u of our choice.

At 1-loop order, we have explicitly

$$\mu^\epsilon \tilde{\lambda}(\mu) = g^\epsilon - \frac{(g^\epsilon)^2}{2}\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \int_0^1 dx$$

$$\times \left[\frac{1}{(k^2+\mu^2 a_s x(1-x))^2} + (a_s \to a_t) + (a_s \to a_u)\right] + \mathcal{O}(g^3)$$

$$= g^{\epsilon} - \frac{(g^{\epsilon})^2}{2}\mu^{-\epsilon}\cdot(4\pi)^{-2+\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})\,\Gamma(1-\frac{\epsilon}{2})^2}{\Gamma(2-\epsilon)} \times \left(a_s^{-\frac{\epsilon}{2}} + a_t^{-\frac{\epsilon}{2}} + a_u^{-\frac{\epsilon}{2}}\right) + \mathcal{O}(g^3).$$

Inverting this relation, we have

$$g^{\epsilon} = \mu^{\epsilon}\left[\tilde{\lambda}(\mu) + \frac{(\tilde{\lambda}(\mu))^2}{2}(4\pi)^{-2+\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})^2}{\Gamma(2-\epsilon)} \times \left(a_s^{-\frac{\epsilon}{2}} + a_t^{-\frac{\epsilon}{2}} + a_u^{-\frac{\epsilon}{2}}\right) + \mathcal{O}(\tilde{\lambda}(\mu)^3)\right]$$

$$= \mu^{\epsilon}\left[\tilde{\lambda}(\mu) + (\tilde{\lambda}(\mu))^2\left(\frac{3}{16\pi^2}\frac{1}{\epsilon} + \underset{\substack{\uparrow \\ \text{depends on} \\ a_s,\, a_t,\, a_u}}{\mathcal{O}(\epsilon^0)}\right) + \mathcal{O}(\tilde{\lambda}(\mu)^3)\right]$$

in MS scheme,

$\lambda(\mu)$ is defined s.t.

$$= \mu^{\epsilon}\left[\lambda(\mu) + (\lambda(\mu))^2\cdot\frac{3}{16\pi^2}\cdot\frac{1}{\epsilon} + \mathcal{O}(\lambda(\mu)^3)\right]$$

- Working in MS scheme has the advantage of certain technical simplifications, such as the nontrivial perturbative contributions to the β-functions are strictly of order ϵ^0.

To see this, let us inspect the RGE in MS.

$$g_I^\epsilon = \mu^{-\delta_I(\epsilon)}\left[\lambda_I(\mu) + \sum_{n\geq 1}\frac{1}{\epsilon^n}K_I^{(n)}(\lambda(\mu))\right]$$

$$\Rightarrow \quad 0 \equiv \frac{d\, g_I^\epsilon}{d\log\mu} = -\delta_I\, \mu^{-\delta_I}\left(\lambda_I + \sum_{n\geq 1}\epsilon^{-n}K_I^{(n)}\right)$$

$\circledast$
$$+\mu^{-\delta_I}\left(\beta_I^\epsilon + \sum_{n\geq 1}\epsilon^{-n}\frac{\partial K_I^{(n)}}{\partial\lambda_J}\beta_J^\epsilon\right)$$

where $\beta_I^\epsilon \equiv \dfrac{d\lambda_I(\mu)}{d\log(\mu)}$ is the β-fn in $(d-\epsilon)$-dim.

A priori expectation: β_I^ϵ finite but all-order expansion in ϵ, i.e.

$$\beta_I^\epsilon = \sum_{m=0}^{\infty}\beta_I^{(m)}(\lambda)\cdot\epsilon^m .$$

$$\xrightarrow{\epsilon\to 0} \beta_I^{(0)}(\lambda) \equiv \beta_I(\lambda),$$ true β-fn of the QFT.

Organizing $\circledast$ in powers of ϵ:

$$-\left(\delta_I^{(0)}+\epsilon\,\delta_I^{(1)}\right)\left(\lambda_I + \sum_{n\geq 1}\epsilon^{-n}K_I^{(n)}\right) + \sum_{m\geq 0}\epsilon^m\beta_I^{(m)}$$

$$+\sum_{n\geq 1}\sum_{m\geq 0}\epsilon^{m-n}\frac{\partial K_I^{(n)}}{\partial\lambda_J}\beta_J^{(m)} = 0 .$$

e.g. order ϵ^1:

$$-\delta_I^{(1)}\lambda_I + \beta_I^{(1)} + \sum_{n\geq 1} \frac{\partial K_I^{(n)}}{\partial \lambda_J} \beta_J^{(n+1)} = 0.$$

order ϵ^2:

$$\beta_I^{(2)} + \sum_{n\geq 1} \frac{\partial K_I^{(n)}}{\partial \lambda_J} \beta_J^{(n+2)} = 0, \quad \text{etc.}$$

Note that with increasing n, $K_I^{(n)}$ begins at higher order in λ. These relations can be satisfied in perturbation theory only if

$$\beta_I^{(m)} = 0, \quad m \geq 2$$

$$\beta_I^{(1)} = \delta_I^{(1)} \lambda_I .$$

Thus, $\beta_I^{\epsilon} = \beta_I^{(0)}(\lambda) + \epsilon\, \delta_I^{(1)} \lambda_I$

as claimed.

[The remaining equation at order $\epsilon^{\leq 0}$ determine $K_I^{(n)}$ in terms of $\beta_I^{(0)}(\lambda) \equiv \beta_I(\lambda)$, through the recursive relations

$$\left(\beta_J \frac{\partial}{\partial \lambda_J} - \delta_I^{(0)}\right) K_I^{(n)} = \delta_I^{(1)}\left(1 - \lambda_J \frac{\partial}{\partial \lambda_J}\right) K_I^{(n+1)}, \quad n \geq 1.$$

and $\beta_I - \delta_I^{(0)}\lambda_I = \delta_I^{(1)}\left(1 - \lambda_J \frac{\partial}{\partial \lambda_J}\right) K_I^{(1)}$.]

Recall that the stress-energy tensor $T_{\mu\nu}(x)$ is the Noether current associated with translation symmetry :

$S[\phi]$ is invariant under

$$\phi(x) \longmapsto \phi'(x)$$

$$\phi'(x') \equiv \phi(x)$$

for $x'^{\mu} = x^{\mu} + \delta a^{\mu}$

$$\begin{aligned} \delta\phi(x) &= \phi'(x) - \phi(x) \\ &= -\delta a^{\mu}\, \partial_{\mu}\phi(x). \end{aligned}$$

Noether procedure: promote δa^{μ} to $\delta a^{\mu}(x)$

$$\delta\phi(x) = -\delta a^{\mu}(x)\, \partial_{\mu}\phi(x) + \partial_{\mu}\delta a_{\nu}(x)\, D^{\mu\nu}\phi(x)$$

to be chosen

is no longer a symmetry variation -

$$S[\phi + \delta\phi] = S[\phi] - \int d^Dx\, \partial_{\mu}\delta a_{\nu}(x) \cdot T^{\mu\nu}(x)$$

Euclidean convention

stress-energy tensor

Note: the term in $\delta\phi(x)$ proportional to $\partial_{\mu}\delta a_{\nu}(x)$ only affects $T^{\mu\nu}$ by $D^{\mu\nu}\phi(x)$. (EOM)

Now consider the "scaling transformation"

$\delta a^{\mu}(x) = \delta a \cdot x^{\mu}$ (not a symmetry in general)

$$\delta S[\phi] = -\int d^Dx\, \delta a \cdot T^\mu{}_\mu(x)$$

while ϕ transforms by

$$\circledast \qquad \delta\phi(x) = -\delta a\, x^\mu \partial_\mu \phi(x) + \delta a \cdot D^\mu{}_\mu \phi(x).$$

We would like the massless free scalar action

$$S_{kin}[\phi] = \int d^Dx\, \tfrac{1}{2}(\partial_\mu\phi)^2$$ to be invariant

under the above defined scaling transformation

- this is satisfied provided that we choose

$D^{\mu\nu}$ to be such that

$$D^\mu{}_\mu = -\frac{D-2}{2}. \qquad D = d-\epsilon$$

$$\left[\delta S_{kin}[\phi] = \delta a \int d^Dx\, \partial^\mu\phi\left(-\partial_\mu(x^\nu\partial_\nu\phi - \tfrac{D-2}{2}\phi)\right) = 0.\right]$$

$\circledast$ is equivalent to

$$\underset{\substack{\uparrow \\ x' = (1+\delta a)x}}{\phi'(x')} = (1+\delta a)^{-\frac{D-2}{2}}\phi(x).$$

More generally, an "operator" term $\mathcal{O}_I(x)$ made out of $\phi(x)$ and its derivatives of mass dimension $d_I(\epsilon) = d - \epsilon + \delta_I(\epsilon)$ transforms under scaling as

$$\mathcal{O}_I'(x') = (1+\delta a)^{-d_I(\epsilon)}\mathcal{O}_I(x).$$

$$S[\phi] = S_{kin}[\phi] + \int d^{d-\epsilon}x \sum_I g_I^\epsilon \mathcal{O}_I(x) .$$

$$S[\phi'] - S_{kin}[\phi'] = \int d^{d-\epsilon}x \sum_I g_I^\epsilon \mathcal{O}'_I(x)$$

$$= \int d^{d-\epsilon}x' \sum_I g_I^\epsilon \mathcal{O}'_I(x')$$

$$= \int d^{d-\epsilon}x \, (1+\delta a)^{\cancel{d-\epsilon}} \sum_I g_I^\epsilon (1+\delta a)^{-(\cancel{d-\epsilon}+\delta_I)} \mathcal{O}_I(x)$$

$$\Rightarrow \delta S[\phi] = -\int d^{d-\epsilon}x \, \delta a \sum_I \delta_I^{(\epsilon)} g_I^\epsilon \mathcal{O}_I(x)$$

Thus, up to a total derivative,

$$T^\mu{}_\mu(x) = \sum_I \underbrace{\delta_I(\epsilon)\, g_I^\epsilon}_{\circledast} \mathcal{O}_I(x) .$$

It would be more illuminating to re-express the RHS in terms of renormalized quantities.

Recall in MS scheme.

$$g_I^\epsilon = \mu^{-\delta_I(\epsilon)} \left[\lambda_I(\mu) + \sum_{n\geq 1} \epsilon^{-n} K_I^{(n)}(\lambda(\mu))\right]$$

$$\Rightarrow \quad 0 = \frac{dg_I^\epsilon}{d\log\mu} = -\delta_I(\epsilon)\, g_I^\epsilon + \sum_J \underbrace{\frac{d\lambda_J(\mu)}{d\log\mu}}_{\substack{\shortparallel \\ \beta_J + \epsilon\delta_J^{(1)}\lambda_J}} \frac{\partial g_I^\epsilon}{\partial \lambda_J}$$

$$\Rightarrow \circledast \quad \delta_I(\epsilon)\, g_I^\epsilon = \sum_J (\beta_J + \epsilon\delta_J^{(1)}\lambda_J) \frac{\partial g_I^\epsilon}{\partial\lambda_J} .$$

Thus,

$$T^{\mu}{}_{\mu} = \sum_J \left(\beta_J + \epsilon \delta_J^{(1)} \lambda_J\right) \underbrace{\frac{\partial}{\partial \lambda_J} \sum_I g_I^{\epsilon} \mathcal{O}_I}_{\overset{\shortparallel}{[\mathcal{O}_J]_\mu}}$$

"renormalized operator"

Now that the RHS is expressed in terms of the renormalized coupling and operators, we can take $\epsilon \to 0$ limit and write

$$T^{\mu}{}_{\mu} = \sum_J \beta_J \cdot [\mathcal{O}_J]_\mu \, .$$

which holds as an operator identity.

―――――――――――――――――――――――――"

The real surprise of the RG analysis is the existence of nontrivial fixed pts:

$$\beta_J(\lambda) = 0 \quad \text{for all } J \text{ at some nonzero } \lambda_I\text{'s.}$$

$$\Longleftrightarrow \quad T^{\mu}{}_{\mu} = 0 \qquad \text{(up to total derivative)}$$

when this occurs, we have enhanced symmetries due to the Noether currents of the form

$$j_\mu(x) = T_{\mu\nu}(x) \cdot \varepsilon^{\nu}(x) \, ,$$

where $\varepsilon^\mu(x)$ obeys

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu - \frac{2}{d} \eta_{\mu\nu} \partial_\rho \varepsilon^\rho = 0$$

such $\varepsilon^\mu(x)$ is call a "conformal Killing vector"

$$\left[\partial_\mu j^\mu = \partial_\mu (T^{\mu\nu} \varepsilon_\nu) = T^{\mu\nu} \partial_{(\mu} \varepsilon_{\nu)} \propto T^\mu{}_\mu = 0 \right]$$

Examples of CKV:

$$\varepsilon^\mu(x) = \begin{cases} a^\mu & \to \text{translation} \\ b^{[\mu\nu]} x_\nu & \to \text{Lorentz} \\ c\, x^\mu & \to \text{scaling / dilatation} \\ c_\rho (2x^\mu x^\rho - x^2 \eta^{\mu\rho}) & \to \text{special conformal symmetry} \end{cases}$$

A QFT w/ $T^\mu{}_\mu = 0$ is called a "conformal field theory" (CFT).

- Remark: a priori, $\beta_I = 0 \Rightarrow T^\mu{}_\mu = \partial_\mu V^\mu$ for some V^μ. If V^μ is nontrivial, $j_\mu = T_{\mu\nu} x^\nu - V_\mu$ is a conserved current that generates scaling sym, but no special conformal sym. It turns out that while this occurs in certain free (massless) field theories, it does not occur in fully interacting QFTs.

Example of a nontrivial RG fixed point

"ϕ^4 theory in $D = 4 - \epsilon$ dimensions"

now we view ϵ as a small but finite number, and work perturbatively in ϵ, with the hope that the result may be extrapolated to $\epsilon = 1$, or even 2.

"ϵ - expansion"

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{1}{4!} g_0 \phi^4 .$$

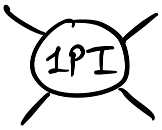
- we will be interested in the possibility that there is no "mass gap" in the spectral density function of $\langle \phi(x)\, \phi(0) \rangle$. In a given scheme, this may be accomplished by tuning the bare mass m_0.

Caution: this may or may not be possible

- an IR question
- what if the effective coupling is strong in the IR? may not be able to trust perturbation theory...

To address this question, we need to analyze the RG equation.

renormalized coupling

$\lambda(\mu) = -\mu^{-\epsilon}(Z(\mu))^2 \cdot$  $\Big|_{\substack{k_i^2=\mu^2 \\ s=t=u=-\frac{4}{3}\mu^2}}$

$$Z(\mu) = \left(1 - \frac{\partial\Sigma}{\partial k^2}\right)\Big|_{k^2=\mu^2}.$$

As already seen, at 1-loop, $Z(\mu) = 1$,

$$\lambda(\mu) = \mu^{-\epsilon}\Big[g_0 - \frac{3}{2} g_0^2 (4\pi)^{-2+\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})^2}{\Gamma(2-\epsilon)} \left(\frac{4}{3}\mu^2\right)^{-\frac{\epsilon}{2}} + \mathcal{O}(g_0^3)\Big]$$

↑ Note: ϵ is no longer infinitesimal!

$\Rightarrow$

$$\frac{d\lambda(\mu)}{d\log\mu} = -\epsilon\mu^{-\epsilon} g_0 - 2\epsilon\mu^{-2\epsilon} g_0^2 \times\left(-\frac{3}{2}\right)\cdot(4\pi)^{-2+\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})^2}{\Gamma(2-\epsilon)} \left(\frac{4}{3}\right)^{-\frac{\epsilon}{2}} + \mathcal{O}(g_0^3)$$

To derive the RGE, we need to re-express the RHS in terms of $\lambda(\mu)$,

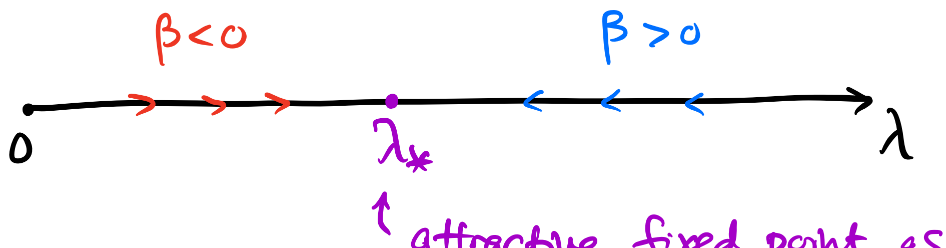
$$\frac{d\lambda(\mu)}{d\log\mu} = -\epsilon\,\lambda(\mu) - \epsilon\big(\lambda(\mu)\big)^2 \underbrace{\times\left(-\frac{3}{2}\right)\cdot(4\pi)^{-2+\frac{\epsilon}{2}}\frac{\Gamma(\frac{\epsilon}{2})\,\Gamma(1-\frac{\epsilon}{2})^2}{\Gamma(2-\epsilon)}\left(\frac{4}{3}\right)^{-\frac{\epsilon}{2}}}_{= \; -\frac{3}{16\pi^2}\cdot\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)} + \mathcal{O}(\lambda(\mu)^3)$$

$$\Rightarrow \quad \beta(\lambda) = -\epsilon\,\lambda + \left(\frac{3}{16\pi^2} + \mathcal{O}(\epsilon)\right)\lambda^2 + \mathcal{O}(\lambda^3).$$

RG fixed point $\beta(\lambda) = 0$

$$\text{at} \quad \lambda = \lambda_* = \frac{16\pi^2}{3}\epsilon + \mathcal{O}(\epsilon^2)$$

$$\frac{d\lambda(\mu)}{d\log\mu} = \beta(\lambda(\mu)). \qquad \text{"RG flow"} \quad \mu \searrow 0.$$



attractive fixed point as $\mu \to 0$

Compare : g_0 has mass dimension ϵ

the "naive" dimensionless coupling

$$\mu^{-\epsilon} g_0 \to \infty \quad \text{as} \quad \mu \to 0$$

while the true coupling strength at scale μ is governed by $\lambda(\mu)$.

$$\lambda(\mu \to 0) = \lambda_*, \quad \text{finite !}$$

(at least for suff. small ϵ)

Q: does such RG fixed point exist for $\epsilon = 1, 2$ (i.e. ϕ^4 theory in $D = 3, 2$)?

- cannot be rigorously established in the ϵ-expansion.
- nonetheless, the answer is YES !

Let us inspect the properties of some field operators at the RG fixed pt.

- $\mathcal{O}_1 = \phi$.

  as we have seen, $\gamma_1 = \frac{1}{2} \frac{d \log Z(\mu)}{d \log \mu}$

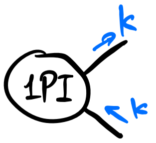
  at 1-loop, $Z(\mu) = 1 \Rightarrow \gamma_1 = 0$

  $\mathcal{O}_1$ has no anomalous dimension at order ε^1.

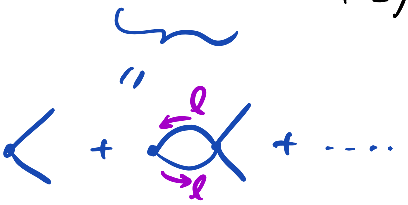
- $\mathcal{O}_2 = \phi^2$.

  consider $\delta S = \int d^{4-\epsilon}x \, \delta g_2 \cdot \phi^2(x)$

  $\delta g_2(\mu) = - Z(\mu) \cdot$



$\Big|_{k^2=\mu^2}$



$\int d^{4-\epsilon}\ell \cdot \frac{1}{(\ell^2)^2}$ IR divergent?!

NOT a good definition of $\delta g_2(\mu)$ ...

Reason: $\int d^D x \langle \mathcal{O}(x) \cdots \rangle$ is IR divergent in perturbation theory when the mass dimension $d_{\mathcal{O}} \leq \frac{D}{2}$.

To define the appropriate renormalized operator in this case, we should consider

$$\delta S = \int d^D x \; \delta g_{\mathcal{O}}(x) \cdot \mathcal{O}(x)$$

$\uparrow$ spatially dependent "coupling"
- a generalization of source J in the generating functional W[J].

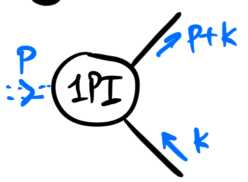
For $\mathcal{O}_2 = \phi^2$, consider

$$\delta S = \int d^{4-\epsilon}x \; \delta g_2(x) \cdot \mathcal{O}_2(x)$$

$$= \int \frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon}} \; \delta \tilde{g}_2(p) \cdot \tilde{\mathcal{O}}_2(-p).$$

Define renormalized coupling

$$\delta \tilde{g}_2(p;\mu) = -Z(\mu) \cdot$$



$$\Big|_{k^2=(p+k)^2=\mu^2}$$

and the renormalized operator $[\tilde{\mathcal{O}}_2]_\mu$ via

$$\delta \tilde{g}_2(p) \cdot \tilde{\mathcal{O}}_2(-p) \equiv \delta \tilde{g}_2(p;\mu) \, [\tilde{\mathcal{O}}_2(-p)]_\mu .$$

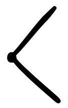
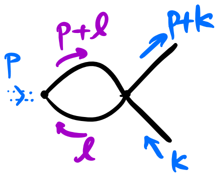
The anomalous dimension γ_2 is defined via

$$\gamma_2(\mu) = \frac{d}{d\log\mu} \log\left(\frac{\delta \tilde{g}_2(p;\mu)}{\delta \tilde{g}_2(p)} \Big|_{p^2=\mu^2} \right).$$

This is such that

$$[\tilde{O}_2(p)]_{\mu=|p|} \propto e^{-\int^{\mu}\frac{d\tilde{\mu}}{\tilde{\mu}}\gamma_2(\tilde{\mu})}\cdot\tilde{O}_2(p)\,. \tag{$\circledast$}$$

Naive perturbation theory at 1-loop:

$\delta\tilde{g}_2(p;\mu) = -\Big[$  $+$ 
$+$ higher order $\Big]_{k^2=\mu^2}$

$$= \delta\tilde{g}_2(p)\left[1 - \frac{g_4}{2}\int\frac{d^{4-\epsilon}\ell}{(2\pi)^{4-\epsilon}}\frac{1}{\ell^2(p+\ell)^2} + O(g_4^2)\right]$$

$$= \delta\tilde{g}_2(p)\cdot\left[1 - \frac{g_4}{2}\cdot(4\pi)^{-2+\frac{\epsilon}{2}}\frac{\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})^2}{\Gamma(2-\epsilon)}|p|^{-\epsilon} + O(g_4^2)\right]$$

$$\log\left(\frac{\overset{\text{renormalized}}{\delta\tilde{g}_2(p;\mu)}}{\underset{\text{bare}}{\delta\tilde{g}_2(p)}}\Bigg|_{p^2=\mu^2}\right) = \underset{\uparrow\ \text{naive pert. expansion}}{-\frac{\overset{\text{bare}}{g_4}\mu^{-\epsilon}}{2}(4\pi)^{-2+\frac{\epsilon}{2}}\frac{\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})^2}{\Gamma(2-\epsilon)}} + O(g_4^2)$$

$$\gamma_2(\mu) = \frac{d}{d\log\mu}\log\left(\frac{\delta\tilde{g}_2(p;\mu)}{\delta\tilde{g}_2(p)}\Bigg|_{p^2=\mu^2}\right)$$

$$= \epsilon\,\overset{\text{bare}}{g_4}\mu^{-\epsilon}\cdot\left(\frac{1}{16\pi^2}\frac{1}{\epsilon} + O(\epsilon^0)\right) + O(g_4^2)$$

$$= \overset{\text{renormalized}}{\lambda_4(\mu)}\cdot\left(\frac{1}{16\pi^2} + O(\epsilon)\right) + O(g_4(\mu)^2) \quad \text{RG-improved}$$

At the RG fixed point,

$$\lambda_4(\mu) = \lambda_{4*} = \frac{16\pi^2}{3}\epsilon + \mathcal{O}(\epsilon^2),$$

$$\Rightarrow \gamma_2(\mu) = \gamma_{2*} = \frac{\epsilon}{3} + \mathcal{O}(\epsilon^2).$$

✪ $$\Rightarrow \left[\tilde{\mathcal{O}}_2(p)\right]_{\mu=|p|} = \left(\frac{\mu}{\mu_0}\right)^{-\gamma_{2*}} \cdot \tilde{\mathcal{O}}_2(p)$$

↗ some UV regulator scale

↖ bare operator

Implication on Green functions

- at the RG fixed point, correlators of renormalized operators cannot depend on any dimensionful parameter, and must be well-defined (i.e. finite) functions of the momenta involved.

e.g. $\mathcal{O}_2(x) = \phi^2(x)$ has mass dimension $2-\epsilon$.

$$\tilde{\mathcal{O}}_2(p) = \int d^{4-\epsilon}x \; e^{-ip\cdot x}\mathcal{O}(x)$$

has mass dimension -2.

Same for $\left[\tilde{\mathcal{O}}_2(p)\right]_{\mu=|p|}$

unlike $\tilde{\mathcal{O}}_2(p)$, $\left[\tilde{\mathcal{O}}_2(p)\right]_{\mu=|p|}$ has finite correlators.

Green functions of $[\tilde{\mathcal{O}}_2(p)]_{\mu=|p|}$ cannot depend on any dimensionful parameter apart from p. By dimension analysis, we have

e.g. $$\langle [\tilde{\mathcal{O}}_2(p)]_{\mu=|p|} [\tilde{\mathcal{O}}_2(q)]_{\mu=|q|} \rangle$$
$$\propto \delta^{4-\epsilon}(p+q) \cdot |p|^{-\epsilon} .$$

using $$[\tilde{\mathcal{O}}_2(p)]_{\mu=|p|} = \left(\frac{\mu}{\mu_0}\right)^{-\gamma_{2*}} \tilde{\mathcal{O}}_2(p) ,$$

$\Rightarrow$

$$\mu_0^{2\gamma_*} \langle \tilde{\mathcal{O}}_2(p)\, \tilde{\mathcal{O}}_2(q) \rangle \propto \delta^{4-\epsilon}(p+q) \cdot |p|^{-\epsilon+2\gamma_{2*}} .$$

$$\Updownarrow$$

$$\mu_0^{2\gamma_*} \langle \mathcal{O}_2(x)\, \mathcal{O}_2(0) \rangle \propto \int d^{4-\epsilon}p \; e^{ip\cdot x} \, |p|^{-\epsilon+2\gamma_{2*}}$$

$$\propto \frac{1}{|x|^{2\Delta}} .$$

$$\Delta = 2 - \epsilon + \gamma_*$$

Interpretation: $\mu_0^{\gamma_{2*}} \mathcal{O}_2(x)$ is a well-defined local (field) operator in the QFT (CFT), with finite correlation functions.

Δ is the mass dimension of $\mu_0^{\gamma_{2*}}\mathcal{O}_2(x)$, and is a scaling dimension intrinsic to this operator in the CFT.

- Remark 1: while $\mu_0^{\gamma_{2*}}\mathcal{O}_2(x)$ is a local op.,

$$[\mathcal{O}_2(x)] := \int \frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon}}\, e^{ip\cdot x}\, [\tilde{\mathcal{O}}_2(p)]_{\mu=|p|}$$

$$= \int \frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon}}\, e^{ip\cdot x}\, \underbrace{\left(\frac{|p|}{\mu_0}\right)}_{p\text{-dependent factor}}{}^{-\gamma_*}\, \tilde{\mathcal{O}}_2(p)$$

is NOT a local operator.

- Remark 2: while we write $\mathcal{O}_2(x) = \phi^2(x)$ as a term in the bare Lagrangian density, the corresponding field operator cannot be viewed as simply

$$\underbrace{\lim_{x'\to x} \hat{\phi}(x')\hat{\phi}(x)}_{\substack{\text{ill-defined: Green function} \\ \langle\hat{\phi}(x')\hat{\phi}(x)\cdots\rangle \text{ singular as } x'\to x.}}$$

The true operator relation takes the form

$$\mu_0^{\gamma_{2*}}\mathcal{O}_2(x) \propto \lim_{x'\to x} |x'-x|^{2\gamma_{1*}-\gamma_{2*}}\left[\hat{\phi}(x')\hat{\phi}(x) - \langle\hat{\phi}(x')\hat{\phi}(x)\rangle\right].$$

More examples of operators in $(4-\epsilon)$-dimensional massless ϕ^4 theory:

- the E.O.M. implies an operator relation of the form $\mathcal{O}_3(x) = \phi^3(x) \propto \Box\phi(x)$

  $\mathcal{O}_3(x)$ is not independent from $\mathcal{O}_1(x)$

  "descendant operator"

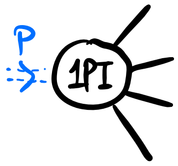
  $$\Delta_{\phi^3} = \Delta_\phi + 2.$$

- $\mathcal{O}_4(x) = \phi^4(x)$

  we may consider $\delta S = \int d^{4-\epsilon}x\, \delta g_4(x) \cdot \phi^4(x)$

  and extract γ_4 from

  $$\delta \tilde{g}_4(p;\mu) = \delta\Big[-(Z(\mu))^2$$

  $$\Big|_{k_i \sim \mathcal{O}(\mu)} \Big].$$

  $$\propto \delta\tilde{g}_4(p) \cdot \mu^{\gamma_4}$$

  for $p \sim \mathcal{O}(\mu)$ at RG fixed point.

- In this case, can set $p=0$, no IR divergence in agreement with earlier definition of anomalous dimension based on x-independent deformations of coupling.

$$\frac{d\,\delta g_4(\mu)}{d\log\mu} = \underbrace{\frac{\partial\beta(\lambda_4(\mu))}{\partial\lambda_4(\mu)}}_{\substack{\shortparallel \\ \gamma_4}} \cdot \delta g_4(\mu)$$

At RG fixed pt, $\gamma_{4*} = \left.\frac{\partial\beta(\lambda_4)}{\partial\lambda_4}\right|_{\lambda_{4*}}$

$$= \left(-\epsilon + \frac{3}{8\pi^2}\lambda_4 + \cdots\right)\Big|_{\lambda_4 = \frac{16\pi^2}{3}\epsilon + O(\epsilon^2)}$$

$$= \epsilon + O(\epsilon^2)$$

[Attractive RG fixed pt $\Longleftrightarrow$ $\gamma_* = \left.\frac{\partial\beta(\lambda)}{\partial\lambda}\right|_{\lambda_*} > 0$]

$$\Delta_{\phi^4} = 4-\epsilon+\gamma_{4*} > d = 4-\epsilon$$

"irrelevant operator"

Q: Does the RG fixed pt of ϕ^4 theory in $(4-\epsilon)$-dim really exist for $\epsilon = 1, 2$?

A: Yes!

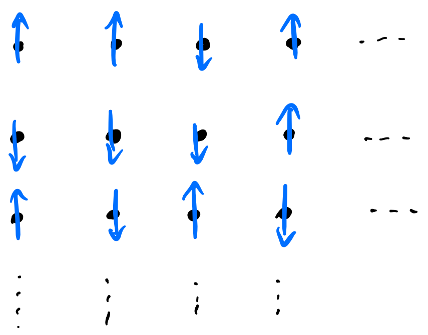
	Δ_ϕ	Δ_{ϕ^2}	Δ_{ϕ^4}
$d=4-\epsilon$	$1-\frac{\epsilon}{2}+O(\epsilon^2)$	$2-\frac{2}{3}\epsilon+O(\epsilon^2)$	$4+O(\epsilon^2)$
$d=3$ $(\epsilon=1)$	0.518...	1.412...	3.829...
$d=2$ $(\epsilon=2)$	0.125	1	4

The (statistical) Ising model

- a system of spins on D-dimensional lattice in thermal equilibrium

e.g. D=2 square lattice Λ

$x = (i, j)$, $i, j \in \mathbb{Z}$.



Hilbert space

$$\mathcal{H} = \bigotimes_{x \in \Lambda} V_x .$$

$$V_x = \text{span}\{\overset{S_x = 1}{|\uparrow\rangle}, \overset{S_x=-1}{|\downarrow\rangle}\} \simeq \mathbb{C}^2 .$$

Hamiltonian

$$H = - \sum_{\langle x x' \rangle} S_x S_{x'}$$

neighboring sites x, x'

i.e. $x = (i, j)$

$x' = (i \pm 1, j)$ or $(i, j \pm 1)$

[This Hilbert space and Hamiltonian are unrelated to those of the QFT that will emerge later]

In thermal equilibrium, the probability of the system being in any energy eigenstate $|n\rangle$ is governed by the Boltzmann distribution

$$P_n = \frac{1}{Z} e^{-\beta E_n} .$$

$\beta = \frac{1}{T}$

T = temperature

(in units where Boltzmann's constant is set to 1)

$$Z = \sum_n e^{-\beta E_n}$$

[Recall that Boltzmann's distribution maximizes the entropy $S = -\sum_n P_n \log P_n$ for given energy expectation value $\langle E \rangle = \sum_n P_n E_n$. and that the entropy is non-decreasing under probability evolution of the form (H-theorem)

$$\dot{P}_n = \sum_m \nu_{nm} (P_m - P_n) , \qquad \nu_{nm} = \nu_{mn} \geq 0$$]

Thermal expectation value of an operator $\hat{O}$

$$\langle \hat{O} \rangle = \frac{1}{Z} \mathrm{Tr}_{\mathcal{H}} (e^{-\beta \hat{H}} \hat{O})$$

$$Z = \mathrm{Tr}_{\mathcal{H}} e^{-\beta \hat{H}}$$

In Ising model, take $\hat{O} = S_{x_1} \cdots S_{x_n}$

$\rightsquigarrow$ spin correlation function $\langle S_{x_1} \cdots S_{x_n} \rangle$

- In the low temperature limit $T = \frac{1}{\beta} \to 0$, dominated by ground states (2-fold degeneracy)

$$S_x = +1 \quad \text{for all } x \in \Lambda$$
$$\text{or } \; = -1 \quad \text{for all } x \in \Lambda .$$

"Spontaneous magnetization"

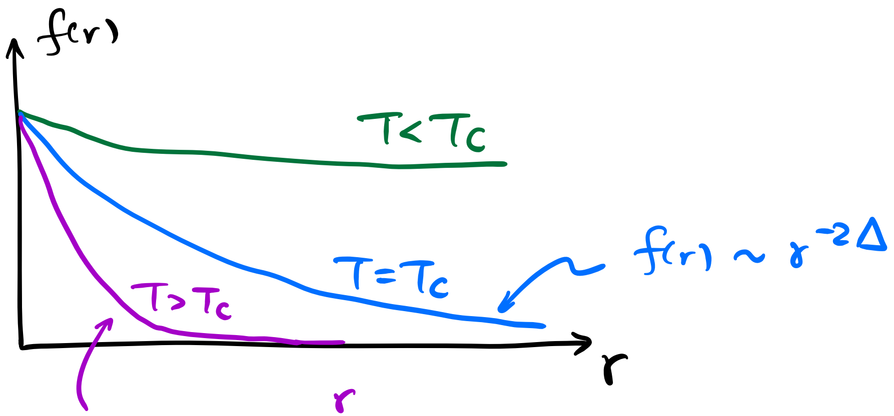
- At finite T, thermal fluctuations tend to wash out correlation. In fact, spontaneous magnetization disappears for $T \geq T_c$

"Curie temperature"

- spin correlator

$$\langle S_{0,0} \, S_{N,M} \rangle \equiv f_{N,M} \approx f(r)$$
$$r = \sqrt{N^2 + M^2}$$

for $N, M \gg 1$ and T close to T_c



$f(r) \sim r^{-2\Delta}$

$f(r) \sim e^{-\frac{r}{\xi(T)}}$, $\xi(T) \sim (T - T_c)^{-\nu}$ "correlation length"

$D=2$: $\Delta = \frac{1}{8}$, $\nu = 1$

$D=3$: $\Delta = 0.518\ldots$, $\nu = 0.62998\ldots$

– what does this have to do with QFT?

<u>Claim</u> 1: In the limit $T \to T_c$, spin correlators at large distances, e.g. $\langle S_{x_1} \cdots S_{x_n} \rangle$ for $|x_{ij}|$ of order $\xi(T) \sim (T-T_c)^{-\nu}$, become Euclidean Green functions of a local operator $\sigma(x)$ in a D-dimensional (Poincaré-invariant) QFT, called "Ising field theory" (IFT).

More precisely,

some renormalization factor

$$\lim_{\substack{T\to T_c,\, L\to\infty \\ L/\xi(T)\to m \ (\text{fixed})}} \left(Z_T(L)\right)^{-\frac{n}{2}} \langle S_{[Lx_1]} \cdots S_{[Lx_n]} \rangle_{\text{Ising},T}$$

$[\cdot] \equiv$ nearest lattice point

$$= \langle \sigma(x_1) \cdots \sigma(x_n) \rangle_{\text{IFT},\, m}$$

mass gap of IFT

Claim 2 : At $T = T_c$, where $\xi(T_c) = \infty$, long distance correlators of the Ising model are captured by IFT with mass gap $m = 0$, known as the "Ising CFT". Furthermore,

Ising CFT $=$ RG fixed point of D-dimensional massless ϕ^4 theory.

$\sigma(x) \propto \phi(x)$ "spin field".

$\Delta =$ scaling dimension of $\phi(x)$

$D - \frac{1}{\nu} =$ scaling dimension of $\phi^2(x)$.

- Why ?

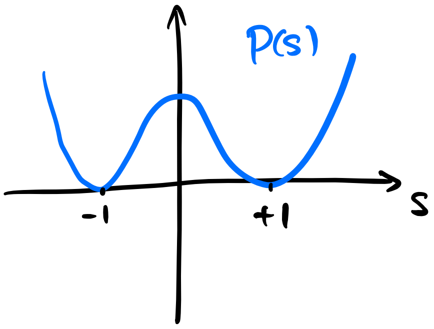
"universality" : details of Ising lattice (e.g. square vs hexagon) and Hamiltonian are unimportant at long distances, for T close to T_c.

- To understand, we first rewrite Ising model in the form of a lattice scalar field theory.

In fact, we can consider a slightly generalized version of Ising model:

replace $$Z = \sum_{S_x = \pm 1,\, x\in\Lambda} e^{-\beta H[\{S_x\}]}$$

with
$$\tilde{Z} = \int \prod_{x\in\Lambda} \left(dS_x\, e^{-P(S_x)}\right) \cdot e^{-\beta H[\{S_x\}]}$$
$$\equiv \int \prod_{x\in\Lambda} dS_x\, e^{-S[\{S_x\}]} ,$$



This looks like a Euclidean path integral, where the "Euclidean action" S is

⊛
$$S[\{S_x\}] = \beta H[\{S_x\}] + \sum_{x\in\Lambda} P(S_x)$$
$$= \underbrace{\frac{\beta}{2}\sum_{\langle xx'\rangle}(S_x - S_{x'})^2}_{\text{"kinetic term"}} + \sum_x \underbrace{\left(P(S_x) - \beta D\, S_x^2\right)}_{\text{"potential term"}}$$

We can rewrite this action in a way that is more familiar from the field theory perspective:

$$S_x \equiv \int_B \frac{d^D\vec{k}}{(2\pi)^D}\, \tilde{\phi}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

"Brillouin zone"

B: $-\pi \leq k^\mu \leq \pi$, $\mu = 1, \dots, D$.

$\tilde{\phi}(\vec{k})$ periodic under $k^\mu \to k^\mu + 2\pi$

$|k^\mu| \leq \pi$ can be viewed as a UV cutoff.

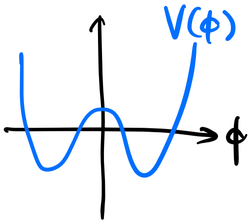
$$\sum_{\langle x x' \rangle} (S_x - S_{x'})^2 = \int_B \frac{d^D\vec{k}}{(2\pi)^D}\, \tilde{\phi}(-\vec{k})\, \tilde{\phi}(\vec{k}) \sum_{\mu=1}^{D} \left| e^{ik_\mu} - 1 \right|^2$$

$$= \int_B \frac{d^D\vec{k}}{(2\pi)^D}\, \tilde{\phi}(-\vec{k})\, \tilde{\phi}(\vec{k}) \left(\vec{k}^2 - \frac{1}{12} \sum_{\mu=1}^{D} k_\mu^4 + \dots \right)$$

The lattice action $\circledast$ can be written as

$$S[\phi] = \int d^Dx \left[\frac{\beta}{2} \sum_{\mu=1}^{D} (\partial_\mu \phi)^2 + V(\phi) \underbrace{- \frac{\beta}{12} \sum_{\mu=1}^{D} (\partial_\mu \phi)^4 + \dots}_{\text{break } O(D)\text{-symmetry, but irrelevant at long distances}} \right]$$

UV cutoff: $\tilde{\phi}(k)$ restricted to $|k_\mu| \leq \pi$.



- the (generalized) statistical Ising model is equivalent to a D-dimensional Euclidean scalar field theory with SO(D)-breaking higher-derivative kinetic terms and a non-Euclidean-invariant UV cutoff.

- at distances $L \gg$ lattice spacing, the Poincaré symmetry breaking effects are negligible.

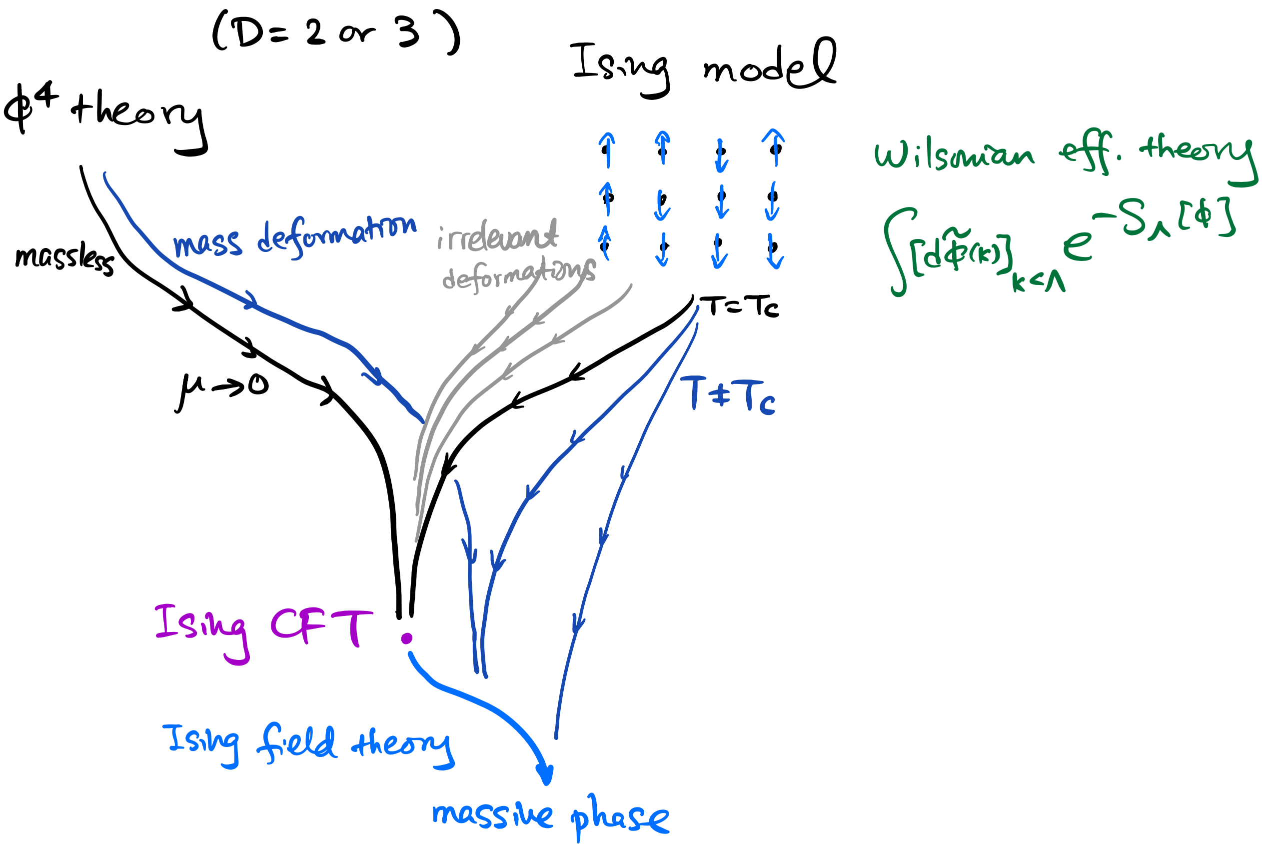
- on the other hand, at generic β (or T), at large distances there is also no nontrivial spin correlation (apart from spontaneous mag. for $\beta > \beta_c = \frac{1}{T_c}$)

Nontrivial long distance correlation arises when $\beta \to \beta_c$. In particular, at $\beta = \beta_c$, long distance spin correlators are the same (up to rescaling of variables) as the RG fixed point of D-dimensional ϕ^4 theory, for $D = 2, 3$.

Note that the Ising path integral a priori differs from that of ϕ^4 theory in two important ways:

(1) the lattice action $S[\phi]$ contain many more couplings, ϕ^6, $(\partial\phi)^4$, ...

(2) the lattice path integral comes with a finite UV cutoff $\Lambda \sim \frac{1}{a}$, a = lattice spacing.



The universality of Ising field theory can be understood from the fact that the Ising CFT has only two nontrivial local operators of scaling dimension $\Delta \lesssim D$

$$\left.\begin{array}{lll} \text{“spin field”} \ \sigma(x) & \longleftrightarrow & \phi(x) \\ \text{“energy operator”} \ \varepsilon(x) & \longleftrightarrow & \phi^2(x) \end{array}\right\} \text{description in } \phi^4 \text{ theory}$$

Other local deformations amounts to insert $e^{-\Delta S}$ in correlators, where

$$\Delta S = \int d^D x \sum_I \Delta g_I \cdot \mathcal{O}_I .$$

$\Delta_I > D$

the dependence on Δg_I

$\sim \Delta g_I \cdot \mu^{\Delta_I - D}$ disappear in IR $(\mu \to 0)$

μ = mass scale of physical observable of interest

OTOH, the $\mathbb{Z}_2$-invariant ε deformation

$$\Delta S = \int d^D x \ \delta m^2 \cdot \varepsilon(x) ,$$

some function of T for $T \approx T_c$

Expect generically

$$\delta m^2 \propto (T - T_c) \quad \text{for } T \text{ close to } T_c$$

Effective mass dimension of δm^2 is $D - \Delta_\varepsilon$.

$\Rightarrow$ correlation length

$$\xi(T) \propto (\delta m^2)^{-\frac{1}{D-\Delta_\varepsilon}}$$

$$\propto (T - T_c)^{-\underset{\uparrow}{\frac{1}{D-\Delta_\varepsilon}}}$$

one of the critical exponents

Wilsonian effective field theory

work in Euclidean signature

illustrated with the example of a single scalar field
(straightforward generalization to multiple spinning fields)

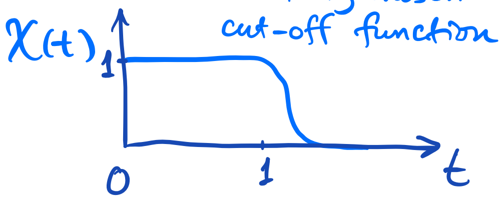
The Wilsonian effective action $S_\Lambda[\phi]$ is a functional of the form

$$S_\Lambda[\phi] = S_{kin,\Lambda}[\phi] + L_\Lambda[\phi].$$

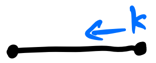
where $S_{kin,\Lambda}$ is the "kinetic term with UV cutoff":

$$S_{kin,\Lambda}[\phi] = \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\, \tilde{\phi}(-k)\, \frac{k^2+m^2}{\chi\left(\frac{k^2}{\Lambda^2}\right)}\, \tilde{\phi}(k)$$

a suitably chosen cut-off function



In perturbation theory, treating L_Λ as interaction, the propagator of ϕ w.r.t. $S_{kin,\Lambda}$ is



$$= \frac{\chi(k^2/\Lambda^2)}{k^2+m^2}$$

- agrees with $\frac{1}{k^2+m^2}$ for $|k|<\Lambda$, falls off to 0 sufficiently fast for $|k|>\Lambda$.

L_Λ is a functional of the general form

$$L_\Lambda[\phi] = \sum \frac{1}{n!} \int \prod_{i=1}^{n-1} \frac{d^D k_i}{(2\pi)^D} L_\Lambda^{(n)}(k_1, \cdots, k_{n-1})$$
$$\times \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_{n-1})\, \tilde{\phi}(-k_1 - \cdots - k_{n-1})$$

Green functions computed from the path integral

$$Z_\Lambda[J] = \int [D\phi]\, e^{-S_\Lambda[\phi] - \int d^D x\, J(x)\phi(x)} .$$

Idea: want to move $\Lambda \rightsquigarrow \Lambda' < \Lambda$, replacing L_Λ with $L_{\Lambda'}$, such that

$$Z_{\Lambda'}[J] = Z_\Lambda[J]$$

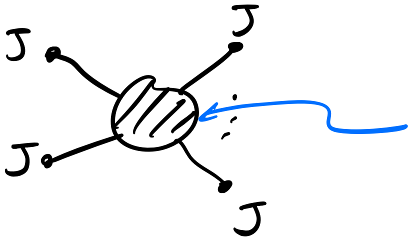
for $\tilde{J}(k)$ supported at $|k| < \Lambda'$.

i.e. Wilsonian effective theories defined at different "floating cutoffs" Λ vs Λ' produce the same low energy/momentum observables.

Compare: progator of $S_{kin,\Lambda}$ vs $S_{kin,\Lambda'}$

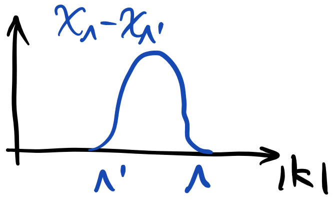
$$\frac{\chi_\Lambda(k)}{k^2+m^2} = \frac{\chi_{\Lambda'}(k)}{k^2+m^2} + \frac{\chi_\Lambda(k) - \chi_{\Lambda'}(k)}{k^2+m^2}$$

where $\chi_\Lambda(k) \equiv \chi(k^2/\Lambda^2)$.



replace every propagator

with equivalently



↑ ↑

view as propagators of two independent fields ϕ' and $\hat\phi$

⇝ integrate out $\hat\phi$ to produce an effective theory of ϕ'-field.

More precisely, we can write

$$Z[J] = \int [D\phi']\, e^{-S_{kin,\Lambda'}[\phi']} \int [D\hat\phi]\, e^{-\underset{\sim}{\hat S_{kin}}[\hat\phi]}$$

$$\times\; e^{-L_\Lambda[\phi'+\hat\phi] - \int J\cdot(\phi' + \cancel{\hat\phi})}$$

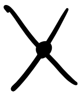
drop out because $\tilde J(k)$ is supported at $|k|<\Lambda'$ by assumption

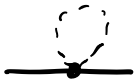
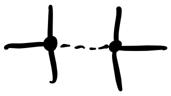
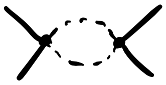
$$\hat S_{kin}[\hat\phi] = \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\, \tilde{\hat\phi}(-k)\,\tilde{\hat\phi}(k)\cdot \frac{k^2+m^2}{\chi_\Lambda(k)-\chi_{\Lambda'}(k)}.$$

$$Z[J] = \int [D\phi']\, e^{-S_{kin,\Lambda'}[\phi'] - L_{\Lambda'}[\phi']}$$

provided that

$$e^{-L_{\Lambda'}[\phi']} = \int [D\hat{\phi}]\, e^{-\hat{S}_{\rm kin}[\hat{\phi}] - L_\Lambda[\phi'+\hat{\phi}]}$$

e.g. $L_\Lambda \supset$  + ...

$L_{\Lambda'} \supset$  +  +  + ...

If we take $\Lambda' = \Lambda - \delta\Lambda$, to first order in $\delta\Lambda$, the variation $\delta L_\Lambda[\phi] = L_\Lambda[\phi] - L_{\Lambda'}[\phi]$ can be expressed in a form analogous to RGE:

$$\Lambda\frac{\partial}{\partial\Lambda}L_\Lambda[\phi] = \int d^Dk \, \underbrace{\frac{(2\pi)^D}{k^2+m^2}\Lambda\frac{\partial}{\partial\Lambda}\chi\left(\frac{k^2}{\Lambda^2}\right)}_{\text{supported near } |k|=\Lambda}$$

$$\times\frac{1}{2}\left[\underbrace{\frac{\delta L_\Lambda}{\delta\tilde{\phi}(k)}\cdot\frac{\delta L_\Lambda}{\delta\tilde{\phi}(-k)}}_{} - \underbrace{\frac{\delta^2 L_\Lambda}{\delta\tilde{\phi}(k)\,\delta\tilde{\phi}(-k)}}_{}\right]$$

"Wilson - Polchinski RGE"

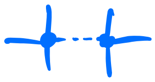
e.g. massless scalar ϕ in $D=4$

Suppose $L_\Lambda[\phi] \supset \int d^4x \,(g_4\phi^4 + g_6\phi^6)$

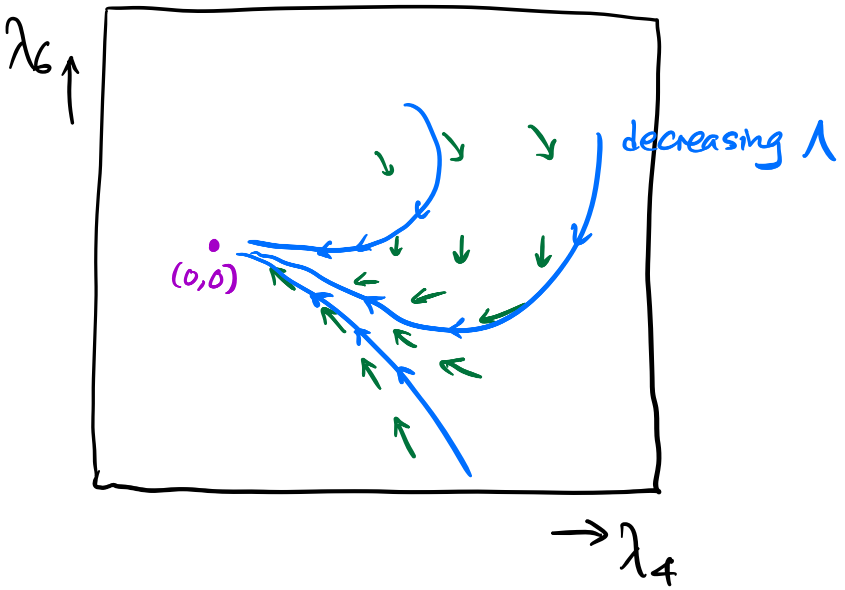
dimensionless couplings

$$\lambda_4 \equiv g_4, \qquad \lambda_6 \equiv \Lambda^2 g_6$$

$$\text{part of wilsonian RGE}\left\{\begin{array}{l} \Lambda \dfrac{d\lambda_4}{d\Lambda} = -a\cdot\lambda_6 + \cdots\cdots \\[2em] \Lambda \dfrac{d\lambda_6}{d\Lambda} = 2\lambda_6 + b\cdot\lambda_4^2 + \cdots\cdots \end{array}\right.$$



Naive truncation to (λ_4, λ_6),
RG flow to IR ($\Lambda \to 0$) looks like



For small $\lambda_6 \sim \mathcal{O}(\lambda_4^2)$,

$$\frac{d}{d\log\Lambda}\left(\lambda_6 + \frac{b}{2}\lambda_4^2\right) = 2\left(\lambda_6 + \frac{b}{2}\lambda_4^2\right) + \mathcal{O}(\lambda_4^3) \Rightarrow \lambda_6 + \frac{b}{2}\lambda_4^2 \propto \Lambda^2$$

as $\Lambda \to 0$

With λ_6 quickly approching $-\frac{b}{2}\lambda_4^2$,

$$\frac{d}{d\log\Lambda}\lambda_4 \approx \frac{ab}{2}\lambda_4^2 + \mathcal{O}(\lambda_4^3)$$

as expected in 1PI RGE.

- In the Wilsonian effective theory framework, $S_\Lambda[\phi]$ at large Λ may or may not exist; RG flow is defined only for $\Lambda \searrow$.

On the other hand, we can also rephrase renormalizability in the Wilsonian language, as follows.

The regularized path integral of a renormalizable QFT defined with a UV cutoff Λ_0 may also be viewed as a special case of a Wilsonian path integral, where the Wilsonian eff. action $S_{\Lambda_0}[\phi]$ coincides with the bare action.

In particular, $L_{\Lambda_0}[\phi]$ contains the interaction terms whose coeffizients are the bare couplings. The Wilson-Polchinski RGE then constructs $S_\Lambda[\phi]$ for $\Lambda < \Lambda_0$. Schematically,

$$L_{\Lambda_0}[\phi] = \int g_I^{\text{bare}} \mathcal{O}_I[\phi]$$

same set of operators

$$\rightsquigarrow L_\Lambda[\phi] = \int g_I(\Lambda)\, \mathcal{O}_I[\phi] + \text{many more terms.}$$

where $g_I(\Lambda)$ can be viewed as physical couplings, as physical observables such as Green functions at momenta $|k| < \Lambda$ are computed as finite functions of $g_I(\Lambda)$ via the Wilsonian path integral with floating cutoff Λ.

We can denote $L_\Lambda[\phi]$ that results from the UV action as

$$L_\Lambda(g_I^{bare}, \Lambda_0)[\phi].$$

Renormalizability is equivalent to the statement that the limit

$$\lim_{\Lambda_0 \to \infty} L_\Lambda(g_I^{bare}, \Lambda_0) \Big|_{\substack{g_I^{bare} \text{ chosen s.t.} \\ g_I(\Lambda) = g_I \text{ fixed}}} \quad \text{exists}.$$

- We will illustrate this through the toy model of a truncated Wilsonian RGE

$D=4$ ϕ^4 theory:

$$L_\Lambda[\phi] \supset \int g_4(\Lambda)\phi^4 + g_6(\Lambda)\phi^6$$

$$\lambda_4(\Lambda) := g_4(\Lambda), \quad \lambda_6(\Lambda) := \Lambda^2 g_6(\Lambda)$$

toy model RGE:

$$\begin{cases} \dfrac{d\lambda_4}{d\log\Lambda} = \beta_4(\lambda_4, \lambda_6) \\ \dfrac{d\lambda_6}{d\log\Lambda} = 2\lambda_6 + \underbrace{\beta_6(\lambda_4, \lambda_6)}_{\text{higher order than } \lambda_6 \text{ for small } \lambda_4, \lambda_6.} \end{cases}$$

At UV cutoff scale Λ_0,

$$\lambda_4(\Lambda_0) = \lambda_4^{\text{bare}}$$

$$\lambda_6(\Lambda_0) = 0 \quad \text{(by assumption)}$$

At physical scale Λ_R,

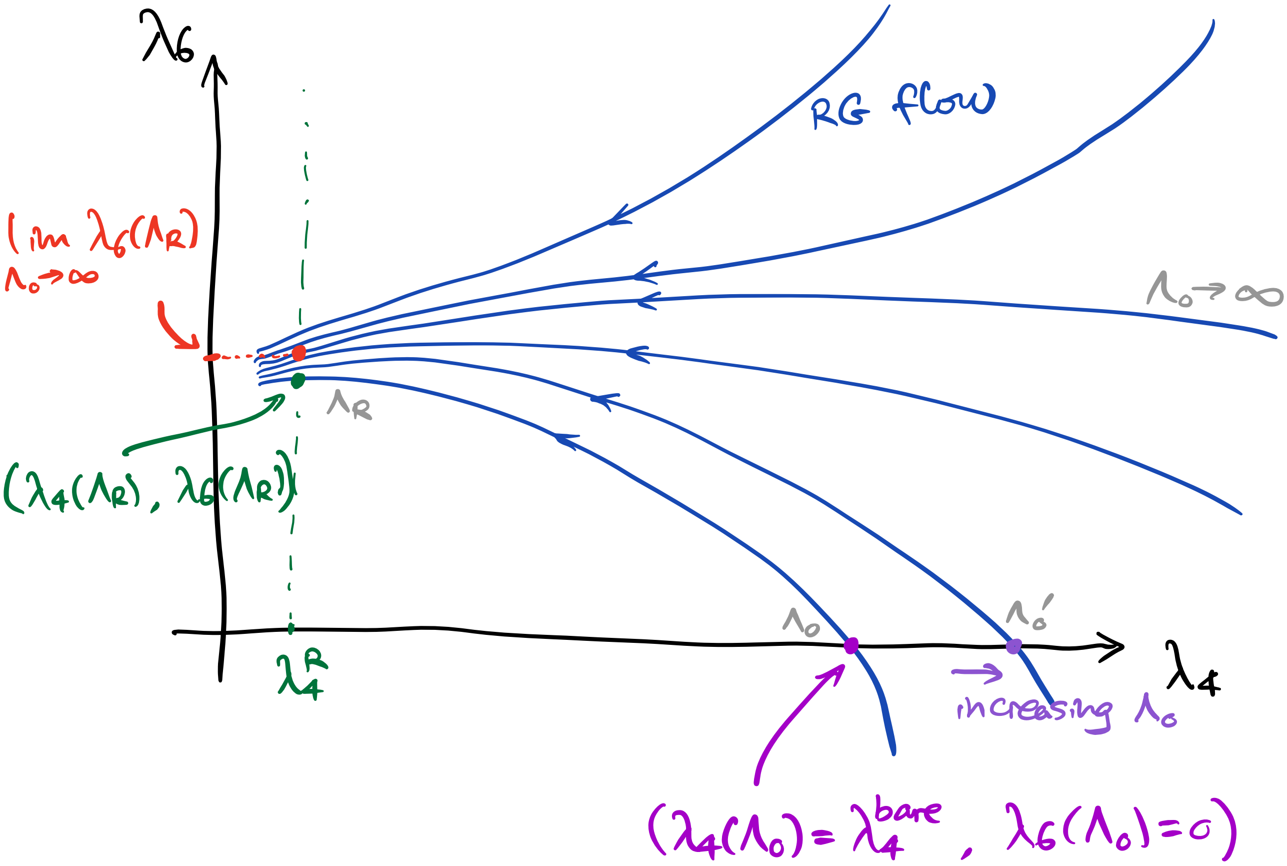
$$\lambda_4(\Lambda_R), \quad \lambda_6(\Lambda_R)$$

both nontrivial and determined by λ_4^{bare}.

"renormalizability":

Does $\displaystyle \lim_{\Lambda_0 \to \infty} \lambda_6(\Lambda_R) \Big|_{\lambda_4(\Lambda_R) = \lambda_4^R \text{ fixed}}$ exist?

This limiting procedure can be illustrated with the following picture:



[Polchinski '84 : "renormalization and effective Lagrangians"]

- $\lim_{\Lambda_0 \to \infty} \lambda_6(\Lambda_R)$ exists provided that $\frac{\partial \beta_6}{\partial \lambda_6}$ and $\frac{\partial \beta_4}{\partial \lambda_4}$ are sufficiently small along RG trajectory, and that $\beta_4(\lambda_4(\Lambda), \lambda_6(\Lambda))$ does not vary too fast with Λ.

  [i.e. parametrically small anomalous dimension]

This explains why naively-power-counting renormalizable theories are actually (perturbatively) renormalizable.

Note: same analysis applies to the situation where we allow finite nonzero $\lambda_6(\Lambda_0)$

$$\Longleftrightarrow \quad g_6^{\text{bare}} = \Lambda_0^{-2} \lambda_6(\Lambda_0)$$

as is the case in a typical lattice QFT action (Λ_0^{-1} = lattice spacing)

[This is the logical foundation of lattice QFT.]

Yang-Mills theory

- We begin by formulating a classical field theory of multiple gauge fields

$$A_{a\mu}$$

internal "gauge" index

spacetime vector index $\mu = 0, 1, \cdots, D-1$

- Simplest possible nonlinear infinitesimal gauge transformation

$$\delta A_{a\mu}(x) = \partial_\mu \underbrace{\zeta_a(x)} + \underbrace{f^{bc}{}_a}_{\text{"structure constants"}} A_{b\mu}(x) \underbrace{\zeta_c(x)} .$$

$\zeta_a(x)$ are abitrary functions of x^μ.

Such gauge transformation close among themselves if we can organize $A_{a\mu}$ into a Lie-algebra-valued vector field

$$A_\mu(x) \equiv \sum_a A_{a\mu} \underbrace{t^a}_{\text{(abstract) generators of a Lie algebra}}$$

where t^a's obey

$$[t^a, t^b] = \sum_c i f^{ab}{}_c t^c .$$

Now we can express the infinitesimal gauge transformation as

$$\delta A_\mu(x) = \partial_\mu \underset{\uparrow}{\zeta(x)} - i\,[A_\mu(x), \zeta(x)]$$

$$\zeta(x) \equiv \sum_a \zeta_a(x)\, t^a$$

or better, the finite form of gauge transf.

$$A_\mu(x) \mapsto A'_\mu(x) = g(x)\, A_\mu(x)\, g^{-1}(x) + i\, g(x)\, \partial_\mu g^{-1}(x).$$

where

$$g(x) \equiv e^{i \sum_a \underbrace{\zeta_a(x)}_{\text{finite}} t^a}$$

Note: due to Baker-Campbell-Hausdorff.

$$\log(e^X e^Y) = X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]] + \cdots$$

The product $g \cdot h$ is well-defined given $g = e^{i\zeta_a t^a}$, $h = e^{i\eta_a t^a}$, and the Lie algebra commutator $[t^a, t^b] = i f^{ab}{}_c\, t^c$.

In particular, $g^{-1}(x) = e^{-i \sum_a \zeta_a(x) t^a}$.

Furthermore, using

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \dots + \frac{1}{n!}\underbrace{[X,[X,\dots,[X}_{n\ X's},Y]] + \dots$$

we can express $g(x) A_\mu g^{-1}(x)$ entirely as a linear combination of t^a's.

The same is true for $g(x)\partial_\mu g^{-1}(x)$ via BCH.

- Thus $A_\mu'(x)$ takes the form $A'_{a\mu}(x) t^a$ as needed.

Define "field strength"

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$
$$\equiv \sum_a F_{a\mu\nu} t^a.$$

Unlike in Abelian gauge theory, $F_{\mu\nu}(x)$ is generally NOT gauge-invariant. However, $F_{\mu\nu}$ transforms in a simple way under gauge transformation:

writing $F_{\mu\nu} = i\,[\partial_\mu - iA_\mu ,\ \partial_\nu - iA_\nu]$

and using

$$\partial_\mu - iA'_\mu = g(x)\underset{\uparrow}{\,}(\partial_\mu - iA_\mu)\underset{\uparrow}{\,}g^{-1}(x)$$

multiply as diff. operators

we easily see that

$$F'_{\mu\nu}(x) = g(x)\cdot F_{\mu\nu}(x)\cdot g^{-1}(x)$$

To construct a gauge-invariant Lagrangian density, we need a notion of "trace" on the Lie algebra:

$$\mathrm{tr}(t^a t^b) \equiv d^{ab} ,$$

that is symmetric under a ↔ b, and has the "cyclicity" property

$$\mathrm{tr}([t^c, t^a]\, t^b) = \mathrm{tr}(t^a\, [t^b, t^c])$$

It follows that

$$\mathrm{tr}(g\,X\,Y\,g^{-1}) = \mathrm{tr}(X\,Y)$$

for any X, Y that are elements of the Lie algebra.

Note: $$f^{abc} \equiv \sum_e f^{ab}{}_e \, d^{ec}$$

$$= -i\,\mathrm{tr}([t^a, t^b]\, t^c)$$

is completely anti-symmetric w.r.t. a, b, c.

- Yang-Mills Lagrangian density

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$ is gauge-invariant

- So far, we have not specified the normalization of tr, or d^{ab}.

  to construct a unitary theory, it will be necessary to take $(d^{ab}) \succ 0$

  (positive definite matrix)

[turns out that such tr exists if and only if the Lie algebra takes the form

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_i,$$ each $\mathfrak{g}_i$ is either $\mathbb{R}$ (Abelian)

or "simple, compact" of type ABCDEFG.]

- if so, can choose a linear basis for t^a such that

$$\mathrm{tr}\,(t^a t^b) \equiv d^{ab} = \underset{\substack{\uparrow \\ \text{default convention}}}{\frac{1}{2}}\,\delta^{ab}.$$

Example: A_1 a.k.a. $\mathfrak{su}(2)$,

$$t^a = \frac{1}{2}\sigma^a \,, \quad a = 1,2,3.$$

tr is ordinary trace of 2×2 matrices. ✓

More examples of simple, compact Lie algebra:

$$\mathfrak{g} = A_{N-1} \text{ a.k.a. } \mathfrak{su}(N)$$

$\simeq$ space of $N\times N$ Hermitian traceless matrices.

$$\zeta(x) \equiv \zeta_a(x)\, t^a \in \mathfrak{g}.$$

$$g(x) = e^{i\zeta(x)} \in G = SU(N).$$

$N\times N$ unitary matrizes with det $= 1$.

Note: in $D=4$, there is another gauge-invariant, Poincaré-invariant, and power-counting-renormalizable term one can include in the Yang-Mills Lagrangian:

"θ-term"

$$\Delta \mathcal{L} = \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}(F_{\mu\nu} F_{\rho\sigma})$$

- $\Delta\mathcal{L}$ is odd under parity or time-reversal
- can be written as a total derivative of an expression in A_μ, does not affect E.O.M. nor quantum perturbation theory

(e.g. no Feynman vertex corresponding to θ-term)

- However, A_μ need not vanish as $x \to \infty$ for a field config. of finite action, and thus $\Delta S = \int d^4x \, \Delta\mathcal{L}$ can be non-vanishing for an admissible field config. that contributes to the path integral.
- non-perturbative effects depend on θ. (will return to this)

- Yang-Mills theory with matter fields

matter field $\psi(x)$ transforms under a finite gauge transformation parameterized by $g(x) = e^{i\zeta_a(x) t^a} \in G$

as

$$\psi(x) \longmapsto \psi'(x) = \underbrace{\rho(g(x))}_{\text{matrix}} \cdot \underbrace{\psi(x)}_{\text{column vector}}$$

ρ associates a Lie group element $g \in G$ to a matrix, such that $\rho(g \cdot h) = \rho(g) \cdot \rho(h)$

$$\rho(\underset{\substack{\uparrow \\ \text{group} \\ \text{multiplication}}}{g \cdot h}) = \rho(g) \underset{\substack{\uparrow \\ \text{matrix} \\ \text{multiplication}}}{\cdot} \rho(h)$$

- A representation R of the group G is a map

$$\rho_R : G \longrightarrow \underbrace{GL(V)}_{\substack{\text{set of invertible linear maps} \\ V \to V}}$$

such that

$$\rho_R(g \cdot h) = \rho_R(g) \cdot \rho_R(h) \qquad \forall\, g, h \in G.$$

- correspondly, a representation of the Lie algebra $\mathfrak{g}$

$$\tilde{\rho}_R : \mathfrak{g} \to \underbrace{\mathrm{End}(V)}_{\text{set of linear maps } V \to V}$$

is related by

$$\rho_R(g = e^{i\zeta_a t^a}) = e^{i\zeta_a \tilde{\rho}_R(t^a)}$$

In particular,

$$\tilde{\rho}_R([t^a, t^b]) = [\tilde{\rho}_R(t^a), \tilde{\rho}_R(t^b)] .$$

Define $t^a_R \equiv \tilde{\rho}_R(t^a)$,

$$[t^a_R, t^b_R] = i f^{ab}{}_c \, t^c_R .$$

Example : $G = SU(N) \longleftrightarrow \mathfrak{g} = \mathfrak{su}(N)$

- "fundamental representation" $R = \square$ (notation)
  a.k.a. "defining rep."

$V = \mathbb{C}^N$,

$$\rho_\square(g) = g \quad \text{as a unitary } N\times N \text{ matrix acting on } V.$$

$$\Updownarrow$$

$$\tilde{\rho}_\square(t^a) = t^a .$$

- anti-fundamental rep. $R = \bar{\square}$.

  $V = \mathbb{C}^N$.

  $$\rho_{\bar{\square}}(g) = g^*$$ (complex conjugate, without transpose)

  $$\Updownarrow$$

  $$\tilde{\rho}_{\bar{\square}}(t^a) = -(t^a)^*.$$

- adjoint rep. $R = \text{adj}$.

  $$V = \mathfrak{g} \simeq \mathbb{R}^{N^2-1}$$

  $\rho_{adj}(g)$ is a map $V \to V$ that acts on $v \in V$ a Hermitian traceless matrix by

  $$(\rho_{adj}(g))(v) = g v g^{-1}$$

  $$\Updownarrow$$

  $$(\tilde{\rho}_{adj}(t^a))(t^b) = [t^a, t^b] = i f^{ab}{}_c\, t^c.$$

  In other words, acting on the basis t^a of $\mathfrak{g}$, $\tilde{\rho}_{adj}(t^a) \equiv t^a_{adj}$ is the $(N^2-1)\times(N^2-1)$ matrix

  $$(t^a_{adj})_c{}^b = i f^{ab}{}_c.$$

A matter field $\psi(x)$ in representation R transforms under gauge transf by

$$\psi(x) \rightsquigarrow \psi'(x) = \rho_R(g(x))\cdot\psi(x)$$

gauge covariant derivative

$$D_\mu\psi(x) \equiv \partial_\mu\psi(x) - i\,A_{a_\mu}(x)\,t^a_R\cdot\psi(x)$$

under gauge transf.

$$D_\mu\psi(x) \rightsquigarrow (D_\mu\psi)'(x) = \rho_R(g(x))\cdot D_\mu\psi(x)$$

QCD : fermionic spinor fields

$\psi_{I\alpha}(x)$ in $\square$ of $G = SU(N)$

components $\psi_{Ii\alpha}(x)$,

gauge index

$i = 1,\dots,N$

$I = 1,\dots,N_f$

flavor index

$\bar\psi_I^{\ \alpha}(x)$ in $\bar\square$ of $SU(N)$.

Lagrangian density (most general Poincaré-inv't power-counting renormalizable in $D=4$)

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

$$\mathcal{L}_\psi\left\{ -\bar\psi_I\gamma^\mu D_\mu\psi_I - \underset{\uparrow}{m_{IJ}}\bar\psi_I\psi_J - \underset{\uparrow}{\tilde m_{IJ}}\bar\psi_I\gamma_5\psi_J \right.$$

Hermitian mass matrices

Alternatively, define

$$\psi_{IL} = \frac{1+\gamma_5}{2}\psi_I$$

$$\psi_{IR} = \frac{1-\gamma_5}{2}\psi_I \, ,$$

$$\mathcal{L}_\psi = -\bar{\psi}_{IL}\not{D}\psi_{IL} - \bar{\psi}_{IR}\not{D}\psi_{IR}$$

$$-\underbrace{M_{IJ}}_{\text{complex mass matrix}}\bar{\psi}_{IL}\psi_{JR} - M^*_{IJ}\bar{\psi}_{JR}\psi_{IL} \, .$$

Quantization of Yang-Mills theory

- canonical quantization proceeds the same way as in Abelian gauge theory; one encounters 1st class constraints (vanishing Poisson bracket) that ought to be eliminated by choice of gauge, after which one can proceed with canonical quantization prescription.

  However, one often encounters ambiguities in IR regularization due to unfixed residual gauge redundancies, as well as complications due to Lorentz-non-covariance of gauge conditions.

- We will adopt a more systematic covariant approach to quantization by gauge fixing the path integral,

  a la Faddeev-Popov, and

  Becchi-Rouet-Stora-Tyutin (BRST)

Begin with a "formal" path integral of a gauge theory

$$Z \sim \int D\phi \; e^{iS[\phi]} ,$$

where ϕ collectively denotes gauge and possibly matter fields.

We denote a gauge transf. by

$$\phi \longmapsto \phi^{\zeta}$$

under which the action and the measure are invariant:

$$S[\phi^{\zeta}] = S[\phi] .$$

$$D\phi^{\zeta} = D\phi$$

Here ζ represents an element of the group $\mathcal{G}$ of gauge transformations.

[Note: $\mathcal{G}$ is typically ∞-dimensional, not to be confused with the "gauge group" G of Yang-Mills theory:

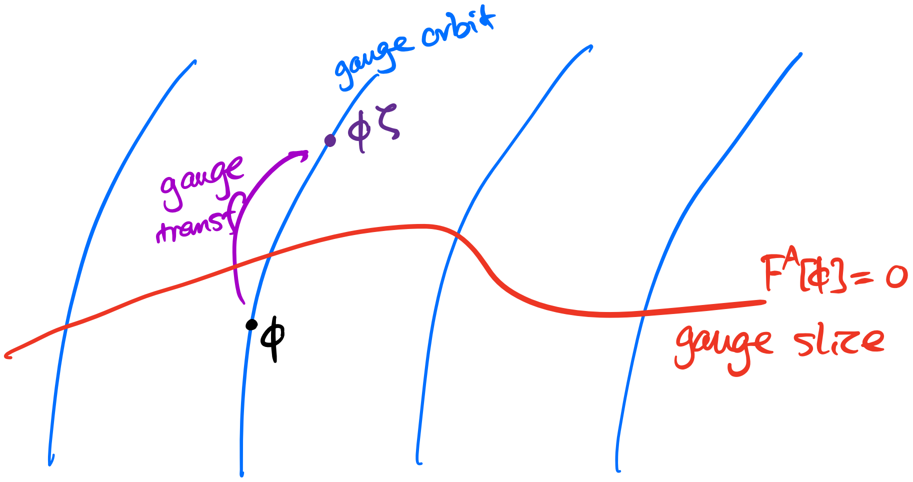
$$\mathcal{G} \simeq \{ g(x) : G\text{-valued functions on } \mathbb{R}^{1,D-1} \text{ spacetime} \} .$$
]

We denote by ζ^{α} coords on $\mathcal{G}$.

[e.g. in YM theory, $\zeta \equiv g(x) = e^{i\zeta_a(x) t^a}$
we may take $\zeta^{(a,x)} \equiv \zeta_a(x)$, $a=1,\dots,\dim G$, $x \in \mathbb{R}^{1,D-1}$]

Idea: $\phi \sim \phi^\zeta$ represent "equivalent" field configurations. We wish to avoid such overcounting by restricting the functional integral through a gauge-fixing condition

$$F^A[\phi] = 0 .$$



but we should do so in a way that respects gauge invariance (of observables).

This is achieved using

$$\circledast \qquad 1 = \int_{\mathcal{G}} D\zeta \, \delta(F^A[\phi^\zeta]) \cdot \det\left(\frac{\partial F^A[\phi^\zeta]}{\partial \zeta^\alpha}\right)$$

Inserting the RHS into the path integral, we have

$$Z \sim \int_{\mathcal{G}} D\zeta \int D\phi \; \delta(F^A[\phi^\zeta]) \det \frac{\partial F^A[\phi^\zeta]}{\partial \zeta^\alpha} \, e^{iS[\phi]}$$

$$= \int_{\mathcal{G}} D\zeta \int D\phi^\zeta \; \delta(F^A[\phi^\zeta]) \det \frac{\partial F^A[\phi^\zeta]}{\partial \zeta^\alpha} \, e^{iS[\phi^\zeta]}$$

$$\overset{\phi \to \phi^{\zeta^{-1}}}{=\!=} \int_{\mathcal{G}} D\zeta \int D\phi \; \delta(F^A[\phi])$$

exchange

$$\times \det \frac{\partial F^A[\phi^{\zeta' \circ \zeta^{-1}}]}{\partial \zeta'^\alpha}\bigg|_{\zeta'=\zeta} \cdot \; e^{iS[\phi]}$$

The factor involving ζ in the ϕ-integrand is

$$\int_{\mathcal{G}} D\zeta \det \frac{\partial F^A[\phi^{\zeta' \circ \zeta^{-1}}]}{\partial \zeta'^\alpha}\bigg|_{\zeta'=\zeta}$$

$$= \int_{\mathcal{G}} \underbrace{D\zeta \left(\det \frac{\partial (\zeta'' \circ \zeta)^\alpha}{\partial \zeta''^\beta}\right)^{-1}\bigg|_{\zeta''=id}}_{[D\zeta]_H \text{ Haar measure on } \mathcal{G}} \qquad \zeta' \equiv \zeta'' \circ \zeta$$

$$\times \underbrace{\det \left(\frac{\partial F^A[\phi^{\zeta''}]}{\partial \zeta''^\alpha}\right)\bigg|_{\zeta''=id}}_{\Delta_{FP}[\phi] \text{ "Faddeev-Popov" det.}}$$

The Haar measure $[D\zeta]_H$ on $\mathcal{G}$ has defining property

$$[D(\zeta\circ\zeta_1)]_H = [D\zeta]_H$$

$$\forall \text{ fixed } \zeta_1 \in \mathcal{G}.$$

Formally, $\int_{\mathcal{G}} [D\zeta]_H =$ "volume of $\mathcal{G}$".

The appropriate definition of gauge theory is

$$Z = \frac{1}{\int_{\mathcal{G}} [D\zeta]_H} \int D\phi \, e^{iS[\phi]}$$

$$(*) \qquad = \int D\phi \, \delta(F^A[\phi]) \cdot \Delta_{FP}[\phi] \, e^{iS[\phi]}.$$

The Faddeev-Popov determinant can be written as

$$\Delta_{FP}[\phi] = \det \frac{\partial F^A[\phi^\zeta]}{\partial \zeta^\alpha}\bigg|_{\zeta=id}$$

$$= \int Db_A Dc^\alpha \cdot e^{i b_A \cdot \frac{\partial F^A[\phi^\zeta]}{\partial \zeta^\alpha}\big|_{\zeta=id} \cdot c^\alpha}$$

(b_A, c^α) are Grassmann-odd variables

"Faddeev-Popov ghosts"

Note that in the field theory context,

$\alpha = (a, x)$ takes continuous values,

and A is an index that loosely speaking

has the same "number of degrees of freedom" as α.

Both c^α and b_A will be field variables in

the sense that they depend on spacetime coords,

e.g. $\quad c^\alpha \equiv c_a(x)$.

and hence $\quad \int Db_A \, Dc^\alpha \ldots$

is a Grassmann-odd functional integral.

The notation $f_\alpha g^\alpha$ stands for

$$\sum\!\!\!\!\!\!\int \underset{\substack{\uparrow \\ \text{a suitable measure}}}{d\alpha} \, f_\alpha \, g^\alpha$$

e.g. $\quad \alpha = (a, x)$,

$$f_\alpha \, g^\alpha \equiv \sum_a \int d^D x \; f_a(x) \, g^a(x) .$$

In $\circledast$, the gauge condition is implemented

through the insertion of δ-functional

$$\delta(F^A[\phi]) = \int DB_A \, e^{\,i B_A F^A[\phi]}$$

where B_A are commuting (field) variables.

Therefore, the gauge-fixed path integral can be written equivalently as

$$Z = \int D\phi\, DB_A\, Db_A\, Dc^\alpha$$

$$\times\, e^{i\left(S[\phi] + B_A F^A[\phi] + b_A \frac{\partial F^A[\phi^\xi]}{\partial \xi^\alpha}\Big|_{\xi=id} \cdot c^\alpha\right)}$$

- The total action

$$S[\phi, B, b, c] = S[\phi] + S_{G.F.}$$

$$\nearrow$$

$$B_A F^A[\phi] + b_A \frac{\partial F^A[\phi^\xi]}{\partial \xi^\alpha}\Big|_{\xi=id} \cdot c^\alpha$$

is NOT gauge invariant, as $S_{G.F.}$ is not gauge invariant by construction.

- On the other hand, there is an emergent fermionic global symmetry, known as the BRST symmetry:

$$\delta_B S[\Phi] \equiv S[\Phi + \delta_B \Phi] - S[\Phi]$$

$$= 0,$$

$$\Phi \equiv (\phi, B, b, c)$$

$$\delta_B \phi = \underbrace{\epsilon\, c^\alpha \delta_\alpha \phi}_{}\,, \quad \text{where} \quad \delta_\alpha \phi \equiv \frac{\partial \phi^\zeta}{\partial \zeta^\alpha}\Big|_{\zeta = id}$$

same as the infinitesimal gauge variation of ϕ, except that the gauge parameter $\delta\zeta^\alpha$ is replaced by ϵc^α.

here ϵ is Grassmann-odd, and hence ϵc^α is Grassmann-even as usual.

$$\delta_B B_A = 0.$$

$$\delta_B b_A = -\epsilon B_A\,,$$

$$\delta_B c^\alpha = -\frac{1}{2}\epsilon f^\alpha{}_{\beta\gamma} c^\beta c^\gamma .$$

$f^\alpha{}_{\beta\gamma}$ are structure constants of $\mathcal{G}$:

$$[\delta_\beta, \delta_\gamma] = f^\alpha{}_{\beta\gamma}\, \delta_\alpha .$$

Given any functional $F[\Phi]$, we write

$$\delta_B F[\Phi] \equiv i\epsilon\, Q_B \cdot F[\Phi] .$$

where Q_B takes the form of a first-order functional differential operator.

- Key property:

$$Q_B^2 = 0.$$

for instance, $Q_B \cdot \phi = -i c^\alpha \delta_\alpha \phi$

$$Q_B^2 \cdot \phi = Q_B \cdot (-i c^\alpha \delta_\alpha \phi)$$

$$= -i(Q_B \cdot c^\alpha)\, \delta_\alpha \phi + i c^\alpha Q_B \cdot (\delta_\alpha \phi)$$

viewed as a functional of ϕ

$$= -i\left(\frac{i}{2} f^\alpha{}_{\beta\gamma} c^\beta c^\gamma\right) \delta_\alpha \phi$$

$$+ i c^\alpha (-i c^\beta \delta_\beta \cdot (\delta_\alpha \phi))$$

$$= \frac{1}{2} f^\alpha{}_{\beta\gamma} c^\beta c^\gamma \delta_\alpha \phi + c^\alpha c^\beta \delta_\beta \delta_\alpha \phi$$

$$= 0. \quad (\text{due to } c^\alpha c^\beta = -c^\beta c^\alpha)$$

Similarly, can verify $Q_B^2 \cdot c^\alpha = 0$ using Jacobi identity for $f^\alpha{}_{\beta\gamma}$:

$$f^\alpha{}_{\beta\gamma} f^\gamma{}_{\rho\sigma} = 0.$$

Observe that

$$S_{G.F.} \equiv B_A F^A[\phi] + b_A \delta_\alpha F^A[\phi] c^\alpha$$

$$= -i\, Q_B \cdot (b_A F^A[\phi]).$$

Consequently, $Q_B \cdot S_{G.F.} = Q_B^2 (\cdots) = 0,$

and since $Q_B \cdot S[\phi] = -ic^\alpha \underbrace{\delta_\alpha S[\phi]}_{\overset{\shortparallel}{0}} = 0$,

the full gauged fixed action

$$S[\phi, B, b, c] \quad \text{is} \quad \text{BRST-invariant.}$$

i.e. δ_B is a symmetry.

As usual in the path integral quantization, a quantum state $|\Psi\rangle$ is represented by a wave functional

$$\Psi[\phi, B, b, c|_{\text{fixed time } x^0}]$$

Q_B is promoted to an operator $\hat{Q}_B$, defined by

$$\hat{Q}_B |\Psi\rangle := |\underset{\uparrow}{Q_B \cdot} \Psi\rangle .$$

Q_B-variation of the wave functional Ψ at fixed time x^0

The insertion of a functional of field variable $F[\Phi]$ in the path integral corresponds to acting on the state with an operator $\hat{F}$.

then $\quad Q_B \cdot F[\Phi] \rightsquigarrow [\hat{Q}_B, \hat{F}]$

Definition: an admissible ("physical") state $|\Psi\rangle$ is one that obeys

$$\hat{Q}_B |\Psi\rangle = 0 .$$

Consequence: the transition amplitude $\langle \Psi_f | U(t_f, t_i) | \Psi_i \rangle$ between a pair of admissible states $|\Psi_i\rangle$, $|\Psi_f\rangle$ is invariant with respect to deformations of the gauge fixing functional $F^A[\phi]$.

$$\langle \Psi_f | U(t_f, t_i) | \Psi_i \rangle$$

$$= \int [D\Phi(t)]_{t_i \leq t \leq t_f} \, \Psi_f^*[\Phi(t_f)] \, e^{iS[\Phi]} \, \Psi_i[\Phi(t_i)]$$

gauge fixed action

$\Phi = (\phi, B, b, c)$

Under a change of gauge condition $F^A[\phi] = 0$, i.e. replacing

$$F^A[\phi] \to \tilde{F}^A[\phi] = F^A[\phi] + \underbrace{\delta}_{\text{some arbitrarily infinitesimal functional}} F^A[\phi]$$

the action $S[\Phi]$ changes by

$$\bar{\delta} S[\Phi] = \bar{\delta} S_{G.F.} = -i Q_B \cdot (b_A \bar{\delta} F^A[\phi]).$$

$$\bar{\delta}\left(\langle \Psi_f | U(t_f, t_i) | \Psi_i \rangle\right)$$

$$= \int [D\Phi]\, \Psi_f^* e^{iS} \Psi_i \cdot Q_B (b_A \bar{\delta} F^A[\phi])$$

$$\underset{\underset{Q_B \cdot \Psi_f = 0 = Q_B \cdot \Psi_i \text{ by assumption}}{\uparrow}}{=} \int [D\Phi]\; Q_B \cdot (\Psi_f^* e^{iS} \Psi_i \cdot b_A \bar{\delta} F^A[\phi])$$

$= 0$ (assuming the measure $[D\Phi]$ is also BRST-invariant.)

Note that a state of the form $\hat{Q}_B |\chi\rangle$ is admissible, but in a "trivial" manner. In fact, the inner product between any admissible $|\Psi\rangle$ and $\hat{Q}_B |\chi\rangle$ vanishes:

$$\langle \Psi | \hat{Q}_B | \chi \rangle = \underbrace{(\langle \Psi | \hat{Q}_B)}_{\overset{\shortparallel}{0}} |\chi\rangle$$

BRST quantization: we declare that an admissible state $|\Psi\rangle$ (i.e. $\hat{Q}_B|\Psi\rangle = 0$) and $|\Psi\rangle + \hat{Q}_B|\chi\rangle$ represent the same physical state.

That is, the Hilbert space of physical states, in the conventional QM sense, is

$$\mathcal{H}_{phys} = \frac{\mathrm{Ker}\,\hat{Q}_B}{\mathrm{Im}\,\hat{Q}_B}$$

"BRST cohomology"

Applying Faddeev-Popov procedure to the path integral of Yang-Mills theory, with the gauge-fixing functional

$$F^{(a,x)}[A_\mu] = \partial_\mu A^\mu_a(x),$$

under the infinitesimal gauge variation

$$\delta_\zeta A^\mu_a(x) = \partial^\mu \zeta_a(x) + f^{bc}{}_a A^\mu_b(x) \zeta_c(x),$$

the gauge condition varies by

$$\delta_\zeta F^{(a,x)}[A_\mu] = \partial_\mu (\partial^\mu \zeta_a + f^{bc}{}_a A^\mu_b \zeta_c)$$

Faddeev-Popov ghosts:

$c_a(x)$ in correspondence with gauge parameters $\zeta_a(x)$

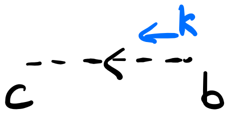
$b^a(x)$ in correspondence with gauge conditions $F^{(a,x)}[A]$

The ghost action is

$$S_{gh} = \int d^4x \; b^a(x)\, \partial_\mu (\partial^\mu c_a(x) + f^{bc}{}_a A^\mu_b(x)\, c_c(x))$$

$$= -\int d^4x \; \partial_\mu b^a (\partial^\mu c_a + f^{bc}{}_a A^\mu_b c_c).$$

The kinetic term for b^a, c_a is identical to that of a complex scalar, except for the Grassmann-odd property of b, c.



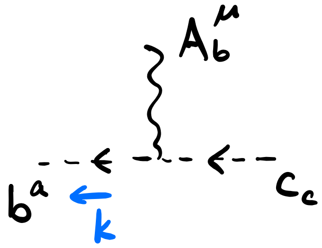
$$\langle\Omega| T\, \hat{b}^a(x)\, \hat{c}_{a'}(y) |\Omega\rangle$$

$$= -\langle\Omega| T\, \hat{c}_{a'}(y)\, \hat{b}^a(x) |\Omega\rangle$$

$$= \delta^a_{a'} \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-y)} \cdot \frac{-i}{k^2 - i\epsilon}$$

Note: as in fermion Feynman rules, a ghost loop comes with a factor of -1 due to Grassmann-oddness.

interaction vertex



$$= -k_\mu f^{bc}{}_a$$

[unlike in scalar QED, there is no vertex of the form ]

The gauge-fixing part of the action $S_{G.F.}$ contains

$$B_A F^A[\phi] \rightsquigarrow \int d^4x \, B^a(x)\, \partial_\mu A^\mu_a(x) .$$

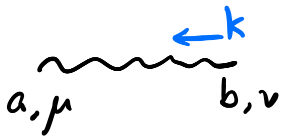
We are free to add the BRST-exact term

$$\frac{g^2}{2}\xi \int d^4x \, B^a(x) B^a(x) .$$

Integrating out $B^a(x)$ in the path integral results in

$$\Delta S = -\frac{1}{2g^2\xi}\int d^4x \, (\partial_\mu A^\mu_a)^2 .$$

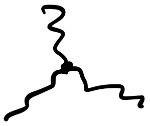
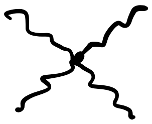
$\Rightarrow$ A_μ propagator analogous to Maxwell theory in ξ-gauge.



$\langle \Omega | T \, \hat{A}_{a\mu}(x) \, \hat{A}_{b\nu}(y) \, | \Omega \rangle$

$$= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \, g^2 \, \frac{-i\delta_{ab}}{k^2 - i\epsilon}\left[\eta_{\mu\nu} + (\xi-1)\frac{k_\mu k_\nu}{k^2}\right].$$

due to our normalization convention of Yang-Mills kinetic term

YM interaction vertices  

- 1PI RG analysis of QCD (with massless quarks)

Strategy: Calculate the 1PI effective action $\Gamma[A_\mu, \dots]$ at 1-loop order, using

- dimensional regularization scheme $D = 4 - \epsilon$

and

- minimal subtraction as renormalization scheme that defines the renormalized Yang-Mills coupling $g(\mu)$.
- extract RGE for $g(\mu)$.

Naive expectation: assuming quark fields are massless, all power-counting renormalizable couplings are controlled by a single coupling constant g due to gauge-invariance.

This expectation is essentially correct, but not obviously so:

in path integral quantization, gauge invariance is ~~not manifest~~ replaced by BRST invariance.

Thereby, the possible counter terms to be included in the (bare) gauge-fixed action are restricted by BRST-invariance (as opposed to gauge invariance).

A related complication is that the 1PI eff. action is not only gauge non-invariant, but is generally not BRST-invariant either. This is loosely speaking due to the fact that BRST variations are nonlinear in the fields in general, and $\langle Q_B \cdot \phi \rangle_{J_{\phi_0}} \neq Q_B \cdot \phi \big|_{\phi=\phi_0}$.

[See Weinberg section 17.2]

Nonetheless, a simplification occurs for $\Gamma[A, \ldots]$ computed in the "background field gauge".

In evaluating Γ, we begin by expanding

$$A_\mu(x) = \underbrace{\tilde{A}_\mu(x)}_{\text{bkground field}} + \underbrace{A'_\mu(x)}_{\text{fluctuations}}$$

and similarly for ψ, b, c

add a source term $\Delta S = \int d^4x\, j^\mu(x) A'_\mu(x) + \cdots$ such that $\langle A'_\mu(x) \rangle = 0$, etc. and integrate out $A'_\mu(x), \cdots$ to obtain $\Gamma[\tilde{A}, \cdots]$:

$$e^{i\Gamma[\tilde{A}, \tilde{\psi}, \cdots]} = \int DA' D\psi' Db' Dc'$$

$$\times \exp\Big[i \Big(S[\tilde{A} + A', \tilde{\psi} + \psi', \cdots] + S_{G.F.}[A', \psi', \cdots ; \tilde{A}]$$

$$+ \int j^\mu_A A'_\mu + j_\psi \cdot \psi' + \cdots \Big) \Big]$$

The gauge condition for $A', \cdots$ may be chosen to depend on $\tilde{A}$. A particularly "nice" choice is

$$F^{(a,x)}[A'] = \tilde{D}_\mu A'^\mu_a(x)$$

↑ gauge-covariant derivative (in adjoint rep) w.r.t. bkgrd field $\tilde{A}_\mu$

$$\equiv \partial_\mu A'^\mu_a + f^{bc}{}_a \tilde{A}_{b\mu} A'^\mu_e$$

* Observe that $F^{(a,x)}[A']$ is covariant w.r.t. the following transformation:

$$\delta_{NEW} \tilde{A}^\mu_a = \partial^\mu \zeta_a + f^{bc}{}_a \tilde{A}^\mu_b \zeta_c$$

← "gauge-like"

✪

$$\delta_{NEW} A'^\mu_a = f^{bc}{}_a A'^\mu_b \zeta_c$$

as if $\tilde{A}$ were gauge fields and A' were matter fields transforming in adjoint representation.

Consequently, $S_{G.F.}$ is invariant w.r.t. the transformation ✪ together with

$$\delta_{NEW}\, \tilde{b}^a = -f^{ab}{}_c\, \tilde{b}^c \zeta_b .$$

$$\delta_{NEW}\, \tilde{c}_a = f^{bc}{}_a\, \tilde{c}_b \zeta_c .$$

likewise with

$\bar{b} \rightsquigarrow b'$

$\bar{c} \rightsquigarrow c'$

and the full gauge-fixed action is invariant provided that the matter fields also transf. according to

$$\delta_{NEW}\, \tilde{\psi} = i \zeta_a t^a_R \cdot \tilde{\psi} .$$

likewise with $\tilde{\psi} \rightarrow \psi'$.

Key point: δ_{NEW} does not mix $\tilde{A}, \cdots$ with $A', \ldots$

After integrating out $A', \cdots$, we obtain $\Gamma[\tilde{A}, \cdots]$ that is invariant w.r.t. the δ_{NEW} transformations of $\tilde{A}, \cdots$, which take the form of gauge transformations as if b^a, c_a were also matter fields in the adjoint representation!

Conclusion: in the background field gauge, the 1PI effective action $\Gamma[\tilde{A}, \tilde{\psi}, \tilde{b}, \tilde{c}]$ is invariant under a gauge-like transformation δ_{NEW}. This constrains the possible terms in Γ similarly to how gauge-invariance constrains the Yang-Mills action.

In particular, the operators of mass dim ≤ 4 appearing in Γ, up to field rescaling, are controlled by a single renormalized gauge coupling $g(\mu)$. To determine the RGE for $g(\mu)$, it suffices to consider $\Gamma[\tilde{A}]$ with $\tilde{\psi}, \tilde{b}, \tilde{c}$ set to zero.

Gauge-fixed action of massless QCD $(\theta = 0)$ in bkgrd field gauge with $\tilde{A}$ turned on:

$$S[A', \underbrace{\psi, b, c}_{\text{drop the superscript } '}; \tilde{A}]$$

$$= \int d^4x \Big[-\frac{1}{4g^2} F_{a\mu\nu} F^{a\mu\nu} - \bar{\psi}_I \gamma^\mu D_\mu \psi_I$$

$$-\frac{1}{2\xi g^2}\left(\widetilde{D}_\mu A'^{\mu}_a\right)^2 - \left(\widetilde{D}_\mu b^a\right)\left(\underset{\uparrow}{D^\mu} c_a\right)\Bigg]$$

full gauge-covariant derivative

$$= \int d^4x \Bigg[-\frac{1}{4g^2}\Big(\widetilde{F}_{a\mu\nu} + \widetilde{D}_\mu A'_{a\nu} - \widetilde{D}_\nu A'_{a\mu}$$

$$+ f^{bc}{}_a A'_{b\mu} A'_{c\nu}\Big)^2 - \bar{\Psi}_I \gamma^\mu\left(\widetilde{D}_\mu - i A'_{a\mu} t^a_R\right)\psi_I$$

$$-\frac{1}{2\xi g^2}\left(\widetilde{D}_\mu A'^{\mu}_a\right)^2 - \widetilde{D}_\mu b^a\left(\widetilde{D}^\mu c_a + f^{bc}{}_a A'^{\mu}_b c_c\right)\Bigg].$$

The 1PI effective action of massless QCD in background field gauge (restricted to the case of vanishing $\tilde{\psi}$, $\tilde{b}$, $\tilde{c}$)

$$e^{i\Gamma[\tilde{A}]} = \int DA' D\psi Db Dc$$

$$\times \exp\left(iS[A',\psi,b,c;\tilde{A}] + i\int j_{a\mu} A'^{\mu}_{a}\right)\Bigg|_{\substack{j \text{ chosen} \\ \text{s.t.} \\ \langle A' \rangle = 0}}$$

The 1-loop contribution $\Gamma^{\text{1-loop}}[\tilde{A}]$ comes from performing the Gaussian functional integral with the terms in the action that are quadratic in the fluctuation fields A', ψ, b, c.

$$S^{(2)}[A',\psi,b,c;\tilde{A}]$$

$$= \int d^4x \Big[-\frac{1}{4g^2}(\tilde{D}_\mu A'_{a\nu} - \tilde{D}_\nu A'_{a\mu})^2 - \frac{1}{2g^2} f^{bc}{}_{a} \tilde{F}^{\mu\nu}_{a} A'_{b\mu} A'_{c\nu}$$

$$- \frac{1}{2\xi g^2}(\tilde{D}_\mu A'^{\mu}_{a})^2 - \bar{\psi}_I \gamma^\mu \tilde{D}_\mu \psi_I - (\tilde{D}_\mu b^a)(\tilde{D}_\mu c_a) \Big]$$

$$\equiv \int d^4x \Big[-\frac{1}{4g^2} A'^{\mu}_{a} (\mathbb{D}_A)_{a\mu,b\nu} A'^{\nu}_{b} + \bar{\psi}_I \mathbb{D}_\psi \psi_I + b^a \mathbb{D}_{gh} c_a \Big].$$

where

$$(\mathbb{D}_A)_{a\mu,b\nu} \equiv \left(-\tilde{\tilde{D}}_\rho \tilde{\tilde{D}}^\rho \eta_{\mu\nu} + \tilde{\tilde{D}}_\nu \tilde{\tilde{D}}_\mu - \frac{1}{\xi}\tilde{\tilde{D}}_\mu \tilde{\tilde{D}}_\nu\right)_{ab} + f^{ab}{}_c\, \tilde{\tilde{F}}_{c\mu\nu}\,,$$

↑ we will choose $\xi = 1$ below.

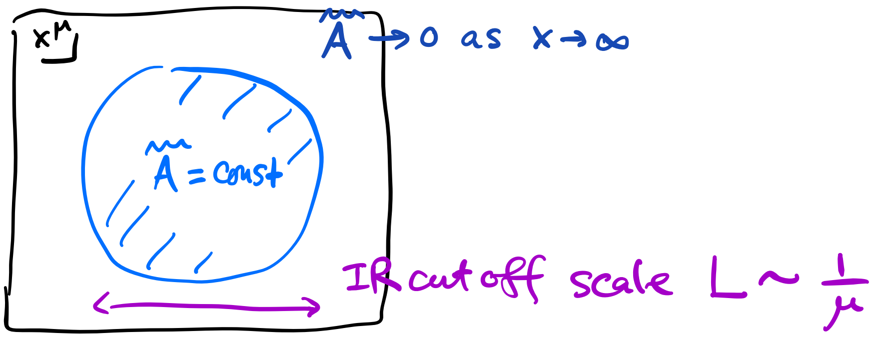
$$\mathbb{D}_\psi = -\gamma^\mu \tilde{\tilde{D}}_\mu\,, \qquad \mathbb{D}_{gh} = \tilde{\tilde{D}}_\mu \tilde{\tilde{D}}^\mu\,.$$

↑ acts on each Ψ_I, $I=1,\dots,N_f$

$$i\Gamma^{1\text{-loop}}[\tilde{\tilde{A}}] = \log \frac{(\det \mathbb{D}_\psi)^{N_f}(\det \mathbb{D}_{gh})}{(\det \mathbb{D}_A)^{\frac{1}{2}}}\,.$$

$\rightsquigarrow$ $-\Gamma^{1\text{-loop}}[\tilde{\tilde{A}}]$ in Euclidean signature

The evaluation of these functional determinants for general $\tilde{\tilde{A}}$ is complicated. For our purpose of extracting the renormalized coupling $g(\mu)$ in MS scheme, it suffices to consider nearly constant $\tilde{\tilde{A}}_{a\mu}$. More precisely, we will evaluate $\Gamma^{1\text{-loop}}[\tilde{\tilde{A}}]$ on bkgrd (Euclidean) gauge field config. of the form:



$$-\Gamma^{1\text{-loop}}[\tilde{A}] = N_f \,\mathrm{Tr} \log \mathbb{D}_\psi + \mathrm{Tr} \log \mathbb{D}_{gh} - \frac{1}{2} \mathrm{Tr} \log \mathbb{D}_A$$

where each Tr is defined over the appropriate configuration space of field variables.

For kinetic operator $\mathbb{D}_\alpha{}^\beta$ acting on a set of field variables $\varphi_\alpha(x)$.

$$\mathrm{Tr}\, f(\mathbb{D}) = \int d^4x \, \langle x | \overset{\text{trace over internal } \alpha\beta \text{ indizes}}{\mathrm{tr}}\, f(\mathbb{D}) | x \rangle$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4} \langle x | \mathrm{tr}\, f(\mathbb{D}) | k \rangle \overbrace{\langle k | x \rangle}^{e^{-ik\cdot x}}$$

For constant $\tilde{A}$, the relevant differential operator $\mathbb{D}$ has no explicit x-dependence, thus

$$\mathbb{D}_\alpha{}^\beta |k\rangle = M_\alpha{}^\beta(k) \, |k\rangle$$

$$\Rightarrow \mathrm{Tr}\, f(\mathbb{D}) = \int d^4x \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\, f(M(k)) .$$

And thus

$$-\Gamma^{1\text{-loop}}[\tilde{A}] = \int d^4x \int \frac{d^4k}{(2\pi)^4} \Big[N_f \,\mathrm{tr} \log M_\psi(k) + \mathrm{tr} \log M_{gh}(k) - \frac{1}{2} \mathrm{tr} \log M_A(k) \Big] .$$

e.g. $M_\psi(k) = i(\not{k} - \gamma^\mu \tilde{A}_{a\mu} t^a_R)$

- a matrix of size $4 \cdot \dim R$ acting on $\psi_{i\alpha}$,

$\alpha = 1, \dots, 4$ spinor index

$i = 1, \dots, \dim R$ gauge index

$$\int \frac{d^4k}{(2\pi)^4} \operatorname{tr} \log M_\psi(k)$$

$$= (\tilde{A}\text{-independent} \quad \text{reg. scheme dependent} \quad \text{constant})$$

$$- \int \frac{d^4k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} \left(\frac{\not{k}}{k^2} \not{\tilde{A}} \right)^n$$

↑ trace over spinor and gauge indices

↗

UV divergence occurs for $n \leq 4$

(dimensional regularization)

IR divergence occurs for $n \geq 4$

(cutoff at $L \sim \frac{1}{\mu}$)

Compare to bare Euclidean YM Lagrangian

$$\mathcal{L}_E = \frac{1}{2g^2} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu})$$

$\supset$ quartic term $-\frac{1}{2g^2} \operatorname{tr}\left([A_\mu, A_\nu][A^\mu, A^\nu]\right)$.

We can extract $g(\mu)$ from the $\tilde{A}^4$ term in $\Gamma[\tilde{A}]$.

Working in dim reg. $D = 4-\epsilon$.

$$\int \frac{d^Dk}{(2\pi)^D}\, \mathrm{tr}\log M_\psi(k)\Big|_{\tilde{A}^4}$$

$$= -\frac{1}{4}\,\mathrm{tr}_R(\tilde{A}_{\mu_1}\tilde{A}_{\mu_2}\tilde{A}_{\mu_3}\tilde{A}_{\mu_4})$$

$$\times \int \frac{d^Dk}{(2\pi)^D}\ \underbrace{\frac{\mathrm{Tr}(\not{k}\gamma^{\mu_1}\not{k}\gamma^{\mu_2}\not{k}\gamma^{\mu_3}\not{k}\gamma^{\mu_4})}{(k^2)^4}}_{\parallel}$$

$$\underbrace{\frac{k^{\nu_1}k^{\nu_2}k^{\nu_3}k^{\nu_4}}{(k^2)^4}}_{\downarrow}\ \mathrm{Tr}(\gamma_{\nu_1}\gamma^{\mu_1}\ldots\gamma_{\nu_4}\gamma^{\mu_4})$$

by Euclidean sym, replace with

$$\frac{1}{(k^2)^2}\,\frac{1}{D(D+2)}\left(\underline{\delta^{\nu_1\nu_2}\delta^{\nu_3\nu_4} + (2\leftrightarrow 3) + (2\leftrightarrow 4)}\right)$$

Contract w/ $\mathrm{Tr}(\gamma_{\nu_1}\gamma^{\mu_1}\ldots)$.

$$\left(\delta^{\nu_1\nu_2}\delta^{\nu_3\nu_4} + (2\leftrightarrow 3) + (2\leftrightarrow 4)\right)\mathrm{Tr}(\gamma_{\nu_1}\gamma^{\mu_1}\ldots\gamma_{\nu_4}\gamma^{\mu_4})$$

$$= 2\cdot\mathrm{Tr}(\underbrace{\gamma_\alpha\gamma^{\mu_1}\gamma^\alpha}\gamma^{\mu_2}\underbrace{\gamma_\beta\gamma^{\mu_3}\gamma^\beta}\gamma^{\mu_4}) + \mathrm{Tr}(\gamma_\alpha\gamma^{\mu_1}\gamma_\beta\gamma^{\mu_2}\gamma^\alpha\gamma^{\mu_3}\gamma^\beta\gamma^{\mu_4})$$

$$= 2(2-D)^2\,\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_4}) + 2\delta^{\mu_2\alpha}\,\mathrm{Tr}(\gamma_\alpha\gamma^{\mu_1}\gamma_\beta\gamma^{\mu_3}\gamma^\beta\gamma^{\mu_4})$$

$$-2\delta_\beta^\alpha\,\mathrm{Tr}(\gamma_\alpha\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^\beta\gamma^{\mu_4}) + 2\delta^{\mu_1\alpha}\,\mathrm{Tr}(\gamma_\alpha\gamma_\beta\gamma^{\mu_2}\gamma^{\mu_3}\gamma^\beta\gamma^{\mu_4})$$

$$-D\,\mathrm{Tr}(\gamma^{\mu_1}\gamma_\beta\gamma^{\mu_2}\gamma^{\mu_3}\gamma^\beta\gamma^{\mu_4})$$

$$= \left(2(2-D)^2 - 2(2-D)\right)\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_4}) + 2(2-D)\,\mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\gamma^{\mu_4}) +$$

$$+(2-D)\,\mathrm{Tr}(\gamma^{\mu_1}\underbrace{\gamma_\beta\gamma^{\mu_2}\gamma^{\mu_3}\gamma^\beta}_{\substack{\parallel \\ 2\gamma^{\mu_3}\gamma^{\mu_2}-(2-D)\gamma^{\mu_2}\gamma^{\mu_3}}}\gamma^{\mu_4})$$

$$=\left(2(2-D)^2-2(2-D)-(2-D)^2\right)\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_4})$$

$$+2(2-D)\,\mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\gamma^{\mu_4})+2(2-D)\,\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_3}\gamma^{\mu_2}\gamma^{\mu_4})$$

$$=8\,\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_4})-4\,\mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\gamma^{\mu_4})-4\,\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_3}\gamma^{\mu_2}\gamma^{\mu_4})+\mathcal{O}(\epsilon)$$

Further contraction with $\mathrm{tr}_R(\tilde{A}_{\mu_1}\cdots\tilde{A}_{\mu_4})$ gives

$$\mathrm{tr}_R(\tilde{A}_{\mu_1}\cdots\tilde{A}_{\mu_4})\cdot\underbrace{4\,\mathrm{Tr}([\gamma^{\mu_1},\gamma^{\mu_2}][\gamma^{\mu_3},\gamma^{\mu_4}])}_{\substack{\parallel \\ -64(\delta^{\mu_1\mu_3}\delta^{\mu_2\mu_4}-\delta^{\mu_1\mu_4}\delta^{\mu_2\mu_3})}}+\mathcal{O}(\epsilon)$$

$$=-32\cdot\mathrm{tr}_R([\tilde{A}_{\mu_1},\tilde{A}_{\mu_2}][\tilde{A}^{\mu_1},\tilde{A}^{\mu_2}])+\mathcal{O}(\epsilon)$$

Thus, we have

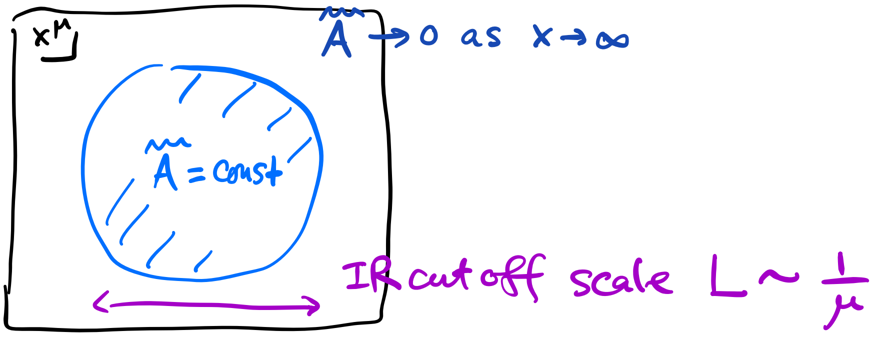
$$\int\frac{d^Dk}{(2\pi)^D}\,\mathrm{tr}\log M_\psi(k)\Big|_{\tilde{A}^4}$$

$$=\left[\frac{1}{3}\mathrm{tr}_R([\tilde{A}_\mu,\tilde{A}_\nu][\tilde{A}^\mu,\tilde{A}^\nu])+\mathcal{O}(\epsilon)\right]$$

$$\cdot\int\frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}}\frac{1}{(k^2)^2}$$

↖ IR-divergent

The IR divergence is of course expected, since we have assumed constant $\tilde{A}$ without μ-dependence. The renormalized YM coupling should be extracted from $\Gamma[\tilde{A}]$ evaluated on a bkgrd field config. $\tilde{A}$ of momentum scale μ, e.g.



To extract the μ-dependence, it suffices to modify the earlier calculation by introducing an IR cutoff $|k| > \mu$, namely

$$\int d^D x \int \frac{d^D k}{(2\pi)^D} \operatorname{tr} \log M_\psi(k) \Big|_{\tilde{A}^4}$$

$$= \int d^D x \, \frac{1}{3} \operatorname{tr}_R([\tilde{A}_\mu, \tilde{A}_\nu][\tilde{A}^\mu, \tilde{A}^\nu])$$

$$\cdot \int_{|k|>\mu} \frac{d^{4-\epsilon} k}{(2\pi)^{4-\epsilon}} \frac{1}{(k^2)^2} \quad + \text{ finite}$$

$$= \int d^Dx \; \frac{1}{3}\, tr_R([\tilde{A}_\mu, \tilde{A}_\nu][\tilde{A}^\mu, \tilde{A}^\nu]) \frac{1}{8\pi^2} \frac{\mu^{-\epsilon}}{\epsilon}$$

$$+ \text{finite}$$

Completing this term via the formal gauge inv. (δ_{NEW} symmetry) of $\Gamma[\tilde{A}]$ in the bkgrd field gauge, we obtain a contribution

$$\int d^Dx \left[-\frac{1}{3}\, tr_R(\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}) \cdot \frac{1}{8\pi^2} \frac{\mu^{-\epsilon}}{\epsilon} + \text{finite} \right]$$

from each of the N_f flavors of quark fields.

Likewise, the contributions from the F-P ghost determinant and the functional det. for A' are

$$\mathrm{Tr}\log D_{gh} = \int d^Dx \; \frac{1}{12}\, tr_{adj}(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}) \cdot \frac{1}{8\pi^2} \frac{\mu^{-\epsilon}}{\epsilon}$$

$$+ \text{finite}$$

$$\mathrm{Tr}\log \mathbb{D}_A = \int d^Dx \; \left(-\frac{5}{3}\right) tr_{adj}(\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}) \cdot \frac{1}{8\pi^2} \frac{\mu^{-\epsilon}}{\epsilon}$$

$$+ \text{finite}$$

[HW exercise]

Comparing to the expected form of Euclidean 1PI eff. action evaluated on such $\tilde{A}$,

$$\Gamma[\tilde{A}] = \frac{1}{4g^2(\mu)} \int d^4x \, \tilde{F}_{a\mu\nu} \tilde{F}^{a\mu\nu}$$

$$+ \left(\text{higher order terms in } g(\mu) \text{ or in } \tilde{A} \right)$$

$$-\frac{1}{4g^2(\mu)} = -\frac{1}{4g_0^2} + \frac{b}{8\pi^2} \frac{\mu^{-\epsilon}}{\epsilon} + \text{finite}, \qquad ✪$$

where

$$b = -\frac{N_f}{3} \underset{\substack{\uparrow \\ \text{quadratic Casimir}}}{C(R)} + \left(-\frac{1}{2} \cdot \left(-\frac{5}{3} \right) + \frac{1}{12} \right) C(\text{adj})$$

$$\operatorname{tr}_R(t^a t^b) = C(R) \cdot \delta^{ab}$$

$$\operatorname{tr}_{\text{adj}}(t^a t^b) = i f^{ac}{}_d \cdot i f^{bd}{}_c$$

$$= C(\text{adj}) \, \delta^{ab}$$

For $G = SU(N)$,

$R = \square$ or $\overline{\square}$, $C(R) = \frac{1}{2}$

$C(\text{adj}) = N$.

Thus, $$b = \frac{11}{12} C(\text{adj}) - \frac{N_f}{3} C(R)$$

$$= \frac{11}{12} N - \frac{1}{6} N_f .$$

In our usual notation, $\lambda^2(\mu) \equiv \mu^{\epsilon} g^2(\mu)$,

We can invert ✪ to write

drop in MS scheme

$$\frac{1}{g_0^2} = \mu^{-\epsilon}\left[\frac{1}{\lambda^2(\mu)} + \frac{b}{2\pi^2}\frac{1}{\epsilon} + \cancel{\text{finite}} + \text{higher order in } \lambda(\mu)\right]$$

$\Rightarrow$ RGE

$$0 = \frac{d}{d\log\mu}\frac{1}{g_0^2}$$

$$= \mu^{-\epsilon}\left[\frac{d}{d\log\mu}\frac{1}{\lambda^2(\mu)} - \epsilon\left(\frac{1}{\lambda^2(\mu)} + \frac{b}{2\pi^2}\frac{1}{\epsilon}\right) + \text{higher order in } \lambda\right]$$

$$\xrightarrow{\epsilon\to 0} \quad \frac{d}{d\log\mu}\frac{1}{\lambda^2(\mu)} = \frac{b}{2\pi^2} + \mathcal{O}(\lambda^2).$$

Or equivalently,

$$\frac{d}{d\log\mu}\lambda(\mu) = \beta(\lambda(\mu)),$$

$$\beta(\lambda) = -\frac{b}{4\pi^2}\lambda^3 + \mathcal{O}(\lambda^5).$$

1-loop β-fn of QCD.

Solution to the 1-loop RGE

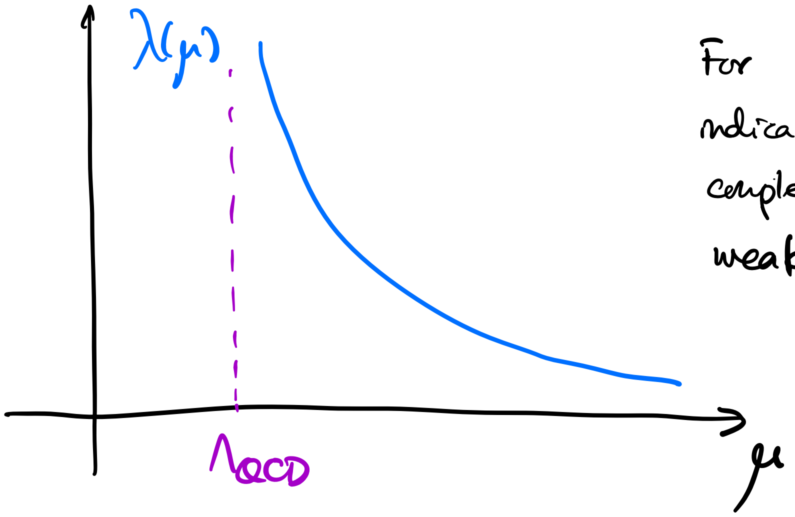
$$\frac{d}{d\log\mu}\frac{1}{\lambda^2(\mu)} = \frac{b}{2\pi^2} \qquad \left(b = \frac{11}{12}N - \frac{1}{6}N_f\right)$$

$$\Rightarrow \quad \frac{1}{\lambda^2(\mu)} = \frac{1}{\lambda^2(\mu_0)} + \frac{b}{2\pi^2}\log\frac{\mu}{\mu_0}$$

i.e. $$\lambda^2(\mu) = \frac{1}{\frac{b}{2\pi^2}\log\frac{\mu}{\Lambda_{QCD}}}$$

where $\Lambda_{QCD} \equiv \mu_0 e^{-\frac{2\pi^2}{b\lambda^2(\mu_0)}}$

is a characteristic mass scale



For $b>0$, the 1-loop RGE indicates that QCD is strongly coupled in the IR, and weakly coupled in the UV

"asymptotic freedom"

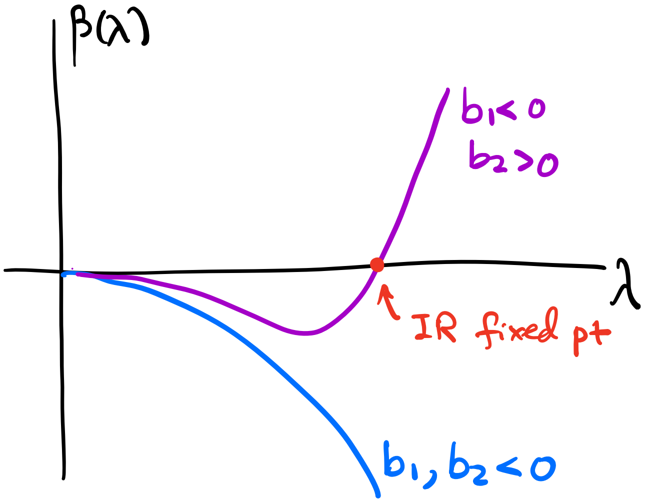
- What happens in the IR?

  - as $\lambda(\mu)$ becomes large, higher loop corrections to the RGE become important

$$\beta(\lambda) = b_1 \lambda^3 + b_2 \lambda^5 + \cdots$$

$$\left(b_1 = -\frac{b}{4\pi^2}\right)$$

[recall that both b_1 and b_2 are scheme-independent]



The 2-loop RGE would predict an IR fixed point when $b_1 < 0$, $b_2 > 0$.

Further, if $|b_2| \ll |b_1|$, fixed point value of λ is small, can trust 2-loop approximation near the fixed pt?

Such an IR fixed point of massless QCD occurs when

$$\frac{34 N^3}{13 N^2 - 3} < N_f < \frac{11}{2} N \qquad \circledast$$

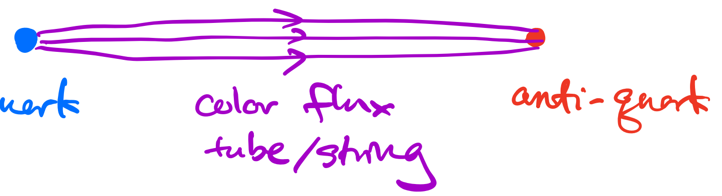
and is believed to give rise to a CFT

"Banks – Zaks fixed point"

- Not relevant for real world QCD, where the number N_f of light quarks is outside of the window ⊛

- expectation: for N_f not too large, the strong coupling in IR leads to confinement, namely that there are no particles created by A_μ (gluons) or ψ (quarks). The particles are created by gauge-invariant local operators like $\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$, and are color neutral. e.g. mesons, baryon, glueballs.

- the gluons and quarks exist as "jets" or "particles with (color flux) strings attached"

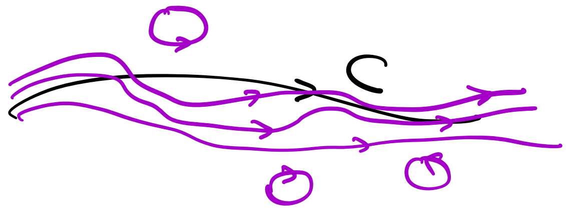


can created such a state with non-local gauge invariant operator $\bar{\psi}(x_2)\, W_R(C)\, \psi(x_1)$

where $$W_R(C) = \underset{\uparrow \text{ path-ordered exponential}}{\mathcal{P}}\, e^{i\int_C A_{a\mu}(x)\, t_R^a\, dx^\mu}$$

C is a path in spacetime going from x_1 to x_2.

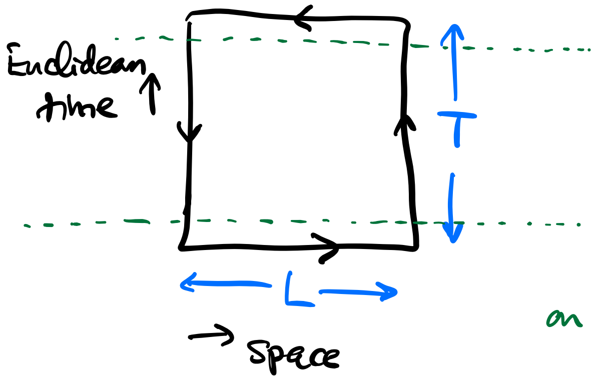
$W_R(C)|\Omega\rangle$ is generally a state corresponding to a color flux string extended roughly along C with all sorts of excitations:



In pure Yang-Mills theory, i.e. QCD without quarks, the color string created by $W_R(C)$ for $R = \square$ or $\bar{\square}$ cannot break; it carries a $\mathbb{Z}_N$-valued "string charge". The latter is a charge of a "1-form symmetry" known as the "center symmetry".

The flux string has tension $\gamma \sim \Lambda_{QCD}^2$.

For a closed path in Euclidean spacetime of the form



a state of flux tube of length L that begins & ends on color sources ("classical quarks")

$$\langle W_R(C) \rangle \overset{\text{large } T}{\sim} e^{-V(L)\cdot T}$$

confinement: $V(L) \to \infty$ as $L \to \infty$

expect for $L, T \gg \Lambda_{QCD}^{-1}$,

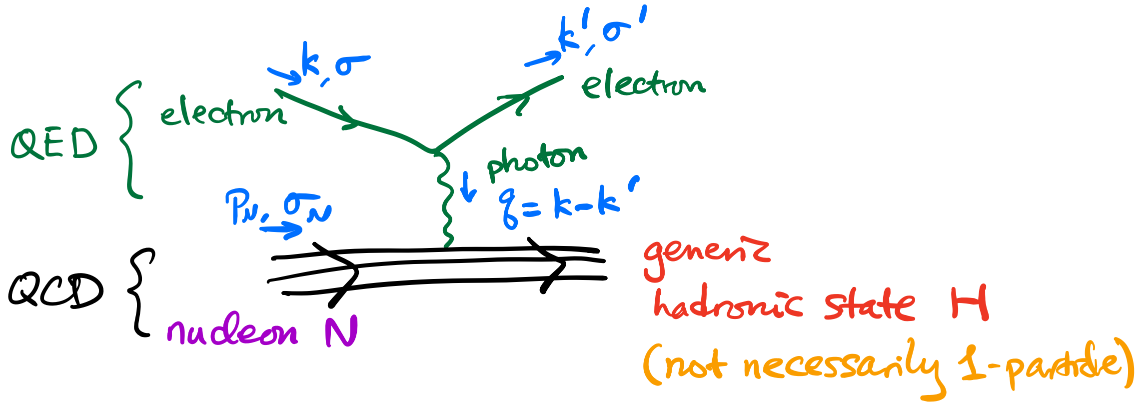
$$V(L) \to \underset{\uparrow \text{ string tension}}{\gamma} \cdot L$$

Q: What is perturbative QCD good for?

- $\lambda(\mu)$ small at large μ.
- can reliably calculate Green function of gauge-invariant (local) operators (e.g. $\mathrm{tr} F_{\mu\nu}^2$) at short distances / large momentum.

Experimental signature of asymptotic freedom

- deep inelastic scattering (DIS)



To leading order in the electric charge (with respect to the photon), the coupling of QCD to E-M is described by the term

$$\Delta\mathcal{L} = A^{EM}_{\mu} J^{\mu}$$

in the Lagrangian density, where

$$J^{\mu} = i \sum_{I=1}^{N_f} \underbrace{e_I}_{\text{electric charge of the I-th type of quark}} \bar{\Psi}_I \gamma^{\mu} \Psi_I$$

Scattering amplitude of

electron + nucleon N

→ electron + generic hadronic state H

at leading order in electric charge,

$$\mathcal{A}(k',\sigma';H\,|\,k,\sigma;N)$$

$$= \bar{u}_{\sigma'}(k')\, e\gamma_\mu\, u_\sigma(k)\, \frac{-i}{q^2 - i\epsilon}\, \underbrace{\langle H|\, J^\mu(0)\,|N\rangle}_{\text{a form factor in QCD}}$$

$$q = k - k'$$

An observable of interest - the differential cross section (in nucleon N rest frame) that includes all possible hadronic states H within a given range of energy transfer $\nu \equiv -q\cdot \frac{p_N}{m_N}$:

$$\left.\frac{d^2\sigma}{d\Omega\, d\nu}\right|_{\substack{\text{avg. over}\\ e^- \,\&\, N \text{ spins}}} = \frac{1}{v_e}\, |\vec{k}'|^2 \frac{d|\vec{k}'|}{d\nu} \sum_H (2\pi)^4 \delta^4(p_H - p_N - q)$$

$$\times \frac{1}{4} \sum_{\sigma,\sigma_N} \sum_{\sigma'} \left|\mathcal{A}(k',\sigma';H|k,\sigma;N)\right|^2$$

$$= (\text{known factor})_{\mu\nu} \cdot \boxed{\begin{aligned} &\frac{1}{2}\sum_{\sigma_N}\sum_H (2\pi)^4 \delta^4(p_H - p_N - q) \\ &\times \langle N|J^\nu(0)|H\rangle \langle H|\, J^\mu(0)\,|N\rangle \end{aligned}}$$

$$\parallel$$

$$\int d^4x\, e^{-iq\cdot x}\, \frac{1}{2}\sum_{\sigma_N} \langle N|\, \underbrace{J^\nu(x)\, J^\mu(0)}_{\text{NOT time-ordered.}}\, |N\rangle$$

$$W^{\mu\nu}(q,P_N) \equiv \frac{P_N^0}{m_N}\int d^4x\, e^{-iq\cdot x}\,\frac{1}{2}\sum_N \langle N|J^\nu(x)J^\mu(0)|N\rangle$$

is a Lorentz covariant quantity that obeys current conservation $q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$.

$\Rightarrow$ the only possible expression is $\qquad \nu \equiv -\frac{q\cdot P}{m_N}$

$$W^{\mu\nu}(q,p) = \left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) W_1(\nu,q^2)$$

$$+ \left(p^\mu - \frac{p\cdot q}{q^2}q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}q^\nu\right) W_2(\nu,q^2).$$

properties:

- $W^{\mu\nu} = W^{\nu\mu} = (W^{\mu\nu})^*$
- $W_1,\ W_2 \geq 0$
- assuming N is the lightest hadron with the given quantum numbers, i.e. $-P_H^2 \geq m_N^2 \Rightarrow \nu \equiv -\frac{q\cdot P_N}{m_N^2} \geq \frac{q^2}{2m_N}$

$\Rightarrow$ $W^{\mu\nu}(q,p)$ is supported at $\nu \geq \frac{q^2}{2m_N}$.

- related to time-ordered expectation value via

$$T^{\mu\nu}(q,P_N) \equiv \frac{P_N^0}{m_N}\int d^4x\, e^{-iq\cdot x}\,\frac{1}{2}\sum_{\sigma_N}\langle N|\, T\, J^\nu(x)\, J^\mu(0)\,|N\rangle$$

$$= \frac{P_N^0}{m_N}\frac{1}{2}\sum_{\sigma_N}\int d^4x \int_{-\infty}^{\infty}\frac{d\omega}{\omega+i\epsilon}\Big[e^{-iq\cdot x+i\omega n\cdot x}\langle N|J^\nu(x)J^\mu(0)|N\rangle$$

$$+e^{-iq\cdot x-i\omega n\cdot x}\langle N|J^\mu(0)J^\nu(x)|N\rangle\Big]$$

here n^μ is a const. timelike vector, e.g. $n^\mu=(1,\vec{0})$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{\omega+i\epsilon}\left[W^{\mu\nu}(q-\omega n, P_N) + W^{\nu\mu}(-q-\omega n, P_N)\right]$$

$$\Rightarrow \mathrm{Re}\, T^{\mu\nu}(q,P_N) = \frac{1}{2}\Big[W_N^{\mu\nu}(q,P_N) + \underbrace{W_N^{\nu\mu}(-q,P_N)}_{\text{vanishes for } \nu>0,\ q^2>0}\Big]$$

The behavior in the regime of high momentum transfer $|q| \gg \Lambda_{QCD}$ ("deep inelastic") is governed by the property of the product operator $J^\mu(x)J^\nu(0)$ at $|x| \ll \Lambda_{QCD}^{-1}$

Operator product expansion (OPE):

$$\mathcal{O}_i(x)\,\mathcal{O}_j(0) \approx \sum_\ell F_{ij}^\ell(x)\,\mathcal{O}_\ell(0)$$

valid at least for small x

or in momentum space

$$\int d^Dx\, e^{-iq\cdot x}\, T\left[\mathcal{O}_i(x)\,\mathcal{O}_j(0)\right] \approx \sum_\ell U_{ij}^\ell(q)\,\mathcal{O}_\ell(0)$$

In QCD,

WLOG symmetric traceless

$$\int d^4x\, e^{-iq\cdot x}\, T\left[J^\mu(x)\, J^\nu(0)\right]$$

$$\sim \Big[\eta^{\mu\nu} f_1(q^2)\, q_{\mu_1}\cdots q_{\mu_s}\, \mathcal{O}^{\mu_1\cdots\mu_s}(0) + f_2(q^2)\, q_{\mu_1}\cdots q_{\mu_s}\, \mathcal{O}^{\mu\nu\mu_1\cdots\mu_s}(0)\Big]\Big|_{\text{projection to } q^\perp}.$$

In $q^2\to\infty$ limit, RHS dominated by the operator $\mathcal{O}^{\mu_1\cdots\mu_s}$ of lowest scaling dimension at given (operator) spin s, e.g. "twist-2 operators"

$$\mathcal{O}^{\mu_1\cdots\mu_s} = \begin{cases} \bar{\psi}_I\, \gamma^{(\mu_1}\overleftrightarrow{D}^{\mu_2}\cdots\overleftrightarrow{D}^{\mu_s)}\psi_J \Big|_{\text{traceless}} \\ F_{a\,\nu}{}^{(\mu_1}\overleftrightarrow{D}^{\mu_2}\cdots\overleftrightarrow{D}^{\mu_{s-1}} F_a{}^{\mu_s)\nu}\Big|_{\text{traceless}} \end{cases}$$

The anomalous dimension of $\mathcal{O}^{\mu_1\cdots\mu_s}$ controls the scaling behavior of $W^{\mu\nu}(q,P_N)$ at large q^2 and $\nu\ \left(= -\frac{q\cdot P_N}{m_N}\right)$:

$$W^{\mu\nu}(q,P_N) \sim \underbrace{\omega^s}_{\text{"Bjorken scaling"}} \left(\log\frac{|q|}{\Lambda_{QCD}}\right)^{-\frac{2\pi^2}{b}\tau_1}$$

where $\omega \equiv \frac{2m_N\nu}{q^2}$,

$$\gamma_{\mathcal{O}}(\mu) = 2+s+\underbrace{\tau_1 g^2(\mu)}_{\text{1-loop anomalous dim.}}+\cdots$$

chiral (axial) anomaly

warm up example: D=2 massless Dirac fermion

$$S = -\int d^2x \, \bar{\Psi} \gamma^\mu \partial_\mu \Psi .$$

here $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$, $\gamma^{\mu=0,1}$ are 2×2 matrices that obey $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $(\gamma^\mu)^\dagger = \gamma_\mu$, and $\bar{\Psi} \equiv i\Psi^\dagger \gamma^0$.

There is an obvious $U(1)$ global symmetry under which Ψ transforms as

$$\Psi \mapsto e^{i\alpha}\Psi . \qquad \alpha = \text{any constant}$$

corresponding Noether current

$$j^\mu = i\bar{\Psi}\gamma^\mu \Psi$$

obeys conservation law $\partial_\mu j^\mu = 0$

- at the classical level, $\partial_\mu j^\mu = 0$ follows from EOM
- at the quantum level, $\partial_\mu \hat{j}^\mu = 0$ holds as an operator equation, i.e.

  $$\langle \partial_\mu \hat{j}^\mu(x) \, \hat{\mathcal{O}}(y) \cdots \rangle = 0$$

  modulo possible contact terms supported at $y = x$, etc.

There is also a "chiral" symmetry which we denote by $U(1)_A$:

$$\psi \longmapsto e^{i\alpha\gamma}\psi ,$$

where $\gamma = \gamma^0\gamma^1$ is the $D=2$ chirality matrix

$(\gamma^2 = \mathbb{1},\ \gamma^\dagger = \gamma)$

Note that while the kinetic term $\bar\psi\gamma^\mu\partial_\mu\psi$ is invariant under $U(1)_A$, a mass term $\propto \bar\psi\psi$ would have broken $U(1)_A$ (hence our restriction to the massless case).

Corresponding conserved Noether current

$$j^\mu_A = i\bar\psi\gamma^\mu\gamma\psi .$$

Let us investigate the conservation law of j^μ_A in the context of the time-ordered 2-point function

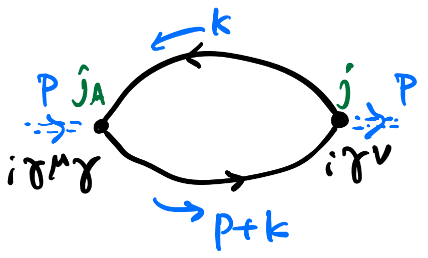
$$\langle\Omega|T\,\hat{j}^\mu_A(x)\,\hat{j}^\nu(y)|\Omega\rangle$$

↖ non-chiral $U(1)$

In momentum space,

$$\int d^2x\, e^{-ip\cdot x}\langle\Omega|T\,\hat{j}^\mu_A(x)\,\hat{j}^\nu(0)|\Omega\rangle$$

is computed by the Feynman diagram



$$= \underset{\text{fermion loop}}{-} \int \frac{d^2k}{(2\pi)^2} \frac{\mathrm{Tr}\left[i\gamma^\nu (\not{p}+\not{k})\, i\gamma^\mu \gamma \not{k} \right]}{(k^2 - i\epsilon)((p+k)^2 - i\epsilon)}$$

We will focus on

$$\partial_\mu \langle \Omega | T\, \hat{j}_A^\mu(x)\, \hat{j}^\nu(0) | \Omega \rangle \qquad \circledast$$

naively $= 0$ due to conservation of $\hat{j}_A^\mu$,

but what about contact terms supported at $y=x$?

The Fourier transform of $\circledast$ is

$$\int d^2x\, e^{-ip\cdot x}\, \partial_\mu \langle \Omega | T\, \hat{j}_A^\mu(x)\, \hat{j}^\nu(0) | \Omega \rangle$$

$$= i \int \frac{d^2k}{(2\pi)^2} \frac{\mathrm{Tr}\left(\gamma^\nu (\not{p}+\not{k}) \not{p} \gamma \not{k} \right)}{(k^2 - i\epsilon)((p+k)^2 - i\epsilon)}$$

$$\underset{\text{shift } k \to k - xp}{\overset{\text{Feyn. trick}}{=\!=}} i \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{\mathrm{Tr}\left(\gamma^\nu (\not{k} + \not{p}(1-x)) \not{p} \gamma (\not{k} - \not{p}x) \right)}{\left(k^2 + p^2 x(1-x) - i\epsilon \right)^2}$$

$$= i\int_0^1 dx \int \frac{d^2k}{(2\pi)^2}\, \frac{\boxed{\mathrm{Tr}(\gamma^\nu \not{k}\not{p}\gamma\not{k})} - x(1-x)p^2\,\boxed{\mathrm{Tr}(\gamma^\nu\gamma\not{p})}}{(k^2+p^2x(1-x)-i\epsilon)^2}$$

$\circledast$

potential log divergence?

$-2\epsilon^{\mu\nu}p_\mu$

$(\epsilon^{01}=1=-\epsilon^{01})$

- use dim reg (even in free theory!)

$D = 2-\epsilon$, but maintain $\gamma = \gamma^{01}$

$$\mathrm{Tr}(\gamma^\nu \not{k}\not{p}\gamma\not{k}) = \underbrace{k_\mu k_\rho}_{\substack{\text{replace with}\\ \frac{1}{D}k^2\eta_{\mu\rho}}} \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\not{p}\gamma)$$

$$\rightarrow \frac{k^2}{D}\mathrm{Tr}(\underbrace{\gamma_\mu\gamma^\nu\gamma^\mu}_{=(2-D)\gamma^\nu}\not{p}\gamma) = \frac{2-D}{D}k^2\,\overbrace{\mathrm{Tr}(\gamma^\nu\not{p}\gamma)}^{+2\epsilon^{\mu\nu}p_\mu =}$$

would have vanished in $D=2$!

Thus, $\circledast$ is evaluated to be

$$i\int_0^1 dx \underbrace{\int \frac{d^{2-\epsilon}k}{(2\pi)^{2-\epsilon}}}_{\text{Wick rotate}}\, \frac{\frac{\epsilon}{2-\epsilon}k^2 + x(1-x)p^2}{(k^2+p^2x(1-x)-i\epsilon)^2}\,(2\epsilon^{\mu\nu}p_\mu)$$

$$= -\int_0^1 dx\left[\overbrace{\frac{1}{2\pi}\frac{1}{\epsilon}\cdot\frac{\epsilon}{2}}^{\text{non-vanishing!!}} + \frac{1}{4\pi} + \mathcal{O}(\epsilon)\right]\cdot(2\epsilon^{\mu\nu}p_\mu)$$

$$= -\frac{1}{\pi}\epsilon^{\mu\nu}p_\mu .$$

In position space, this result amounts to

$$\partial_\mu \langle\Omega| T \hat{j}_A^\mu(x)\, \hat{j}^\nu(y) |\Omega\rangle = \frac{i}{\pi} \epsilon^{\mu\nu} \partial_\mu \delta^2(x-y).$$

[Analogous result holds for Euclidean 2-pt function

$$\left\langle \partial_\mu j_A^\mu(x)\, j^\nu(y) \right\rangle = -\frac{1}{\pi} \epsilon^{\mu\nu} \partial_\mu \delta^2(x-y).$$
]

In contrast, the analogous computation without the γ insertion gives

$$\int d^2x\, e^{-ip\cdot x}\, \partial_\mu \langle\Omega| T \hat{j}^\mu(x)\, \hat{j}^\nu(0) |\Omega\rangle$$

non-chiral U(1)

$$= -\int_0^1 dx \left[\frac{1}{2\pi} \frac{1}{\epsilon} \cdot \frac{\epsilon}{2} - \frac{1}{4\pi} + \mathcal{O}(\epsilon) \right] \cdot \mathrm{Tr}\underbrace{(\gamma^\nu \not{p})}_{= 2p^\nu}$$

$$\doteq 0$$ ← dim reg is needed to arrive at this answer!

- What is the significance of such contact terms?

- consider massless Dirac fermion coupled to a (background) U(1) gauge field A_μ.

$$S = -\int d^2x\, \bar{\psi}\, \gamma^\mu D_\mu \psi$$

$$D_\mu = \partial_\mu - i\underbrace{A_\mu}_{\text{bkgrd field}}$$

can be viewed as free massless Dirac fermion theory deformed by

$$\Delta S = \int d^2x\, \hat{j}^\mu(x) A_\mu(x) .$$

In the new theory (i.e. with bkgrd field A_μ turned on), time-ordered Green functions involving $\partial_\mu \hat{j}^\mu_A(x)$ are related to the original ($A_\mu = 0$) theory by

$$\partial_\mu \langle T\, \hat{j}^\mu_A(x) \cdots \rangle_{\text{new}}$$

$$= \underbrace{\partial_\mu \langle T\, \hat{j}^\mu_A(x)\, e^{i\int d^2y\, \hat{j}^\nu(x) A_\nu(x)}}_{\text{contains (only) contact term}} \cdots \rangle_{\text{old}}$$

$$i\, \partial_\mu \hat{j}^\mu_A(x) \int d^2y\, \hat{j}^\nu(y) A_\nu(y) \qquad \text{[time-ordering implicit]}$$

$$= -\frac{1}{\pi} \int d^2y\, \epsilon^{\mu\nu} \partial_\mu \delta^2(x-y) A_\nu(y) = \frac{1}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}(x) .$$

Time-ordered correlators of $\partial_\mu \hat{j}^\mu_A(x)$ would vanish if it weren't for this integrated contact term. Consequently, we can identify

$$\partial_\mu \hat{j}^\mu_A(x) = \frac{1}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}(x)$$

as an operator equation in the new theory.

Note: under the transformation $\psi \mapsto e^{i\alpha\gamma}\psi$, the action S with nontrivial bkgrd field A_μ is still invariant. However, $\hat{j}^\mu_A(x)$ is no longer a conserved current.

– chiral anomaly !

- Why is there anomaly ?
  - our quantization procedure, or equivalently regularization scheme, does not respect the $U(1)_A$ symmetry of the action.

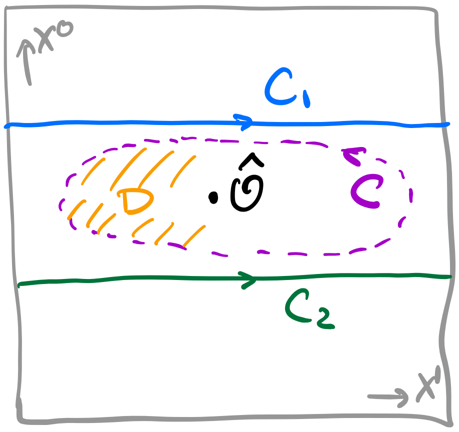
$$\int \underbrace{[D\psi]}_{\text{NOT invariant under } U(1)_A} e^{iS[\psi;A]}$$

- OTOH, the quantum violation of $U(1)_A$ is "mild", in the following sense.

  - Suppose a $U(1)$-symmetry corresponds to a conserved current $\hat{j}^\mu$, $\partial_\mu \hat{j}^\mu(x) = 0$, $\hat{Q} = \int dx^1\, j^0(x^0, x^1)$ is the Noether charge. The symmetry transformation of a local operator $\hat{\mathcal{O}}(x)$ is given by

$$\hat{\mathcal{O}}(x) \longmapsto \hat{\mathcal{O}}'(x) = U(\alpha)\, \hat{\mathcal{O}}(x) (U(\alpha))^{-1},$$

with $U(\alpha) = e^{-i\alpha \hat{Q}}$.



$$\hat{\mathcal{O}}' = T e^{-i\alpha \underbrace{\int_{C_1 - C_2} j^0 dx^1}} \cdot \hat{\mathcal{O}}$$

can replace with

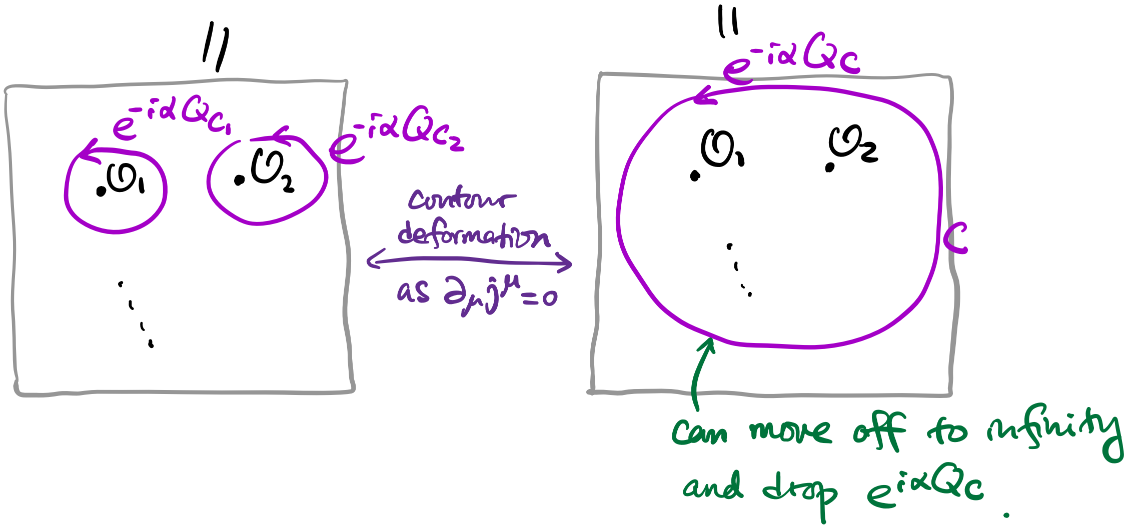
$$\hat{Q}_C = \int_C j^\mu \cdot n_\mu \, ds \overset{\text{Stokes}}{=} \int_D \partial_\mu j^\mu \, d^2x$$

[Similarly in Euclidean signature, $\hat{\mathcal{O}}' = e^{-i\alpha \hat{Q}_C} \hat{\mathcal{O}}$]

Green functions are invariant under the symmetry transformation in the sense

(either time-ordered or Euclidean)

$$\langle \mathcal{O}_1' \, \mathcal{O}_2' \cdots \rangle = \langle \mathcal{O}_1 \, \mathcal{O}_2 \cdots \rangle$$



Now for the anomalous chiral "symmetry" $U(1)_A$, which acts by

$$\psi \mapsto \psi' = e^{i\alpha\gamma}\psi .$$

$$\mathcal{O} \longmapsto \mathcal{O}'$$

$$\hat{\mathcal{O}}'(x) = T\, e^{-i\alpha \hat{Q}_C}\, \hat{\mathcal{O}}(x) .$$

$$\hat{Q}_C = \oint_C j_A^\mu \cdot n_\mu \, ds .$$

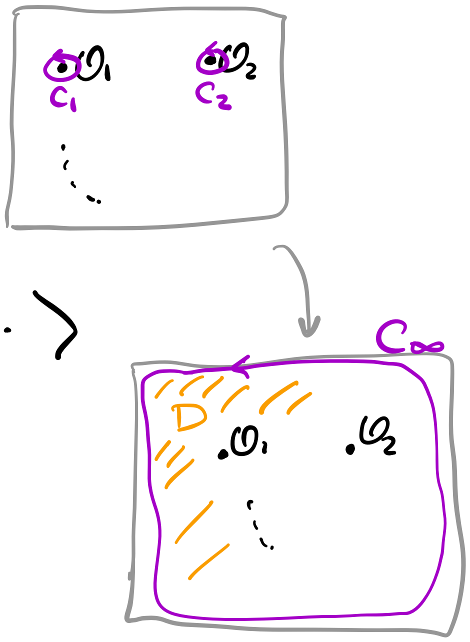
where C is an infinitesimal contour enclosing x.

The same contour deformation argument now leads to

$$\langle T\, \hat{\mathcal{O}}_1' \,\hat{\mathcal{O}}_2' \cdots \rangle$$

$$\|$$

$$\langle T\, \underbrace{e^{-i\alpha(\hat{Q}_{C_1}+\hat{Q}_{C_2}+\cdots)}}_{\substack{\| \\ e^{-i\alpha(\hat{Q}_\infty - \Delta Q)}}} \hat{\mathcal{O}}_1\, \hat{\mathcal{O}}_2 \cdots \rangle$$



where

$$\Delta Q = \hat{Q}_\infty - (\hat{Q}_{C_1} + \hat{Q}_{C_2} + \cdots)$$

$$= \int_{D = \text{entire spacetime}} d^2x\, \partial_\mu \hat{j}^\mu_A$$

$$= \frac{1}{2\pi} \int d^2x\, \epsilon^{\mu\nu} F_{\mu\nu}(x) \,.$$

$$\Rightarrow \langle T\, \mathcal{O}_1' \,\mathcal{O}_2' \cdots \rangle = e^{i\alpha\Delta Q} \langle T\, \mathcal{O}_1 \mathcal{O}_2 \cdots \rangle$$

The effect of this chiral "symmetry" transf. amounts to shifting the action by

$$\Delta S = \alpha \Delta Q$$

$$= \frac{\alpha}{2\pi} \int d^2x\, \epsilon^{\mu\nu} F_{\mu\nu}(x)$$

[D=2 analog of the Yang-Mills θ-term]

Now we turn to $D=4$ massless Dirac fermion

$$\mathcal{L} = -\bar{\psi}\gamma^\mu\partial_\mu\psi.$$

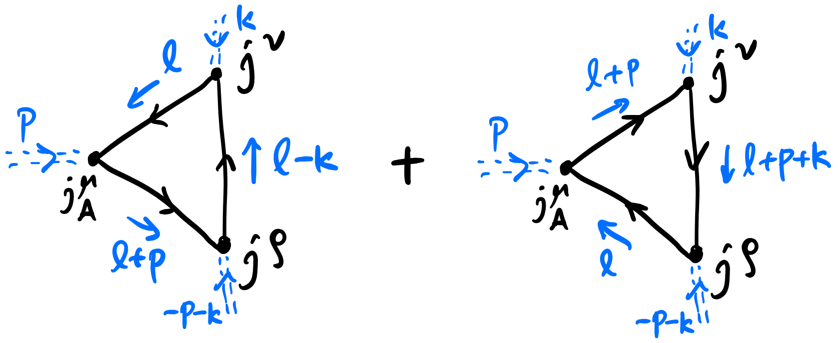
$$U(1): \quad j^\mu = i\bar{\psi}\gamma^\mu\psi$$

$$U(1)_A: \quad j^\mu_A = i\bar{\psi}\gamma^\mu\gamma_5\psi, \qquad \gamma_5 \equiv i\gamma^{0123}.$$

We will consider the 3-point function

$$\langle\Omega|T\,\hat{j}^\mu_A(x)\,\hat{j}^\nu(y)\,\hat{j}^\rho(z)\,|\Omega\rangle$$

[or its Euclidean version $\langle j^\mu_A(x)\,j^\nu(y)\,j^\rho(z)\rangle$]



particularly the contact term in

$$\int d^4x\, e^{-ip\cdot x}\int d^4y\, e^{-ik\cdot y}\,\frac{\partial}{\partial x^\mu}\langle\Omega|T\,\hat{j}^\mu_A(x)\,\hat{j}^\nu(y)\,\hat{j}^\rho(0)\,|\Omega\rangle$$

$$= \underset{\text{fermion loop}}{\underset{\uparrow}{-}}\, ip_\mu\int\frac{d^4l}{(2\pi)^4}\,\frac{\mathrm{Tr}\left[\overbrace{i\gamma^\mu\gamma_5\not{l}}^{j^\mu_A}\;\overbrace{i\gamma^\nu(\not{l}-\not{k})}^{j^\nu}\;\overbrace{i\gamma^\rho(\not{l}+\not{p})}^{j^\rho}\right]}{(l^2-i\epsilon)((l-k)^2-i\epsilon)((l+p)^2-i\epsilon)}$$

$$+\;(\nu\leftrightarrow\rho,\; k\leftrightarrow -p-k)$$

Wick rot^n

$$-i\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathrm{Tr}\left[\not{p}\,\gamma_5\not{\ell}\,\gamma^\nu(\not{\ell}-\not{k})\gamma^\rho(\not{\ell}+\not{p})\right]}{\ell^2\,(\ell-k)^2\,(\ell+p)^2}$$

$$+\,(\nu\leftrightarrow\rho\,,\;k\leftrightarrow -p-k)\qquad ⊛$$

potential log divergence ?

dim reg: $D = 4-\epsilon$

we can split the D-dimensional loop momentum ℓ^μ according to

$$\ell^\mu = \underbrace{\ell_{\parallel}^\mu}_{\text{“}D=4\text{”}} + \underbrace{\ell_\perp^\mu}_{\text{“extra }(-\epsilon)\text{ dimension”}}$$

while maintaining $\gamma_5 = i\gamma^{0123}$. Consequently, $\not{\ell}_\parallel$ anti-commutes with γ_5, whereas $\not{\ell}_\perp$ commutes with γ_5.

For instance,

$$\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathrm{Tr}\left[\boxed{\not{p}\,\gamma_5}\not{\ell}\,\gamma^\nu(\not{\ell}-\not{k})\gamma^\rho(\not{\ell}+\not{p})\right]}{\ell^2\,(\ell-k)^2\,(\ell+p)^2}$$

$$\not{p}\,\gamma_5 = \underbrace{(\not{\ell}+\not{p})\gamma_5}_{①} + \underbrace{\gamma_5\not{\ell}}_{②} - 2\gamma_5\not{\ell}_\perp$$

$$① \rightarrow \int\frac{\cancel{(\ell+p)^2}\,\mathrm{Tr}(\gamma_5\not{\ell}\,\gamma^\nu(\not{\ell}-\not{k})\gamma^\rho)}{\ell^2(\ell-k)^2\cancel{(\ell+p)^2}}$$

$$② \rightarrow \int\frac{\cancel{\ell^2}\,\mathrm{Tr}(\gamma_5\gamma^\nu(\not{\ell}-\not{k})\gamma^\rho(\not{\ell}+\not{p}))}{\cancel{\ell^2}(\ell-k)^2(\ell+p)^2}$$

} after adding $(\nu\leftrightarrow\rho,\ k\leftrightarrow -p-k)$ they cancel

This leaves

$$\circledast = 2i \int \frac{d^D\ell}{(2\pi)^D} \frac{\boxed{\mathrm{Tr}\left[\gamma_5 \not{\ell}_\perp \not{\ell} \gamma^\nu (\not{\ell}-\not{k})\gamma^\rho(\not{\ell}+\not{p})\right]}}{\ell^2(\ell-k)^2(\ell+p)^2}$$

expand $\ell = \ell_\perp + \ell_{\parallel}$

$$+ (\nu \leftrightarrow \rho,\ k \leftrightarrow -k-p)$$

$$-\ell_\perp^4 \underset{0}{\cancel{\mathrm{Tr}(\gamma_5\gamma^\nu\gamma^\rho)}} + \ell_\perp^2 \,\mathrm{Tr}\Big[\gamma_5\gamma^\nu(\not{\ell}_\parallel-\not{k})\gamma^\rho(\not{\ell}_\parallel+\not{p})$$

$$+\gamma_5\not{\ell}_\parallel\gamma^\nu\gamma^\rho(\not{\ell}_\parallel+\not{p}) + \gamma_5\not{\ell}_\parallel\gamma^\nu(\not{\ell}_\parallel-\not{k})\gamma^\rho\Big]$$

$\mathrm{Tr}(\gamma_5\gamma^{0123}) = -4i$

$$= \ell_\perp^2\cdot(-4i)\,\epsilon^{\nu\rho\alpha\beta}\Big[-(\ell_\parallel-k)_\alpha(\ell_\parallel+p)_\beta$$

$$+\ell_{\parallel\alpha}(\ell_\parallel+p)_\beta - \ell_{\parallel\alpha}(\ell_\parallel-k)_\beta\Big]$$

$$= \ell_\perp^2(-4i)\,\epsilon^{\nu\rho\alpha\beta}k_\alpha p_\beta$$

$$\circledast = 8\,\epsilon^{\nu\rho\alpha\beta}k_\alpha p_\beta \underbrace{\int\frac{d^D\ell}{(2\pi)^D}\frac{\ell_\perp^2}{\ell^2(\ell-k)^2(\ell+p)^2}}_{\text{Feynman trick}}$$

$$+(\nu\leftrightarrow\rho,\ k\leftrightarrow -k-p)$$

$$\to \int\frac{d^D\ell}{(2\pi)^D}\frac{\ell_\perp^2}{(\ell^2+\mu^2)^3} \qquad \ell_\perp^2 \longrightarrow \frac{D-4}{D}\ell^2$$

$$= \frac{1}{(2\pi)^4}\cdot 2\pi^2\cdot\frac{1}{\epsilon}\,\frac{-\epsilon}{4} + O(\epsilon)$$

$$= -\frac{1}{32\pi^2}$$

$$= \frac{1}{2\pi^2}\,\epsilon^{\nu\rho\alpha\beta}p_\alpha k_\beta .$$

That is,

$$\int d^4x\, e^{-ip\cdot x} \int d^4y\, e^{-ik\cdot y} \frac{\partial}{\partial x^\mu} \langle \Omega | T\, \hat{j}^\mu_A(x)\, \hat{j}^\nu(y)\, \hat{j}^\rho(0) | \Omega \rangle$$

$$= \frac{1}{2\pi^2} \epsilon^{\nu\rho\alpha\beta} p_\alpha k_\beta .$$

Fourier transforming back to position space,

$$\Rightarrow \quad \partial_\mu \langle \Omega | T\, \hat{j}^\mu_A(x)\, \hat{j}^\nu(y)\, \hat{j}^\rho(0) | \Omega \rangle$$

$$= -\frac{1}{2\pi^2} \epsilon^{\nu\rho\alpha\beta} \partial_\alpha \delta^4(x)\, \partial_\beta \delta^4(y) .$$

Now consider a new theory: the massless Dirac fermion coupled to a (background) U(1) gauge field A_μ:

$$\Delta S = \int d^4x\, A_\mu(x)\, \hat{j}^\mu(x)$$

$$\partial_\mu \langle T\, \hat{j}^\mu_A(x) \cdots \rangle_{\text{new}}$$

$$= \underbrace{\partial_\mu \langle T\, \hat{j}^\mu_A(x)\, e^{i\int d^4y\, A_\nu(y)\, \hat{j}^\nu(y)}}_{\text{contains the (only) contact term}} \cdots \rangle_{\text{old}}$$

(time-ordering implicit)

$$\partial_\mu \hat{j}^\mu_A(x) \left(-\frac{1}{2}\int d^4y\, d^4z\; \hat{j}^\nu(y)\, \hat{j}^\rho(z)\, A_\nu(y)\, A_\rho(z)\right)$$

$$= \frac{1}{4\pi^2}\epsilon^{\nu\rho\alpha\beta}\partial_\alpha A_\nu(x)\, \partial_\beta A_\rho(x)$$

$$= -\frac{1}{16\pi^2}\epsilon^{\nu\rho\alpha\beta} F_{\mu\nu} F_{\rho\sigma}.$$

Once again, the chiral "symmetry" is anomalous: we have the operator equation

$$\partial_\mu \hat{j}^\mu_A(x) = -\frac{1}{16\pi^2}\epsilon^{\nu\rho\alpha\beta} F_{\mu\nu} F_{\rho\sigma}.$$

Remark: even though we simply extracted this anomaly from a specific contact term that arise at order $\mathcal{O}(A^2)$, the result is in fact exact, and can be reproduced by inspecting the non-invariance of the (suitably regularized) path integral measure $[D\psi]$ with respect to the chiral "symmetry" transformation.

[See Weinberg section 22.2]

So far, we have seen that the chiral symmetry of a classical action of massless fermions may be rendered a non-symmetry in the quantum theory when the fermion fields are coupled to gauge fields (either as background or dynamical fields).

This notion of anomaly does not indicate any inconsistency of the QFT. Rather, it implies that the associated Noether current $\hat{j}_A^\mu$, while still a well-defined local operator, is not conserved, but rather obeys a modified conservation relation, e.g.

$$\partial_\mu \hat{j}_A^\mu = -\frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}.$$

There is a different notion of anomaly, called " 't Hooft anomaly", which refers to a true global symmetry of a QFT that would become anomalous if coupled to a dynamical gauge field via

$$\Delta S = \int d^Dx \; j^\mu(x) A_\mu(x),$$

leading to an inconsistent gauge theory (that doesn't exist as a QFT).

As an example, consider again in $D=4$ a theory of free massless Dirac fermion:

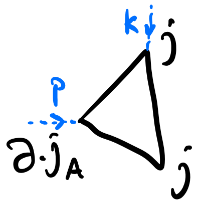
$$\mathcal{L} = -\bar\psi \partial\!\!\!/ \psi,$$

with symmetry currents

$$j_\mu = i\bar\psi\gamma_\mu\psi, \quad j^A_\mu = i\bar\psi\gamma_\mu\gamma_5\psi.$$

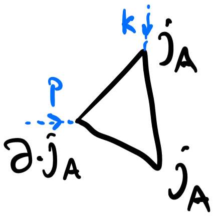
Previously, we analyzed (time-ordered)

$$\langle \partial_\mu j^\mu_A(x) \, j^\nu(y) \, j^\rho(z) \rangle$$



$$\sim \int \frac{d^D\ell}{(2\pi)^D} \frac{\mathrm{Tr}\left(\gamma_5 \ell\!\!\!/_\perp \ell\!\!\!/ \gamma^\nu (\ell\!\!\!/ - k\!\!\!/) \gamma^\rho (\ell\!\!\!/ + p\!\!\!/)\right)}{\ell^2 (\ell-k)^2 (\ell+p)^2}$$

Now let us consider $\langle \partial_\mu j^\mu_A(x) \, j^\nu_A(y) \, j^\rho_A(z) \rangle$



$$\sim \int \frac{d^D\ell}{(2\pi)^D} \frac{\mathrm{Tr}\left(\gamma_5 \ell\!\!\!/_\perp \ell\!\!\!/ \gamma^\nu \gamma_5 (\ell\!\!\!/ - k\!\!\!/) \gamma^\rho \gamma_5 (\ell\!\!\!/ + p\!\!\!/)\right)}{\ell^2 (\ell-k)^2 (\ell+p)^2}$$

$$\mathrm{Tr}\left(\gamma_5 \not{\ell}_\perp \not{\ell}\, \gamma^\nu \gamma_5 (\not{\ell}-\not{k})\, \gamma^\rho \gamma_5 (\not{\ell}+\not{p})\right)$$

$$= \ell_\perp^2 (-4i)\, \epsilon^{\nu\rho\alpha\beta} \Big[-(\ell_{\parallel}-k)_\alpha (\ell_{\parallel}+p)_\beta$$

$$+ (-1)\, \ell_{\parallel\alpha} (\ell_{\parallel}+p)_\beta - \ell_{\parallel\alpha} (\ell_{\parallel}-k)_\beta \Big]$$

$$= \ell_\perp^2 (-4i)\, \epsilon^{\nu\rho\alpha\beta} (k-2\ell)_\alpha\, p_\beta$$

write $\ell_\alpha \equiv \underbrace{\frac{\ell_\alpha + (\ell-k)_\alpha + (\ell+p)_\alpha}{3}} + \frac{(k-p)_\alpha}{3}$

$$\int d^D\ell \, \frac{\ell_\perp^2}{\ell^2 (\ell-k)^2 (\ell+p)^2} (\cdots)$$

is invariant under permutation on $(k, p, -k-p)$ ⇝ no finite contribution which must be linear in k, p.

Thus we can make the replacement

$$\epsilon^{\nu\rho\alpha\beta} (k-2\ell)_\alpha p_\beta \rightarrow \epsilon^{\nu\rho\alpha\beta} \left(k - 2\,\frac{k-p}{3}\right)_\alpha p_\beta$$

$$= \frac{1}{3} \cdot \epsilon^{\nu\rho\alpha\beta} k_\alpha p_\beta ,$$

giving the result

$$\partial_\mu \langle \Omega | T \hat{j}_A^\mu(x)\, \hat{j}_A^\nu(y)\, \hat{j}_A^\rho(z) | \Omega \rangle$$

$$= \frac{1}{3} \cdot \left(-\frac{1}{2\pi^2}\right) \epsilon^{\nu\rho\alpha\beta}\, \partial_\alpha \delta^4(x-z)\, \partial_\beta \delta^4(y-z) .$$

Interpretation: if we couple the fermion fields to a background U(1) gauge field via the axial current:

$$\Delta S = \int d^4x \; j_A^{\mu}(x) \, A_\mu(x)$$

Consider the effection action $\Gamma[A]$

$$e^{i\Gamma[A]} = \int D\psi \; e^{i(S_0[\psi] + \Delta S[\psi, A])}$$

under a gauge variation,

$$\Gamma[A + d\zeta] - \Gamma[A]$$

$$= \int d^4x \; \partial_\mu \zeta(x) \left\langle j_A^{\mu}(x) \right\rangle_{new}$$

$$= - \int d^4x \; \zeta(x) \left\langle \partial_\mu j_A^{\mu}(x) \; e^{i\int d^4y \, j_A^{\nu}(y) A_\nu(y)} \right\rangle_{old}$$

$$= \frac{1}{2} \int d^4x \; \zeta(x) \int d^4y \, d^4z \;\; A_\nu(y) \, A_\rho(z)$$

$$\cdot \left\langle \partial_\mu j_A^{\mu}(x) \; j_A^{\nu}(y) \; j_A^{\rho}(z) \right\rangle$$

$$= \int d^4x \; \zeta(x) \cdot \frac{1}{3} \cdot \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x) \, .$$

- $\Gamma[A]$ is NOT gauge invariant, cannot promote A_μ to a dynamical gauge field !

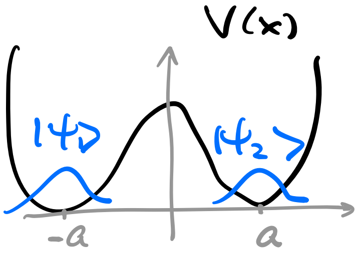
- Nonetheless, the 't Hooft anomaly is a useful characterization of the symmetry generated by $\hat{j}_A$ intrinsic to the original (ungauged) QFT.

"Spontaneously symmetry breaking"

- it is possible for the vacuum state of a QFT to not be invariant with respect to global symmetry.

- slightly surprising considering the following toy model : QM with 1 d.o.f.

$$H = p^2 + V(x)$$



$V(x) = V(-x)$ with two degenerate minima $x = \pm a$

H-eigenstates $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle \pm |\psi_2\rangle)$

Typically, $|\psi_+\rangle$ is the true ground state ($\mathbb{Z}_2$-invariant), while $|\psi_-\rangle$ has exponentially small excitation energy : $\Delta E \sim e^{-\frac{1}{\hbar}S_0}$,

S_0 is the action of a Euclidean "tunnelling" solution $x(\tau)$:

$$-\partial_\tau^2 x + V''(x) = 0$$
$$x(\pm\infty) = \pm a .$$

- In a general QFT with this type of potential, however, tunnelling between different minima is suppressed by $e^{-\frac{1}{\hbar}S_0}$,
  $S_0 \propto V$ the spatial volume $(=\infty)$.

- Suppose there is a $U(1)$ global symmetry represented by the unitary operator

$$U(\alpha) = e^{-i\alpha\hat{Q}}.$$

(assuming $[U(\alpha), \hat{P}_\mu] = 0$)

If a vacuum state $|\Omega\rangle$ is not invariant,

i.e. $U(\alpha)|\Omega\rangle \neq |\Omega\rangle$

$$\Longleftrightarrow \quad \hat{Q}|\Omega\rangle \neq 0,$$

we obtain a family of vacua by the symmetry transformation

$$U(\alpha)|\Omega\rangle \equiv |\Omega, \alpha\rangle.$$

<u>Claim</u>: there is a basis of vacua $|\Omega_i\rangle$ such that for any local operator $\hat{A}(x)$,

$$\langle\Omega_i|\hat{A}(x)|\Omega_j\rangle = 0 \quad \text{unless } i=j$$

"superselection sectors"

This is a consequence of causality:

for any two local operators $\hat{A}(x)$, $\hat{B}(y)$,

$$\langle\Omega_i|\,\hat{A}(x)\,\hat{B}(y)\,|\Omega_j\rangle$$

$$= \sum_k \langle\Omega_i|\hat{A}(0)|\Omega_k\rangle\langle\Omega_k|\hat{B}(0)|\Omega_j\rangle$$

$$+ \underbrace{\boxed{\int_{\substack{\text{excited}\\\text{states}}} d\alpha\, \langle\Omega_i|\hat{A}(0)|\alpha\rangle\langle\alpha|\hat{B}(0)|\Omega_j\rangle\, e^{iP_\alpha\cdot(x-y)}}}_{0}$$

in the limit $|x-y|\to\infty$

For spacelike separation, i.e. $(x-y)^2>0$, $[\hat{A}(x),\hat{B}(y)]=0$. taking $(x-y)^2\to\infty$ leads to

$$A_{ik}B_{kj} = B_{ik}A_{kj},$$

where $A_{ij}\equiv\langle\Omega_i|\hat{A}(0)|\Omega_j\rangle$.

$\Rightarrow$ can diagonalize the "vacuum matrices" (A_{ij}) and (B_{ij}) simultaneously

More generally, can find a basis of vacua $|\Omega_i\rangle$ such that A_{ij} is diagonal for any local operator $\hat{A}(x)$. Cluster property of Green functions holds in each vacuum $|\Omega_i\rangle$.

Suppose the vacuum state $|\Omega\rangle$ is not invariant with respect to symmetry generated by

$$\hat{Q} = \int d^{D-1}\vec{x}\, \hat{j}^0(t, \vec{x}),$$

i.e. $\hat{Q}|\Omega\rangle \neq 0.$

Claim: $\hat{j}^\mu(x)|\Omega\rangle$ contains the 1-particle state of a massless scalar particle, known as the "Nambu-Goldstone boson".

One way to see this is to consider a charged scalar field operator $\hat{\phi}(x)$, say of charge -1:

$$[\hat{Q}, \hat{\phi}(x)] = -\hat{\phi}(x).$$

[Note: $\hat{\phi}^\dagger(x)$ has the opposite charge
$$[\hat{Q}, \hat{\phi}^\dagger(x)] = \hat{\phi}^\dagger(x)$$]

and suppose that $\langle\phi\rangle \equiv \langle\Omega|\hat{\phi}(x)|\Omega\rangle$
$\neq 0$

[Note: if the symmetry is preserved by $|\Omega\rangle$, i.e. $\hat{Q}|\Omega\rangle = 0$, then
$$\langle\phi\rangle = -\langle\Omega|[\hat{Q}, \hat{\phi}(0)]|\Omega\rangle = 0$$]

Then by the same derivation of Källén -Lehmann spectral representation, we have

$$\langle\Omega|[\hat{j}^\mu(x), \hat{\phi}(y)]|\Omega\rangle$$

$$= \frac{\partial}{\partial x_\mu}\int_0^\infty d\mu^2\, \rho_\phi(\mu^2)\,[\Delta_+(x-y;\mu^2) - \Delta_+(y-x;\mu^2)]$$

where $\Delta_+(x;\mu^2) \equiv \int \frac{d^Dk}{(2\pi)^{D-1}} e^{ik\cdot x}\,\theta(k^0)\,\delta(k^2+\mu^2)$ as before, and $\rho_\phi(\mu^2)$ is defined via

$$\int d\alpha\, \langle\Omega|\hat{j}^\mu(0)|\alpha\rangle\langle\alpha|\hat{\phi}(0)|\Omega\rangle\,\delta^D(k-P_\alpha)$$

$$= \frac{i}{(2\pi)^{D-1}}\, k^\mu\, \rho_\phi(-k^2)\,\theta(k^0),$$

$$\int d\alpha\, \langle\Omega|\hat{\phi}(0)|\alpha\rangle\langle\alpha|\hat{j}^\mu(0)|\Omega\rangle\,\delta^D(k-P_\alpha)$$

$$= \frac{i}{(2\pi)^{D-1}}\, k^\mu\, \tilde{\rho}_\phi(-k^2)\,\theta(k^0),$$

and $\tilde{\rho}_\phi = -\rho_\phi$ by causality $\left(\begin{array}{l}[\hat{j}(x), \hat{\phi}(y)] = 0 \\ \text{for } (x-y)^2 > 0\end{array}\right)$

Now $\partial_\mu \hat{j}^\mu(x) = 0 \Rightarrow \mu^2\rho_\phi(\mu^2) = 0.$

* This does NOT imply $\rho_\phi = 0$!

$$\int d^{D-1}\vec{x}\,\langle\Omega|[\hat{j}^0(x),\hat{\phi}(y)]|\Omega\rangle = \langle\Omega|[Q,\hat{\phi}]|\Omega\rangle = -\langle\phi\rangle$$

$$\Vert \qquad \text{free to set } x^0 = y^0$$

$$\int d^{D-1}\vec{x}\int d\mu^2\,\rho_\phi(\mu^2)\,\underbrace{\left(-\frac{\partial}{\partial x^0}\right)\left[\Delta_+(x-y;\mu^2)-\Delta_+(y-x;\mu^2)\right]\Big|_{x^0=y^0}}_{\overset{\Vert}{i\,\delta^{D-1}(\vec{x}-\vec{y})}}$$

$$= i\int d\mu^2\,\rho_\phi(\mu^2).$$

$$\Rightarrow \quad \rho_\phi(\mu^2) = i\langle\phi\rangle\,\delta(\mu^2).$$

This contribution to $\rho_\phi(\mu^2)$ can only come from some state $|\alpha\rangle$ with

$$\underbrace{\langle\Omega|\hat{j}^\mu(0)|\alpha\rangle}_{\neq 0}\;\underbrace{\langle\alpha|\hat{\phi}(0)|\Omega\rangle}_{\neq 0} \neq 0.$$

$\Rightarrow$ $|\alpha\rangle$ cannot be vacuum

$\Rightarrow$ $|\alpha\rangle$ can only be a scalar (spinless) particle

$$|\alpha\rangle \rightsquigarrow |B,\vec{k}\rangle, \qquad k^0 = |\vec{k}|$$

"Nambu-Goldstone boson"

Lorentz invariance

constant

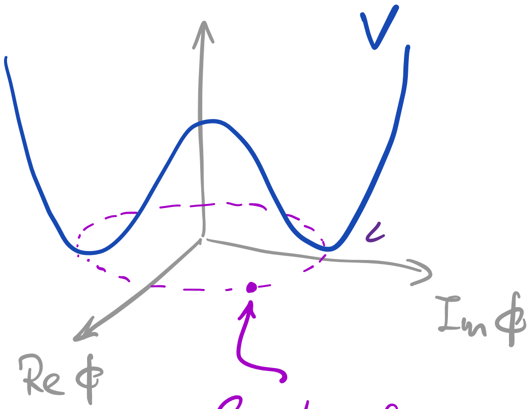
$$\Rightarrow \quad \langle\Omega|\hat{j}^\mu(0)|B,\vec{k}\rangle = i\,\frac{F\cdot k^\mu}{(2\pi)^{\frac{D-1}{2}}\sqrt{2k^0}}$$

$$\langle B,\vec{k}|\hat{\phi}(0)|\Omega\rangle = \frac{Z_\phi^{\frac{1}{2}}}{(2\pi)^{\frac{D-1}{2}}\sqrt{2k^0}}$$

with $\boxed{\,iF\cdot Z_\phi^{\frac{1}{2}} = -\langle\phi\rangle\,}$

A simple model of SSB

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - V(|\phi|)$$



a family of U(1)-sym breaking vacua with $\langle \phi \rangle \neq 0$

U(1) current

$$j_\mu = -i \phi^* \overleftrightarrow{\partial_\mu} \phi$$

Reparameterize the field variable:

$$\phi(x) \equiv r(x)\, e^{i\theta(x)}$$

$$\Rightarrow \mathcal{L} = -\partial_\mu r \partial^\mu r - V(r) - r^2 \partial_\mu \theta \partial^\mu \theta$$

$$j_\mu = 2r^2 \partial_\mu \theta,$$

U(1) charge $Q = \int d\vec{x}\, j^0 = -\int d\vec{x}\, \underbrace{2r^2\dot{\theta}}_{\Pi_\theta}$

$$[\hat{Q}, \hat{\theta}(x)] = i,$$

$\hat{Q}$ generates shift symmetry on $\theta(x)$

$V(r)$ minimized at $r = v\ (>0)$.

write $r(x) \equiv v + \rho(x)$.

ρ massive

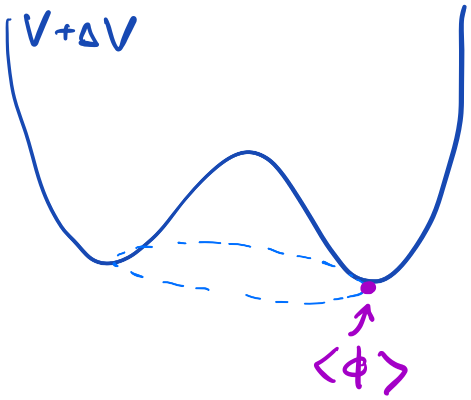
$$\mathcal{L} = -\partial_\mu\rho\,\partial^\mu\rho - \tfrac{1}{2}V''(v)\,\rho^2 + \text{const}$$
$$- v^2\,\partial_\mu\theta\,\partial^\mu\theta + (\text{interaction terms involving } \rho,\ \partial_\mu\theta)$$

θ massless, creates NG boson

A key feature of SSB: if we deform the theory in such a way that the symmetry is explicitly broken slightly, e.g. add to scalar potential

$\Delta V(\phi) = -\varepsilon\,\mathrm{Re}(\phi)$, $\varepsilon > 0$,

there is a unique vacuum no matter how small ε is.



In the limit $\varepsilon \to 0$, this vacuum state approaches a specific vacuum in the family of degenerate vacua of the undeformed theory (in this example, it is the one in which $\langle\phi\rangle$ is real and positive)

Back to QCD with massless quarks

(non-anomalous) global symmetry

$$U(1)_{\text{non-chiral}} \times \underset{\substack{\rotatebox{90}{$\in$} \\ U_L}}{SU(N_f)_L} \times \underset{\substack{\rotatebox{90}{$\in$} \\ U_R}}{SU(N_f)_R}$$

$$\psi_{IL} \to (U_L)_I{}^J \psi_{JL}, \quad \psi_{IR} \to (U_R)_I{}^J \psi_{JR}.$$

It is believed that the vacuum of QCD is only invariant under

$$U(1)_{\text{non-chiral}} \times SU(N_f)_{\text{non-chiral}}$$

An operator that transforms nontrivially under the chiral $SU(N_f)$, whose vacuum expectation value could indicate the spontaneous breaking of chiral symmetry, is the "quark mass operator"

$$\Phi_J{}^I = \bar{\psi}_L^I \psi_{JR}$$

$$\langle \Phi_J{}^I \rangle \equiv A_J{}^I \neq 0$$

Under $SU(N_f)_L \underset{\rotatebox{90}{$\in$}}{\times} SU(N_f)_R$

(U_L, U_R) represented by unitary operator $\hat{U}$

$$\hat{\Phi}_J{}^I \longmapsto \hat{U}\,\hat{\Phi}_J{}^I\,\hat{U}^{-1}$$
$$= (U_R)_J{}^M\,\hat{\Phi}_M{}^K\,(U_L^\dagger)_K{}^I$$

Given a vacuum state $|\Omega\rangle$,

$\hat{U}|\Omega\rangle \equiv |\Omega\rangle_U$ is a new vacuum state.

$$ {}_U\langle\Omega|\,\hat{\Phi}_J{}^I\,|\Omega\rangle_U = \langle\Omega|\,\hat{U}^{-1}\,\hat{\Phi}_J{}^I\,\hat{U}\,|\Omega\rangle$$
$$= (U_R^{-1}\cdot A\cdot U_L)_J{}^I .$$

We can choose (U_L, U_R) such that $U_R^{-1} A\, U_L$ is diagonal, and thus WLOG assume that A is itself diagonal.

Assuming that the vacuum is invariant under a "non-chiral" $SU(N_f)$, $A_J{}^I$ must be proportional to identity matrix, i.e. $A_J{}^I = A\,\delta_J{}^I$.

In this vacuum, the unbroken non-chiral $SU(N_f)$ corresponds to (U_L, U_R) with $U_L = U_R$.

Remark 1: the non-chiral $SU(N_f)$ cannot be spontaneous broken in the absence of θ-angle. This can be argued by deforming to the theory with nonzero quark masses, and consider the Euclidean path integral computation of the vev

$$\langle \Phi_J{}^I \rangle = \frac{1}{Z}\int \underbrace{[DA]_{G.F.}\, e^{-S_{YM}[A]}}_{\substack{\text{defined w/ UV regulator} \\ \text{real, positive when } \theta=0}}$$

$$\times \underbrace{\int D\psi\, e^{-\int \bar{\psi}(\not{D}_A+m)\psi}\, \Phi_J{}^I}_{= \underbrace{\det(\not{D}_A+m)}_{\text{positive!}} \cdot \underbrace{\langle \bar{\psi}_L{}^I \psi_{JR} \rangle_A}_{\substack{\text{trace of fermion propagator} \\ \text{in background gauge field}}}} .$$

eigenvalues of Euclidean $\not{D}_A$ come in pairs $\pm i\lambda$, $\lambda \in \mathbb{R}$ → positive!

can be shown to uniformly approach $\propto \delta_J{}^I$ in the limit where all quark masses are equal for any Euclidean gauge field configuration

[Vafa, Witten 1984]

Remark 2: $\langle \Phi_J{}^I \rangle = A\, \delta_J{}^I$ may a priori be complex. One may attempt to rotate its phase with axial $U(1)_A$ transformation on ψ_L, ψ_R. but recall that $U(1)_A$ is anomalous, and that a $U(1)_A$ "non-symmetry" rotation amounts to shifting the θ-angle.

To summarize, QCD with N_f massless Dirac fermions (quarks) has global symmetry

$$U(1) \times \underbrace{SU(N_f)_L \times SU(N_f)_R}_{\substack{\text{only the non-chiral} \\ \text{"diagonal" } SU(N_f) \\ \text{is preserved by vacuum}}}$$

the broken chiral $SU(N_f)$ generators are in correspondence with $N_f^2 - 1$ species of Nambu-Goldstone bosons, a.k.a. "pions".

In particular, since $\Phi_J{}^I \equiv \bar{\psi}_L^I \psi_{JR}$ transforms nontrivially under the chiral $SU(N_f)$, $\hat{\Phi}_J{}^I |\Omega\rangle$ contains (i.e. has nonzero overlap with) 1-pion states.

We now seek an "effective field theory" of pions, in which pions are created by fundamental scalar fields that appear as field variables in an (effective) Lagrangian.

Assume: $SU(N_f) \times SU(N_f)$ global symmetry spontaneously broken to diagonal $SU(N_f)$.

Consider:

an $SU(N_f)$-valued scalar field $U(x)$ transforms under $(U_L, U_R) \in SU(N_f) \times SU(N_f)$ by

$$U(x) \longmapsto U_R\, U(x)\, U_L^{-1} .$$

SSB by vev

$$\langle \Omega | U(x) | \Omega \rangle = U_0 \in SU(N_f).$$

The symmetries preserved by $|\Omega\rangle$ correspond to (U_L, U_R) that obey

$$U_R\, U_0\, U_L^{-1} = U_0 ,$$

$$\text{i.e.} \quad U_L = U_0^{-1}\, U_R\, U_0$$

This is what we mean by the diagonal $SU(N_f)$.

We will assume that there is a Wilsonian effective Lagrangian of $U(x)$, that comes with a floating cutoff Λ along with a suitable regularization scheme. As in any Wilsonian eff. Lagrangian, there can be infinitely many couplings that involve arbitrarily many derivatives & powers of fields, that are Λ-dependent.

We expect that such a description is valid so long as

$$\Lambda \lesssim \Lambda_{QCD},$$

so that the pions are the only possible excitations at energies below Λ.

In the low energy limit ⟷ slowly varying fields, Lagrangian is dominated by terms with fewest number of derivatives. Also, no potential term is possible by our symmetry assumption.

At 2-derivative order, the only possible Lagrangian is

$$\mathcal{L} = -\frac{\overset{\text{some constant}}{F^2}}{16} \operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)$$

Possible higher order terms:

$$\left(\operatorname{Tr} \partial_\mu U \partial^\mu U^\dagger\right)^2 ,$$

$$\operatorname{Tr}\left(\partial_\mu U \partial_\nu U^\dagger\right) \operatorname{Tr}\left(\partial^\mu U \partial^\nu U^\dagger\right),$$

$$\operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger\right),$$

. . - - . -- - - .

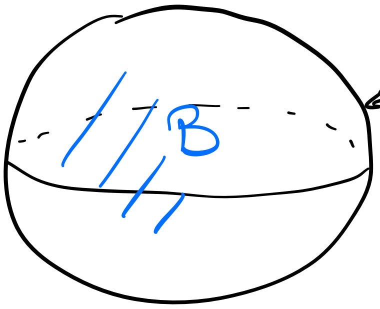
Remark: $SU(N_f) \times SU(N_f)$ -invariant terms can be built out of traces of products of derivatives of U and $U^\dagger$ alternatingly.

There is an exceptional term, known as the Wess-Zumino-Witten term, that is not of the above form, that exists for $N_f \geq 3$. It takes the form

$$\Delta S_{WZW} = n \cdot \int_B \omega$$

$$\omega = -\frac{i}{240\pi^2}\,\mathrm{Tr}\left((U^{-1}dU)^5\right)$$

extended to a 5-dim'l auxiliary space B.



spacetime = ∂B

$\int_B \omega$ turns out to be well-defined up to shifts by integer multiple of 2π, for $N_f \geq 3$, in which case $e^{i\Delta S_{wzw}}$ is well-defined for integer n. Consideration of gauging $U(1) \subset SU(N_f)_{diag}$ which leads to anomaly in chiral $SU(N_f)$ can be used to fix $n = N$.

colors

[See Weinberg sections 19.8 & 22.7]

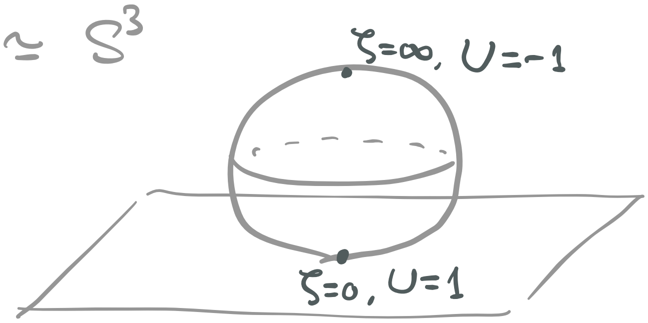
Specializing to $N_f = 2$ case: $U(x) \in SU(2)$

as seen in HW, we can parameterize $U(x)$ in terms of unconstrained field variables

$$\vec{\zeta}(x) = (\zeta_1(x), \zeta_2(x), \zeta_3(x)) ,$$

$$U(x) = \frac{(1 + i\,\vec{\zeta}\cdot\vec{\sigma})^2}{1+\zeta^2} .$$

[$\vec{\zeta}$ can be viewed as stereographic coordinates on $SU(2) \simeq S^3$



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It is straightforward to verify that the chiral Lagrangian is expressed in terms of $\vec{\zeta}$ as

$$\mathcal{L} = -\frac{F^2}{16}\,\mathrm{Tr}(\partial_\mu U\, \partial^\mu U^{-1})$$

$$= -\frac{F^2}{2}\,\frac{\partial_\mu\vec{\zeta}\cdot\partial^\mu\vec{\zeta}}{(1+\zeta^2)^2} .$$

"non-linear sigma model"

Under infinitesimal $SU(2)_L \times SU(2)_R$ symmetry transformation

$$U \longmapsto U_R\, U\, U_L^{-1}, \qquad U_{L/R} = e^{i\vec{\theta}_{L/R}\cdot\frac{\vec{\sigma}}{2}}$$

$$\qquad = U + i\vec{\theta}_R\cdot\frac{\vec{\sigma}}{2}\,U - i\,U\,\vec{\theta}_L\cdot\frac{\vec{\sigma}}{2}$$

define $\quad \vec{\theta}_{V/A} \equiv \frac{1}{2}\left(\vec{\theta}_R \pm \vec{\theta}_L\right)$.

$$\Rightarrow \quad \delta\vec{\xi} = \underbrace{-\vec{\theta}_V\times\vec{\xi}}_{\substack{\text{non-chiral sym}\\ \text{acts linearly}}} + \underbrace{\frac{1}{2}\vec{\theta}_A(1-\xi^2) + \vec{\xi}(\vec{\theta}_A\cdot\vec{\xi})}_{\substack{\text{chiral symmetry}\\ \text{acts nonlinearly}}}$$

It is convenient to work with the quantity

$\vec{D}_\mu \equiv \dfrac{\partial_\mu\vec{\xi}}{1+\xi^2}$ which transforms as

$$\delta\vec{D}_\mu = -\vec{\theta}_V\times\vec{D}_\mu - (\vec{\xi}\times\vec{\theta}_A)\times\vec{D}_\mu .$$

The higher derivative corrections to the chiral Lagrangian can be organized as

$$\mathcal{L} = -\frac{F^2}{2}\vec{D}_\mu\cdot\vec{D}^\mu - \frac{C_4}{4}(\vec{D}_\mu\cdot\vec{D}^\mu)^2$$

$$\qquad - \frac{C_4'}{4}(\vec{D}_\mu\cdot\vec{D}_\nu)(\vec{D}^\mu\cdot\vec{D}^\nu) + \cdots$$

We can put the kinetic term in canonical normalization by redefining the "pion" field

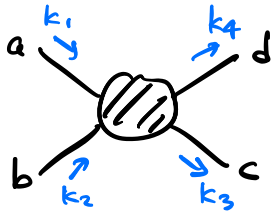
$$\vec{\pi} \equiv F\vec{\xi} \, ,$$

and the effective Lagrangian is expanded in $\vec{\pi}$ as

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\vec{\pi})^2 + \frac{1}{F^2}\vec{\pi}^2(\partial_\mu\vec{\pi})^2 + \cdots$$
$$- \frac{c_4}{4F^4}((\partial_\mu\vec{\pi})^2)^2 - \frac{c_4'}{4F^4}(\partial_\mu\vec{\pi}\cdot\partial_\nu\vec{\pi})^2 + \cdots$$

Note that F has mass dimension 1, and the dimensionless effective coupling would be $\sim \frac{\text{energy}}{F}$.

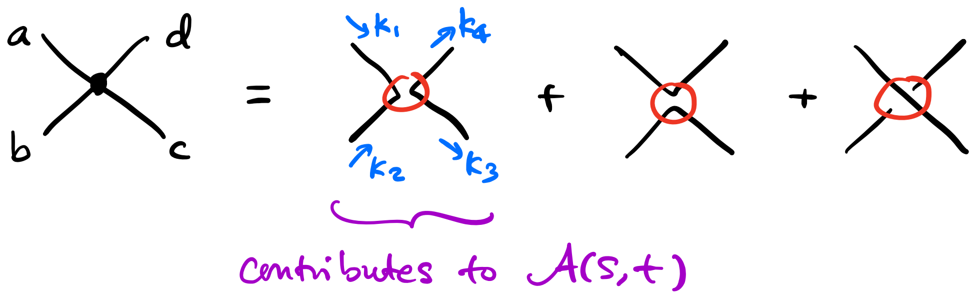
Let us inspect the $2\to2$ scattering amplitude of pions in this framework, aka "chiral perturbation theory":



$s = -(k_1+k_2)^2$
$t = -(k_1-k_4)^2$
$u = -(k_1-k_3)^2$

$$\mathcal{A}_{cd|ab} = \delta_{ab}\delta_{cd}\mathcal{A}(s,t)$$
$$+ \delta_{ac}\delta_{bd}\mathcal{A}(u,t)$$
$$+ \delta_{ad}\delta_{bc}\mathcal{A}(t,s)$$

- at order $\frac{E^2}{F^2}$, the only contribution comes from the tree diagram



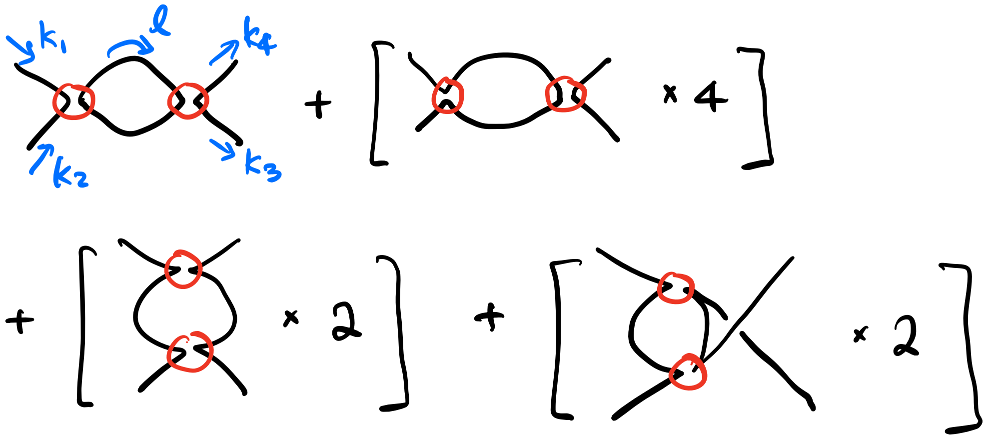
$$\frac{4}{F^2}(-k_1\cdot k_2 - k_3\cdot k_4) = \frac{4s}{F^2}.$$

- Next, at order $\frac{E^4}{F^4}$, there are two types of contributions:

I. 1-loop diagram built out of 2-deriv vertices

II. tree diagram built out of 4-deriv vertices

It suffices to consider diagrams that contribute to $\delta_{ab}\delta_{cd}A(s,t)$:



e.g. the first diagram above evaluates to

sum/ loop index $\quad$ Wilsonian cutoff

$$\left(\frac{4}{F^2}\right)^2 \cdot \frac{\overset{\downarrow}{3}}{\underset{\nearrow}{2}} \int^{\Lambda} \frac{d^4\ell_E}{(2\pi)^4} \frac{(-k_1\cdot k_2-\ell\cdot(k_1+k_2-\ell))(-k_3\cdot k_4-\ell\cdot(k_1+k_2-\ell))}{\ell^2\,(k_1+k_2-\ell)^2}$$

Symmetry factor

the loop momentum integral contains

- power "divergence" $\sim \Lambda^4 \,,\ \Lambda^2 s$
- log "divergence" $\sim s^2 \log\frac{\Lambda^2}{-s}$

In the Wilsonian framework, these are not divergences as we take Λ to be finite ($\lesssim \Lambda_{QCD}$).

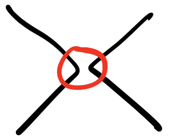
However, we should adopt a Wilsonian cutoff scheme that respects the underlying (global) symmetries. A priori, a simple cutoff on the Fourier modes of $\vec{\pi}(x)$ may not respect the nonlinearly realized chiral SU(2).

For instance, the appearance of Λ^4 contribution to $\mathcal{A}(s,t)$ would generate an effective potential that violates the chiral SU(2) symmetry. This problem can be fixed by either working with a symmetry-preserving scheme (e.g. dim reg), or by including suitable local counter terms to cancel the power "divergences".

Modulo the power "divergences", which should be cancelled in the "correct" scheme or equivalently with the appropriate counter terms, the 1-loop contribution to $\mathcal{A}(s,t)$ at order $\frac{E^4}{F^4}$ is

$$\frac{1}{F^4}\left[\frac{s^2}{2\pi^2}\log\frac{\Lambda^2}{-s}+\frac{u^2-s^2+3t^2}{12\pi^2}\log\frac{\Lambda^2}{-t}\right.$$
$$\left.+\frac{t^2-s^2+3u^2}{12\pi^2}\log\frac{\Lambda^2}{-u}\right].$$

In addition, we also have at this order the contribution from tree diagrams involving the 4-derivative vertices with coefficients C_4, C_4':



$$-\frac{C_4}{2}s^2-\frac{C_4'}{2}(t^2+u^2).$$

We can now absorb the remaining Λ-dependence into C_4, C_4' via

$$C_4 = C_4(\mu)+\frac{2}{3\pi^2}\log\frac{\Lambda^2}{\mu^2},$$
$$C_4' = C_4'(\mu)+\frac{4}{3\pi^2}\log\frac{\Lambda^2}{\mu^2},$$

giving the final result

$$\mathcal{A}(s,t) = \frac{4s}{F^2} + \frac{1}{F^4}\Big[\frac{s^2}{2\pi^2}\log\frac{\mu^2}{-s} + \frac{u^2-s^2+3t^2}{12\pi^2}\log\frac{\mu^2}{-t} + \frac{t^2-s^2+3u^2}{12\pi^2}\log\frac{\mu^2}{-u} - \frac{C_4(\mu)}{2}s^2 - \frac{C_4'(\mu)}{2}(t^2+u^2)\Big] + \mathcal{O}\left(\frac{1}{F^6}\right)$$

Here μ is an arbitrary finite renormalization scale. while we do not know the value of C_4 and C_4' based on our symmetry consideration, we see that remarkably chiral perturbation theory still makes unambiguous predictions on the structure of the low-energy/momentum expansion of $\mathcal{A}(s,t)$!

Remark 1: In real world QCD, the quark fields are massive, although up & down quarks are light, and their mass terms may be viewed as a small deformation of QCD (with gauge group SU(3)] with $N_f=2$ flavors of massless quarks, by the term

$$\Delta\mathcal{L} = -m_u \bar{u} u - m_d \bar{d} d$$

where we have denoted $\psi_1 = u$, $\psi_2 = d$.

In terms of $\Phi_J{}^I = \bar{\psi}_L{}^I \psi_{JR}$, we can write

$$\Delta\mathcal{L} = -\frac{m_u+m_d}{2} \operatorname{tr}(\Phi + \Phi^\dagger) - \frac{m_u-m_d}{2} \operatorname{tr}(\sigma_3(\Phi + \Phi^\dagger)).$$

What is its effect on pion mass?

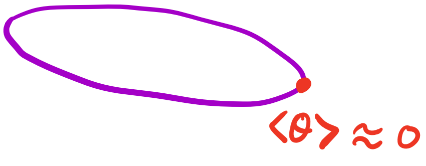
Consider the example of spontaneously broken $U(1)$:

effective Lagrangian

$$\mathcal{L} = -\frac{R^2}{2}(\partial_\mu \theta)^2$$

$\downarrow$ $U(1)$-breaking deformation

$$\Delta\mathcal{L} = u \cos\theta$$



$$\mathcal{L} + \Delta\mathcal{L} = -\frac{R^2}{2}(\partial_\mu\theta)^2 - \frac{u}{2}\theta^2 + \dots$$

The (pseudo-) Nambu-Goldstone boson created by $\theta(x)$ now acquire a mass m,

$$\boxed{m^2 \approx \frac{u}{R^2}}$$ (to leading order at small u)

The same analysis is applicable to deformation of the chiral Lagrangian of pions, except that we should be careful in comparing the normalization of kinetic terms:

recall that $U(x) = \dfrac{(1+i\vec{\zeta}\cdot\vec{\sigma})^2}{1+\zeta^2}$ transforms under $SU(2)_L \times SU(2)_R$ the same way as $\Phi_J{}^I$, namely $\Phi \mapsto U_R \Phi U_L^{-1}$.

In particular, the $U(1) \subset SU(2)$ corresponding to $\vec{\zeta} = (0, 0, \zeta_3)$ is parameterized by

$$U = 1 + 2i\zeta_3\sigma_3 + \cdots$$

which should be identified with $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$

i.e. $\theta = 2\zeta_3 + \cdots$

Comparing the effective Lagrangian

$$\mathcal{L} = -\frac{F^2}{2}(\partial_\mu\zeta_3)^2 + \cdots$$

we should identify $R = \frac{F}{2}$.

This leads to the pion mass term

$-\frac{1}{2}m_{ab}^2\pi_a\pi_b$, with the mass matrix

$$m^2_{ab} = \delta_{ab}\cdot\frac{\langle \Delta\mathcal{L}\rangle}{(F/2)^2}$$

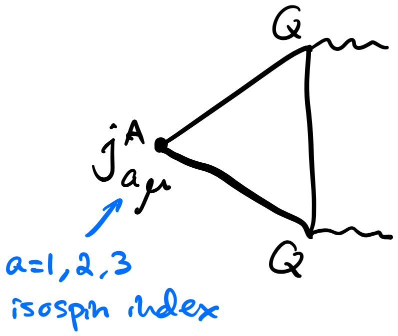
$$= \delta_{ab}\cdot\frac{4}{F^2}(m_u+m_d)\left\langle -\frac{\bar{\Psi}\Psi}{2}\right\rangle$$

[real world pion masses

$m_{\pi^0} \approx 135$ MeV

$m_{\pi^\pm} \approx 140$ MeV]

Remark 2: still in the approximation of vanishing u, d quark masses, the coupling of QCD to electromagnetic U(1) gauge field leads to a new anomaly in the chiral SU(2) global symmetry.



electric charge matrix

$$Q = e\begin{pmatrix}\overset{u}{\frac{2}{3}} & 0\\ 0 & \underset{d}{-\frac{1}{3}}\end{pmatrix}$$

$$\partial_\mu j^{A\,\mu}_a = -\frac{\overset{\#\text{ colors}}{N}}{16\pi^2}\underbrace{\mathrm{tr}\left(\frac{\sigma_a}{2}Q^2\right)}_{\text{non-vanishing only for } a=3}\epsilon^{\mu\nu\rho\sigma}\overbrace{F_{\mu\nu}F_{\rho\sigma}}^{\text{electromagnetic field strength}}$$

$$\mathrm{tr}\left(\frac{\sigma_3}{2}Q^2\right) = \frac{e^2}{6}.$$

This anomaly must also be present in the pion effective theory based on the chiral Lagrangian. Recall $j^A_{a\mu}$ is such that

$$\vec{\theta}_A \cdot \vec{j}_A^{\,\mu} = - \frac{\partial \mathcal{L}}{\partial \partial_\mu \vec{\xi}} \cdot \delta_A \vec{\xi}$$

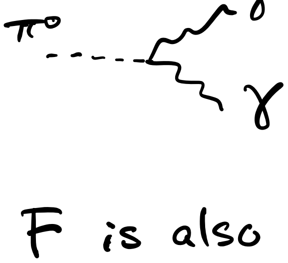
$$\Rightarrow \quad \vec{j}_A^{\,\mu} = -\frac{1}{2}(1-\xi^2) \frac{\partial \mathcal{L}}{\partial \partial_\mu \vec{\xi}} - \vec{\xi} \left(\vec{\xi} \cdot \frac{\partial \mathcal{L}}{\partial \partial_\mu \vec{\xi}} \right)$$

$$= \frac{F}{2} \partial_\mu \vec{\pi} + \dots$$

The anomaly due to coupling to E-M can be reproduced by the term

$$\Delta S = \int d^4x \, \frac{N \cdot e^2}{48\pi^2 F} \, \epsilon^{\mu\nu\rho\sigma} \overbrace{F_{\mu\nu} F_{\rho\sigma}}^{\text{E-M}} \underset{\text{“}\pi_0\text{-field”}}{\pi_3(x)}$$

in the effective action, leading to the decay of π^0 to a pair of photons:



$$\Gamma(\pi^0 \to 2\gamma) \sim \frac{N^2 \alpha^2}{F^2} m^3_{\pi^0}$$

F is also known as “pion decay constant”.