Physics 253b   Problem set 1

Due Wednesday February 15, 2023

Problem 1. In class we discussed the poles in the analytic continuation of the scattering amplitude that are associated with bound states and resonances respectively. In this problem you will investigate the analytic structure of the scattering amplitude in the simplest non-trivial model of quantum scattering: a non-relativistic particle scattering off of a δ-function potential in front of a hard wall, described by the Hamiltonian $H = \frac{p^2}{2m} + V(x)$ with

$$V(x) = \begin{cases} \infty, & x < 0 \\ \lambda\, \delta(x - a), & x > 0 \end{cases} \tag{1}$$

Here $a > 0$ and the coefficient λ may take either sign.

(a) Calculate the scattering amplitude $S(E)$ (as a function of the energy E) defined as usual through the asymptotic form of the (in-state) wave function

$$\psi^{in}(x) \sim e^{-ikx} + S(E)e^{ikx}, \qquad x \gg 0. \tag{2}$$

(b) Now consider the analytic continuation of $S(E)$ to the complex E-plane, and find the locations of the poles that correspond to bound states and resonances respectively, in the case of negative λ as well as the case of positive λ. Describe what happens to the positions of these poles if we dial λ continuously from negative to positive values.

Problem 2. Consider a 4-dimensional relativistic quantum field theory with two massive scalar fields ϕ_1, ϕ_2 of identical mass m, weakly coupled through a massless scalar field φ, described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}m\phi_1^2 - \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m\phi_2^2 - \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}g\varphi\phi_1^2 - \frac{1}{2}g\varphi\phi_2^2. \tag{3}$$

In this problem you will study the spectrum of bound states of a ϕ_1-particle with a ϕ_2-particle, in the ladder approximation, through the Bethe-Salpeter amplitude

$$\Phi_B(x_1, x_2) = \langle \Omega | \mathbf{T} \hat{\phi}_1(x_1) \hat{\phi}_2(x_2) | B \rangle. \tag{4}$$

For $|B\rangle$ a momentum eigenstate of four-momentum P^μ, with $P^2 = -M^2$, normalized in the usual convention for 1-particle states $\langle B, \vec{P} | B, \vec{P'} \rangle = \delta^3(\vec{P} - \vec{P'})$, we can write Φ_B as

$$\Phi_B(x_1, x_2) = \int \frac{d^4 k}{(2\pi)^4} e^{i(\frac{P}{2}+k)\cdot x_1 + i(\frac{P}{2}-k)\cdot x_2} \frac{\widetilde{\Phi}_B(k, P)}{(2\pi)^{\frac{3}{2}} \sqrt{2P_B^0}}. \tag{5}$$

Note: as we will be primarily interested in the mass spectrum, namely admissible values of M, the normalization convention of $\widetilde{\Phi}_B(k, P)$ will be inessential.

(a) Show that in the ladder approximation, the Bethe-Salpeter equation can be written as

$$((\frac{P}{2} + k)^2 + m^2)((\frac{P}{2} - k)^2 + m^2)\widetilde{\Phi}_B(k, P) = g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{-i}{(q-k)^2 - i\epsilon} \widetilde{\Phi}_B(q, P). \tag{6}$$

By analytic continuation, we may equivalently work with the Wick rotated version of the equation,

$$((k_E + \frac{iM}{2}\hat{e}_4)^2 + m^2)((k_E - \frac{iM}{2}\hat{e}_4)^2 + m^2)\chi(k_E) = g^2 \int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{(q_E - k_E)^2} \chi(q_E), \tag{7}$$

where we have set $P = (M, \vec{0})$, $\hat{e}_4$ stands for the unit vector in the x_4-direction in the Euclidean space $\mathbb{R}^4$, and the inner product of k_E, q_E etc. are understood to be Euclidean inner products. The Wick rotated Bethe-Salpeter amplitude is now denoted $\chi(k_E)$.

(b) Show that with the following (clever) change of variables

$$\begin{aligned} k_E^\mu &= \sqrt{m^2 - \frac{M^2}{4}} \left(\hat{e}_4^\mu + 2\frac{\widetilde{k}^\mu - \hat{e}_4^\mu}{(\widetilde{k} - \hat{e}_4)^2} \right), \\ \chi(k_E) &\equiv ((\widetilde{k} - \hat{e}_4)^2)^3 \widetilde{\chi}(\widetilde{k}), \end{aligned} \tag{8}$$

the equation (7) can be put in the manifestly $SO(4)$ invariant form

$$(m^2(\widetilde{k}^2+1)^2 - M^2\widetilde{k}^2)\widetilde{\chi}(\widetilde{k}) = g^2 \int \frac{d^4\widetilde{q}}{(2\pi)^4} \frac{\widetilde{\chi}(\widetilde{q})}{(\widetilde{q}-\widetilde{k})^2}. \tag{9}$$

For the purpose of solving this equation, from now on the tilde on the variables k and q will be omitted.

(c) By the $SO(4)$ rotational symmetry of (7), without loss of generality we can consider a solution with definite rotational quantum numbers, which is proportional to the $SO(4)$ spherical harmonics $\mathcal{Y}_{n\ell m}(\hat{k})$,[1]

$$\widetilde{\chi}(k) = R_n(k^2)\mathcal{Y}_{n\ell m}(\hat{k}). \tag{12}$$

Show that the equation for $R_n(k^2)$ reduces to[2]

$$(m^2(x+1)^2 - M^2 x)R_n(x) = \frac{g^2}{16\pi^2(n+1)} \int_0^\infty dy \left[x^{-\frac{n}{2}-1} y^{\frac{n}{2}+1}\theta(x-y) + x^{\frac{n}{2}} y^{-\frac{n}{2}}\theta(y-x)\right] R_n(y). \tag{13}$$

(d) if we assume that the binding energy $2m - M$ is small, the factor $(m^2(x+1)^2 - M^2x)$ multiplying $R_n(x)$ on the LHS of (13) is small at $x = 1$, and we expect $R_n(x)$ to be sharply peaked at $x = 1$. Thus, assuming the RHS of (13) is dominated by the integral near $y = 1$, find an approximate expression for $R_n(x)$ and solve for the possible values of M under this approximation.

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[1] Here $n = 0, 1, 2, \cdots$ is a quantum number that labels the total $SO(4)$ angular momentum, while $0 \le \ell \le n$ labels the total angular momentum within the (x_1, x_2, x_3) subspace, and $m = -\ell, -\ell+1, \cdots, \ell$ labels the angular momentum associated with rotations in the (x_1, x_2)-plane. (*Incidentally,* $(n+1, \ell, m)$ *are analogous to the familiar quantum numbers of the non-relativistic hydrogen atom which admits an accidental* $SO(4)$ *symmetry.*)

Some useful formulae include the decomposition of the Euclidean massless propagator in $SO(4)$ spherical harmonics

$$\frac{1}{(q-k)^2} = \frac{2\pi^2}{k_>^2} \sum_{n=0}^{\infty} \left(\frac{k_<}{k_>}\right)^n \frac{1}{n+1} \sum_{\ell,m} \mathcal{Y}_{n\ell m}(\hat{k})\mathcal{Y}^*_{n\ell m}(\hat{q}), \tag{10}$$

where $k_> \equiv \max(|k|, |q|)$, $k_< \equiv \min(|k|, |q|)$, $\hat{k} \equiv k^\mu/|k|$, $\hat{q}^\mu \equiv q^\mu/|q|$, and the orthogonality condition for the $SO(4)$ spherical harmonics

$$\int d^3\hat{q}\, \mathcal{Y}^*_{n\ell m}(\hat{q})\mathcal{Y}_{n'\ell' m'}(\hat{q}) = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}. \tag{11}$$

[2] In principle, a suitable normalizability condition should also be imposed on the Bethe-Salpeter amplitude.