Physics 253b Problem set 2

Due Wednesday March 1, 2023

Problem 1. In Lagrangian mechanics, the configuration space need not admit a linear (i.e. vector space) structure, and different parameterizations of the configuration space via generalized coordinates lead to equivalent Euler-Lagrange equations. A basic example of this is the three-dimensional rigid body, whose rotational configuration space is $SO(3)$, and one may formulate its Euler-Lagrange equation in terms of Euler angles, or any other parameterization of $SO(3)$. In this problem you will investigate the Lagrangian formulation of a quantum field theory in which the fields parameterize a nonlinear configuration space.

Consider a theory of three scalar fields $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ in four-dimensional spacetime with the Lagrangian density

$$\mathcal{L} = -\frac{M^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger), \qquad U(x) \equiv \exp\left[i\vec{\sigma}\cdot\vec{\phi}(x)\right]. \tag{1}$$

Here $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices, M is a constant parameter, and $U(x)$ is a 2×2 unitary matrix.[1]

(a) Expand the Lagrangian density (1) to quartic order in $\vec{\phi}$ and derive the Feynman rules of up to 4-point vertices. Compute the tree-level $2 \to 2$ scattering amplitude of a pair of particles created by ϕ_a, ϕ_b into a pair of particles created by ϕ_c, ϕ_d, for arbitrary a, b, c, d. You should express your answer in terms of the independent Mandelstam variables, say s and t.

(b) We could consider an alternative parameterization of the $SU(2)$ matrix $U(x)$,

$$U(x) \equiv \frac{1}{1+\vec{\zeta}(x)\cdot\vec{\zeta}(x)}(1 + i\vec{\sigma}\cdot\vec{\zeta}(x))^2, \tag{2}$$

in terms of the scalar fields $\vec{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$. Expand (1) to quartic order in $\vec{\zeta}$ and calculate the same amplitude as in **(a)** but now for particles created by ζ fields. How does your result compare to that of **(a)**?

[1] One may view $U(x)$ as a field that takes value in $SU(2)$, the space of 2×2 unitary matrices. Different $\vec{\phi}$'s that give the same U, e.g. $\vec{\phi} = (v, 0, 0)$ versus $\vec{\phi} = (v+2\pi, 0, 0)$, are viewed as the *same* field configuration. In principle, the path integral measure should be defined in such a way that $U(x)$ is the fundamental variable, where $\vec{\phi}(x)$ is merely a local coordinate system on the configuration space. This subtlety is inconsequential for the consideration of perturbation theory in this problem.

(c) Expand the above tree-level $2 \to 2$ amplitude in terms of the partial wave amplitudes $S_\ell(s)$ for all non-negative integer ℓ. How does your answer compare with the unitarity bound on $S_\ell(s)$?

Problem 2. Recall that the QED Lagrangian density can be expressed in terms of the renormalized vector potential $A_\mu(x)$ and electron field $\psi_\alpha(x)$ as

$$\mathcal{L} = -\frac{1}{4} Z_A F_{\mu\nu} F^{\mu\nu} - Z_\psi \bar{\psi}(\gamma^\mu D_\mu + m_0)\psi, \tag{3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $D_\mu \psi = \partial_\mu \psi + ieA_\mu \psi$. Here e is the unit (physical) electric charge, m_0 is the bare (not physical) mass, Z_A and Z_ψ are the field renormalization constants, as usual defined such that the 1-photon component of $F_{\mu\nu}(x)|\Omega\rangle$ and the 1-electron component of $\psi_\alpha(x)|\Omega\rangle$ are normalized as in free field theories.[2] In this problem we will carry out perturbation theory by splitting (3) as

$$\begin{aligned} &\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}, \\ &\mathcal{L}_{\text{free}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}(\gamma^\mu \partial_\mu + m_R)\psi, \\ &\mathcal{L}_{\text{int}} = -ieA_\mu \bar{\psi}\gamma^\mu \psi - \frac{1}{4}(Z_A - 1) F_{\mu\nu} F^{\mu\nu} - (Z_\psi - 1)\bar{\psi}(\gamma^\mu D_\mu + m_R)\psi - Z_\psi \delta m \bar{\psi}\psi, \end{aligned} \tag{4}$$

and view $\mathcal{L}_{\text{free}}$ as the free theory Lagrangian density, treating all terms in $\mathcal{L}_{\text{int}}$ as interactions that are of order e or higher. Here $m_R = m_0 - \delta m$ is a finite quantity that we refer to as the renormalized mass, which a priori need not agree with the physical mass m of the electron. For the rest of this problem, we will *choose* m_R to agree with the physical mass m.

The time-ordered 2-point function of the electron field takes the form

$$\langle \Omega | \mathbf{T} \hat{\psi}_\alpha(x) \hat{\bar{\psi}}^\beta(y) | \Omega \rangle = -i \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \left((i\not{k} + m - i\epsilon - \Sigma(k))^{-1} \right)_\alpha{}^\beta, \tag{5}$$

where the self-energy $\Sigma(k)$ is a matrix with spinor indices. Note that the self-energy of the electron field is a priori gauge-dependent, since $\hat{\psi}_\alpha(x)$ is not a gauge-invariant operator. You can adopt Feynman gauge for this problem.

Calculate $\Sigma(k)$ at order e^2, and determine Z_ψ as well as δm at this order. You can regularize the UV divergence using dimensional regularization (setting spacetime dimension $D = 4 - \epsilon$), and regularize the IR divergence by cutting off the photon loop momentum at a small mass scale μ. You should find in your final result that $\Sigma(k)$ is free of UV divergence (due to the way we defined Z_ψ and δm), but it does contain an IR divergence.

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[2] Recall that we have calculated Z_A at 1-loop order in 253a.

Problem 3. Consider scalar QED defined by the action

$$S[A,\varphi] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - |D_\mu \varphi|^2 - m^2 |\varphi|^2 \right), \tag{6}$$

where φ is a complex scalar field, $D_\mu \varphi \equiv \partial_\mu \varphi - iqA_\mu \varphi$. We will define an effective action $S_{\text{eff}}[A]$ (this is similar to, but not the same as, the 1PI effective action introduced in class) by performing the functional integral over φ,

$$e^{iS_{\text{eff}}[A]} = \int [D\varphi D\varphi^*] e^{iS[A,\varphi]}. \tag{7}$$

Evidently, the RHS of (7) can be evaluated as a Gaussian integral, resulting in the effective action (up to an inconsequential constant shift)

$$S_{\text{eff}}[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \right) + i \log\det\left(-D^\mu D_\mu + m^2 \right), \tag{8}$$

where $\det(\cdots)$ stands for the determinant of a linear functional.

(a) Using $\log\det = \text{tr}\log$, and the formula

$$\log \frac{H}{\mu} = -\int_0^\infty \frac{ds}{s} \left(e^{-sH} - e^{-s\mu} \right), \tag{9}$$

calculate the RHS of (8) in the case of a constant magnetic field $F_{12} \equiv B_3$. From the result, deduce the correction to the energy density of a constant magnetic field due to coupling to the charged scalar field.

(b) Extend your result of **(a)** to the case of a constant electric field $F_{01} \equiv E_1$ by a suitable analytic continuation.

(c) If you are sufficiently careful with the analytic continuation in **(b)**, you should find that in the presence of a constant electric field, S_{eff} is complex. Calculate the imaginary part of S_{eff} in this case, and from the result deduce the probability of the electric field decaying via pair production of charged particles per unit spatial volume per unit time.[3]

[3] Note that this effect is non-perturbative and could not have been seen at any given finite order in perturbation theory with respect to the electric charge nor the electric field.