Physics 253b Problem set 3

Due Friday March 24, 2023

Problem 1. In class we considered ϕ^4 theory in $4-\epsilon$ dimensions, where ϵ is viewed as a small but finite expansion parameter, and found an interacting infrared RG fixed point. In this problem you will investigate an analogous RG fixed point of the perturbative ϕ^3 theory in $6-\epsilon$ dimensions, whose bare Euclidean action takes the form

$$S_E = \int d^{6-\epsilon}x \left[\frac{1}{2} \left(\partial_\mu \phi\right)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{1}{3!} g_0 \phi^3 \right]. \tag{1}$$

The bare mass m_0 will be chosen such that there is no mass gap in the spectrum of states created by $\hat{\phi}(x)$ acting on the vacuum.

As introduced in class, the dimensionless renormalized running coupling $\lambda(\mu)$ is defined as

$$\lambda(\mu) = -\mu^{\delta(\epsilon)} (Z(\mu))^{\frac{3}{2}} \Gamma^{(3)}(k_1, k_2)|_{k_1^2 = k_2^2 = (k_1+k_2)^2 = \mu^2}. \tag{2}$$

Here μ is a mass scale variable, $-\delta(\epsilon)$ is the mass dimension of g_0, $\Gamma^{(3)}(k_1, k_2)$ is computed by the amputated 1PI diagram with momenta $k_1, k_2, -k_1-k_2$ assigned to the three external lines, and $Z(\mu)$ is related to the self-energy $\Sigma(k)$ via

$$Z(\mu) = \left(1 - \frac{\partial \Sigma}{\partial k^2} \right)^{-1} \Bigg|_{k^2 = \mu^2}. \tag{3}$$

(a) Determine the beta function $\beta(\lambda)$ appearing in the RG equation $\frac{d\lambda(\mu)}{d \log \mu} = \beta(\lambda(\mu))$ at 1-loop order.

(b) You should find an interacting RG fixed point at a complex value of λ. Determine the latter to the leading nontrivial order in the ϵ-expansion.

Problem 2. Consider a theory of massless fermionic spinor fields in two-dimensional spacetime, defined by the bare (Lorentzian) action

$$S[\psi,\bar{\psi}] = \int d^2x \left[-\sum_{i=1}^{N} \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i + \frac{1}{2} g_0 \left(\sum_{i=1}^{N} \bar{\psi}_i \psi_i \right)^2 \right]. \tag{4}$$

Here ψ_i ($i = 1, ..., N$) are a set of N fermionic fields that transform as 2-component complex spinors with respect to the 1+1 dimensional Lorentz symmetry. The 2D version of gamma matrices, γ^μ ($\mu = 0, 1$), are 2×2 constant matrices that obey $-(\gamma^0)^2 = (\gamma^1)^2 = 1$, $\{\gamma^0, \gamma^1\} = 0$. We have also defined $\overline{\psi}_i \equiv i\psi_i^\dagger \gamma^0$ as usual.

The chirality matrix is defined as $\gamma \equiv \gamma^0\gamma^1$ (this is the 2D analog of γ_5 we have seen in 4D). Observe that (4) is invariant under the "chiral symmetry" $\psi_i \mapsto \gamma\psi_i$, while a mass term for the fermion fields would have violated the chiral symmetry.

(a) The renormalized coupling $g(\mu) = g_0 +$ (loop corrections) is defined through the four-fermion 1PI diagram with external momenta set at mass scale μ as usual, and obeys an RG equation of the form $\frac{dg(\mu)}{d\log\mu} = \beta(g(\mu))$. Compute the beta function $\beta(g)$ at 1-loop order. Is this theory weakly or strong coupled at low energies?

Next, we consider an equivalent formulation of the same theory by writing the path integral as

$$Z = \int [D\bar{\psi}_i D\psi_i D\sigma] \; e^{i\widetilde{S}[\psi,\bar{\psi},\sigma]}, \tag{5}$$

where $\sigma(x)$ is a scalar field, and the new action $\widetilde{S}$ is

$$\widetilde{S}[\psi,\bar{\psi},\lambda] = \int d^2x \left[-\sum_{i=1}^{N} \bar{\psi}_i \partial^\mu \gamma_\mu \psi_i - \sum_{i=1}^{N} \sigma \bar{\psi}_i \psi_i - \frac{1}{2g_0}\sigma^2 \right]. \tag{6}$$

Integrating out $\sigma(x)$ gives back the original fermion theory. On the other hand, integrating out $\psi_i, \bar{\psi}_i$ produces an effective action $S_{\text{eff}}[\sigma]$,

$$e^{iS_{\text{eff}}[\sigma]} = \int [D\bar{\psi}_i D\psi_i] \; e^{i\widetilde{S}[\psi,\bar{\psi},\sigma]}. \tag{7}$$

(b) Evaluate $S_{\text{eff}}[\sigma]$ for a constant field configuration $\sigma(x) = \sigma_0$. The answer should take the form $S_{\text{eff}} = -\int d^2x \, V_{\text{eff}}(\sigma_0)$, where the "effective potential" $V_{\text{eff}}(\sigma_0)$ is hopefully finite when expressed as a function of the renormalized coupling $g(\mu)$ defined in part (a).

(c) You may minimize V_{eff} with respect to σ_0 to find an approximate[1] vacuum expectation value of the field $\sigma(x)$. Note that in the formulation (5), the chiral symmetry acts by $\psi_i \mapsto \gamma\psi_i$, $\sigma \mapsto -\sigma$. Is $V_{\text{eff}}(\sigma)$ invariant under this symmetry? What about the vacuum expectation value of σ?

[1]Strictly speaking, the vacuum expectation value $\langle\sigma\rangle$ is determined by the minimizing the potential term in the 1PI effective action with respect to σ. In the limit of large N, the 1PI effective action for σ reduces to S_{eff}.