# Physics 253b   Problem set 4

Due Friday March 31, 2023

Consider the massless $\phi^4$ theory in $D = 4 - \epsilon$ dimensions, defined by the bare Euclidean Lagrangian density

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{1}{4!}g_0\phi^4, \tag{1}$$

where the bare mass $m_0$ is chosen such that the mass gap vanishes. As discussed in class, an appropriate definition of the dimensionless renormalized coupling $\lambda(\mu)$ is

$$\lambda(\mu) = -\mu^{-\epsilon}(Z(\mu))^2\Gamma^{(4)}(k_1, k_2, k_3)|_{k_i^2=\mu^2,\ s=t=u=-\frac{4}{3}\mu^2}, \tag{2}$$

where $\Gamma^{(4)}(k_1, k_2, k_3)$ is computed by the 1PI diagram with four external momenta $k_1, k_2, k_3, -k_1 - k_2 - k_3$, and $Z(\mu)$ is related to the self-energy $\Sigma(k)$ by

$$Z(\mu) = \left(1 - \frac{\partial \Sigma}{\partial k^2}\right)^{-1}\Bigg|_{k^2=\mu^2}. \tag{3}$$

The renormalized operator $[\phi]_\mu$ at mass scale $\mu$ is defined as

$$[\phi]_\mu = (Z(\mu))^{-\frac{1}{2}}\phi. \tag{4}$$

In the RG-improved perturbation theory, we compute Green functions at momenta scale $\mu$, or equivalently distance scale $\mu^{-1}$, through the perturbative expansion of Green functions of $[\phi]_\mu$ with respect to $\lambda(\mu)$, truncated to a certain order in $\lambda(\mu)$, and then substitute $\lambda(\mu)$ with its value determined through the RG equation truncated to the relevant loop order.

Recall that in class, we have studied the RG equation for $\lambda(\mu)$ at 1-loop order, and found the RG fixed point for $\lambda$ up to order $\epsilon$ in the $\epsilon$-expansion. We have also seen that at 1-loop order, $\Sigma(k)$ is independent of $k$ and thus does not contribute to $Z(\mu)$.

**(a)** Calculate $\Sigma(k)$ (in the absence of mass gap) at 2-loop order in the naive perturbation theory, and then determine $Z(\mu)$ in the RG-improved perturbation theory by expressing

$$\gamma_\phi(\mu) \equiv \frac{1}{2}\frac{d \log Z(\mu)}{d \log \mu} \tag{5}$$

as a function of $\lambda(\mu)$, keeping up to order $\lambda^2$ terms, and using the solution for $\lambda(\mu)$ from the 1-loop RGE of the latter.

**(b)** Calculate the momentum space 2-point function of the renormalized operator $[\widetilde{\phi}(p)]_\mu$

$$\langle [\widetilde{\phi}(p)]_\mu [\widetilde{\phi}(q)]_\mu \rangle \tag{6}$$

in the RG-improved perturbation theory, with the choice $\mu = |p|$ and the expansion of (6) in $\lambda(\mu)$ truncated to order $\lambda^2$.

**(c)** How does the 2-point function $\langle \phi(x)\phi(0)\rangle$ behave at large $|x|$? Determine the scaling dimension of the operator $\phi(x)$ at the infrared RG fixed point to order $\epsilon^2$.