Physics 253b Problem set 5

Due Friday April 7, 2023

Problem 1. Consider a 0-dimensional analog of Yang-Mills theory where the spacetime is replaced by a point and the gauge field variable[1] is replaced by an $N \times N$ unitary matrix U, with the Euclidean action

$$S(U) = -\lambda \operatorname{Tr}(U + U^{-1}). \tag{1}$$

The $U(N)$ "gauge" transformation on U takes the form

$$U \mapsto gUg^{-1}, \tag{2}$$

where g is an arbitrary $N \times N$ unitary matrix. The analog of the path integral for the partition function is

$$Z = \frac{1}{\mathrm{vol}(U(N))} \int_{U(N)} dU\, e^{-S(U)}, \tag{3}$$

where dU is the Haar measure on $U(N)$, with the defining property $dU = d(gU) = d(Ug)$ for any unitary g, and $\mathrm{vol}(U(N)) \equiv \int_{U(N)} dU$. Note that the overall normalization convention of the Haar measure drops out of (3), and that Z is unambiguously defined as a function of λ.

(a) Any unitary matrix U can be brought to a diagonal one by a gauge transformation of the form (2), and thus we can partially fix the gauge redundancy by imposing the condition

$$U_{ab} = \delta_{ab} e^{i\theta_a}, \quad a, b = 1, \cdots, N, \tag{4}$$

where $\theta_1, \cdots, \theta_N$ are unconstrained real variables. Following the Faddeev-Popov procedure, find the Faddeev-Popov determinant Δ associated with the gauge condition (4) as a function of $\vec{\theta} = (\theta_1, \cdots, \theta)$, such that (3) is equal to

$$Z = C_N \prod_{a=1}^{N} \int_0^{2\pi} d\theta_a\, \Delta(\vec{\theta})\, e^{2\lambda \sum_{a=1}^{N} \cos\theta_a}, \tag{5}$$

for some λ-independent constant C_N.

(b) In the case $N = 2$, we can parameterize the $U(2)$ matrix U as

$$U \equiv e^{i\alpha} \frac{(1 + i\vec{\sigma} \cdot \vec{x})^2}{1 + \vec{x}^2}, \tag{6}$$

[1] More precisely, the unitary matrix U here is the analog of a closed Wilson loop in Yang-Mills theory.

and write the Haar measure as

$$dU = f(\alpha, \vec{x}) d\alpha d^3\vec{x}. \tag{7}$$

Determine the function $f(\alpha, \vec{x})$ (up to overall normalization), and verify that (3) is equivalent to (5) by directly comparing the integration over $(\alpha, \vec{x})$ in the former to the integration over $(\theta_1.\theta_2)$ in the latter (numerically if you are so inclined).

Problem 2. In 253a we introduced a gauged fixed form of the free Maxwell theory, whose action is

$$S_{\text{naive}} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right]. \tag{8}$$

Following the recipe of the BRST formalism introduced in class, we can now understand this through the BRST-invariant action

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + B \partial_\mu A^\mu + \frac{\xi}{2} B^2 + b\, \partial^\mu \partial_\mu c \right], \tag{9}$$

where the BRST variations of the field variables $A_\mu(x)$, $B(x)$, $b(x)$, $c(x)$ take the form

$$\begin{aligned} \delta_B A_\mu(x) &= \epsilon \partial_\mu c(x), \\ \delta_B B(x) &= 0, \\ \delta_B b(x) &= -\epsilon B(x), \\ \delta_B c(x) &= 0, \end{aligned} \tag{10}$$

or equivalently, we may perform the Gaussian functional integration over $B(x)$, leaving the action

$$\widetilde{S} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + b\, \partial^\mu \partial_\mu c \right], \tag{11}$$

and the BRST variations of the field variables now take the form

$$\begin{aligned} \delta_B A_\mu(x) &= \epsilon \partial_\mu c(x), \\ \delta_B b(x) &= \epsilon \frac{1}{\xi} \partial_\mu A^\mu(x), \\ \delta_B c(x) &= 0. \end{aligned} \tag{12}$$

In this problem, you will analyze the Hilbert space of physical states in the BRST formalism, in the Feynman gauge $\xi = 1$.

(a) You should observe that in the Feynman gauge, the part of the action involving A_μ takes the form identical to that of 4 massless free scalars. Describe the full space of states in terms of a Fock basis constructed using creation operators for A_μ and b, c fields.

(b) Find an explicit expression of the BRST operator $\widehat{Q}_B$ in terms of the creation and annihilation operators for A_μ, b, c.

(c) Show that the BRST cohomology classes describing the physical 1-particle state are precisely in correspondence with the 1-photon states of either positive or negative helicity, nothing more and nothing less.