Physics 253b Problem set 6

Due Wednesday April 26, 2023

In this problem you will finish the computation of the one-loop β-function of QCD with $SU(N)$ gauge fields and N_f flavors of massless Dirac fermions in a representation R, as was analyzed in class through the 1PI effective action computed in the background field gauge via dimensional regularization (in $D = 4 - \epsilon$ spacetime dimensions), with the renormalized Yang-Mills coupling $g(\mu)$ defined in the minimal subtraction scheme.

In the background field gauge, the total (bare) action including Faddeev-Popov ghosts is

$$
\begin{aligned}
S[A', \psi, b, c; \bar{A}] = \int d^D x \Big[&- \frac{1}{4g^2} \left(\widetilde{F}_{a\mu\nu} + \widetilde{D}_\mu A'_{a\nu} - \widetilde{D}_\nu A'_{a\mu} + f^{bc}{}_a A'_{b\mu} A'_{c\nu} \right)^2 - \overline{\psi}_i \gamma^\mu \left(\widetilde{D}_\mu - i A'_{a\mu} t^a_R \right) \psi_i \\
&- \frac{1}{2\xi g^2} \left(\widetilde{D}_\mu A'^\mu_a \right)^2 - \left(\widetilde{D}_\mu b^a \right) \left(\widetilde{D}^\mu c_a + f^{bc}{}_a A'^\mu_b c_c \right) \Big],
\end{aligned}
\tag{1}
$$

where the original gauge field is written as $A_{a\mu}(x) = \widetilde{A}_{a\mu}(x) + A'_{a\mu}(x)$ with $\widetilde{A}_{a\mu}(x)$ being a background gauge field, while $A'_{a\mu}(x), \psi_i(x), b^a(x)$ and $c_a(x)$ are fluctuation fields to be integrated out in obtaining the 1PI effective action.[1] $\widetilde{D}_\mu$ is the gauge covariant derivative with respect to the background gauge field $\widetilde{A}$, in the adjoint representation when acting on A', b, c, and in the representation R when acting on ψ.

The one-loop contribution to the *Euclidean* 1PI effective action $\Gamma^{1-\text{loop}}[\widetilde{A}]$ is computed through the functional determinant

$$
e^{-\Gamma^{\text{1-loop}}[\bar{A}]} = \frac{(\det \mathbb{D}_\psi)^{N_f} (\det \mathbb{D}_{\text{gh}})}{(\det \mathbb{D}_A)^{\frac{1}{2}}}
\tag{2}
$$

for a Euclidean background gauge field $\widetilde{A}_{a\mu}(x)$, where

$$
\begin{aligned}
&\mathbb{D}_\psi = -\gamma^\mu \widetilde{D}_\mu, \quad \text{(in the representation } R) \\
&\mathbb{D}_{\text{gh}} = \widetilde{D}_\mu \widetilde{D}^\mu, \quad \text{(in the adjoint representation)} \\
&[\mathbb{D}_A]_{a\mu, b\nu} = (-\widetilde{D}_\rho \widetilde{D}^\rho \eta_{\mu\nu} + \widetilde{D}_\nu \widetilde{D}_\mu - \frac{1}{\xi} \widetilde{D}_\mu \widetilde{D}_\nu) \delta_{ab} + f^{ab}{}_c \widetilde{F}_{c\mu\nu}.
\end{aligned}
\tag{3}
$$

For a constant background field configuration $\widetilde{A}$, the functional determinant (2) is formally

[1] As usual, in deriving the 1PI effective action, linear terms in the fluctuation fields are added to the bare action so that the tadpoles i.e. 1-point functions of fluctuation fields vanish.

evaluated as

$$\Gamma^{\text{1-loop}}[\widetilde{A}] = -\int d^D x \int \frac{d^D k}{(2\pi)^D}\left[N_f \mathrm{tr}\,\log \mathbb{M}_\psi(k) + \mathrm{tr}\,\log \mathbb{M}_{\mathrm{gh}}(k) - \frac{1}{2}\mathrm{tr}\,\log \mathbb{M}_A(k)\right], \tag{4}$$

where $\mathbb{M}(k)$ is the momentum space version of the kinetic operator $\mathbb{D}$, e.g.

$$\mathbb{M}_\psi(k) = i\gamma^\mu \otimes (k_\mu - \widetilde{A}_{a\mu} t_R^a), \tag{5}$$

where t_R^a is the generator of $SU(N)$ in the representation R. As explained in class, the IR divergence in the k-integral on the RHS of (4) is cut off at the renormalization scale μ, corresponding to a background field $\widetilde{A}$ that is constant in a domain of size $L \sim 1/\mu$ but falls off outside of the domain.

In class, we have computed the contribution from the functional determinant of ψ to the singular part of $\widetilde{A}^4$ term in the Euclidean one-loop effective Lagrangian,

$$-\int_{|k|>\mu} \frac{d^D k}{(2\pi)^D} \mathrm{Tr}\,\log \mathbb{M}_\psi(k)\Big|_{\widetilde{A}^4} \sim \frac{1}{3}\mathrm{tr}_R(\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu})\frac{1}{8\pi^2}\frac{\mu^{-\epsilon}}{\epsilon}, \tag{6}$$

where on the RHS we have omitted terms that are finite in the $\epsilon \to 0$ limit as they do not affect the one-loop beta function (and do not appear in the one-loop renormalized coupling in the minimal subtraction scheme either).

Your task is to compute the analogous contribution from the functional determinant of A' and the Faddeev-Popov ghosts b, c, in the $\xi = 1$ gauge. Show that their respective contributions to the singular part of the Euclidean effective Lagrangian is

$$\begin{aligned}
-\int_{|k|>\mu} \frac{d^D k}{(2\pi)^D} \mathrm{Tr}\,\log \mathbb{M}_{\mathrm{gh}}(k)\Big|_{\widetilde{A}^4} &\sim -\frac{1}{12}\mathrm{tr}_{\mathrm{adj}}(\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu})\frac{1}{8\pi^2}\frac{\mu^{-\epsilon}}{\epsilon}, \\
+\frac{1}{2}\int_{|k|>\mu} \frac{d^D k}{(2\pi)^D} \mathrm{Tr}\,\log \mathbb{M}_A(k)\Big|_{\widetilde{A}^4} &\sim \frac{5}{6}\mathrm{tr}_{\mathrm{adj}}(\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu})\frac{1}{8\pi^2}\frac{\mu^{-\epsilon}}{\epsilon},
\end{aligned} \tag{7}$$

which result in the one-loop β-function claimed in the lecture. Here $\mathrm{tr}_{\mathrm{adj}}$ stands for trace in the adjoint representation.