Conformal Field Theory

- What is CFT, and why?

recipes for local ("UV-complete") QFT

- QM system equipped with a Hilbert space $\mathcal{H}$ of physical states

- Poincaré symmetry $U(\Lambda, a)$ generated by $\hat{P}_\mu$, $\hat{J}_{\mu\nu}$

  energy-momentum ($\hat{P}_\mu$); angular momentum + boosts ($\hat{J}_{\mu\nu}$)

  Poincaré-invariant vacuum $|\Omega\rangle$

- local "field" operators $\hat{\phi}(x)$ that obey Poincaré-covariance and microcausality

- In particular, a local stress-energy tensor operator $\hat{T}_{\mu\nu}$,

$$\partial_\mu \hat{T}^{\mu\nu} = 0 \qquad \text{(Conservation)}$$

$$\hat{T}^{\mu\nu} = \hat{T}^{\nu\mu} \qquad \text{(Lorentz sym)}$$

such that

$$\hat{P}^{\mu} = \int d^{D-1}\vec{x}\; T^{0\mu}$$

$$\hat{J}^{\mu\nu} = \int d^{D-1}\vec{x}\; \left(x^{\mu} T^{0\nu} - x^{\nu} T^{0\mu}\right)$$

Generally, $\hat{T}_{\mu\nu}(x)$ transforms as a symmetric tensor under Lorentz sym

i.e. $U(\Lambda)\, \hat{T}_{\mu\nu}(x) \left(U(\Lambda)\right)^{-1} = \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu}\, \hat{T}_{\rho\sigma}(\Lambda \cdot x)$

the trace $\hat{T}^{\mu}{}_{\mu}(x)$ is a scalar operator, can expand on a basis of scalar field operators $\mathcal{O}_I(x)$:

$$\hat{T}^{\mu}{}_{\mu}(x) = \sum_I \underbrace{\beta_I}_{\text{“}\beta\text{-functions”}} \mathcal{O}_I(x)$$

In a Lagrangian description, $\mathcal{O}_I(x)$ are constructed as renormalized version of "bare operators" $\mathcal{O}_I^{\text{bare}}(x)$ made out of product of "elementary" fields & their derivatives, with

$$\mathcal{L} \supset \sum_I g_I^{\text{bare}} \mathcal{O}_I^{\text{bare}}$$

$g_I(\mu)$ renormalized coupling

defined in a given renorm. scheme

e.g. minimal subtraction

under variations of coupling,

$$\delta \mathcal{L} = \sum_I \delta g_I^{\text{bare}} \mathcal{O}_I^{\text{bare}}$$

$$\equiv \sum_I \delta g_I(\mu) \, [\mathcal{O}_I]_\mu$$

$$\frac{d}{d\log\mu} g_I(\mu) = \beta_I(g_J(\mu))$$

same β-fn as in

$$\hat{T}^\mu{}_\mu = \sum_I \beta_I [\mathcal{O}_I]_\mu .$$

(up to total derivatives)

The β-functions control how the renormalized "physical" couplings change with energy/momentum scale μ.

scale-invariant "RG fixed point"

$$\Leftrightarrow \quad \hat{T}^\mu{}_\mu = 0 .$$

Example of an RG fixed pt

recall the statistical Ising model

- a system of spins on a D-dimensional lattice Λ in thermal equilibrium

e.g. $D=2$ square lattice

$$\begin{array}{ccccc} \uparrow & \downarrow & \uparrow & \uparrow & \cdots \\ \uparrow & \uparrow & \downarrow & \downarrow & \cdots \\ \downarrow & \downarrow & \uparrow & \downarrow & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

$x = (i,j)\,,\ i,j \in \mathbb{Z}$

$$\tilde{\mathcal{H}} = \bigotimes_{x\in\Lambda} V_x \,,$$

unrelated to the Hilbert space & Hamiltonian of underlying QFT

$$V_x = \mathrm{span}\{\underset{S_x=1}{|\uparrow\rangle},\ \underset{S_x=-1}{|\downarrow\rangle}\}$$

$$\tilde{H} = -\sum_{\langle x x'\rangle} S_x S_{x'}$$

neighboring sites x, x'

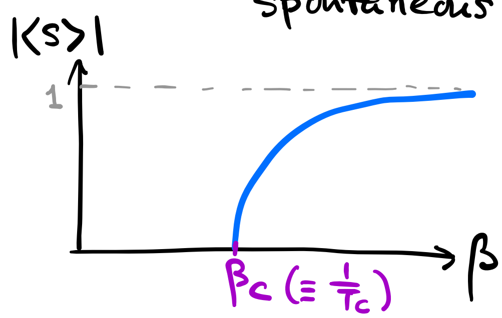
Observables: thermal expectation value

$$\langle \hat{\mathcal{O}} \rangle = \frac{1}{Z}\,\mathrm{Tr}_{\tilde{\mathcal{H}}}\left(e^{-\beta\tilde{H}}\,\hat{\mathcal{O}}\right)$$

$$Z = \mathrm{Tr}_{\tilde{\mathcal{H}}}\, e^{-\beta\tilde{H}}$$

$\beta = \frac{1}{T}$ inverse temperature

"spontaneous magnetization"

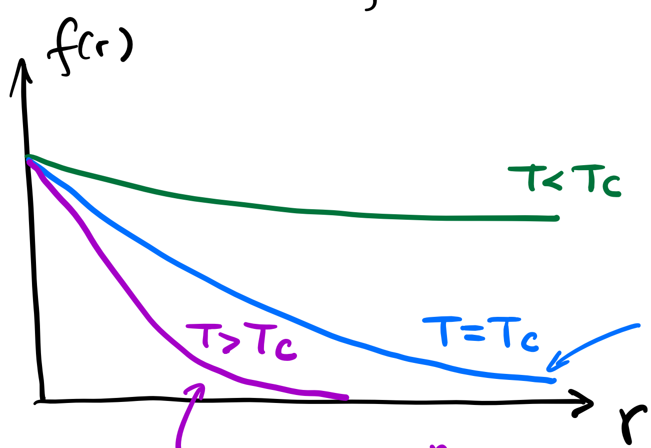


spin correlator

$$\langle S_{0,0} \, S_{N,M} \rangle \equiv f_{N,M} \approx f(r)$$

$$r = \sqrt{N^2+M^2}$$

for $r \gg 1$ and T close to T_c



$f(r) \sim r^{-2\Delta}$

$f(r) \sim e^{-\frac{r}{\xi(T)}}$

$\xi(T) \sim (T-T_c)^{-\nu}$ "correlation length"

$D=2$: $\Delta = \frac{1}{8}$, $\nu = 1$

$D=3$: $\Delta = 0.518\ldots$, $\nu = 0.62998\ldots$

The statistical Ising model may be equivalently formulated as a D-dim'l Euclidean scalar field theory with a single field variable $\phi(x)$, a "double-well-like" potential $V(\phi)$, with $SO(D)$-breaking higher-derivative kinetic terms and a non-Euclidean-invariant UV cutoff.

Key claim: at $T = T_c$, where $\xi(T_c) = \infty$, long distance correlators of the Ising model are captured by a CFT, known as the "Ising CFT". Furthermore,

Ising CFT $=$ IR RG fixed point of D-dimensional massless ϕ^4 theory for $D = 2, 3$

Δ = scaling dimension of $\phi(x)$

$D - \frac{1}{\nu}$ = scaling dimension of $\phi^2(x)$.

[calculated in ϵ-expansion for $D = 4 - \epsilon$ in 253b]

A CFT is a QFT whose stress-energy tensor $\hat{T}_{\mu\nu}(x)$ is traceless, namely

$$\hat{T}^{\mu}{}_{\mu}(x) = 0 .$$

Consequence : more conserved currents

$\rightsquigarrow$ symmetries

consider

$$J_\mu(x) \equiv T_{\mu\nu}(x)\, \varepsilon^{\nu}(x)$$

for some chosen vector field $\varepsilon^{\mu}(x)$.

$$? \quad \partial_\mu J^\mu = 0 \iff \partial_\mu \left(T^{\mu\nu} \varepsilon_\nu \right) = 0$$

$$\iff T^{\mu\nu} \partial_\mu \varepsilon_\nu = 0.$$

Satisfied provided $T^{\mu}{}_{\mu} = 0$

and $\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu \propto \eta_{\mu\nu}$

$$\iff \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu - \frac{2}{D} \eta_{\mu\nu} \partial_\rho \varepsilon^\rho = 0.$$

"Conformal Killing equation"

Q: What are the solutions, i.e. "conformal Killing vectors" ?

A: $\varepsilon_\mu(x) = a_\mu + b_{\mu\nu} x^\nu + 2(c\cdot x) x_\mu - c_\mu x^2$

translation — Lorentz + dilatation — "special conformal transf"

for any constant a_μ, c_μ,

and $b_{\mu\nu}$ anti-symmetric or $b_{\mu\nu} \propto \eta_{\mu\nu}$

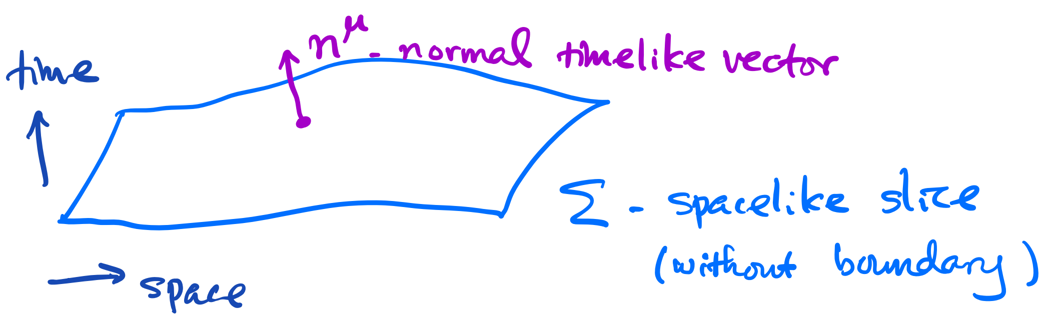
[Note: D=2 case special
→ more CKVs, will revisit]

Given a CKV $\varepsilon^\mu(x)$, we can construct the conserved (Noether) current operator

$$\hat{J}_\mu(x) = \hat{T}_{\mu\nu}(x)\, \varepsilon^\nu(x)$$

⇝ conserved Noether charge operator

$$\hat{Q}_\varepsilon = \int_\Sigma d\sigma\, n^\mu \hat{J}_\mu(x)$$



It follows from $\partial_\mu \hat{J}^\mu(x) = 0$ and Stoke's theorem that $\hat{Q}_\varepsilon$ is independent of the choice of Σ.

• Geometric interpretation of conformal sym.

$\hat{Q}_\varepsilon$ generates an infinitesimal sym transf.

$$\hat{U} = e^{i\hat{Q}_\varepsilon}.$$

The case $\varepsilon_\mu(x) = a_\mu + b_{[\mu\nu]} x^\nu$ corresponds to Poincaré symmetry. In particular, such sym transfs compose just like the infinitesimal coordinate transf. $x^\mu \to \tilde{x}^\mu = x^\mu + \varepsilon^\mu(x)$, namely

$$[\hat{Q}_{\varepsilon_1}, \hat{Q}_{\varepsilon_2}] = -i\,\hat{Q}_{[\varepsilon_1,\varepsilon_2]}$$

Commutator of vector fields:

$$[\varepsilon_1, \varepsilon_2]^\mu \equiv \varepsilon_1^\nu \partial_\nu \varepsilon_2^\mu - (1 \leftrightarrow 2).$$

This relation is expected to extend to all CKVs.

It follows that infinitesimal conformal transf's can be labeled by infinitesimal coord transf.

$$x^\mu \to \tilde{x}^\mu = x^\mu + \varepsilon^\mu(x)$$

and by composing them we can also label a finite conformal transf. by a certain finite coord. transf. $x^\mu \to \tilde{x}^\mu = f^\mu(x)$

what kind of function?

Observe that for $f^\mu(x) = x^\mu + \varepsilon^\mu(x)$,

infinitesimal

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial \tilde{x}^\nu}{\partial x^\sigma} \eta_{\mu\nu}$$

$$= \eta_{\rho\sigma} + \partial_\rho \varepsilon_\sigma + \partial_\sigma \varepsilon_\rho + \cancel{\mathcal{O}(\varepsilon^2)}$$

$$= e^{2\omega(x)} \eta_{\rho\sigma} \quad \text{for some } \omega(x).$$

Such a relation is preserved under composition:

$$x^\mu \to \tilde{x}^\mu \to \tilde{\tilde{x}}^\mu$$

$$\frac{\partial \tilde{\tilde{x}}^\mu}{\partial x^\rho} \frac{\partial \tilde{\tilde{x}}^\nu}{\partial x^\sigma} \eta_{\mu\nu} = \underbrace{\frac{\partial \tilde{\tilde{x}}^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{\tilde{x}}^\nu}{\partial \tilde{x}^\beta} \eta_{\mu\nu}}_{= e^{2\tilde{\omega}(x)} \eta_{\alpha\beta}} \frac{\partial \tilde{x}^\alpha}{\partial x^\rho} \frac{\partial \tilde{x}^\beta}{\partial x^\sigma}$$

$$= e^{2\tilde{\omega}(x) + 2\omega(x)} \eta_{\rho\sigma}$$

Conclusion: finite conformal transformations are labeled by coord transf $x^\mu \to \tilde{x}^\mu = f^\mu(x)$, such that

$$(\star) \qquad \frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial \tilde{x}^\nu}{\partial x^\sigma} \eta_{\mu\nu} = e^{2\omega(x)} \eta_{\rho\sigma} \quad \text{for some } \omega.$$

In $D>2$ dimensions, it is easy to write down <u>all</u> finite conformal transformations.

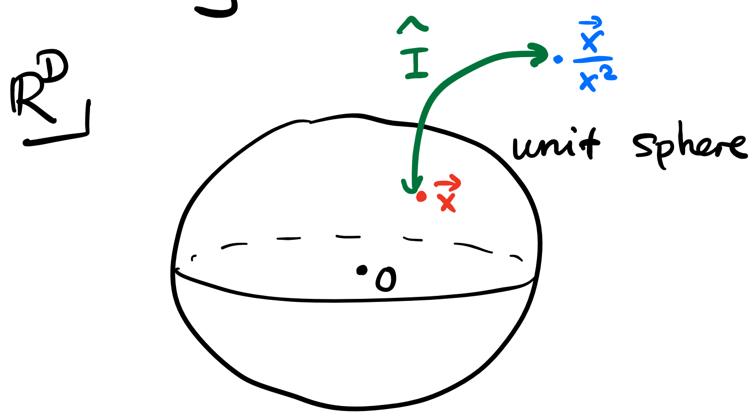
- Consider $\hat{I}: x^\mu \longmapsto \tilde{x}^\mu = \dfrac{x^\mu}{x^2}$ "inversion"

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho} = \frac{\delta^\mu_\rho}{x^2} - 2\frac{x^\mu x_\rho}{(x^2)^2} \qquad \text{(in Euclidean signature)}$$

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho}\frac{\partial \tilde{x}_\mu}{\partial x^\sigma} = \left(\frac{\delta^\mu_\rho}{x^2} - 2\frac{x^\mu x_\rho}{(x^2)^2}\right)\left(\frac{\delta_{\mu\sigma}}{x^2} - \frac{2x_\mu x_\sigma}{(x^2)^2}\right)$$

$$= \frac{\delta_{\rho\sigma}}{(x^2)^2} - \cancel{\frac{4x_\rho x_\sigma}{(x^2)^3}} + \cancel{\frac{4x^2 x_\rho x_\sigma}{(x^2)^4}} \propto \delta_{\rho\sigma} \ \checkmark$$

$\hat{I}$ is a conformal transformation in the sense of ⊛, even though it is not continuously connected to identity:



On the other hand,

$$\hat{I}\cdot(\text{translation})\cdot\hat{I} = \text{"special conformal transformation"}$$

is continuously connected to identity.

$$x^\mu \mapsto \frac{x^\mu}{x^2} \mapsto \frac{x^\mu}{x^2} + a^\mu$$

$$\mapsto \frac{\frac{x^\mu}{x^2} + a^\mu}{\left|\frac{\vec{x}}{x^2} + \vec{a}\right|^2} = \frac{x^\mu + a^\mu x^2}{1 + 2a\cdot x + a^2 x^2}$$

its infinitesimal form is

$$x^\mu \mapsto x^\mu + a^\mu x^2 - 2(a\cdot x)\, x^\mu + \mathcal{O}(a^2)$$

as seen previously from CKV.

- The Poincaré symmetries together with the inversion $\hat{I}$ generate <u>all</u> conformal transf's in $D>2$ dimensions.

- <u>Conformal vs Weyl</u> invariance

A classical field theory action

$$S[\phi] = \int d^D x\, \mathcal{L}(\phi, \partial_\mu \phi, \cdots)$$

can be extended to a covariant action $S[\phi; g]$ of the field $\phi(x)$ propagating in a curved spacetime metric $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$

e.g. free scalar (in Euclidean signature)

$$S[\phi] = \int d^Dx \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 \right)$$

$\rightsquigarrow$ covariant action

$$S[\phi; g] = \int d^Dx \sqrt{\det g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 + R(g)(\cdots) \right)$$

under metric variation

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$$

$$S[\phi; g + \delta g] - S[\phi] \equiv \delta_g S$$

$$= \frac{1}{2} \int d^Dx \sqrt{\det g} \, \delta g^{\mu\nu} \underbrace{T_{\mu\nu}}_{\text{stress-energy tensor}}.$$

When restricted to $g_{\mu\nu} = \delta_{\mu\nu}$ (or $\eta_{\mu\nu}$), this $T_{\mu\nu}$ agrees with the Noether current for translation symmetry (ex).

A local QFT may also be defined in curved spacetime, at least for small deformation away from flat space, i.e. $g_{\mu\nu} = \delta_{\mu\nu} + \delta g_{\mu\nu}$, via the insertion of

$$e^{-\delta_g S} = e^{-\frac{1}{2}\int d^D x \, \delta g^{\mu\nu} T_{\mu\nu}(x)}$$

in correlation functions.

now interpreted as a local operator

- Caution: potential singularities in colliding $T_{\mu\nu}$ insertions beyond 1^{st} order in δg, due to OPE as well as contact terms. (will revisit in the context of Weyl anomaly)

For a CFT, $T^\mu{}_\mu(x) = 0$ (modulo ϕ-E.O.M.)

$\Rightarrow$ $S[\phi; g]$ is invariant under $g \to g + \delta g$ when $\delta g_{\mu\nu} \propto g_{\mu\nu}$.

Equivalently, $S[\phi; g]$ is invariant

under $g_{\mu\nu}(x) \longrightarrow e^{2\omega(x)} g_{\mu\nu}(x)$

"Weyl transformation"

↑ acts on $g_{\mu\nu}$, possibly also on ϕ.

- A CFT would be Weyl-invariant when covariantly coupled to a background metric $g_{\mu\nu}$.

Example: free massless scalar field in D-dimensions

covariantized Euclidean action

$$S_0 = \int d^Dx \sqrt{\det g}\, g^{\mu\nu} \tfrac{1}{2} \partial_\mu\phi \partial_\nu\phi$$

w.r.t. variation of the background metric,

$$\delta_g S_0 = \int \Big(\delta\sqrt{\det g} \cdot \tfrac{1}{2}(\partial\phi)^2 + \sqrt{\det g}\, \delta g^{\mu\nu} \cdot \tfrac{1}{2} \partial_\mu\phi \partial_\nu\phi \Big)$$

$$= \frac{1}{2}\int \sqrt{\det g}\, \delta g^{\mu\nu} \underbrace{\left(\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2} g_{\mu\nu} (\partial\phi)^2\right)}_{= T_{\mu\nu}}$$

verify from Noether's procedure (HW) $\longrightarrow T_{\mu\nu}$

Note: $\quad T^{\mu}{}_{\mu} = \left(1-\frac{D}{2}\right)(\partial\phi)^2$

$\neq 0$ in $D>2$?

Is the free massless scalar field theory a CFT ?

Consider "improvement term"

$$S_1 = \int d^Dx \sqrt{\det g}\ \frac{1}{2} R(g)\, \phi^2$$

$$\delta_g S_1 = \frac{1}{2}\int \sqrt{\det g}\ \phi^2 \left(-\frac{1}{2}\delta g^{\mu\nu} g_{\mu\nu} R + \delta R\right)$$

$$\delta R = R_{\mu\nu}\,\delta g^{\mu\nu} + \nabla_\rho\left(g^{\mu\nu}\delta\Gamma^{\rho}_{\nu\mu} - g^{\mu\rho}\delta\Gamma^{\nu}_{\nu\mu}\right)$$

$$\overset{(ex)}{=} \frac{1}{2}\int \sqrt{\det g}\ \delta g^{\mu\nu}\left(-\partial_\mu\partial_\nu\phi^2 + g_{\mu\nu}\Box\phi^2\right)$$

$+$ (terms that vanish for $g_{\mu\nu} = \delta_{\mu\nu}$)

- Even though S_1 vanishes in flat spacetime, $\delta_g S_1$ would contribute to $T_{\mu\nu}$ in flat spacetime !

$$S = S_0 + a S_1$$

$$\Rightarrow T_{\mu\nu} \overset{\text{flat spacetime}}{=\!=} \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\delta_{\mu\nu}(\partial\phi)^2$$
$$+ a\left(-\partial_\mu\partial_\nu\phi^2 + \delta_{\mu\nu}\Box\phi^2\right)$$

"improvement term"

does not contribute to $P^\mu = \int d^{D-1}\vec{x}\; T^{0\mu}$.

but contributes to the trace

$$T^\mu{}_\mu = \left(1-\frac{D}{2}\right)(\partial\phi)^2 + a(D-1)\underbrace{\Box\phi^2}_{= 2(\phi\Box\phi + (\partial\phi)^2)}$$

$$= \left(1-\frac{D}{2} + 2(D-1)a\right)(\partial\phi)^2$$
$$+ 2(D-1)\,a\,\phi\underbrace{\Box\phi}_{\overset{\text{by E.O.M.}}{=} 0}$$

$\overset{\text{E.O.M.}}{\approx} 0$ if and only if $a = \dfrac{D-2}{4(D-1)}$.

In this case,

$$S = \int d^Dx \sqrt{\det g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{a}{2}R\phi^2\right)$$

is Weyl-invariant, in the sense that S is invariant under $\delta g_{\mu\nu} = 2\delta\omega\, g_{\mu\nu}$ and a suitable choice of $\delta\phi$ (due to the term $\propto$ EOM in $T^\mu{}_\mu$)

"conformal coupled scalar"

The statement $T^\mu{}_\mu = 0$ holds as an operator equation

$$\hat{T}^\mu{}_\mu(x) = 0$$

in the corresponding free massless scalar QFT.

———————————— "

Basic Consequences of Conformal Symmetry in QFT

1. Conformal invariance includes scaling i.e. dilatation invariance, generated by the Noether current

$$j_\mu = T_{\mu\nu} x^\nu .$$

This is possible only if there is no characteristic mass scale intrinsic to the theory.

[For instance, the 4D Yang-Mills theory is classically conformally invariant, but the quantum Yang-Mills theory has a characteristic mass scale Λ_{QCD} (as well as non-vanishing β-functions, hence $T^\mu{}_\mu \neq 0$), not a CFT.]

In such QFTs, there is no obvious notion of asymptotic states/particles, nor the S-matrix.

2. The physical observables include correlation functions of local operators.

i.e. $\hat{\mathcal{O}}_I(x)$ that obey microcausality

$$[\hat{\mathcal{O}}_I(x), \hat{\mathcal{O}}_J(y)] = 0 \quad \text{for } (x-y)^2 > 0,$$
$$x, y \in \mathbb{R}^{1,D-1}$$

Different versions of correlators:

- Wightman function

$$\langle \Omega | \hat{\mathcal{O}}_{I_1}(x_1) \cdots \hat{\mathcal{O}}_{I_n}(x_n) \rangle$$

- Time-ordered Green function

$$\langle \Omega | T \, \hat{\mathcal{O}}_{I_1}(x_1) \cdots \hat{\mathcal{O}}_{I_n}(x_n) | \Omega \rangle$$

- Euclidean Green function

$$\langle \mathcal{O}_{I_1}(x_1^E) \cdots \mathcal{O}_{I_n}(x_n^E) \rangle$$

$W \leftrightarrow E$ related by analytic continuation

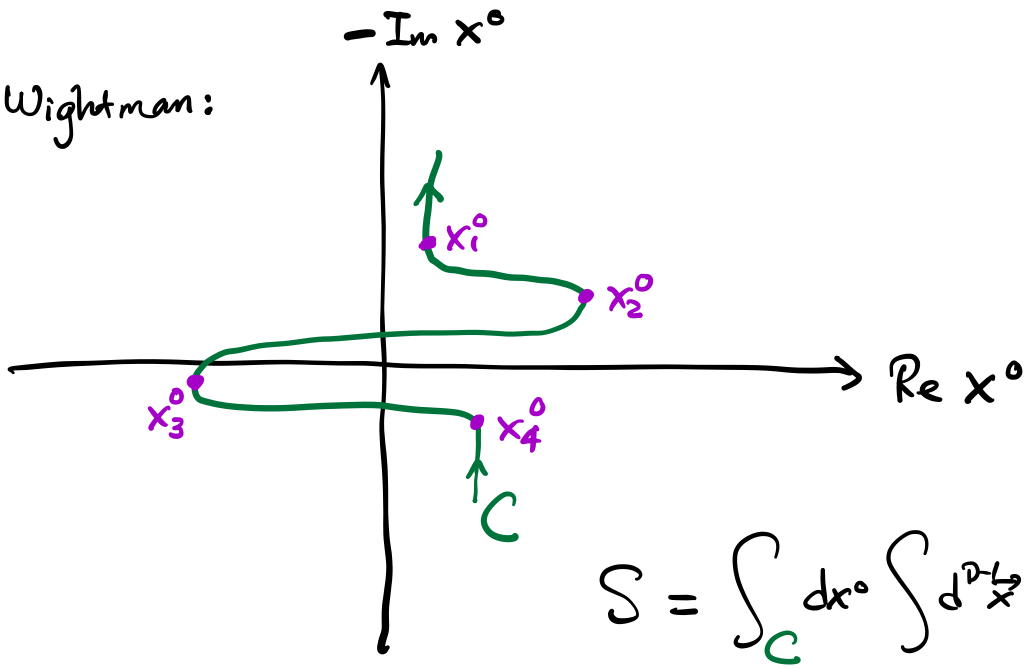
$$x_k^0 = -i \, x_k^D \qquad x^E \equiv (x^1, \cdots, x^D)$$

while maintaining

$$\mathrm{Im}\, x_1^0 < \mathrm{Im}\, x_2^0 < \cdots < \mathrm{Im}\, x_n^0$$

In path integral formulation

Wightman:

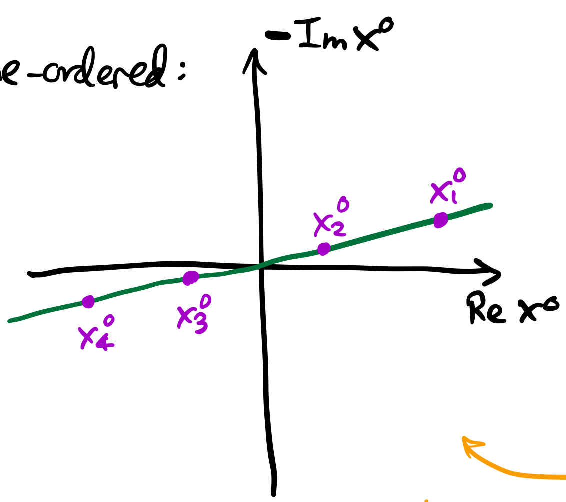


$$S = \int_C dx^0 \int d^{D-1}\vec{x}\, \mathcal{L}$$

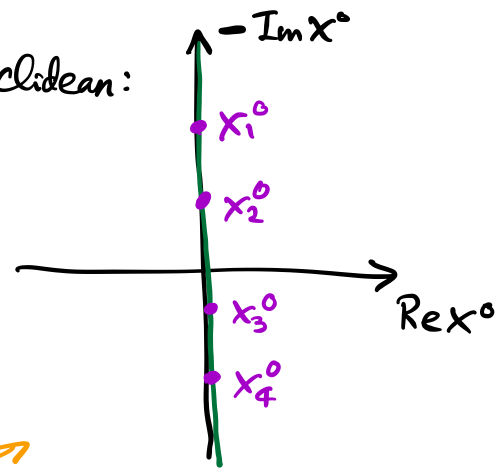
$$\langle \Omega | \hat{\mathcal{O}}_{I_1}(x_1) \cdots \hat{\mathcal{O}}_{I_n}(x_n) | \Omega \rangle$$

$$= \int D\phi \; e^{i \int_C dx^0 \int d\vec{x}\, \mathcal{L}} \; \mathcal{O}_{I_1}(x_1) \cdots \mathcal{O}_{I_n}(x_n)$$

time-ordered:



Euclidean:



"Wick rotation"

3. State/operator correspondence

- Assume that we can covariantize a CFT, e.g. via a Weyl-invariant action $S[\phi; g]$

background metric

Starting with Euclidean spacetime $\mathbb{R}^D$

$$ds^2 = \sum_{\mu=1}^{D} dx^\mu dx^\mu$$

$$= dr^2 + r^2 \underbrace{d\Omega^2}$$ in polar coord.

line element on unit sphere S^{D-1}

there is a Weyl transformation that changes the (background) metric to

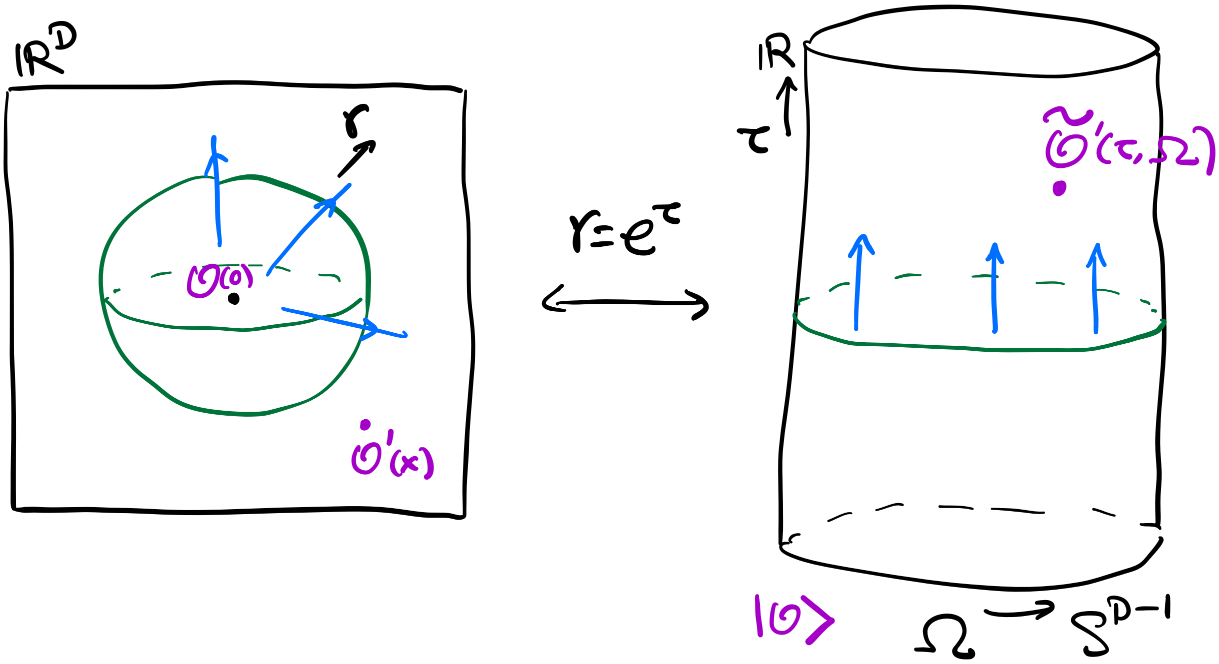
$$d\tilde{s}^2 = \frac{1}{r^2} ds^2$$

$$= d\tau^2 + d\Omega^2 , \qquad \tau \equiv \log r$$

This is the metric of $\mathbb{R} \times S^{D-1}$.

– "the (Euclidean) cylinder"

$$\mathbb{R}^D \xleftrightarrow{\text{Weyl}} \mathbb{R} \times S^{D-1}$$



What does this equivalence mean?

- The CFT on $\mathbb{R}^D$, characterized by the data of its spectrum of local operators and their (Euclidean) correlation functions, is equivalent to a CFT on $\mathbb{R} \times S^{D-1}$ (with suitable curvature coupling) with the same spectrum of local operators and identical (up to operator re-definition) correlation fn's.

Expect:

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{\mathbb{R}^D} = \langle \tilde{\mathcal{O}}_1(\tau_1, \Omega_1) \cdots \tilde{\mathcal{O}}_n(\tau_n, \Omega_n) \rangle_{\mathbb{R} \times S^{D-1}}$$

will understand the map $\mathcal{O} \to \tilde{\mathcal{O}}$ later

- The Weyl transf. is singular at $r=0$ $(\tau=-\infty)$

$\mathcal{O}(x=0)$ is mapped to a state $|\mathcal{O}\rangle$ of the CFT on $\underbrace{\mathbb{R}}_{\text{Eucl. time}} \times \underbrace{S^{D-1}}_{\text{space}}$

Expect:

$$\langle \mathcal{O}(0)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{\mathbb{R}^D}$$

$$= \langle \Omega | \, \hat{\tilde{\mathcal{O}}}_1(\tau_1, \Omega_1) \cdots \hat{\tilde{\mathcal{O}}}_n(\tau_n, \Omega_n) \, | \mathcal{O} \rangle$$

ground state on S^{D-1}

(assuming $\tau_1 > \tau_2 > \cdots > \tau_n$)

- Scaling transformation corresponds to the CKV

$$r \frac{\partial}{\partial r} \longleftrightarrow \frac{\partial}{\partial \tau}$$

i.e. time-translation on $\mathbb{R} \times S^{D-1}$

dilatation on $\mathbb{R}^D \longleftrightarrow$ Hamiltonian on S^{D-1}

- will elaborate on the meaning of this later.

Example: conformally coupled free scalar

$$S = \int d^D x \sqrt{\det g} \left(\tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + \tfrac{1}{2} a \, R(g) \, \phi^2 \right)$$

$$a = \frac{D-2}{4(D-1)}$$

on $\mathbb{R} \times S^{D-1}$, $\quad R(g) = (D-1)(D-2)$

$$\rightsquigarrow \quad S = \int d^D x \sqrt{\det g} \left(\tfrac{1}{2} (\partial \phi)^2 + \tfrac{1}{2} m^2 \phi^2 \right)$$

mass $\quad m = \sqrt{a \cdot R} = \dfrac{D-2}{2}$

(same value as the mass/scaling dimension of the field operator $\hat{\phi}$ on $\mathbb{R}^D$)

<u>Ex</u>: starting with the Euclidean action of a free massless scalar field $S_0[\phi]$ on $\mathbb{R}^D$, perform the coordinate change $x^\mu \rightsquigarrow (r, \Omega)$ together with a suitable (coord-dependent) rescaling of the field $\phi(x) \rightsquigarrow \tilde{\phi}(r, \Omega)$, show that $S_0[\phi] = S[\tilde{\phi}]$

on $\mathbb{R} \times S^{D-1}$, with conformal mass term.

What are the local operators?

$$\phi, \quad \partial_\mu \phi, \quad \partial_\mu \partial_\nu \phi, \quad \ldots$$

and their products – must regularize

e.g. $:\phi^2(x): \; \equiv \lim_{y \to x} \phi(x)\phi(y) - \underbrace{\phi(x)\phi(y)}_{G(x,y)}$

Green function

So that correlation functions like $\langle :\phi^2(x): \; \phi(y)\, \phi(z) \ldots \rangle$ are well-defined

Redundancy: E.O.M. $\partial^\mu \partial_\mu \phi(x) = 0$

$\Box \phi(x)$ vanishes as a local operator.

So are $:\phi \Box \phi:$, $:(\Box\phi)^2:$, etc.

It is illuminating to enumerate all linearly independent local operators via a sort of "operator partition function"

$$Z(q) \equiv \mathrm{Tr}_{\substack{\text{space of all} \\ \text{local operators} \\ \text{at a given point} \\ (\text{say } x=0)}} \; q^{\Delta}$$

scaling dimension

e.g. $\Delta_\phi = \frac{D-2}{2}$

$\Delta_{\partial_\mu \phi} = \frac{D-2}{2} + 1$

etc.

A complete basis of local operators in the free massless scalar field theory consist of "unordered words"

$$:\underbrace{\phi\ \partial_\mu\phi\ \partial_\nu\phi}_{\text{"letters"}}\cdots:$$

"letters", subject to $\Box\phi = 0$

Letter partition function (counting ϕ, $\partial_\mu\phi$, ...)

$$f(q) = \mathrm{Tr}_{\text{letters}}\, q^{\Delta}$$

$$= \underbrace{q^{\frac{D-2}{2}}}_{\Delta_\phi} \frac{1-q^2}{(1-q)^D}$$

remove $\Box\phi$

D different $\partial_\mu\phi$'s

$$\equiv \sum_{\Delta} a_\Delta\, q^\Delta .$$

The partition function of unordered words is related by

$$Z(q) = \prod_{\Delta} \frac{1}{(1-q^\Delta)^{a_\Delta}}$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f(q^n)\right).$$

The state/operator map would identify each local operator with a state of the conformally coupled scalar field theory on S^{D-1}.

Is this true ??

single particle partition function

$$\mathrm{Tr}_{\text{1-particle}}\, e^{-\beta \hat{H}}$$

relativistic particle of mass m

$$\hat{H} = \sqrt{-\nabla^2 + m^2}$$

Laplacian on S^{D-1}

acting on wave function $\psi \in L^2(S^{D-1})$

e.g. $D=3$, $L^2(S^2)$ is spanned by wave functions labeled by (ℓ, m)

$$\ell = 0, 1, 2, \cdots, \qquad m = -\ell, -\ell+1, \cdots, \ell$$

Laplacian $-\nabla^2 = \ell(\ell+1)$,

conformal mass $m = \frac{D-2}{2} = \frac{1}{2}$.

$$\hat{H} = \sqrt{\ell(\ell+1) + \tfrac{1}{4}} = \ell + \tfrac{1}{2}$$

$\Rightarrow$ single-particle partition function

$$\sum_{\ell=0}^{\infty} (2\ell+1)\, e^{-\beta(\ell+\frac{1}{2})} = f(e^{-\beta})$$

$f(q) = \dfrac{q^{\frac{1}{2}}(1-q^2)}{(1-q)^3}$ ✓

In QM, given a symmetry generator $\hat{Q}$ and an operator $\hat{\mathcal{O}}$, the infinitesimal symmetry variation of the operator is

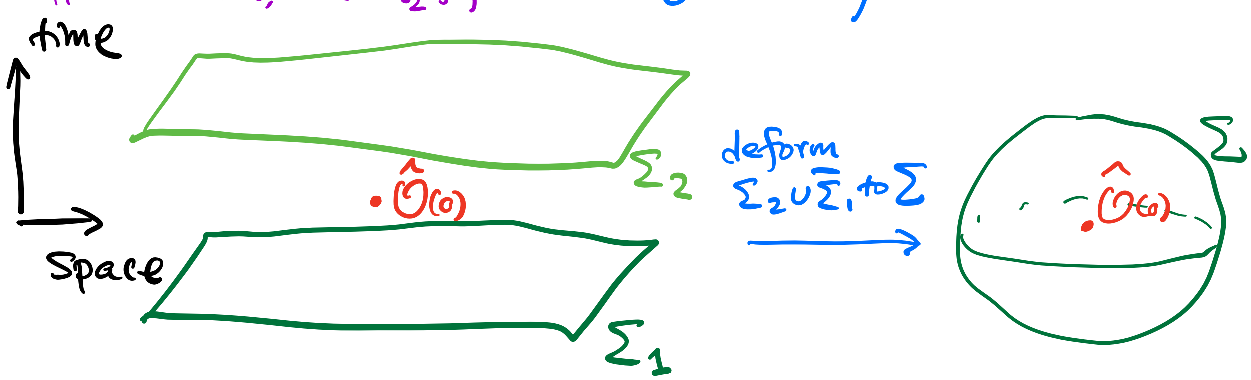
$$\delta_\epsilon \hat{\mathcal{O}} = i\epsilon\,[\hat{Q}, \hat{\mathcal{O}}].$$

In QFT, $\hat{Q}$ is obtained by integrating Noether current along a spatial slice:

$$\hat{Q}(\Sigma) = \int_\Sigma * \hat{j} = \int_\Sigma d\sigma\, \hat{j}_\mu n^\mu$$

$$\hat{j} \equiv \hat{j}_\mu(x)\, dx^\mu$$

(also applies to -Im time, with $Q = -i\int_\Sigma * j$)



$$[\hat{Q}, \hat{\mathcal{O}}(0)] = \hat{Q}(\Sigma_2)\hat{\mathcal{O}}(0) - \hat{\mathcal{O}}(0)\hat{Q}(\Sigma_1)$$

$$= T\, \hat{Q}(\Sigma)\, \hat{\mathcal{O}}(0)$$

time-ordering

Viewed as local operators inserted in a Euclidean Green function, we may also write

$$[Q, \mathcal{O}(0)] = Q(\Sigma) \cdot \mathcal{O}(0)$$

where Σ is a closed hypersurface in $\mathbb{R}^D$ that encloses 0.

Implication on correlation functions:

Assuming vacuum is sym-invariant, i.e.

$$\hat{Q} \, |\Omega\rangle = 0 ,$$

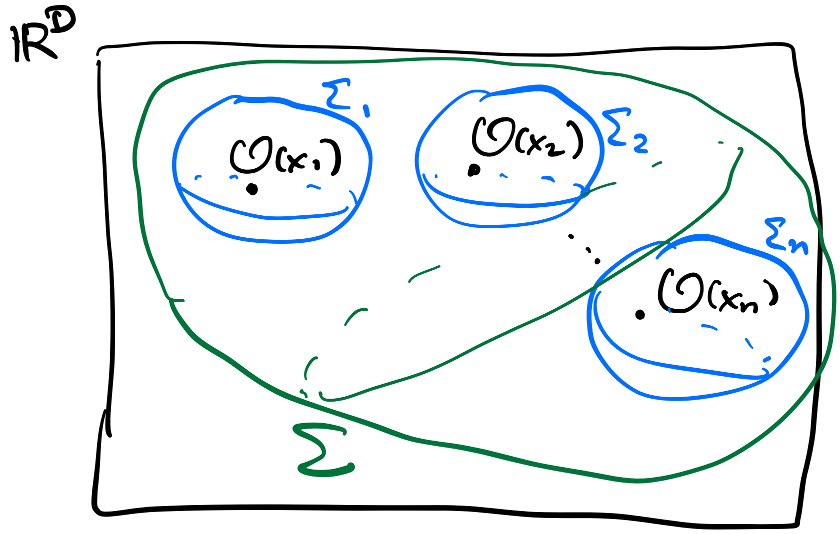
then

$$\langle \Omega | \, \delta_\epsilon \hat{\mathcal{O}}_1(x_1) \, \hat{\mathcal{O}}_2(x_2) \cdots \hat{\mathcal{O}}_n(x_n) | \Omega \rangle$$
$$+ \langle \Omega | \, \hat{\mathcal{O}}_1(x_1) \, \delta_\epsilon \hat{\mathcal{O}}_2(x_2) \cdots \hat{\mathcal{O}}_n(x_n) | \Omega \rangle$$
$$+ \cdots + \langle \Omega | \, \hat{\mathcal{O}}_1(x_1) \cdots \hat{\mathcal{O}}_{n-1}(x_{n-1}) \, \delta_\epsilon \hat{\mathcal{O}}_n(x_n) | \Omega \rangle = 0 ,$$

or for Euclidean Green function

$$\sum_{i=1}^{n} \langle \mathcal{O}_1(x_1) \cdots \underbrace{\delta_\epsilon \mathcal{O}_i(x_i)}_{= \, i\epsilon \, Q(\Sigma_i) \cdot \mathcal{O}_i(x_i)} \cdots \mathcal{O}_n(x_n) \rangle = 0 .$$

Σ_i encloses x_i



$\Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$

$\downarrow$ deform

Σ

The symmetry inv. of correlator amounts to

$$\langle Q(\Sigma)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = 0.$$

Σ encloses all of $x_1, \ldots, x_n$

"Ward identities"

Given conformal Killing vector $\varepsilon^\mu(x)$.

$\rightsquigarrow$ conformal symmetry generator

$$Q_\varepsilon = \int_\Sigma d\sigma\, n^\mu\, T_{\mu\nu}(x)\, \varepsilon^\nu(x).$$

Algebra obeyed by the Q_ε's ?

It is illuminating to inspect the CKVs under operator/state mapping

$$\mathbb{R}^D \xleftrightarrow{|x| = e^\tau} \mathbb{R}\times S^{D-1}$$

x^μ $\qquad$ (τ, Ω)

$\mathcal{O}(0)$ $\qquad$ $|\mathcal{O}\rangle \in \mathcal{H}_{S^{D-1}}$

inversion

$\hat{I}: x^\mu \mapsto \frac{x^\mu}{x^2}$ $\quad = \quad$ time reversal $(\tau,\Omega)\mapsto(-\tau,\Omega)$

dilatation

$\hat{D}$: CKV $x^\mu\partial_\mu$ $\quad = \quad$ time translation ∂_τ

rotation

$\hat{J}_{\mu\nu}$: CKV $x_\mu\partial_\nu - x_\nu\partial_\mu$ $\quad = \quad$ SO(D) rotation of S^{D-1}

translation

$$\hat{P}_\mu : \text{CKV } \partial_\mu = \left(\hat{n}_\mu \frac{\partial}{\partial r} + \frac{\delta_{\mu\nu} - \hat{n}_\mu \hat{n}_\nu}{r}\frac{\partial}{\partial \hat{n}_\nu}\right)$$

$$= e^{-\tau}\left(\hat{n}_\mu \partial_\tau + (\delta_{\mu\nu} - \hat{n}_\mu\hat{n}_\nu)\frac{\partial}{\partial \hat{n}_\nu}\right)$$

???

Special conformal transf.

$$\hat{K}_\mu : \text{CKV } \underbrace{\hat{I}\,\partial_\mu\,\hat{I}}_{x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu} = e^{\tau}\left(-\hat{n}_\mu \partial_\tau + (\delta_{\mu\nu} - \hat{n}_\mu\hat{n}_\nu)\frac{\partial}{\partial \hat{n}_\nu}\right)$$

Note that on the Lorentzian cylinder parameterize by (t, Ω), $t = -i\tau$

$\hat{P}_\mu$ and $\hat{K}_\mu$ are associated with (complex) CKV's that are complex conjugates of one another, and hence $\hat{P}_\mu^\dagger = \hat{K}_\mu$ as operators on $\mathcal{H}_{S^{D-1}}$.

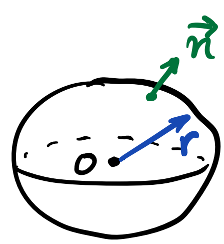
- Let us inspect an example of conformal generator more explicitly:

the dilatation $\hat{D}$ as a linear operator that acts on the space of local operators $\mathcal{O}(0)$ at the origin of $\mathbb{R}^D$ can be constructed as

due to Wick rotation

$$\hat{D} = -i \int_{S_r^{D-1}} d\sigma \, n^\mu \hat{T}_{\mu\nu}(x) x^\nu$$

$$= -i \, r^D \int_{S^{D-1}} d\Omega \, n^\mu n^\nu \hat{T}_{\mu\nu}(r \cdot \vec{n})$$



(independent of r by conservation law)

The Hamiltonian of the CFT on S^{D-1} space, on the other hand, is

$$\hat{H} = \int_{S^{D-1}} d\Omega \, \hat{T}_{tt}$$

$$= -\int_{S^{D-1}} d\Omega \, \hat{T}_{\tau\tau}$$

$t = -i\tau$

τ is the Euclidean time coord. on $\mathbb{R} \times S^{D-1}$

Identification:

$$\hat{D}\Big|_{\mathbb{R}^D} \longleftrightarrow i\hat{H}\Big|_{\text{cylinder}}$$

$$r^D \, \hat{T}_{\mu\nu}(r\vec{n}) \, n^\mu n^\nu \Big|_{\mathbb{R}^D} \xleftrightarrow{r = e^\tau} \hat{T}_{\tau\tau}(\tau, \Omega)\Big|_{\text{cylinder}}$$

weight factor

- the identification between the stress-energy tensor on $\mathbb{R}^D$ vs $\mathbb{R} \times S^{D-1}$ involves more than just the coord. transf of a tensor, but also a suitable weight factor.

With the identification

"space of local operators at the origin"

$$\cong$$

Hilbert space $\mathcal{H}_{S^{D-1}}$ of CFT on $\mathbb{R}_t \times S^{D-1}$

we will think of the conformal charges $\hat{Q}_\epsilon$ either as a linear operation on local operators $\hat{O}(0)$, or as an operator on $\mathcal{H}_{S^{D-1}}$.

Charge	CKV
$\hat{P}_\mu$	∂_μ
$\hat{J}_{\mu\nu}$	$x_\mu \partial_\nu - x_\nu \partial_\mu$
$\hat{D}$	$x^\mu \partial_\mu$
$\hat{K}_\mu$	$x^2 \partial_\mu - 2 x_\mu x^\nu \partial_\nu$

Their commutation relations

$$\left([\hat{Q}_{\varepsilon_1}, \hat{Q}_{\varepsilon_2}] = -i\, \hat{Q}_{[\varepsilon_1, \varepsilon_2]} \right)$$

Poincaré $\Big\{$

$$[\hat{P}_\mu, \hat{J}_{\nu\rho}] = -i\left(\eta_{\mu\nu} \hat{P}_\rho - (\nu \leftrightarrow \rho)\right)$$

$$[\hat{J}_{\mu\nu}, \hat{J}_{\rho\sigma}] = -i\left(\eta_{\nu\rho} \hat{J}_{\mu\sigma} - \eta_{\mu\rho} \hat{J}_{\nu\sigma} - (\rho \leftrightarrow \sigma)\right)$$

$$[\hat{D}, \hat{P}_\mu] = i \hat{P}_\mu$$

$$[\hat{D}, \hat{K}_\mu] = -i \hat{K}_\mu$$

$$[\hat{D}, \hat{J}_{\mu\nu}] = 0$$

$$[\hat{P}_\mu, \hat{K}_\nu] = 2i\left(\eta_{\mu\nu} \hat{D} - \hat{J}_{\mu\nu}\right)$$

Together they generate the "conformal algebra", as a Lie algebra $\simeq SO(D,2)$

$$\mathbb{J}_{\mu\nu} \equiv J_{\mu\nu}, \quad \mathbb{J}_{D,-1} \equiv D$$

$$\mathbb{J}_{D,\mu} \equiv \tfrac{1}{2}(P_\mu - K_\mu), \quad \mathbb{J}_{-1,\mu} \equiv \tfrac{1}{2}(P_\mu + K_\mu)$$

Then

$$[\hat{J}_{MN}, \hat{J}_{PQ}] = -i\left(\eta_{NP}\hat{J}_{MQ} - \eta_{MP}\hat{J}_{NQ} - (P \leftrightarrow Q)\right)$$

$$M, N, P, Q = -1, \underbrace{0, 1, \dots, D-1}_{\text{Lorentz}}, D$$

$$(\eta_{MN}) = \begin{pmatrix} -1 & & & & 0 \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

- States of the CFT on S^{D-1} are organized in representations of the conformal algebra

$$\mathcal{H}_{S^{D-1}} \simeq \bigoplus_i V_i^{irr}$$

irreducible representation of the conformal algebra

each irrep V_i^{irr} is "generated" by a lowest energy state $|\phi_i\rangle$, which by state/operator map also corresponds to a local operator $\hat{\phi}_i(0)$.

Such a state (or operator) ϕ_i is called a "conformal primary (state/operator)".

Properties of <u>conformal primary</u>:

- Suppose $|\phi\rangle$ is a conformal primary, which is also assumed to be an energy eigenstate, with

$$\hat{H}\,|\phi\rangle = \Delta\,|\phi\rangle.$$

$\hat{D} \equiv i\hat{H}$,

$$\hat{D}\,|\phi\rangle = i\Delta\,|\phi\rangle.$$

Since $[\hat{D}, \hat{K}_\mu] = -i\,\hat{K}_\mu$,

$\hat{K}_\mu$ lowers energy by 1.

$\hat{K}_\mu|\phi\rangle \in V^{\text{irr}}$, but $|\phi\rangle$ is the lowest energy state in V^{irr} by assumption

$$\Rightarrow \quad \boxed{\hat{K}_\mu\,|\phi\rangle = 0}$$

(equivalent to definition of primary)

$\hat{P}_\mu$, on the other hand, raises the energy by 1. What is $\hat{P}_\mu |\phi\rangle$?

under state/operator map,

$$|\phi\rangle \longleftrightarrow \hat{\phi}(0)$$

$$\hat{P}_\mu |\phi\rangle \longleftrightarrow \hat{P}_\mu \cdot \hat{\phi}(0) = [\hat{P}_\mu, \hat{\phi}(0)]$$

$$\hat{P}_\mu = \int_{S^{D-1}} d\sigma \, T_{\mu\nu} n^\nu$$

Recall

$$\hat{\phi}(x+a) = e^{-i\hat{P}\cdot a} \, \hat{\phi}(x) \, e^{i\hat{P}\cdot a}$$

$$[\hat{P}_\mu, \hat{\phi}(x)] = i \partial_\mu \hat{\phi}(x).$$

So

$$\hat{P}_\mu |\phi\rangle = i |\partial_\mu \phi\rangle$$

$|\partial_\mu \phi\rangle$ is the state that maps to the local operator $\partial_\mu \hat{\phi}(0)$.

Similarly, suppose $\phi_\alpha(x)$ obey Lorentz transformation

$$U(\Lambda)\,\hat{\phi}_\alpha(x)\,U(\Lambda)^{-1} = (R(\Lambda))_\alpha{}^\beta\,\hat{\phi}_\beta(\Lambda x)$$

where $U(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}\right)$

$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ for infinitesimal $\omega^\mu{}_\nu$

$$R(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$$

as matrices $S^{\mu\nu}$ obey same commutation relations as $\hat{J}^{\mu\nu}$.

i.e. $[S^{\mu\nu}, S^{\rho\sigma}] = -i\left(\eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} - (\rho\leftrightarrow\sigma)\right)$

Then

$$[\hat{J}^{\mu\nu}, \hat{\phi}_\alpha(0)] = -(S^{\mu\nu})_\alpha{}^\beta\,\hat{\phi}_\beta(0)$$

$$\Rightarrow\quad \hat{J}^{\mu\nu}|\phi_\alpha\rangle = -(S^{\mu\nu})_\alpha{}^\beta\,|\phi_\beta\rangle.$$

Thus, we have determined completely how a conformal primary $|\phi\rangle$ transforms under the conformal algebra:

$$\hat{D}\,|\phi\rangle = i\Delta_\phi\,|\phi\rangle$$

scaling dimension of ϕ

$$\hat{K}_\mu\,|\phi\rangle = 0$$

$$\hat{P}_\mu\,|\phi\rangle = i\,|\partial_\mu\phi\rangle$$

$$\hat{J}_{\mu\nu}\,|\phi_\alpha\rangle = -(S_{\mu\nu})_\alpha{}^\beta\,|\phi_\beta\rangle$$

(for scalar ϕ, $S=0$)

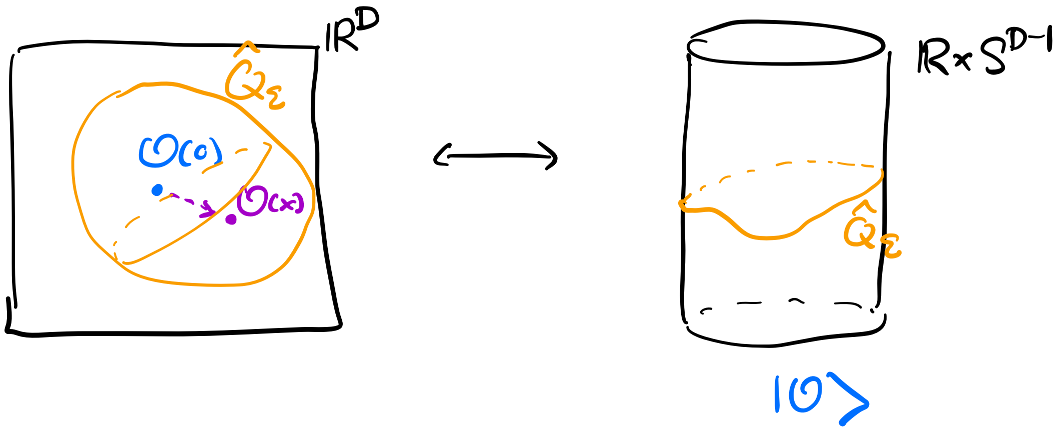
All other states related to ϕ (or ϕ_α for non-scalar primaries) by the conformal algebra are descendants obtained by repeatedly acting on ϕ with $\hat{P}_\mu$'s.

e.g. $\hat{P}_\mu\,|\phi\rangle$, $\hat{P}_\mu\hat{P}_\nu\,|\phi\rangle$, ...

Under state/operator map,

$$\hat{P}_\mu\,|\phi\rangle \longleftrightarrow i\partial_\mu\phi(0)$$

$$\hat{P}_\mu\hat{P}_\nu\,|\phi\rangle \longleftrightarrow i\partial_\mu\, i\partial_\nu\phi(0), \quad \text{etc.}$$



Can view $\mathcal{O}(x)$ as a limit of local operators at the origin,

$$\mathcal{O}(x) = e^{x^\mu \partial_\mu}\, \mathcal{O}(0)$$
$$= \mathcal{O}(0) + x^\mu \partial_\mu \mathcal{O}(0) + \tfrac{1}{2} x^\mu x^\nu \partial_\mu \partial_\nu \mathcal{O}(0) + \ldots$$

$\updownarrow$

state $|\mathcal{O}(x)\rangle = e^{-i x^\mu \hat{P}_\mu} |\mathcal{O}\rangle$.

An infinitesimal conformal transformation $\hat{Q}_\Sigma$ acts on $\mathcal{O}(x)$ as

$$\hat{Q}_\Sigma \cdot \mathcal{O}(x) \longleftrightarrow \text{state } \hat{Q}_\Sigma |\mathcal{O}(x)\rangle$$

$$\shortparallel$$

$$-i \int_\Sigma d\sigma\, n^\mu T_{\mu\nu}(y)\, \varepsilon^\nu(y) \cdot \mathcal{O}(x)$$

for any $\mathcal{O}$ (not necessarily a primary)

How does a primary $\phi(x)$ transform under the conformal symmetry?

- we can deduce $\hat{Q}_\varepsilon \cdot \phi(x)$ from $\hat{Q}_\varepsilon |\phi(x)\rangle$

- since we already know $\hat{Q}_\varepsilon |\phi(0)\rangle$, we can deduce $\hat{Q}_\varepsilon |\phi(x)\rangle$ using $|\phi(x)\rangle = e^{-ix^\mu \hat{P}_\mu} |\phi(0)\rangle$ and the commutation relation between $\hat{P}_\mu$ and the conformal generators.

For instance,

$$\hat{K}_\mu |\phi(x)\rangle = e^{-ix\cdot\hat{P}} \underbrace{e^{ix\cdot\hat{P}} \hat{K}_\mu e^{-ix\cdot\hat{P}}} |\phi(0)\rangle$$

// repeatedly applying conformal algebra relations

$$\cancel{\hat{K}_\mu} - 2x_\mu x\cdot\hat{P} + x^2 \hat{P}_\mu + 2x^\nu \hat{J}_{\mu\nu} - 2x_\mu \hat{D}$$

annihilates $|\phi\rangle$

$$= e^{-ix\cdot\hat{P}} \left(-2x_\mu x\cdot\hat{P} + x^2 \hat{P}_\mu - 2x^\nu S_{\mu\nu} - 2i x_\mu \Delta\right) |\phi(0)\rangle$$

(suppressed Lorentz indices)

The complete set of infinitesimal conformal transformation of primary $\phi(x)$ are:

$$\hat{K}_\mu \cdot \phi(x) = \left(-2i\, x_\mu\, x\cdot\partial + i x^2 \partial_\mu - 2x^\nu S_{\mu\nu} - 2i\, x_\mu \Delta\right)\phi(x)$$

$$\hat{P}_\mu \cdot \phi(x) = i\,\partial_\mu \phi(x)$$

$$\hat{D} \cdot \phi(x) = \left(i\, x^\mu \partial_\mu + i\Delta\right)\phi(x)$$

$$\hat{J}_{\mu\nu} \cdot \phi(x) = \left(i(x_\mu \partial_\nu - x_\nu \partial_\mu) - S_{\mu\nu}\right)\phi(x)$$

What about finite conformal transf.?

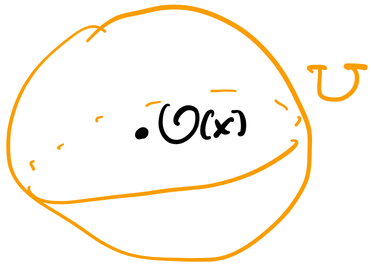
composed out of infinitesimal transf's.

labeled by coord map

$$x^\mu \rightsquigarrow \tilde{x}^\mu(x)$$

such that

$$\delta_{\mu\nu} \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} \propto \delta_{\alpha\beta}$$



$$\mathcal{O}(x) \rightsquigarrow \tilde{\mathcal{O}}(x) = U \cdot \mathcal{O}(x)$$

What is U?

- infinitesimal case $\tilde{x}^\mu = x^\mu + \varepsilon^\mu(x)$

$$U = e^{i\hat{Q}_\varepsilon}$$

$$U \cdot \phi(x) = \phi(x) + \delta\phi(x)$$

where

$$\delta\phi(x) = i\,\hat{Q}_\varepsilon \cdot \phi(x)$$

a linear combination of

$\hat{P}_\mu \cdot \phi$, $\hat{K}_\mu \cdot \phi$, $\hat{D} \cdot \phi$, $\hat{J}_{\mu\nu} \cdot \phi$

For $\varepsilon_\mu(x) = a_\mu + b_{[\mu\nu]} x^\nu + \lambda x_\mu$

$$- 2(c \cdot x) x_\mu + c_\mu x^2$$

$$\hat{Q}_\varepsilon = a \cdot \hat{P} - \tfrac{1}{2} b_{\mu\nu} \hat{J}^{\mu\nu} + \lambda \hat{D} + c \cdot \hat{K}$$

$$\begin{aligned}\delta\phi(x) = \Big(&- a \cdot \partial + b^{\mu\nu} x_\mu \partial_\nu + \tfrac{i}{2} b^{\mu\nu} S_{\mu\nu} \\ &- \lambda\, x \cdot \partial - \lambda \Delta \\ &- x^2\, c \cdot \partial + 2\, c \cdot x\; x \cdot \partial \\ &- 2i\, c^\mu x^\nu S_{\mu\nu} + 2\, c \cdot x\, \Delta \Big)\phi(x)\end{aligned}$$

$$= \Big(- \varepsilon^\mu(x)\, \partial_\mu - \frac{\Delta}{D} \partial_\mu \varepsilon^\mu(x)$$

$$- \frac{i}{2} \partial_\mu \varepsilon_\nu(x)\, S^{\mu\nu} \Big) \phi(x) .$$

Finite form :

$$U \cdot \phi(x) = \tilde{\phi}(x) , \quad \text{such that}$$

$$\textcircled{*} \quad \boxed{\tilde{\phi}(\tilde{x}) = \left(\det \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \right)^{-\frac{\Delta}{D}} \overline{R}\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu} \right) \cdot \phi(x)}$$

where $\overline{R}$ is defined as follows :

$$M^\mu{}_\nu \equiv \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \quad \text{obeys}$$

$$M^T \cdot M \propto I \qquad (\text{Euclidean})$$

$$\text{or} \quad M^T \eta M \propto \eta \qquad (\text{Lorentzian})$$

by assumption of conformal transf.

$$\Lambda \equiv (\det M)^{-\frac{1}{D}} \cdot M$$

$$\text{obeys} \quad \Lambda^T \cdot \Lambda = I \qquad (\text{Euclidean})$$

$$\text{or} \quad \Lambda^T \eta \Lambda = \eta \qquad (\text{Lorentzian})$$

$\overline{R}(M) := R(\Lambda)$, where $R(\Lambda)$ is the representation matrix associated with the

SO(D) rotation or Lorentz transformation $\Lambda^\mu{}_\nu$, namely

$$R(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right)$$

for $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$

with infinitesimal $\omega^\mu{}_\nu$.

(★) is the general conformal transformation property of a primary!

[Conformal transf. of descendants can be deduced from that of the primary and repeated use of the conformal algebra commutation relations.]

As a special case, if the primaries in question transform in a tensor rep. of the Lorentz group, e.g.

$$\phi_{\mu_1 \cdots \mu_s}(x)$$

The conformal transf. (★) can be more explicitly written as

$$U \cdot \phi_{\mu_1 \cdots \mu_s}(x) = \tilde{\phi}_{\mu_1 \cdots \mu_s}(x)$$

with

$$\tilde{\Phi}_{\mu_1\cdots\mu_s}(\tilde{x})$$

$$= \left(\det \frac{\partial \tilde{x}}{\partial x}\right)^{-\frac{\Delta - s}{D}} \frac{\partial x^{\nu_1}}{\partial \tilde{x}^{\mu_1}} \cdots \frac{\partial x^{\nu_s}}{\partial \tilde{x}^{\mu_s}} \phi_{\nu_1\cdots\nu_s}(x)$$

[or equivalently,

$$\tilde{\phi}^{\mu_1\cdots\mu_s}(\tilde{x}) = \left(\det \frac{\partial \tilde{x}}{\partial x}\right)^{-\frac{\Delta + s}{D}} \frac{\partial \tilde{x}^{\mu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial \tilde{x}^{\mu_s}}{\partial x^{\nu_s}} \phi^{\nu_1\cdots\nu_s}(x)$$]

- Consequences of unitarity:

$$\hat{\mathcal{O}}(0) \longleftrightarrow |\mathcal{O}\rangle \in \mathcal{H}_{S^{D-1}}$$

$$\| a^\mu \hat{P}_\mu |\mathcal{O}\rangle \|^2 \geq 0 \quad \text{for any } a^\mu$$

i.e. the Hermitian matrix

$$\langle \mathcal{O} | \hat{P}_\mu^\dagger \hat{P}_\nu |\mathcal{O}\rangle \quad \text{is positive semidefinite}$$

notation "≥ 0"

$\Vert$ recall $\hat{P}_\mu^\dagger = \hat{K}_\mu$

$$\langle \mathcal{O} | \hat{K}_\mu \hat{P}_\nu |\mathcal{O}\rangle$$

Now suppose $\mathcal{O} = \phi$ is a primary i.e. $\hat{K}_\mu |\phi\rangle = 0$

It follows that

$$0 \preceq \langle \phi | \hat{K}_\mu \hat{P}_\nu | \phi \rangle$$
$$= \langle \phi | [\hat{K}_\mu, \hat{P}_\nu] | \phi \rangle$$
$$= -2i \langle \phi | (\delta_{\nu\mu} \hat{D} - \hat{J}_{\nu\mu}) | \phi \rangle$$

$\hat{D}|\phi\rangle = i\Delta_\phi |\phi\rangle$.

$\Rightarrow \quad \Delta_\phi \geq$ any eigenvalue of the matrix

$$\frac{i \langle \phi | \hat{J}_{\mu\nu} | \phi \rangle}{\langle \phi | \phi \rangle}$$

e.g. if ϕ is a scalar primary

$\hat{J}_{\mu\nu} |\phi\rangle = 0 \Rightarrow \Delta_\phi \geq 0$.

· There are more constraints!

Consider

$$\mathcal{M}^{(2)}_{\alpha\beta,\mu\nu} \equiv \langle \phi | K_\alpha K_\beta P_\mu P_\nu | \phi \rangle$$

$\mathcal{M}^{(2)}$ is also positive semidefinite.

Using $[K, P] = \ldots$ repeatedly, and $K_\mu |\phi\rangle = 0$, we have in the case of a scalar primary ϕ,

$$\mathcal{M}^{(2)}_{\alpha\beta,\mu\nu} = 4\left[(\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu})\Delta_\phi(\Delta_\phi+1) - \delta_{\alpha\beta}\delta_{\mu\nu}\Delta_\phi \right] \langle\phi|\phi\rangle.$$

contracting both side with $\delta^{\alpha\beta}\delta^{\mu\nu}$, we derive

notation: spacetime dim $D \to d$ to avoid confusion

$$0 \leq \delta^{\alpha\beta}\delta^{\mu\nu}\mathcal{M}^{(2)}_{\alpha\beta,\mu\nu}$$
$$= 4\left[2d\,\Delta_\phi(\Delta_\phi+1) - d^2\Delta_\phi \right]$$

Two possibilities

(1) $\Delta_\phi = 0 \quad \Rightarrow \quad \hat{P}_\mu |\phi\rangle = i|\partial_\mu\phi\rangle = 0$

i.e. $\partial_\mu \phi(x) = 0$,

$\phi(x) \propto 1$ (identity op.)

(2) $\Delta_\phi \geq \dfrac{d-2}{2}$ "unitarity bound" (for a scalar primary)

Note: the unitarity bound is saturated

$\Delta_\phi = \frac{d-2}{2}$ if and only if

$$\delta^{\alpha\beta}\delta^{\mu\nu}\mathcal{M}^{(2)}_{\alpha\beta,\mu\nu} = 0$$

$$\Longleftrightarrow \quad \hat{P}^\mu \hat{P}_\mu |\phi\rangle = 0$$

i.e. $\partial^\mu \partial_\mu \phi(x) = 0$

a.k.a. massless Klein-Gordon eqn.

"free massless scalar field"

- For non-scalar primaries, e.g. ones in a rank-s symmetric traceless tensor representation of $SO(d)$ rotation

"spin-s" $\quad \phi_{\mu_1 \cdots \mu_s}$

[symmetric in $(\mu_1, \cdots, \mu_s)$, with $\phi_{\mu_1\mu_2\cdots\mu_s}\delta^{\mu_1\mu_2} = 0$]

unitarity bound:

$$\Delta_\phi \geq s + d - 2. \qquad (\text{HW})$$

Now we turn to the constraints of conformal symmetries on correlation functions

(under the assumption that the vacuum is conformal invariant, i.e. the conformal symmetry is not spontaneously broken)

Conformal transformation

$$\mathcal{O}(x) \rightsquigarrow \tilde{\mathcal{O}}(x) = U \cdot \mathcal{O}(x)$$

$$U = e^{i \hat{Q}_\varepsilon(\Sigma)}$$

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \langle \tilde{\mathcal{O}}_1(x_1) \cdots \tilde{\mathcal{O}}_n(x_n) \rangle$$

same coordinates as before transf.

Equivalently, in its infinitesimal form, the conformal Ward identity reads

$$\sum_{i=1}^{n} \langle \mathcal{O}_1(x_1) \cdots \delta \mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n) \rangle = 0 .$$

$$\delta \hat{\mathcal{O}}_i(x_i) = i \left[\hat{Q}_\varepsilon , \hat{\mathcal{O}}_i(x_i) \right] .$$

Example :

Scalar primary $\phi(x)$

under conformal transf.

$$\phi(x) \rightsquigarrow \tilde{\phi}(x).$$

$$\tilde{\phi}(\tilde{x}) = \left(\det \frac{\partial \tilde{x}}{\partial x}\right)^{-\frac{\Delta}{d}} \phi(x)$$

- 1-point function

$$\langle \phi(x) \rangle \equiv \langle \phi \rangle \text{ by translation invariance}$$

under dilatation $\tilde{x}^\mu = \lambda x^\mu$

$$\tilde{\phi}(\lambda x) = \lambda^{-\Delta} \phi(x)$$

$$\Rightarrow \quad \langle \tilde{\phi} \rangle = \lambda^{-\Delta} \langle \phi \rangle$$

$\| $ dilatation symmetry

$\langle \phi \rangle$

$$\Rightarrow \quad \langle \phi \rangle = 0 \quad \text{unless} \quad \Delta = 0$$

$\Downarrow$

$\phi(x) \propto id$

in a unitary CFT

Conclusion: a nontrivial (non-identity) conformal primary field operator has vanishing vacuum expectation value.

- 2-point function

$$\langle \phi_1(x)\, \phi_2(y) \rangle \overset{\text{conf. sym.}}{=\!=} \langle \tilde{\phi}_1(x)\, \tilde{\phi}_2(y) \rangle$$

$$\Big\|$$

$$\langle \tilde{\phi}_1(\tilde{x})\, \tilde{\phi}_2(\tilde{y}) \rangle \cdot \left(\det \frac{\partial \tilde{x}}{\partial x}\right)^{\frac{\Delta_1}{d}} \left(\det \frac{\partial \tilde{y}}{\partial y}\right)^{\frac{\Delta_2}{d}}$$

In particular, if there is a conformal transformation such that

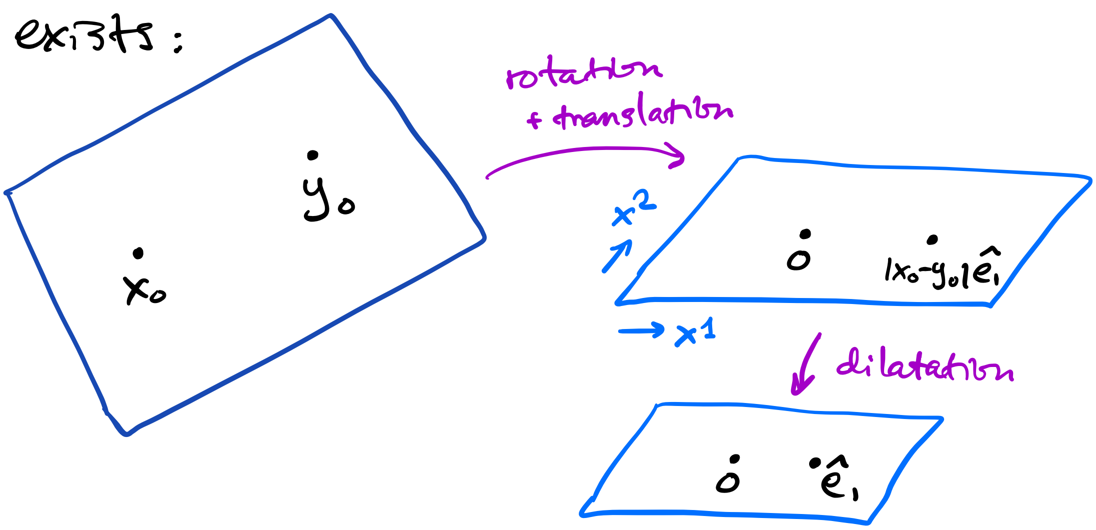
$$x_0 \longmapsto \tilde{x}(x_0) = \hat{e}_1 \qquad \text{(unit vector in } x_1 \text{ direction)}$$

$$y_0 \longmapsto \tilde{y}(y_0) = 0$$

then

$$\langle \tilde{\phi}_1(x_0)\tilde{\phi}_2(y_0) \rangle = \langle \tilde{\phi}_1(\hat{e}_1)\, \tilde{\phi}_2(0) \rangle \cdot \left(\det \frac{\partial \tilde{x}}{\partial x}\Big|_{x_0}\right)^{\frac{\Delta_1}{d}} \left(\det \frac{\partial \tilde{y}}{\partial y}\Big|_{y_0}\right)^{\frac{\Delta_2}{d}}$$

Indeed, such a conformal transformation exists:



By Euclidean / Poincaré symmetry,

we have $\langle \phi_1(x)\, \phi_2(y) \rangle = G(|x-y|)$.

WLOG, can assume $x_0 = \lambda \hat{e}_1$, $y = 0$

$\tilde{x}^\mu(x) = \lambda^{-1} x^\mu$,

so that $\tilde{x}(x_0) = \hat{e}_1$, $\tilde{y}(y_0) = 0$

$$\Rightarrow \quad G(|x_0 - y_0|) = G(\lambda)$$

$$= G(1) \cdot \lambda^{-\Delta_1 - \Delta_2}$$

Thus, we find

some normalization constant

$$\langle \phi_1(x)\, \phi_2(y) \rangle = \frac{N_{12}}{|x-y|^{\Delta_1+\Delta_2}}.$$

This is not the end of the story, however. What about conformal transf. that leave both 0 and $\hat{e}_1$ fixed? e.g. special conformal transf. combined with dilatation

$$x^\mu \rightsquigarrow \tilde{x}^\mu = \lambda \frac{x^\mu + a^\mu x^2}{1 + 2a\cdot x + a^2 x^2}$$

with $a^\mu = (\lambda - 1)\, \delta^\mu_1$.

$$\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \lambda \frac{\delta^\mu{}_\nu + 2a^\mu x_\nu}{1 + 2a\cdot x + a^2 x^2}$$

$$- 2\lambda \frac{(x^\mu + a^\mu x^2)(a^\nu + a^2 x^\nu)}{(1 + 2a\cdot x + a^2 x^2)^2}$$

$$\left.\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right|_0 = \lambda \delta^\mu{}_\nu \,.$$

$$\left.\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right|_{\hat{e}_1} = \lambda \frac{\delta^\mu{}_\nu + 2(\lambda-1)\delta^\mu{}_1 \delta^1{}_\nu}{\lambda^2}$$

$$- 2\lambda \frac{\lambda^2(\lambda-1)\delta^\mu{}_1 \delta^1{}_\nu}{\lambda^4}$$

$$= \lambda^{-1} \delta^\mu{}_\nu .$$

Applying to $\langle \phi(\hat{e}_1)\phi(0)\rangle$, we have

$$G(1) = G(1) . \left(\det \frac{\partial \tilde{x}}{\partial x}\Big|_{\hat{e}_1}\right)^{-\frac{\Delta_1}{d}} \left(\det \frac{\partial \tilde{x}}{\partial x}\Big|_0\right)^{-\frac{\Delta_2}{d}}$$

$$= G(1) . \lambda^{\Delta_1 - \Delta_2}$$

$\Rightarrow$ $N_{12} = 0$ unless $\Delta_1 = \Delta_2$!

[Alternatively, we can derive this result by checking invariance under inversion

$$x^\mu \mapsto \tilde{x}^\mu = \frac{x^\mu}{x^2} .$$

$$\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \frac{\delta^\mu{}_\nu}{x^2} - \frac{2x^\mu x_\nu}{(x^2)^2}, \qquad \det\left(\frac{\partial \tilde{x}}{\partial x}\right) = -(x^2)^{-d}.$$

Assuming ϕ is parity even,

$$\tilde{\phi}(\tilde{x}) = \left|\det\left(\frac{\partial \tilde{x}}{\partial x}\right)\right|^{-\frac{\Delta}{d}} \phi(x).$$

$$\Rightarrow G(|x-y|) = G(|\tilde{x}-\tilde{y}|)\cdot (x^2)^{-\Delta_1}(y^2)^{-\Delta_2}$$

i.e.
$$\frac{N_{12}}{|x-y|^{\Delta_1+\Delta_2}} = \frac{N_{12}}{|\tilde{x}-\tilde{y}|^{\Delta_1+\Delta_2}}\cdot (x^2)^{-\Delta_1}(y^2)^{-\Delta_2}$$

$$= N_{12}\frac{|x|^{\Delta_1+\Delta_2}|y|^{\Delta_1+\Delta_2}}{|x-y|^{\Delta_1+\Delta_2}}(x^2)^{-\Delta_1}(y^2)^{-\Delta_2}$$

$$\Rightarrow N_{12} = 0 \quad \text{unless} \quad \Delta_1 = \Delta_2.$$

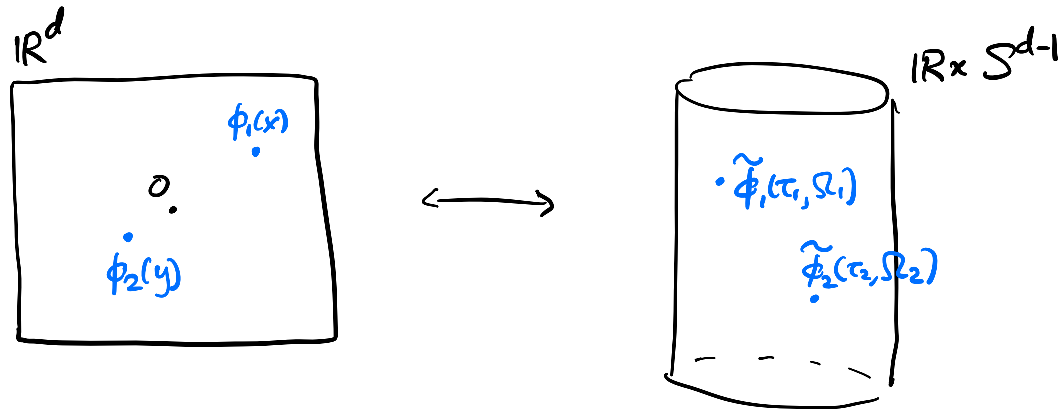
In conclusion, for scalar primaries ϕ_1, ϕ_2, the two-point function

$$\langle \phi_1(x)\,\phi_2(y)\rangle = \frac{N_{12}}{|x-y|^{2\Delta}}$$

for $\Delta_{\phi_1} = \Delta_{\phi_2} = \Delta$,

and vanishes if $\Delta_{\phi_1} \neq \Delta_{\phi_2}$.

Interpretation via state/operator map



The map between operators on $\mathbb{R}^d$ (x^μ) and $\mathbb{R}\times S^{d-1}$ (τ, Ω) can be viewed as a conformal transf.

$$x^\mu \to \tilde{x}^\mu = (\tau, \Omega)$$

$$\det \frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \frac{d\tau}{dr} \cdot \frac{dr\, d^{d-1}\Omega}{d^d x}$$

$$= r^{-d}.$$

field op. on cylinder

$$\Rightarrow \quad \phi(x) = r^{-\Delta} \tilde{\phi}(\tau, \Omega), \qquad r = |x|.$$

Thus we can express the 2-pt function of scalar primaries as

$$\langle \phi_1(x) \phi_2(y) \rangle$$
$$= |x|^{-\Delta_1} |y|^{-\Delta_2} \langle \tilde{\phi}_1(\tau_1, \Omega_1) \tilde{\phi}_2(\tau_2, \Omega_2) \rangle.$$

In other words,

$$\langle \Omega | \tilde{\phi}_1(\tau_1, \Omega_1)\, \tilde{\phi}_2(\tau_2, \Omega_2) | \Omega \rangle$$

$$= N_{12} \frac{|x|^{\Delta} |y|^{\Delta}}{|x-y|^{2\Delta}} \qquad \text{for } \Delta_{\phi_1} = \Delta_{\phi_2} = \Delta$$

Note that the RHS is manifestly time-translation invariant : $\tau_1 \to \tau_1 + u, \quad \tau_2 \to \tau_2 + u,$

$$x \to e^u x, \quad y \to e^u y.$$

as well invariant under time-reversal

$$\tau_1 \to -\tau_1, \quad \tau_2 \to -\tau_2,$$

$$x \to \frac{x}{|x|^2}, \quad y \to \frac{y}{|y|^2}.$$

Consider:

$$\phi_2(0) = \lim_{x \to 0} \phi_2(x) = \lim_{\tau \to -\infty} e^{-\Delta \tau} \tilde{\phi}_2(\tau, \Omega)$$

state

$$|\phi_2\rangle = \lim_{\tau \to -\infty} e^{-\Delta \tau} \tilde{\phi}_2(\tau, \Omega) |\Omega\rangle$$

angular dependence drops out in $\tau \to -\infty$ limit

In a similar manner,

$$\langle\!\langle \phi_1 | \equiv \lim_{\tau \to +\infty} e^{\Delta\tau} \langle \Omega | \tilde{\Phi}_1(\tau, \Omega)$$

"BPZ conjugate"

Note: $\langle\!\langle \phi_1^\dagger | = \langle \phi_1 |$ is the usual Hermitian conjugate of $|\phi_1\rangle$, where $\hat{\Phi}_1^\dagger(0)$ is the Hermitian conjugate of $\hat{\Phi}_1(0)$ as a local field operator.

$$\langle\!\langle \phi_1 | \phi_2 \rangle = \lim_{\substack{\tau_1 \to \infty \\ \tau_2 \to -\infty}} e^{\Delta\tau_1 - \Delta\tau_2} \langle \Omega | \tilde{\Phi}_1(\tau_1, \Omega_1) \tilde{\Phi}_2(\tau_2, \Omega_2) | \Omega \rangle$$

$$= N_{12} .$$

Thus, we see that the 2-pt function coeff. N_{12} has the interpretation of "BPZ inner product" $\langle\!\langle \phi_1 | \phi_2 \rangle$.

- We can find a basis of Hermitian field operators ϕ_i such that $\langle\!\langle \phi_i | \phi_j \rangle = \langle \phi_i | \phi_j \rangle = \delta_{ij}$.

  $i, j = 1, 2, \ldots$

- Two-point functions of non-scalar primaries are constrained analogously by conformal symmetry up to an overall coefficient.

A special example: the stress-energy tensor $T_{\mu\nu}(x)$, symmetric traceless, also obey $\partial_\mu T^{\mu\nu}=0$ $(\Rightarrow \Delta_T = d)$

(HW)

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = \frac{C_T}{|x|^{2d}} I_{\mu\nu,\rho\sigma}(x),$$

where

$$I_{\mu\nu,\rho\sigma}(x) = \frac{1}{2}\left(I_{\mu\rho}(x) I_{\nu\sigma}(x) + (\rho\leftrightarrow\sigma)\right) - \frac{1}{d}\delta_{\mu\nu}\delta_{\rho\sigma},$$

$$I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}.$$

In this case, there is a canonical normalization of $T_{\mu\nu}(x)$, and thus C_T is unambiguously defined and is intrinsic to the CFT.

- 3-point function

For three scalar primaries ϕ_1, ϕ_2, ϕ_3 with scaling dim $\Delta_1, \Delta_2, \Delta_3$.

Consider $\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3) \rangle$

For any three distinct points

$$x_1, x_2, x_3 \in \mathbb{R}^d$$

and $$y_1, y_2, y_3 \in \mathbb{R}^d,$$

we can find a conformal map $x^\mu \rightsquigarrow \tilde{x}^\mu(x)$ such that $$y_i = \tilde{x}(x_i)$$

$$\begin{aligned}
&\langle \phi_1(y_1)\phi_2(y_2)\,\phi_3(y_3) \rangle \\
&= \langle \tilde{\phi}_1(y_1)\,\tilde{\phi}_2(y_2)\,\tilde{\phi}_3(y_3) \rangle \\
&= \prod_{i=1}^{3} \left(\det \frac{\partial \tilde{x}}{\partial x}\Big|_{x_i} \right)^{-\frac{\Delta_i}{d}} \langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3) \rangle
\end{aligned}$$

This fixes the 3-pt function up to its overall normalization.

Result: (ex)

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3) \rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}\, |x_{23}|^{\Delta_2+\Delta_3-\Delta_1}\, |x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}$$

Once we have chosen a basis of $\phi_i(x)$'s, say with $\langle\langle \phi_i | \phi_j \rangle = \delta_{ij}$, the 3-pt fn coefficients C_{ijk} are unambiguously defined.

"structure constants" of the CFT.

A similarly analysis extends to 3-pt functions of arbitrary (non-scalar) primaries, with the extra complication that there can be more than 1 tensor structure involved:

$$\langle \phi_{i\,\alpha}(x_1)\, \phi_{j\,\beta}(x_2)\, \phi_{k\,\gamma}(x_3) \rangle$$

indices of some rep R_1, R_2, R_3 of $SO(d)$

$$= \sum_r C^{(r)}_{ijk}\, I^{(r)}_{\alpha\beta\gamma}(x_{12}, x_{13})$$

a finite sum

structure constants

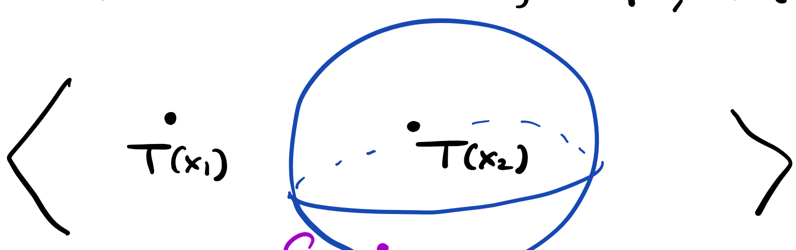
known functions

For instance, the 3-pt function of the stress-energy tensor is constrained by conformal symmetry to be of the form

$$\langle T_{\mu\nu}(x_1)\, T_{\rho\sigma}(x_2)\, T_{\tau\kappa}(x_3)\rangle = \sum_{r=1}^{3} a_r \underbrace{I^{(r)}_{\mu\nu\rho\sigma\tau\kappa}(x_{12}, x_{13})}_{\text{3 known functions}}$$

[Osborn, Petkos hep-th/9307010]

Note: a linear combination of a_1, a_2, a_3 is fixed in terms of C_T, via



$$\hat{Q}_\varepsilon = \int d\sigma\, T_{\mu\nu}(x_3)\, n^\mu \varepsilon^\nu(x_3)$$

$$= \langle T(x_1)\, \underbrace{\hat{Q}_\varepsilon \cdot T(x_2)}_{\text{descendant of } T} \rangle$$

determined by $\langle T(x_1)\, T(x_2)\rangle$ via the conformal algebra.

As another example, the 3-pt function of a pair of scalar primaries ϕ_1, ϕ_2, and the stress-energy tensor $T_{\mu\nu}$ is constrained by conformal symmetry and current conservation to be

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, T_{\mu\nu}(x_3) \rangle$$

$$= \frac{C_{12T}}{|x_{13}|^d\, |x_{23}|^d\, |x_{12}|^{2\Delta-d}} \left(\frac{X_\mu X_\nu}{X^2} - \frac{\delta_{\mu\nu}}{d} \right)$$

where $X^\mu \equiv \dfrac{x_{13}^\mu}{x_{13}^2} - \dfrac{x_{23}^\mu}{x_{23}^2}$,

for $\Delta_{\phi_1} = \Delta_{\phi_2} \equiv \Delta$,

and vanishes if $\Delta_{\phi_1} \neq \Delta_{\phi_2}$.

Furthermore, conformal Ward identity determines C_{12T} in terms of $N_{12} = \langle\langle \phi_1 | \phi_2 \rangle$:

Set $x_2 = 0$, and consider

$$\Big\langle \phi_1(x_1)\, \underbrace{(-i) \int_{S^{d-1}_{|x_3|=r}} d\sigma\, \frac{x_3^\mu}{|x_3|}\, x_3^\nu\, T_{\mu\nu}(x_3)\, \phi_2(0)}_{\hat{D}\cdot\phi_2(0) \,=\, i\Delta\phi_2(0)} \Big\rangle$$

$$= -i\int_{S^{d-1}_{|x_3|=r}} d\sigma \, \frac{x_3^\mu}{|x_3|} x_3^\nu \cdot \left(\frac{X_\mu X_\nu}{X^2} - \frac{\delta_{\mu\nu}}{d} \right) \frac{C_{12T}}{|x_{13}|^d |x_3|^d |x_1|^{2\Delta - d}}$$

$$X_\mu = \frac{(x_{13})_\mu}{x_{13}^2} + \frac{x_{3\mu}}{x_3^2}$$

This expression by construction must be independent of r. Taking $r = |x_3| \to 0$ limit, the RHS becomes

$$-i\int_{S^{d-1}_{\text{unit}}} d\sigma \, n^\mu n^\nu \left(\frac{X_\mu X_\nu}{X^2} - \frac{\delta_{\mu\nu}}{d} \right) \frac{C_{12T}}{|x_1|^{2\Delta}}$$

$$X_\mu \to \frac{x_{3\mu}}{x_3^2} = \frac{n_\mu}{|x_3|}$$

$$= -i \, A_{d-1} \left(1 - \frac{1}{d} \right) \frac{C_{12T}}{|x_1|^{2\Delta}}$$

$(d-1)$-dim'l "area" of the unit S^{d-1}

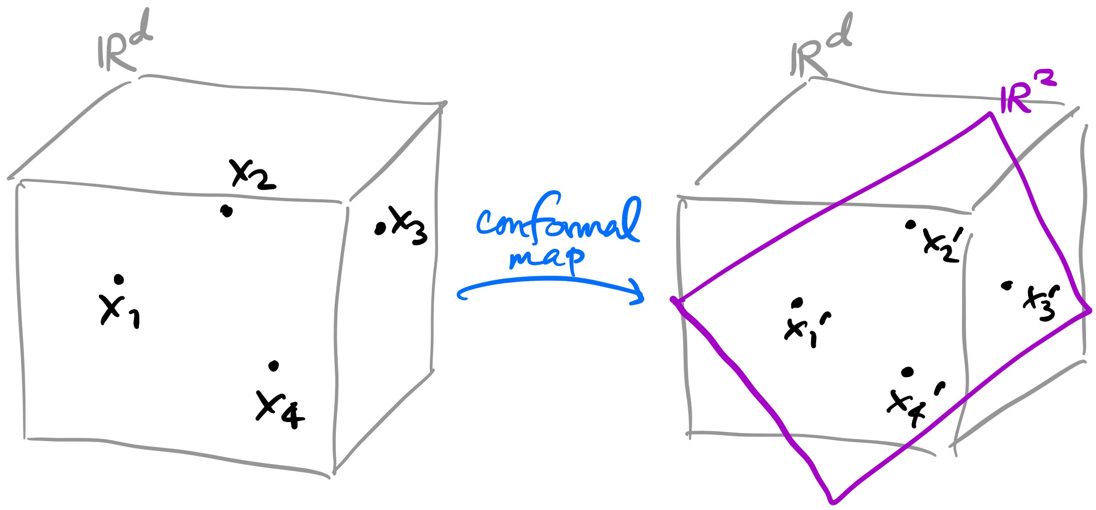
$$A_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

$$\Rightarrow \quad C_{12T} = -\frac{\Delta N_{12}}{\left(1 - \frac{1}{d}\right) A_{d-1}} .$$

- 4-point functions

of scalar primaries $\phi_1, \ldots, \phi_4$

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi(x_3)\, \phi(x_4) \rangle$$



Can find a conformal map that takes 4 arbitrary distinct points $x_1, \ldots, x_4 \in \mathbb{R}^d$ to $x_1', \ldots, x_4'$ that lie in a plane $\mathbb{R}^2 \subset \mathbb{R}^d$.

Within the plane, we can parameterize $x_1', \ldots, x_4'$ in terms of complex coords $z_1, \ldots, z_4 \in \mathbb{C}$. Further conformal map that leaves the plane invariant are of the form

$$(*) \qquad z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0.$$

using (*), we can move z_1, z_3, z_4 to $0, 1, \infty$ respectively, leaving $z_2 \equiv z$ as the only free variable.

More generally, the cross ratio

$$z \equiv \frac{z_{12} z_{34}}{z_{13} z_{24}}$$

is invariant under conformal mapping (*)

Going back to $\mathbb{R}^d$ coords, we can write

$$u \equiv \frac{x_{12}^2 \, x_{34}^2}{x_{13}^2 \, x_{24}^2} = |z|^2$$

and

$$v \equiv \frac{x_{14}^2 \, x_{23}^2}{x_{13}^2 \, x_{24}^2} = |1-z|^2$$

as a pair of independent conformally invariant cross ratios.

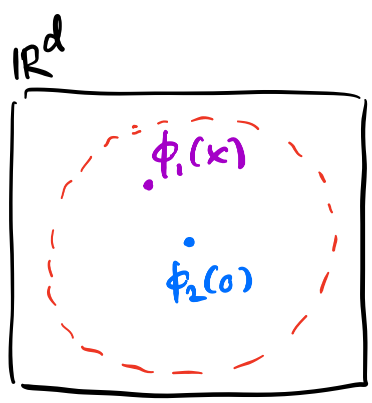
- Conformal Ward identity fixes the 4-point function of scalar primaries up to a single a priori unknown function of u, v :

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3)\, \phi_4(x_4) \rangle$$

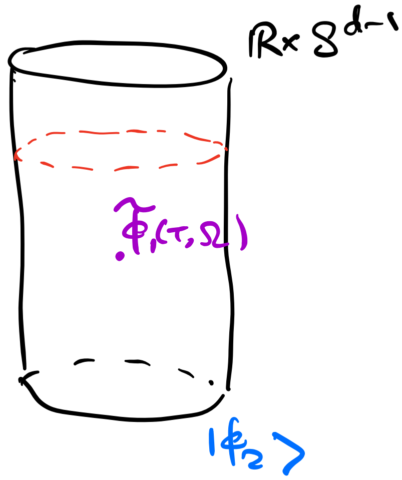
$$= |x_{12}|^{-\Delta_1-\Delta_2}\, |x_{34}|^{-\Delta_3-\Delta_4} \cdot \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_1-\Delta_2} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_3-\Delta_4} \cdot f_{1234}(u,v)$$

This will be the main subject of our investigation !

· The Operator Product Expansion



$\simeq$



$$\Phi_1(\tau,\Omega)\,|\phi_2\rangle \equiv |\Psi\rangle \in \mathcal{H}_{S^{d-1}}$$

What operator does $|\Psi\rangle$ correspond to?

OPE:

$|\Psi\rangle \longleftrightarrow$ a limit of local operators at the origin

i.e.

$$\phi_1(x)\,\phi_2(0) = \lim_{n\to\infty} \Psi_n(0)$$

for some sequence of local operators Ψ_n!

We can expand Ψ_n on a basis of primaries and their descendants (at the origin)

$$\phi_1(x)\,\phi_2(0) = \sum_i \Big[C_{12}^{\;i}(x)\,\Phi_i(0) + \text{descendants of } \Phi_i(0) \Big] \qquad (*)$$

a priori sum over a complete basis of primaries, organized according to increasing scaling dim Δ_i.

We will for now focus on the case where ϕ_1, ϕ_2 are scalar primaries.

The Φ_i's appearing on the RHS, on the other hand, need not be scalar primaries. e.g. a tensor primary $\Phi_{\mu_1 \cdots \mu_s}$ may appear as

RHS of ⊛

$$\supset \underbrace{C^{\mu_1 \cdots \mu_s}(x)}_{\text{What function of } x\,?} \Phi_{\mu_1 \cdots \mu_s}(0) + \text{descendants of } \Phi$$

$SO(d)$ rotation sym

$$\Rightarrow C^{\mu_1 \cdots \mu_s}(x) = (\text{scalar function of } x^2) \cdot x^{\mu_1} x^{\mu_2} \cdots x^{\mu_s} + (\text{products of } \delta^{\mu_i \mu_j} \text{ and } x^{\mu_k})$$

The only Φ's in irreducible reps of $SO(d)$ that can show up are symmetric traceless tensors.

[e.g. an anti-sym tensor $\Phi_{[\mu\nu]}$ vanishes when contracted with $\delta^{\mu\nu}$ or $x^\mu x^\nu$]

Their corresponding rep. of conformal algebra is simply labeled by the "spin" s.

$$\phi_1(x)\,\phi_2(0) = \sum_i C_{12}^{i,\mu_1\cdots\mu_{s_i}}(x)\,\Phi_{i,\mu_1\cdots\mu_{s_i}}(0)$$

$$+ \text{descendants}$$

$$C_{12}^{i,\mu_1\cdots\mu_{s_i}}(x) = \underbrace{\lambda_{12}^{i}}_{\substack{\text{a constant that}\\ \text{depends on the choice}\\ \text{of } \phi_1,\, \phi_2, \text{ and } \Phi_i}} \underbrace{\left.\frac{x^{\mu_1}x^{\mu_2}\cdots x^{\mu_{s_i}}}{|x|^{\Delta_1+\Delta_2-\Delta_{\Phi_i}+s_i}}\right|_{\text{traceless}}}_{\text{fixed by conformal symmetry}}$$

. descendants of $\Phi_{i,\mu_1\cdots\mu_{s_i}}(0)$ appear in the $\phi_1(x)\,\phi_2(0)$ OPE with higher powers of x.

Compare : the 3-pt function

$$\langle \phi_1(x)\,\phi_2(0)\,\Phi_j^{\nu_1\cdots\nu_{s_j}}(y)\rangle$$

$$= \sum_i C_{12}^{i,\mu_1\cdots\mu_{s_i}}(x)\,\underbrace{\langle \Phi_{i,\mu_1\cdots\mu_{s_i}}(0)\,\Phi_j^{\nu_1\cdots\nu_{s_j}}(y)\rangle}_{\propto N_{ij}\,\delta_{s_i s_j}}$$

$$+ (\text{terms that are subleading in } x\to 0 \text{ limit})$$

$$= \underbrace{\sum_i \lambda_{12}^{i} N_{ij}}_{\text{structure constant}} \times \underbrace{F^{\nu_1\cdots\nu_{s_j}}_{\Delta_1,\Delta_2;\Delta_{\Phi_j}}(x,y)}_{\text{fixed by conformal symmetry}}$$

The x-dependence of $\langle \phi_1(x)\, \phi_2(0)\, \Phi_j^{\nu_1 \cdots \nu_{s_j}}(y) \rangle$ also determines the coeffs of the descendants of $\Phi_{i,\mu_1 \cdots \mu_{s_i}}(0)$ that appear in the OPE:

$$\phi_1(x)\, \phi_2(0)$$

$$= \sum_i \lambda_{12}^{\ i}\, \underbrace{\tilde{C}_{12}^{\,i,\mu_1 \cdots \mu_{s_i}}(x, \partial_y)}_{\text{fixed by conformal sym.}}\, \Phi_{i,\mu_1 \cdots \mu_{s_i}}(y) \Big|_{y=0}$$

Example: $S=0$ primaries in the OPE

$$\phi_1(x)\, \phi_1(0) \supset \lambda_{11}^{\ i}\, \tilde{C}_{11}^{\,i}(x, \partial_y)\, \phi_i(y) \Big|_{y=0}$$

Compare the 3-pt fn

$$\langle \phi_1(x)\, \phi_1(0)\, \phi_i(w) \rangle = \lambda_{11}^{\ j}\, \tilde{C}_{11}^{\,j}(x, \partial_y)\, \underbrace{\langle \phi_j(y) \phi_i(w) \rangle}_{= \frac{N_{ij}}{|y-w|^{2\Delta_0}}} \Big|_{y=0}$$

$$\| $$

$$\frac{\lambda_{11i}}{|x|^{2\Delta_1 - \Delta_i}\, |w|^{\Delta_i}\, |x-w|^{\Delta_i}}$$

Expand around $x=0$:

$$\text{LHS} = \boxed{\lambda_{11i}}\ |x|^{\Delta_i - 2\Delta_1}\, |w|^{-2\Delta_i} \left(1 + \Delta_i \frac{x \cdot w}{w^2} + \cdots \right)$$

$$\text{RHS} = \boxed{\lambda_{11}^{\ j} N_{ij}}\ \tilde{C}_{11}^{\,i}(x, -\partial_w)\, |w|^{-2\Delta_i}$$

$$\Rightarrow \tilde{C}_{11}^{\,i}(x, \partial_y) = |x|^{\Delta_i - 2\Delta_1} \left(1 + \tfrac{1}{2} x \cdot \partial_y + \cdots \right)$$

• Representation-theoretic view of OPE

a primary Φ (not necessarily scalar) together with its descendants span a subspace $V_\Phi \subset \mathcal{H}_{CFT\,(\text{on } S^{d-1})}$ that forms an irreducible representation of the conformal algebra $SO(d,2)$ (based on CKV's on the Lorentzian cylinder)

While $\Phi(0) \longleftrightarrow |\Phi\rangle$ is a lowest energy state of V_Φ, $\Phi(x) \longleftrightarrow e^{-ix\cdot\hat{P}}|\Phi\rangle$ is also a state in V_Φ.

Given a pair of (not necessarily scalar) primaries Φ_1, Φ_2, the product operator $\Phi_1(x_1)\,\Phi_2(x_2)$ lives in a subspace $V_{\Phi_1\Phi_2}$ of $\mathcal{H}_{CFT}$, with

homomorphism of representations

$$\begin{array}{ccc} V_{\Phi_1} \otimes V_{\Phi_2} & \longrightarrow & V_{\Phi_1\Phi_2} \\ \wr\wr & & \\ \bigoplus_i V_{R_i} & & \end{array}$$

as representations of $SO(d,2)$

irreps

The OPE can be organized according to such decomposition of $V_{\Phi_1} \otimes V_{\Phi_2}$ into irreps of the conformal group, i.e. primaries V_{Φ_i} and their descendants.

Caution: in general, each irrep V_{R_i} may appear in the tensor product rep. with multiplicity – meaning the primary Φ_i and its descendants may appear through more than 1 linearly independent set of coefficients.

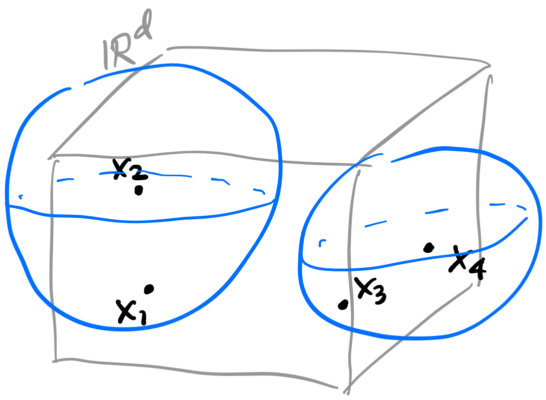
This multiplicity is the same as the number of linearly independent functions of x, y that are allowed by conformal symmetry in the 3-point function

$$\langle \Phi_1(x)\, \Phi_2(0)\, \Phi_i(y) \rangle .$$

[when Φ_1 and Φ_2 are both scalar primaries, Φ_i necessarily a traceless symmetric tensor, this multiplicity is equal to 1.]

· Implication of OPE on the 4-pt function
(focus on scalar primaries for now)

$$\langle \underbrace{\phi_1(x_1)\,\phi_2(x_2)}_{\text{OPE}}\;\underbrace{\phi_3(x_3)\,\phi_4(x_4)}_{\text{OPE}} \rangle$$



$$= \sum_{i,j} \lambda_{12}^{i}\, \lambda_{34}^{j}\, \tilde{C}_{12}^{i,\mu_1\cdots\mu_{s_i}}(x_{12},\partial_{y_1})\, \tilde{C}_{34}^{j,\nu_1\cdots\nu_{s_j}}(x_{34},\partial_{y_2})$$

$$\cdot \underbrace{\langle \Phi_{i,\mu_1\cdots\mu_{s_i}}(y_1)\, \Phi_{j,\nu_1\cdots\nu_{s_j}}(y_2) \rangle}_{\propto N_{ij}\,\delta_{s_i s_j}} \Big|_{y_1=x_2,\; y_2=x_4}$$

$$= \sum_{i,j} \lambda_{12}^{i}\, \lambda_{34}^{j}\, N_{ij}$$

$$\times |x_{12}|^{-\Delta_1-\Delta_2}\, |x_{34}|^{-\Delta_3-\Delta_4} \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_1-\Delta_2} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_3-\Delta_4}$$

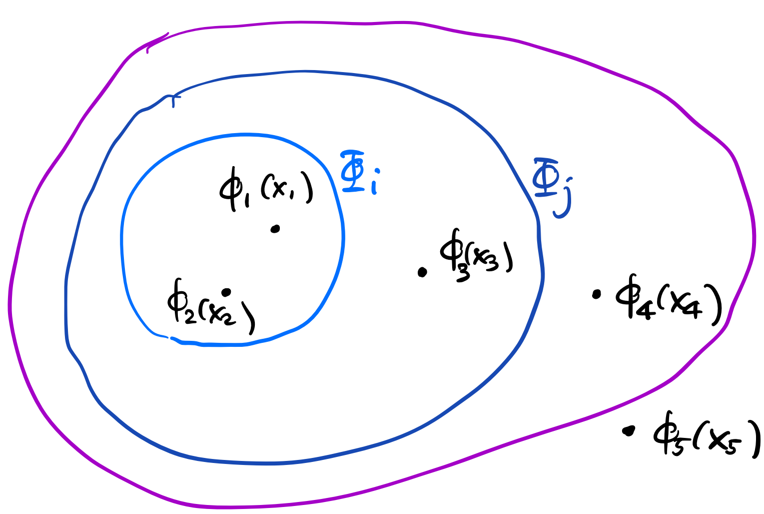
$$\times G_{1234}^{i,s_i}(u,v)$$

- $G^{i,s_i}_{1234}(u,v)$ is called the "conformal block"

  it depends on $\phi_1, \dots, \phi_4$, Φ_i only through their representation of conformal algebra, i.e. Δ_1, Δ_2, Δ_3, Δ_4 and Δ_{Φ_i}, s_i.

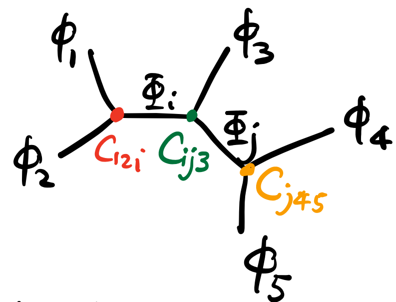
[Note: when Φ_i is a scalar, i.e. $s_i = 0$,
$\lambda_{12}^{\ \ i} N_{ij} \equiv \lambda_{12j}$ is the structure constant appearing in $\langle \phi_1 \phi_2 \phi_j \rangle$, and is symmetric w.r.t. permutation of the three operators ϕ_1, ϕ_2, ϕ_j]

In fact, by repeatedly applying the OPE (assuming convergence - will revisit) we can obtain all Green functions from the data of structure constants $C^{(r)}_{ij\,k}$ that appear in 3-pt functions of primaries:



Can represent such an expansion schematically as

$$\langle \phi_1 \cdots \phi_5 \rangle = \sum_{i,j}$$

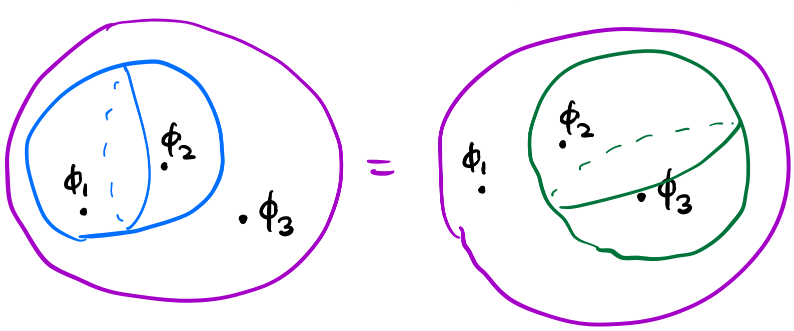


where we have chosen the basis primaries Φ_i to be orthonormal i.e. $N_{ij} = \delta_{ij}$.

"CFT data" - the spectrum of primaries (scaling dimension i.e. "conformal weight" and SO(D) irrep) together with the structure constants $C^{(r)}_{ijk}$ (in an orthonormal basis with $N_{ij} = \delta_{ij}$) determine all Green functions, thereby the CFT completely.

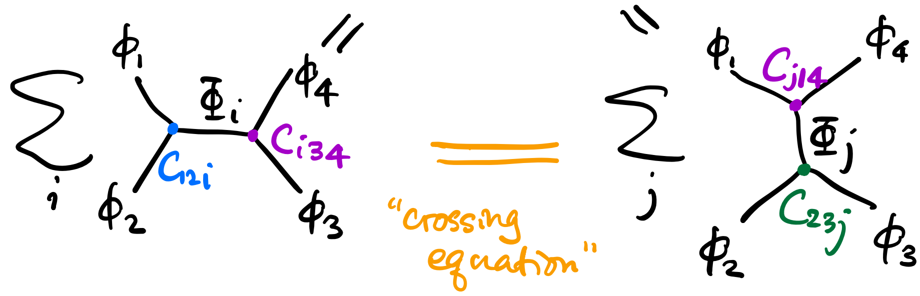
Key consistency condition:

the OPE must be associative



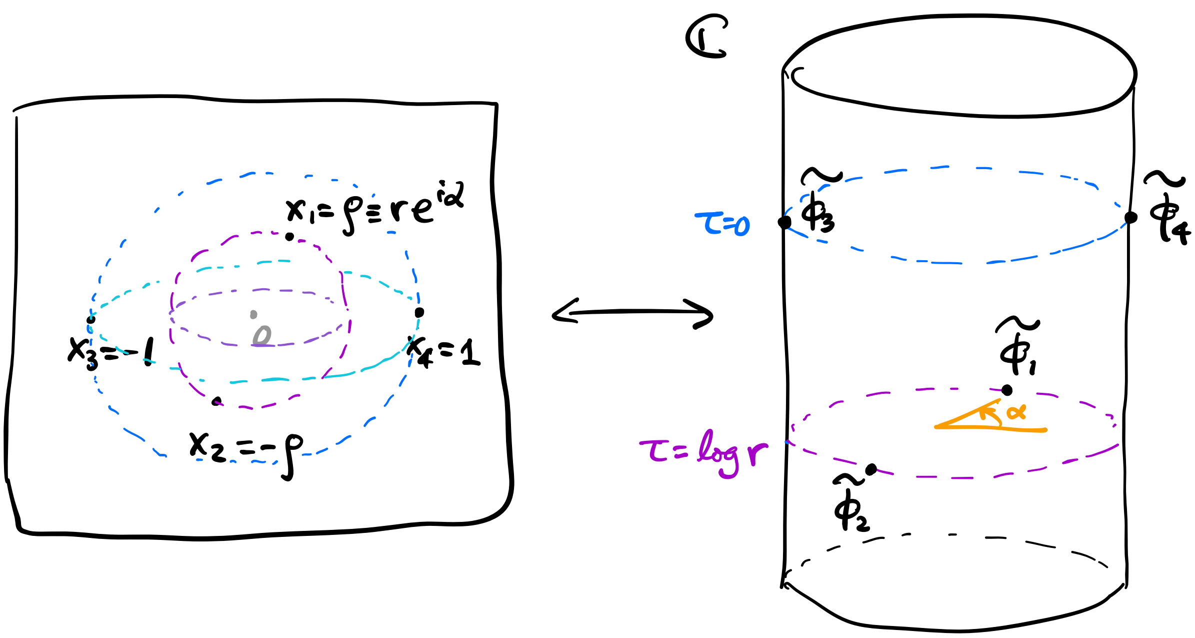
i.e.

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$$



To understand the properties of a conformal block, it is useful to adopt an alternative parameterization of the conformally invariant cross ratios, as follows:

Up to conformal mapping, we may assume
$$x_1, x_2, x_3, x_4 \in \mathbb{R}^2 \subset \mathbb{R}^d$$
$$\mathbb{R}^2 \cong \mathbb{C}$$



$\rho \equiv r e^{i\alpha}$ is related to the cross ratio z via

$$\rho = \frac{z}{(1+\sqrt{1-z})^2} \iff z = \frac{4\rho}{(1+\rho)^2}.$$

Recall that under the $\mathbb{R}^d \to \mathbb{R}\times S^{d-1}$ map, $\phi(x)$ is related to $\tilde{\phi}(\tau,\Omega)$ via

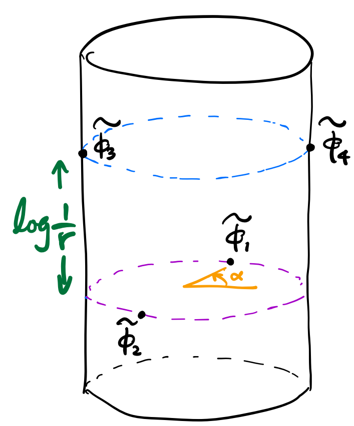
$$\phi(x) = |x|^{-\Delta_\phi}\,\tilde{\phi}(\tau,\Omega)$$

$$\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle = \frac{f(u,v)}{|x_{12}|^{2\Delta_\phi}\,|x_{34}|^{2\Delta_\phi}}$$

$\Rightarrow$

$$\langle \tilde{\phi}(\log r,\alpha)\,\tilde{\phi}(\log r,\alpha+\pi)\,\tilde{\phi}(0,0)\,\tilde{\phi}(0,\pi)\rangle = 2^{-4\Delta_\phi} f(u,v)$$

Omit other angle variables as $\tilde{\phi}$'s are restricted to the equator $S^1 \subset S^{d-1}$



$$= \sum_n \langle\Omega|\,\tilde{\phi}(0,0)\,\tilde{\phi}(0,\pi)\,|n\rangle \cdot r^{E_n}\langle n|\,\tilde{\phi}(0,\alpha)\,\tilde{\phi}(0,\alpha+\pi)\,|\Omega\rangle.$$

Assuming ϕ Hermitian and normalized with $\langle\langle \phi | \phi \rangle = 1$.

Conformal block decomposition

$$f(u,v) = \underbrace{1}_{\text{identity exchange}} + \sum_{\substack{\text{non-identity} \\ \text{primaries } \Phi_i}} \lambda_i^2 \, G_{\Delta_i, s_i}(u,v)$$

$\lambda_i \equiv \lambda_{\phi\phi\Phi_i}$ with Φ_i's orthonormal

Compare:

$$2^{-4\Delta_\phi} f(u,v)$$

$$= \langle \Omega | \tilde{\phi}(0,0) \, \tilde{\phi}(0,\pi) | \Omega \rangle \overbrace{\langle \Omega | \tilde{\phi}(0,\alpha) \, \tilde{\phi}(0,\alpha+\pi) | \Omega \rangle}^{\text{independent of } \alpha}$$

$$+ \sum_{\substack{\text{non-identity} \\ \text{primaries } \Phi_i}} \sum_{n=0}^{\infty} r^{\Delta_i + n} \sum_{\underline{N}, \underline{M}}$$

labeling level n descendants

$$\times \langle \Omega | \tilde{\phi}(0,0) \, \tilde{\phi}(0,\pi) | \Phi_i, \underline{N} \rangle \, G^{\underline{N}\,\underline{M}}$$

$$\times \langle \Phi_i, \underline{M} | \tilde{\phi}(0,\alpha) \, \tilde{\phi}(0,\alpha+\pi) | \Omega \rangle$$

where $\underline{N} = \{\beta; N_1, \ldots, N_d\}$, $\sum_{\mu=1}^{d} N_\mu = n$

β: $SO(d)$ rep index (sym, traceless rep in this case)

$$|\Phi_i, \underline{N}\rangle = \prod_{\mu=1}^{d} \hat{P}_\mu^{N_\mu} |\Phi_{i,\beta}\rangle$$

$$G_{\underline{N}\,\underline{M}} = \langle \Phi_i, \underline{N} | \Phi_i, \underline{M} \rangle$$

"Gram matrix" (assuming $\langle \Phi_i | \Phi_j \rangle \propto \delta_{ij}$)

and $(G^{\underline{N}\,\underline{M}})$ is the inverse matrix of $(G_{\underline{N}\,\underline{M}})$.

[Note: if $G_{\underline{N}\,\underline{M}}$ is degenerate, use a reduced linearly-independent basis of level-n descendants.]

We can isolate the individual conformal block:

$$2^{-4\Delta_\phi} G_{\Delta_i, s_i}(u,v) = \sum_{n=0}^{\infty} A_{\Delta_i, s_i, n}(\alpha)\, r^{\Delta_i + n}$$

where $A_{\Delta_i, s_i, n}(\alpha)$ is given by

$$\sum_{\underline{N},\underline{M}} \langle \Omega | \tilde{\Phi}(0,0)\, \tilde{\Phi}(0,\pi) | \Phi_i, \underline{N} \rangle\, G^{\underline{N}\,\underline{M}}$$

$$\cdot \langle \Phi_i, \underline{M} | \tilde{\Phi}(0,\alpha)\, \tilde{\Phi}(0,\alpha+\pi) | \Omega \rangle$$

$$= \lambda_i^2 \underbrace{A_{\Delta_i, S_i, n}(\alpha)}_{\text{determined by conformal algebra. theory-independent}}$$

Example: $n=0$ case

$$\lambda_i^2 A_{\Delta_i, S_i, 0}(\alpha) = \sum_{\beta,\gamma} \langle \Omega | \tilde{\Phi}(0,0)\, \tilde{\Phi}(0,\pi) | \Phi_{i,\beta} \rangle$$

$$\times\, G^{\beta\gamma} \langle \Phi_{i,\gamma} | \tilde{\Phi}(0,\alpha)\, \tilde{\Phi}(0,\alpha+\pi) | \Omega \rangle$$

$$\Phi_{i,\beta} \sim \Phi_{i(\mu_{i_1} \cdots \mu_{i_S})}$$

symmetric traceless

By SO(D) symmetry,

$$\langle \Phi_{\mu_1 \cdots \mu_S} | \tilde{\Phi}(0,\hat{x})\, \tilde{\Phi}(0,-\hat{x}) | \Omega \rangle$$

$$\propto \hat{x}^{\mu_1} \hat{x}^{\mu_2} \cdots \hat{x}^{\mu_S} \Big|_{\text{traceless}}$$

$$\Rightarrow A_{\Delta,s,0}(\alpha) \propto \left(\hat{x}_1^{\mu_1} \cdots \hat{x}_1^{\mu_s}\big|_{\text{traceless}}\right) \cdot \left(\hat{x}_{4\mu_1} \cdots \hat{x}_{4\mu_s}\big|_{\text{traceless}}\right)$$

$$\equiv P(\hat{x}_1 \cdot \hat{x}_4) = P(\cos\alpha)$$

Some polynomial

e.g.

$$P(\cos\alpha) = \cos(s\alpha) \qquad d=2$$

$$P_s(\cos\alpha) \qquad d=3$$

Legendre polynomial

$$\frac{1}{s+1}\frac{\sin((s+1)\alpha)}{\sin\alpha} \qquad d=4$$

In general d,

$$P(\cos\alpha) = C_s^{(\frac{d}{2}-1)}(\cos\alpha)$$

Gegenbauer polynomial

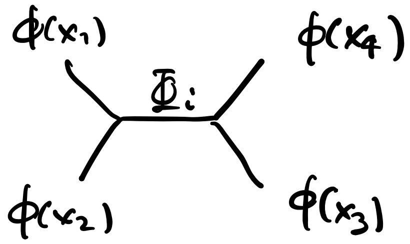
Conformal block

$$G_{\Delta,s}(u,v) = r^\Delta C_s^{(\frac{d}{2}-1)}(\cos\alpha)$$

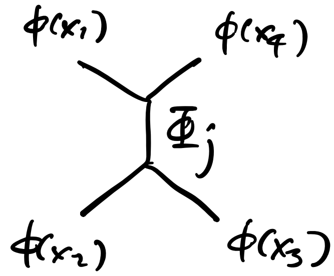
$+$ higher orders in r.

A first attempt at analyzing the crossing equation in the case of the 4-point function of identical scalar primaries ϕ :

$\sum_i$



$= \sum_j$



under $x_2 \leftrightarrow x_4$,

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \longleftrightarrow v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle$$

$$= \frac{f(u,v)}{|x_{12}|^{2\Delta_\phi} |x_{34}|^{2\Delta_\phi}} = \frac{f(v,u)}{|x_{14}|^{2\Delta_\phi} |x_{23}|^{2\Delta_\phi}}$$

multiply both sides by $|x_{13}|^{2\Delta_\phi} |x_{24}|^{2\Delta_\phi}$

$$\Rightarrow \boxed{v^{\Delta_\phi} f(u,v) = u^{\Delta_\phi} f(v,u)} \quad \circledast$$

Using conformal block decomposition

$$f(u,v) = 1 + \sum_{\Phi_i \neq 1} \lambda_i^2 \, G_{\Delta_i, s_i}(u,v)$$

λ_i real, $\lambda_i^2 \geq 0$

we can rewrite the crossing equation in terms of basic CFT data as

$$(v^{\Delta_\phi} - u^{\Delta_\phi}) + \sum_{\Phi_i \neq 1} \lambda_i^2 \Big(v^{\Delta_\phi} G_{\Delta_i, s_i}(u,v) - u^{\Delta_\phi} G_{\Delta_i, s_i}(v,u) \Big) = 0 .$$

Special case: $0 < z = \bar{z} < 1$

$$\rho = \frac{z}{(1+\sqrt{1-z})^2} = r e^{i\alpha}$$

$\Updownarrow$

$\alpha = 0, \quad 0 < r < 1$

recall

$$2^{-4\Delta_\phi} G_{\Delta,s}(u,v) = \sum_{n=0}^{\infty} A_{\Delta,s,n}(\alpha) \, r^{\Delta+n}$$

$A_{\Delta,s,0}(\alpha=0)$ = positive constant

can absorb (Δ_i, s_i) - dependence into the definition of λ_i

Small r approximation

$$G_{\Delta,s}(u,v) \sim r^{\Delta}$$

an approximate crossing equation

$$(*) \quad \left[(1-z)^{2\Delta_\phi} - z^{2\Delta_\phi}\right] + \sum_{\Phi_i \neq 1} \lambda_i^2 \left[(1-z)^{2\Delta_\phi}(r(z))^{\Delta_i} - z^{2\Delta_\phi}(r(1-z))^{\Delta_i}\right] = 0.$$

$$r(z) = \frac{z}{(1+\sqrt{1-z})^2}$$

Note: at crossing-symmetric point

$$z = \tfrac{1}{2} \Rightarrow r(\tfrac{1}{2}) = 3 - 2\sqrt{2} \approx 0.17$$

"somewhat small"

Let us analyze $(*)$ by expanding around $z = \frac{1}{2}$: set $z = \frac{1}{2} + x$, Taylor expand in x

identity-channel crossing block

$$\left[(1-z)^{2\Delta_\phi} - z^{2\Delta_\phi}\right]$$

$$= -\underbrace{\boxed{\Delta_\phi\, 2^{3-2\Delta_\phi}}}_{\text{"}C_{\Delta_\phi} > 0\text{"}}\left(x + \tfrac{4}{3}(\Delta_\phi - 1)(2\Delta_\phi - 1)x^3 + \dots\right)$$

Φ_i - channel crossing block (approx.)

$$\left[(1-z)^{2\Delta_\phi}(r(z))^{\Delta_i} - z^{2\Delta_\phi}(r(1-z))^{\Delta_i}\right]$$

in the limit $\Delta_i \gg \Delta_\phi, \mathcal{O}(1)$

$$\approx \boxed{2^{4+\Delta_i-2\Delta_\phi}(2+\sqrt{2})^{-3-2\Delta_i}(7+5\sqrt{2})\Delta_i}$$

$$\cdot\left(x+\frac{4}{3}\Delta_i^2 x^3+\cdots\right)$$

positive, absorb into $\lambda_i^2 \rightsquigarrow \tilde{\lambda}_i^2$

Approximate crossing equation, in the case $z=\bar{z}=\frac{1}{2}+x$, assuming $\Delta_i \gg \Delta_\phi, 1$

order x: $$-C_{\Delta_\phi}+\sum_{\Phi_i\neq 1}\tilde{\lambda}_i^2=0. \quad (1)$$

order x^3:

$$-C_{\Delta_\phi}\cdot\frac{4}{3}(\Delta_\phi-1)(2\Delta_\phi-1)+\sum_{\Phi_i\neq 1}\tilde{\lambda}_i^2\cdot\frac{4}{3}\Delta_i^2=0. \quad (2)$$

and so forth.

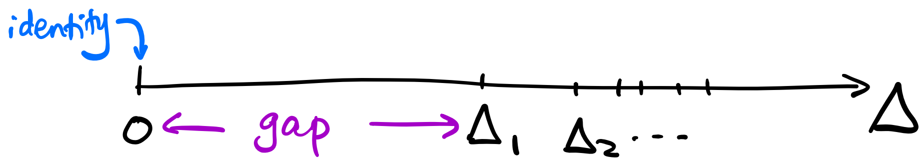
(1) & (2) $\Rightarrow$

$$\sum_{\Phi_i\neq 1}\tilde{\lambda}_i^2\left(\underbrace{\Delta_i^2-(\Delta_\phi-1)(2\Delta_\phi-1)}_{\text{cannot all be positive!}}\right)=0$$

Contradicts the assumption $\Delta_i \gg \Delta_\phi, 1$!

- What did we find ?

  Suppose the spectrum of Φ_i's that appear in $\phi\phi$ CPE looks like this :



  assuming spectral gap $\Delta_{min} \gg \Delta_\phi, 1$

  leads to $\Delta_{min} \lesssim \sqrt{(\Delta_\phi - 1)(2\Delta_\phi - 1)} \sim \mathcal{O}(\Delta_\phi)$

  $\Rightarrow$ $\Delta_{min} \gg \Delta_\phi$ is NOT possible.

There must exists an upper bound on Δ_{min} as a function of Δ_ϕ. To find this bound requires a more careful evaluation of the conformal blocks.

To make progress we need a way to compute conformal blocks efficiently.

"Projective lightcone formalism"

Idea: realize the Euclidean conformal group $SO(d+1,1)$, which a priori acts nonlinearly on $\mathbb{R}^d$, linearly on an auxiliary "spacetime" $\mathbb{R}^{d+1,1}$.

auxiliary coordinates $X^+,\ X^-,\ X^\mu$ $\quad \mu = 1, \ldots, d$

with $(d+2)$-dim'l Minkowskian metric

$$ds^2 = \sum_{\mu=1}^{d} dX^\mu dX^\mu - dX^+ dX^-$$

The $SO(d+1,1)$ Lorentz transformation on $\mathbb{R}^{d+1,1}$ in particular preserves the lightcone

$$X^2 \equiv \sum_{\mu=1}^{d} X^\mu X^\mu - X^+ X^- = 0 .$$

We can identify $\mathbb{R}^d$ with the "Euclidean section" of the lightcone:

$$(X^+, X^-, X^\mu) = (1, x^2, \underset{\mathbb{R}^d}{\underset{\cap}{x^\mu}})$$

Given a scalar primary $\phi(x)$, define formally a field operator $\Phi(X)$ by

$$\Phi(X)\Big|_{(X^+, X^-, X^\mu) = (1, x^2, x^\mu)} = \phi(x)$$

which is further extended to the entire lightcone by

$$\Phi(\lambda X) \equiv \lambda^{-\Delta_\phi} \Phi(X), \quad \text{for any } \lambda \neq 0.$$

[away from the lightcone, we leave $\Phi(X)$ unspecified]

Conformal transformation of $\phi(x)$ associated with the $SO(d+1,1)$ map

$$X \rightsquigarrow X' = \Lambda \cdot X$$

can be equivalently represented via $\Phi(X)$ as

$$\Phi \rightsquigarrow \Phi',$$

with

$$\Phi'(X') = \Phi(X)$$

$$\Updownarrow$$

$$\phi'(x') = \left(\det \frac{\partial x'}{\partial x}\right)^{-\frac{\Delta_\phi}{d}} \phi(x)$$

Suffices to check:

- $x' = \lambda x, \quad \phi'(x') = \lambda^{-\Delta_\phi}\phi(x)$

$$\Updownarrow$$

$$(X'^{+}, X'^{-}, X'^{\mu}) = (\lambda^{-1}X^{+}, \lambda X^{-}, X^{\mu})$$

$$\Phi'(\underbrace{1, \lambda^2 x^2, \lambda x^\mu}_{X'}) = \lambda^{-\Delta_\phi}\Phi(1, x^2, x^\mu) \quad \checkmark$$

$$\rotatebox{90}{$\equiv$}$$

$$\Phi(\underbrace{\lambda, \lambda x^2, \lambda x^\mu}_{X})$$

- $x'^\mu = \dfrac{x^\mu}{x^2}, \quad \phi'(x') = \left|\det\left(\dfrac{\partial x'}{\partial x}\right)\right|^{-\frac{\Delta_\phi}{d}}\phi(x)$

$$= (x^2)^{\Delta_\phi}\phi(x)$$

$$\Updownarrow$$

$$(X'^{+}, X'^{-}, X'^{\mu}) = (X^{-}, X^{+}, X^{\mu})$$

$$\Phi'(\underbrace{1, \tfrac{1}{x^2}, \tfrac{x^\mu}{x^2}}_{X'}) = (x^2)^{\Delta_\phi}\Phi(1, x^2, x^\mu) \quad \checkmark$$

$$\rotatebox{90}{$\equiv$}$$

$$\Phi(\underbrace{\tfrac{1}{x^2}, 1, \tfrac{x^\mu}{x^2}}_{X})$$

We can easily understand conformal sym constraints on correlators of $\phi(x)$ using the $\Phi(X)$ representation, e.g.

- 2-point function

$$\langle \phi_1(x)\,\phi_2(y)\rangle = \langle \Phi_1(X)\,\Phi_2(Y)\rangle$$

SO(d+1,1)-invariant can only be function of $\underbrace{X^2,\ Y^2}_{\text{vanish on L.C.}},\ X\cdot Y$

$$\Phi_i(\lambda X) = \lambda^{-\Delta_i}\Phi_i(X)$$

$$\Rightarrow \quad \langle \Phi_1(X)\,\Phi_2(Y)\rangle = \frac{\tilde{N}_{12}}{(X\cdot Y)^{\Delta_\phi}}$$

indeed,

$$X\cdot Y = -\tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 + x\cdot y$$
$$= -\tfrac{1}{2}(x-y)^2 \quad \checkmark$$

- 3-point function

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle$$
$$\parallel$$
$$\langle \Phi_1(X_1)\,\Phi_2(X_2)\,\Phi(X_3)\rangle = \frac{\tilde{C}_{123}}{(X_1\cdot X_2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} \times (\text{cyclic})}$$

$\checkmark$

• 4-point function

$$\langle \phi_1(x_1) \cdots \phi_4(x_4) \rangle$$

$$= \langle \Phi_1(X_1) \cdots \Phi_4(X_4) \rangle$$

$$= \frac{1}{(X_1\cdot X_2)^{\frac{\Delta_1+\Delta_2}{2}}(X_3\cdot X_4)^{\frac{\Delta_3+\Delta_4}{2}}} \left(\frac{X_2\cdot X_4}{X_1\cdot X_4}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{X_1\cdot X_4}{X_1\cdot X_3}\right)^{\frac{\Delta_3-\Delta_4}{2}}$$

$$\cdot F(u,v)$$

with
$$u = \frac{(X_1\cdot X_2)(X_3\cdot X_4)}{(X_1\cdot X_3)(X_2\cdot X_4)}$$

$$v = \frac{(X_1\cdot X_4)(X_2\cdot X_3)}{(X_1\cdot X_3)(X_2\cdot X_4)}$$

[equivalent to previous definition of u, v]

The CKVs on $\mathbb{R}^d$ can be lifted to $SO(d+1,1)$ Killing vectors on $\mathbb{R}^{d+1,1}$, or rather its lightcone $X^2=0$, as

$$L_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}, \quad A, B = +, -, 1, \dots, d$$

The corresponding conformal generators act on $\Phi(X)$ as

$$\hat{Q}_{AB} \cdot \Phi(X) = L_{AB}\, \Phi(X)$$

same differential operator as the Killing vector field on $\mathbb{R}^{d+1,1}$

The quadratic Casimir of the conformal alg.

$$\hat{C} = -\frac{1}{2} \hat{Q}^{AB} \hat{Q}_{AB}$$

(summed over $A, B = +, -, 1, \cdots, d$)

commutes with each $\hat{Q}_{AB}$, and thus takes the same value on an irreducible representation of the conformal group.

Acting on a primary $|\Phi_{(\mu_1 \cdots \mu_s)}\rangle$ or any of its descendants,

dim Δ, spin s

$\hat{C}$ takes the constant (eigen-)value

$$C_{\Delta,s} = \Delta(\Delta - d) + s(s + d - 2).$$

On the other hand,

$$\hat{C}\,|\phi(x)\rangle \longleftrightarrow \hat{C}\cdot\phi(x)$$

$$\qquad\qquad\equiv -L^2\cdot\Phi(X)$$

projective LC rep. of ϕ

$L^2 \equiv \frac{1}{2}L^{AB}L_{AB}$

Similarly for product operators:

$$\hat{C}\,|\phi_1(x_1)\,\phi_2(x_2)\rangle$$

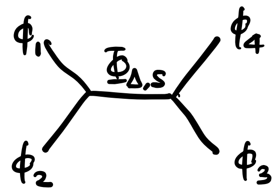
$$\longleftrightarrow -(L_1+L_2)^2\cdot\Phi_1(X_1)\,\Phi_2(X_2)$$

$$(L_i)_{AB} = X_{iA}\frac{\partial}{\partial X_i^B} - X_{iB}\frac{\partial}{\partial X_i^A}$$

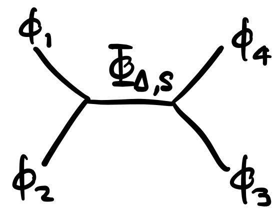
We have organized the OPE $\Phi_1(X_1)\,\Phi_2(X_2)$ according to irreps of the conformal algebra, labeled by (Δ, s); all operators in such an irrep are eigen--states w.r.t. $\hat{C}$ with eigenvalue $C_{\Delta,s}$. It follows that such operators also appear with X_1, X_2 - dependence on which $-(L_1+L_2)^2$ evaluates to $C_{\Delta,s}$.

Applying this to the conformal block decomposition of the 4-point function

$$\langle \Phi_1(X_1) \cdots \Phi_4(X_4) \rangle = \sum_{\Delta, s}$$



(*) $\left[-(L_1+L_2)^2 - C_{\Delta,s}\right]$



$= 0$.

(known prefactor involving powers of $X_i \cdot X_j$) $\times\, G_{\Delta,s}(u,v)$

(*) is a 2nd-order differential equation on

$$G_{\Delta,s}(u,v) \equiv F_{\Delta,s}(z,\bar z).$$

Explicitly in terms of $F_{\Delta,s}$, it is given by

$$\Big[z^2(1-z)\partial_z^2 + \left(\frac{\Delta_1-\Delta_2-\Delta_3+\Delta_4}{2} - 1\right) z^2 \partial_z$$

(¥)
$$+ \frac{(\Delta_1-\Delta_2)(\Delta_3-\Delta_4)}{4} z + (d-2)\frac{|z|^2}{z-\bar z}(1-z)\partial_z$$

$$+ (z \leftrightarrow \bar z,\ \partial_z \leftrightarrow \partial_{\bar z}) \Big] F_{\Delta,s}(z,\bar z) = \frac{1}{2} C_{\Delta,s} F_{\Delta,s}(z,\bar z).$$

• $d=2$ case ($SO(3,1)$ "global" conformal block)

In $z, \bar{z} \to 0$ limit, i.e. $x_2 \to x_1$,

dominated by primary contribution

$$F_{\Delta,s}(z,\bar{z}) \sim z^{\frac{\Delta+s}{2}} \bar{z}^{\frac{\Delta-s}{2}} + \text{c.c.}$$

total power of $z, \bar{z}$ fixed by weight

difference in powers of $z, \bar{z}$ gives the angular dependence $\cos(s\alpha)$ as seen previously.

Incidentally, in $d=2$ the differential operator on $F_{\Delta,s}$ splits into a holomorphic part (involving only z and ∂_z) and an anti-hol. part (involving $\bar{z}$ and $\partial_{\bar{z}}$), and $(*)$ becomes

$$\left[z^2(1-z)\partial_z^2 + \left(\frac{\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4}{2} - 1 \right) z^2 \partial_z + \frac{(\Delta_1-\Delta_2)(\Delta_3-\Delta_4)}{4} z \right] F_{\Delta,s} = \frac{(\Delta+s)(\Delta+s-2)}{4} F_{\Delta,s}.$$

and a similar equation with the replacement $z \to \bar{z}$, $s \to -s$ on the diff. op. & eigenvalue.

The solutions are hypergeometric functions:

define $h \equiv \frac{\Delta+s}{2}$, $\quad \tilde{h} \equiv \frac{\Delta-s}{2}$.

$$
\begin{aligned}
F_{\Delta,s}(z,\bar{z}) \\
= z^h\, {}_2F_1\left(h - \frac{\Delta_1-\Delta_2}{2},\ h+\frac{\Delta_3-\Delta_4}{2};\ 2h;\ z\right) \\
\times \left[\text{same expr. with } z\to\bar{z},\ h\to\tilde{h}\ \right] \\
+\text{c.c.}
\end{aligned}
$$

- $d=4$ case (conformal group $SO(5,1)$)

The differential equation for scalar conformal block $F_{\Delta,s}(z,\bar{z})$ is

$$\left(\hat{\mathcal{D}} + \text{c.c.}\right) F_{\Delta,s} = \frac{1}{2} C_{\Delta,s} F_{\Delta,s}$$

where

$$
\begin{aligned}
\hat{\mathcal{D}} \equiv\ & z^2(1-z)\partial_z^2 + \left(\frac{\Delta_1-\Delta_2-\Delta_3+\Delta_4}{2} - 1\right) z^2 \partial_z \\
& + \frac{(\Delta_1-\Delta_2)(\Delta_3-\Delta_4)}{4} z + 2\frac{|z|^2}{z-\bar{z}}(1-z)\partial_z
\end{aligned}
$$

We can remove the $\bar{z}$-dependence of $\hat{\mathcal{D}}$ using

$$\hat{\mathcal{D}} \frac{1}{z-\bar{z}} = \frac{1}{z-\bar{z}}\left(\hat{\mathcal{D}}' + 2\right)$$

where

$$\hat{\mathcal{D}}' = z^2(1-z)\partial_z^2 + \left(\frac{\Delta_1-\Delta_2-\Delta_3+\Delta_4}{2}+1\right)z^2\partial_z$$
$$-2z\partial_z + \frac{(\Delta_1-\Delta_2+2)(\Delta_3-\Delta_4-2)}{2}z \,.$$

This allows for solving $F_{\Delta,s}$ again in terms of hypergeometric functions: $(d=4)$

$$F_{\Delta,s}(z,\bar{z}) = \frac{1}{z-\bar{z}}\, z^{\frac{\Delta+s}{2}+1}\,\bar{z}^{\frac{\Delta-s}{2}}$$
$$\times\, {}_2F_1\left(\frac{\Delta+s}{2}-\frac{\Delta_1-\Delta_2}{2},\ \frac{\Delta+s}{2}+\frac{\Delta_3-\Delta_4}{2};\ \Delta+s;\ z\right)$$
$$\times\, {}_2F_1\left(\frac{\Delta-s}{2}-\frac{\Delta_1-\Delta_2}{2}-1,\ \frac{\Delta-s}{2}+\frac{\Delta_3-\Delta_4}{2}-1;\ \Delta-s-2;\ \bar{z}\right)$$
$$+\ (z\leftrightarrow\bar{z}) \,.$$

- $d=3$ case

The 3D scalar conformal blocks cannot simply be expressed in terms of hypergeometric functions, and their explicit expressions are more involved. In practice, an efficient way to compute them is through Zamolodchikov recursion relations, which we give without derivation below.

Write

$$G_{\Delta,s}(u,v) \equiv (4r)^{\Delta}\, h_{\Delta,s}(r,\alpha)$$

recall

$$\rho \equiv r e^{i\alpha} = \frac{z}{(1+\sqrt{1-z})^2}$$

$h_{\Delta,s}(r,\alpha)$ obeys the recursion relation

$$h_{\Delta,s} = h_{\infty,s} + \sum_{A} \frac{R_A}{\Delta - \Delta_A^*} (4r)^{n_A}\, h_{\Delta_A^*+n_A,\, s_A}$$

where the index A and R_A, Δ_A^*, n_A, s_A are defined as follows:

$$A \equiv (I, n) \qquad I = 1,2,3, \quad n \geq 1$$
and $n \leq s$ for $I=2$

$I=1$: $\Delta_A^* = 1-s-n$, $n_A = n$, $s_A = s+n$

$I=2$: $\Delta_A^* = s+d-1-n$, $n_A = n$, $s_A = s-n$

$I=3$: $\Delta_A^* = \frac{d}{2} - n$, $n_A = 2n$, $s_A = s$

$$R_{(1,n)} = \frac{-n(-2)^n}{(n!)^2} \left(\frac{\Delta_1-\Delta_2+1-n}{2}\right)_n \left(\frac{\Delta_3-\Delta_4+1-n}{2}\right)_n$$

where $(x)_n \equiv \dfrac{\Gamma(x+n)}{\Gamma(x)}$ is the Pochhammer symbol

$$R_{(2,n)} = \frac{-n \cdot s!}{(-2)^n (n!)^2 (s-n)!} \frac{(d+s-n-2)_n}{(\frac{d}{2}+s-n)_n (\frac{d}{2}+s-n-1)_n}$$
$$\times \left(\frac{\Delta_1-\Delta_2+1-n}{2}\right)_n \left(\frac{\Delta_3-\Delta_4+1-n}{2}\right)_n ,$$

$$R_{(3,n)} = \frac{-n(-)^n (\frac{d}{2}-n-1)_{2n}}{(n!)^2 (\frac{d}{2}+s-n-1)_{2n} (\frac{d}{2}+s-n)_{2n}}$$
$$\times \left(\frac{\Delta_1-\Delta_2-\frac{d}{2}-s-n+2}{2}\right)_n \left(\frac{\Delta_1-\Delta_2+\frac{d}{2}+s-n}{2}\right)_n$$
$$\times \left(\frac{\Delta_3-\Delta_4-\frac{d}{2}-s-n+2}{2}\right)_n \left(\frac{\Delta_3-\Delta_4+\frac{d}{2}+s-n}{2}\right)_n .$$

Finally, we also need the $\Delta=\infty$ case

$$h_{\infty,s}(r,\alpha) = (1-r^2)^{1-\frac{d}{2}}$$
$$\times (r^2-2r\cos\alpha+1)^{-\frac{1-\Delta_1+\Delta_2+\Delta_3-\Delta_4}{2}} (r^2+2r\cos\alpha+1)^{-\frac{1+\Delta_1-\Delta_2-\Delta_3+\Delta_4}{2}}$$
$$\times \frac{s!}{(-2)^s (\frac{d}{2}-1)_s} C_s^{(\frac{d}{2}-1)}(\cos\alpha)$$

Gegenbauer polynomial.

In the case of identical external scalars, $\Delta_1=\Delta_2=\Delta_3=\Delta_4=\Delta_\phi$, and s must be even, some of the R_A's vanish, and it suffices to take $A=(1,2k),\ (2,2k),\ (3,k),$

$k=1,2,3,\dots$

Now we turn to the crossing eqn for $\langle \phi\,\phi\,\phi\,\phi \rangle$:

$$(*) \quad \sum_{\Delta,s} \underbrace{\lambda^2_{\Delta,s}}_{\geq 0} \left(v^{\Delta_\phi} G_{\Delta,s}(u,v) - u^{\Delta_\phi} G_{\Delta,s}(v,u) \right) = 0 .$$

includes id. operator $\Delta = s = 0$, $\lambda_{0,0} = 1$.

Act on $(*)$ with linear functional

$$\partial_z^n \partial_{\bar z}^m \Big|_{z = \bar z = \frac{1}{2}} \qquad (\text{nontrivial for } n+m = \text{odd})$$

we get

$$\sum_{\Delta,s} \lambda^2_{\Delta,s} V^{(n,m)}_{\Delta,s;\Delta_\phi} = 0 .$$

Strategy: we aim to rule out candidate spectrum of (Δ, s) appearing in $\phi\phi$ OPE.

Given a trial spectrum e.g.

$$\mathcal{I} = \{ (0,0), (\Delta_1, s_1), (\Delta_2, s_2), \ldots \}$$

seek real coefficients $a_{n,m}$ such that

$$\sum_{n,m} a_{n,m} V^{(n,m)}_{\Delta,s;\Delta_\phi} \geq 0 \qquad \forall\, (\Delta,s) \in \mathcal{I} .$$

with >0 in the $(\Delta,s)=(0,0)$ case.

If  holds, the crossing eqn implies

$$0 = \sum_{n,m} a_{n,m} \sum_{(\Delta,s)\in I} \lambda^2_{\Delta,s} V^{(n,m)}_{\Delta,s;\Delta_\phi}$$

$$= \sum_{(\Delta,s)\in I} \lambda^2_{\Delta,s} \underbrace{\sum_{n,m} a_{n,m} V^{(n,m)}_{\Delta,s;\Delta_\phi}}_{\geq 0}$$

>0 (due to strictly positive $(0,0)$-term)

$\Rightarrow$ contradiction, thereby I is ruled out.

$(a_{n,m}) \longrightarrow$ vector $\vec{y} = (y_N)$ $N=(n,m)$

$(V^{(n,m)}_{\Delta,s;\Delta_\phi}) \longrightarrow$ matrix $M = (M_{IN})$

$I=(\Delta,s)$
$N=(n,m)$

constraint $M\cdot\vec{y} \geq 0$ (*)

$\uparrow$ meaning each component is ≥ 0.

Linear programming (optimization):

minimize (or maximize)

$$\vec{c} \cdot \vec{y}$$

your choice

subject to ⊛ and a normalization condition, say $\vec{e} \cdot \vec{y} = 1$.

your choice

["Linear Optimization" in mathematica]

Ideally, we want the trial spectrum to include continuous ranges of weight Δ, say

$$I = \{(0,0)\} \cup \{(\Delta, s) : \Delta \geq \Delta_{gap} \text{ (and unitarity bound)}, \; s = 0, 2, 4, \ldots\}$$

Numerical linear programming can only be done with finite matrix M.

The need to approximate a continuum of allowed weights Δ with a discrete subset can be removed with "polynomial matrix program" (PMP), based on the following theorem:

A symmetric matrix $M(x)$ whose entries $M_{ab}(x)$ are polynomial in x (one-variable) of degree d with real coefficients obeys

$$M(x) \succeq 0 \quad \forall\ x \geq 0$$

positive semidefinite

if (obviously) and only if (this is nontrivial)

$$M_{ab}(x) = \sum_{i,j=0}^{\lfloor \frac{d}{2} \rfloor} Y_{ai,bj}\, q_i(x)\, q_j(x) + x \sum_{i,j=0}^{\lfloor \frac{d-1}{2} \rfloor} Z_{ai,bj}\, q_i(x)\, q_j(x)$$

for some real polynomial $q_i(x)$ of degree i and matrices Y, Z that are $\succeq 0$.

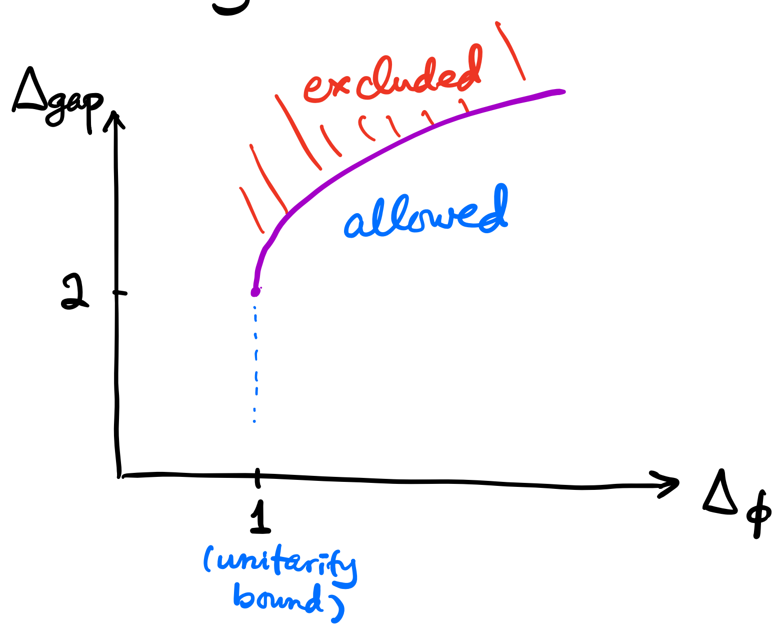
The PMP can be translated into a semidefinite program (SDP)

In practice, one can approximate the conformal blocks with rational functions in Δ, and implement the gap assumption via PMP/SDP. (Still need to truncate on the spin.)

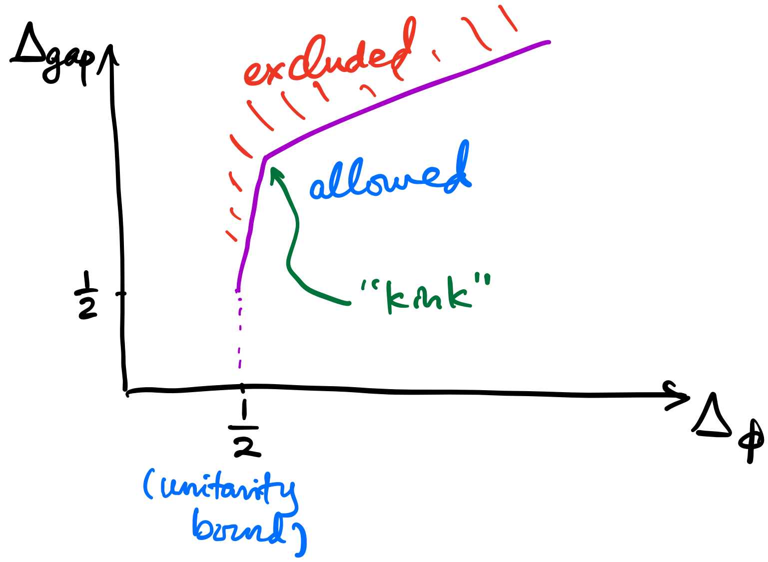
For more details and numerical implementation, see [arXiv: 1502.02033]

Some results [see Mathematica demo for a crude numerical implementation]

Bound on Δ_{gap} in $\phi\phi$ OPE in a 4D unitary CFT:



In a 3D unitary CFT



The kink is (very) close to $(\Delta_\sigma, \Delta_\varepsilon)$ of 3D Ising CFT

$$\Delta_\sigma = 0.518\ldots, \quad \Delta_\varepsilon = 1.412\ldots$$

$\phi = \sigma$, ε saturates (?) bound on Δ_{gap} in $\sigma\sigma$ OPE.

- We can also bound OPE coefficients from the crossing equation, using a slight variant of the above linear optimization.

Writing the crossing equation in the form

$$\sum_{\Delta,s} \lambda_{\Delta,s}^2 \, \alpha(H_{\Delta,s}) = 0,$$

where

$$H_{\Delta,s} = v^{\Delta_\phi} G_{\Delta,s}(u,v) - (u \leftrightarrow v),$$

and α a linear functional say of the form

$$\alpha = \sum a_{n,m} \partial_z^n \partial_{\bar z}^m \Big|_{z=\bar z=\frac{1}{2}},$$

We consider α that obeys

- $\alpha(H_{\Delta,s}) \geq 0 \quad \forall\ (\Delta,s) \in \mathcal{I}'$

  where $\mathcal{I}'$ is the trial spectrum excluding identity i.e. $(\Delta,s) = (0,0)$.

- normalization condition

  $\alpha(H_{\Delta_*,s_*}) = 1$ for a given $(\Delta_*,s_*) \in \mathcal{I}'$

and seek to minize

$$-\alpha(H_{0,0}).$$

It follows from the crossing eqn. that

$$-\alpha(H_{0,0}) = \lambda^2_{\Delta_*,s_*} + \sum_{\substack{(\Delta,s)\in\mathcal{I}' \\ \neq (\Delta_*,s_*)}} \lambda^2_{\Delta,s}\, \alpha(H_{\Delta,s})$$

$$\geq \lambda^2_{\Delta_*,s_*}.$$

The minimal value of $-\alpha(H_{0,0})$ then gives an (optimal) upper bound on the absolute value of the OPE coefficient λ_{Δ_*, S_*}.

For instance, $T_{\mu\nu}$ appears in $\phi\phi$ OPE at $(\Delta_*, S_*) = (d, 2)$, with coefficient

$$\lambda^{T}_{\phi\phi} \underset{\uparrow}{=} \frac{C_{\phi\phi T}}{C_T} = \text{Const.} \frac{\Delta_\phi}{C_T}$$

omitting convention-dependent constant factors

Its contribution to the crossing eqn comes with the factor

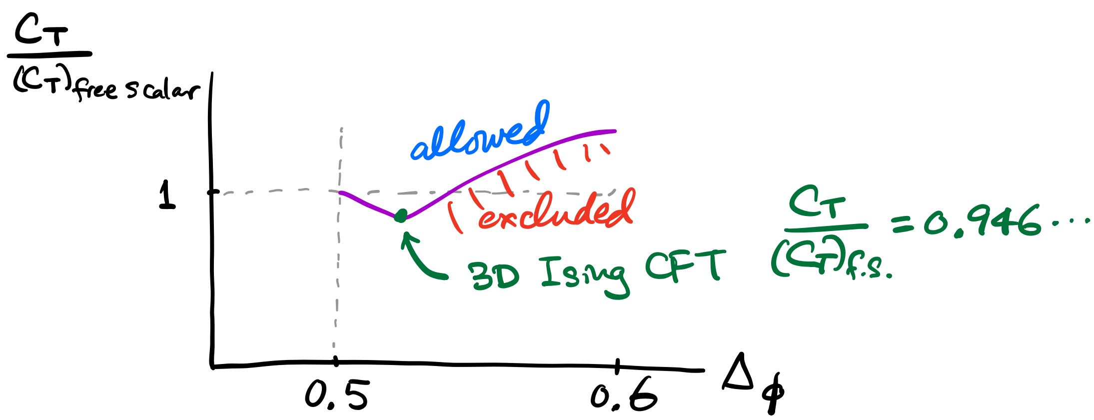
$$\lambda^2_{(d,2)} = (\lambda^{T}_{\phi\phi})^2 C_T = \text{Const.} \frac{\Delta_\phi^2}{C_T}$$

Using linear optimization, we can obtain an upper bound on $\lambda^2_{(d,2)}$

$\Longleftrightarrow$ lower bound on C_T

[for given Δ_ϕ, without or with additional assumptions on trial spectrum I, e.g. a gap assumption]

3D case:



Suggests that the 3D Ising CFT is the "simplest 3D CFT" i.e. with the smallest C_T (loosely speaking counts # d.o.f. as C_T is additive for decoupled sectors.)

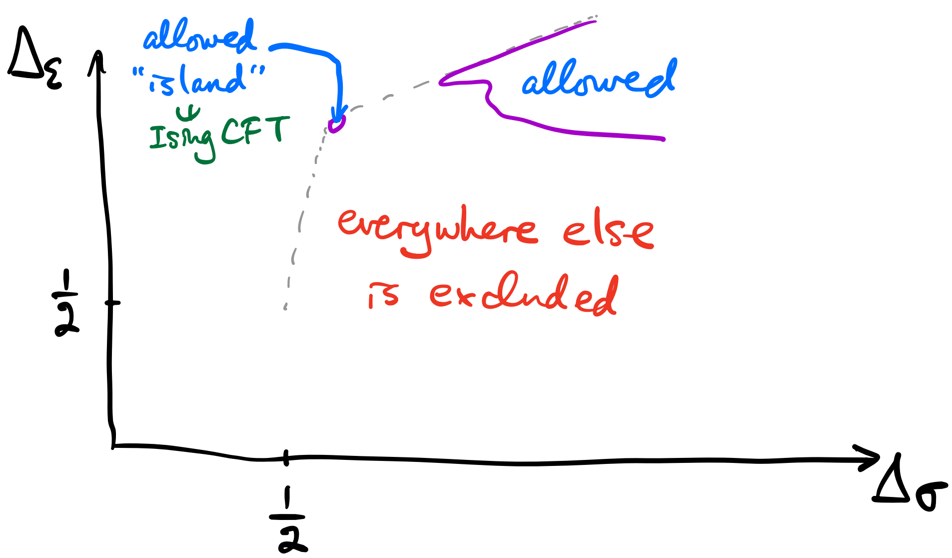
Much strong constraints on CFT data can be obtained by considering systems of crossing equation for more than 1 4-point functions, e.g.

$\langle \sigma\sigma\sigma\sigma \rangle$
$\langle \sigma\sigma\varepsilon\varepsilon \rangle$
$\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle$

$\langle \sigma\sigma\sigma\sigma \rangle$, $\langle \sigma\sigma\varepsilon\varepsilon \rangle$: involve $C_{\sigma\sigma\varepsilon}$

$\langle \sigma\sigma\varepsilon\varepsilon \rangle$, $\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle$: involve $C_{\varepsilon\varepsilon\varepsilon}$

Results from applying SDP to the system of crossing eqns for $\langle \sigma\sigma\sigma\sigma \rangle$, $\langle \sigma\sigma\varepsilon\varepsilon \rangle$, and $\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle$, under the assumption of $\mathbb{Z}_2$-symmetry (under which σ is odd, ε is even) and that there are no other relevant ($\Delta < 3$) $\mathbb{Z}_2$-odd scalar primaries:



Current best result from this subset of crossing eqns:

$$\Delta_\sigma \approx 0.5181489(10)$$

$$\Delta_\varepsilon \approx 1.412625(10)$$

[It is not known where a bigger system of crossing eqns is ultimately needed to pin down the spectrum completely.]

A survey of CFTs

[restricted to unitary, Poincaré-invariant, and local ($\exists\, T_{\mu\nu}$) theories]

Spacetime dimension $d = 2, 3, 4, 5, 6$

[$d=1$ no notion of locality;
$d \geq 7$ interacting CFTs are believed by some to not exist, but no proof]

- free scalars & free fermions
- free Maxwell is a CFT in $d=4$
  scale but not conformal invariant in $d \neq 4$
- $d=2$
  - rational CFTs (typically constructed via WZW/coset models)
  - conformal nonlinear sigma models (known non-free examples requires supersymmetry)
  - "noncompact" CFTs w/ continuous spectrum some of which are soluable (Liouville, Toda, $\mathbb{H}_3^+$-model, "cigar" coset, ...)

- orbifolds
- IR fixed points of RG flows
  (cf. coupled Potts model)

- $d=3$
  - RG fixed points of scalar or fermion Lagrangian field theories (Landau-Ginzburg models)
    e.g. critical $O(N)$ model, Gross-Neveu model
  - Chern-Simon-matter theories
  - RG fixed points of gauge theories
    e.g. massless QED_3 w/ scalar or fermion matter
  - Holographic dual of AdS_4 superstring vacua, some of which have not been identified in field-theoretic descriptions.

- $d=4$
  - Banks - Zaks theories, i.e. RG fixed point of QCD w/ massless quarks in the conformal window,

    $$\frac{34 N_c^3}{13 N_c^2 - 3} \lesssim N_f \lesssim \frac{11}{2} N_c$$

    (based on 2-loop β-function)
  - a zoo of superconformal gauge theories
- $d=5$
  - $\mathcal{N}=1$ superconformal theories whose relevant deformations flow to 5d supersymmetric gauge theories [Seiberg '96]
  - possible non-supersymmetric CFTs related to SU(2) gauge theories by RG flow

    [2001.00023, 2103.15242, ...]
- $d=6$
  - superconformal theories whose existence was inferred from string-theoretic arguments and corroborated by bootstrap results.

2D CFT

Convenient to work in lightcone coordinates

$$x^\pm \equiv x^1 \mp x^0$$

Minkowskian metric is written as

$$ds^2 = dx^+ dx^- = \eta_{\mu\nu} dx^\mu dx^\nu$$

$$\eta_{+-} = \eta_{-+} = \frac{1}{2}$$

$$\eta^{+-} = \eta^{-+} = 2$$

Wick rotation convention

$$x^0 = -i x^2$$

x^2 is Euclidean time

$$x^\pm = x^1 \pm i x^2 \equiv (z, \bar{z})$$

Conformal invariance:

$$T^\mu{}_\mu(x) = 0 \iff T_{+-}(x) = 0$$

or in Euclidean notation

$$T_{z\bar{z}}(z,\bar{z}) = 0.$$

Conservation law

$$\partial_\mu T^{\mu\nu}(x) = 0 \iff \partial_- T_{++} = \partial_+ T_{--} = 0$$

or in Euclidean notation

$$\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0.$$

We often use the shorthand notation

$$T(z) \equiv T_{zz}(z,\bar z), \quad \tilde T(\bar z) \equiv T_{\bar z \bar z}(z,\bar z)$$

already taking into account conservation of stress-energy tensor

Noether currents?

Consider

$$j_\mu \equiv (j_z, j_{\bar z})$$

$$= \big(-\underset{\substack{\uparrow\\ \text{holomorphic}}}{\varepsilon(z)}\, T(z),\ -\underset{\substack{\uparrow\\ \text{anti-holomorphic}}}{\tilde\varepsilon(\bar z)}\, \tilde T(\bar z)\big)$$

$$\partial_\mu j^\mu \equiv 2(\partial_z j_{\bar z} + \partial_{\bar z} j_z) = 0. \quad ✓$$

The analog of $d>2$ conformal algebra, $SO(3,1)$, is generated by

$$\varepsilon(z) = \underbrace{a_0 + a_1 z}_{\text{Poincaré}} + \underbrace{a_2 z^2}_{\text{special conformal transf.}}$$

$$\tilde\varepsilon(\bar z) = \underbrace{\tilde a_0 + \tilde a_1 \bar z}_{\text{Poincaré}} + \underbrace{\tilde a_2 \bar z^2}_{\text{special conformal transf.}}$$

But there are ∞-ly many more generators!

- the full 2D conformal symmetry group is ∞-dimensional.

• $SO(3,1)$ - primaries ("global primaries") are labeled by scaling dimension Δ and rotation quantum number of $SO(d=2)$, i.e. just the (angular) momentum j on S^1 (in our general d notation, the spin is $S = (j\,|)$)

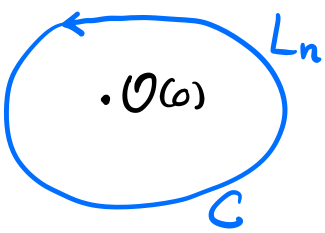
2D conformal weights

$$(h, \tilde{h}) \equiv \left(\frac{\Delta + j}{2}, \frac{\Delta - j}{2} \right)$$

e.g. $T_{\mu\nu}$ has $\Delta = 2$ and $S = 2$

$$\underbrace{T_{zz}}_{\text{weight } (2,0)} \text{ has } j = +2, \quad \underbrace{T_{\bar{z}\bar{z}}}_{\text{weight } (0,2)} \text{ has } j = -2$$

Conformal generators (analogs of $\hat{Q}_\varepsilon$)



take $\varepsilon(z) = -z^{n+1}$ for any integer n,

$$L_n \equiv \oint_C \frac{dz}{2\pi i} z^{n+1} T(z).$$

$\Rightarrow$ we have the OPE

$$T(z) \cdot \mathcal{O}(0) = \sum_{n=-\infty}^{\infty} \frac{1}{z^{n+2}} L_n \cdot \mathcal{O}(0).$$

Similarly $\tilde{L}_n \equiv \oint \frac{d\bar{z}}{-2\pi i} \bar{z}^{n+1} \tilde{T}(\bar{z}).$

Infinitesmal conformal transf. associated with the CKV $\varepsilon(z)\partial_z + \tilde{\varepsilon}(\bar{z})\partial_{\bar{z}}$ is

$$\delta_\varepsilon \mathcal{O}(0) = -\oint_C \frac{dz}{2\pi i}\,\varepsilon(z)\,T(z)\,\mathcal{O}(0) + \oint \frac{d\bar{z}}{2\pi i}\,\tilde{\varepsilon}(\bar{z})\,\tilde{T}(\bar{z})\,\mathcal{O}(0).$$

- The product operator $T(z)\,T(0)$ must depend holomorphically on z for $z \neq 0$

  $\Rightarrow$ $T\,T$ OPE takes the form of a Laurent series:

$$T(z)\,T(0) = \sum_{n=-\infty}^{\infty} z^n\, \underbrace{\Psi_n(0)}_{\text{weight } (h,\tilde{h}) = (4+n,0)}$$

unitarity bound $\Rightarrow$ $h, \tilde{h} \geq 0$

$\Rightarrow$ $n \geq -4$.

$$T(z)\,T(0) = z^{-4}\underbrace{\Psi_{-4}(0)}_{\substack{\text{weight } 0 \\ \Rightarrow \,\propto\, \text{identity op.}}} + z^{-3}\,\Psi_{-3}(0)$$

$$+ z^{-2}\,\Psi_{-2}(0) + z^{-1}\,\Psi_{-1}(0) + \cdots$$

Compare

$$\delta_\varepsilon T(0) = -\oint_C \frac{dz}{2\pi i}\, \varepsilon(z)\, T(z)\, T(0)$$

$$= -\sum_{n\geq -4} \operatorname*{Res}_{z\to 0}\, z^n \varepsilon(z)\, \Psi_n(0)$$

$$= -\frac{1}{6}\partial^3\varepsilon(0)\,\Psi_{-4}(0) - \frac{1}{2}\partial^2\varepsilon(0)\,\Psi_{-3}(0)$$

$$- \partial\varepsilon(0)\,\Psi_{-2}(0) - \varepsilon(0)\,\Psi_{-1}(0).$$

under $SO(3,1)$ "global" conformal transf.,

$\varepsilon(z) = a_0 + a_1 z + a_2 z^2$,

$$\delta_\varepsilon T(0) = -a_2 \underbrace{\cancel{\Psi_{-3}(0)}}_{0} - a_1 \underbrace{\Psi_{-2}(0)}_{2T(0)} - a_0 \underbrace{\Psi_{-1}(0)}_{\partial T(0)}$$

because T is $SO(3,1)$-primary.

We conclude:

central charge

$$T(z)\,T(0) = \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0)$$

$$+ \underbrace{\sum_{n=0}^{\infty} z^n \Psi_n(0)}.$$

This is sometimes denoted $:T(z)\,T(0):$, where $:\cdots:$ simply means subtracting terms in the OPE that would be singular in $z\to 0$ limit.

In a similar way,

$$\tilde{T}(\bar{z})\tilde{T}(0) = \frac{\tilde{c}}{2\bar{z}^4} + \frac{2}{\bar{z}^2}\tilde{T}(0) + \frac{1}{\bar{z}}\bar{\partial}\tilde{T}(0) + \sum_{n=0}^{\infty} \bar{z}^n \tilde{\Psi}_n(0).$$

[$c = \tilde{c}$ in a parity invariant theory, but more generally they need not be equal. When $c \neq \tilde{c}$, there is "gravitational anomaly", meaning that the CFT cannot be deformed to a QFT in curved spacetime while maintaining conservation of $T_{\mu\nu}$.]

On the other hand, the $T\tilde{T}$ OPE is non-singular in the coincidence limit

$$T(z)\tilde{T}(0) = \sum_{n=0}^{\infty} z^n \Phi_n(0)$$

weight $(2+n, 2)$

[Equivalently, $T(z)\tilde{T}(0) = :T(z)\tilde{T}(0):$

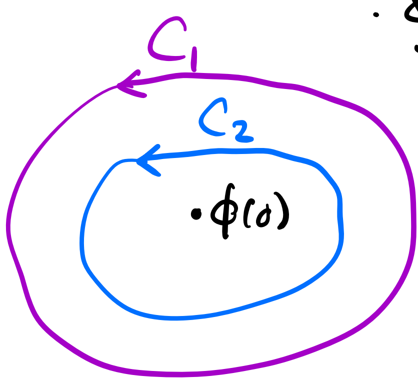
$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^n :\partial^n T \cdot \tilde{T}(0):$$]

What is the algebra of 2D conformal symmetry generated by L_n, $\tilde{L}_n$?

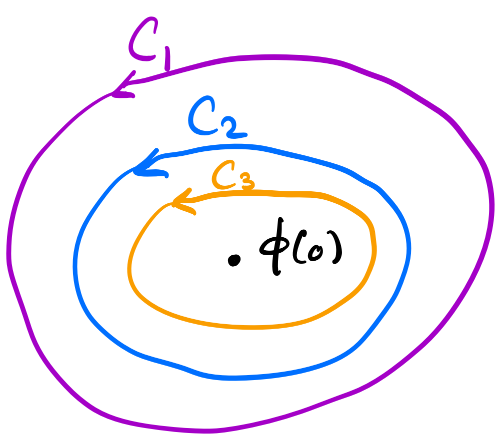
Recall

$$L_n \cdot \phi(0) = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \phi(0).$$

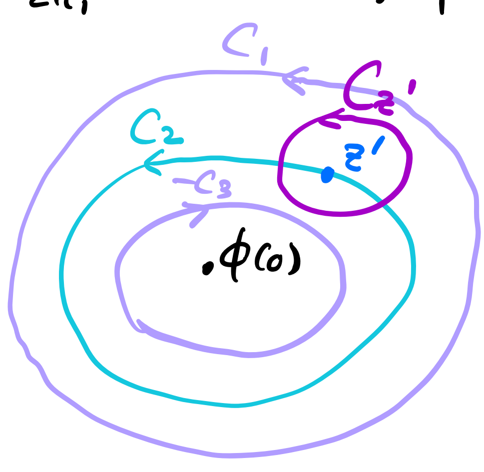
$$L_n L_m \cdot \phi(0) = \oint_{C_1} \frac{dz}{2\pi i} z^{n+1} T(z) \cdot \oint_{C_2} \frac{dz'}{2\pi i} z'^{m+1} T(z') \cdot \phi(0)$$



$$[L_n, L_m] \cdot \phi(0) = \oint_{C_1 - C_3} \frac{dz}{2\pi i} z^{n+1} T(z) \cdot \oint_{C_2} \frac{dz'}{2\pi i} z'^{m+1} T(z') \cdot \phi(0)$$



$\Rightarrow$



$$= \oint_{C_z} \frac{dz'}{2\pi i} z'^{m+1} \underbrace{\oint_{C_{z'}} \frac{dz}{2\pi i} \underbrace{z^{n+1}}_{\substack{\text{analytic} \\ \text{at } z=z'}} T(z) T(z') \phi(0)}$$

only receives residue contribution from singular terms in $T(z)T(z')$ OPE as $z \to z'$:

$$T(z)T(z') = \frac{c}{2(z-z')^4} + \frac{2}{(z-z')^2} T(z') + \frac{1}{z-z'} \partial T(z') + \text{regular}$$

after evaluating residues

$$[L_n, L_m] \cdot \phi(0)$$

$$= \oint_{C_z} \frac{dz'}{2\pi i} z'^{m+1} \Big[\frac{c}{12}(n+1)n(n-1) z'^{n-2}$$

$$+ 2(n+1) z'^{n} T(z') + z'^{n+1} \partial T(z') \Big] \cdot \phi(0)$$

$$\Rightarrow \quad [L_n, L_m] = (n-m) L_{n+m}$$

$$+ \frac{c}{12}(n^3 - n)\delta_{n,-m}$$

(acting on any $\phi(0)$)

Similarly,

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \frac{\tilde{c}}{12}(n^3-n)\delta_{n,-m}.$$

$$[L_n, \tilde{L}_m] = 0.$$

Note that $L_0, L_{\pm 1}$ and $\tilde{L}_0$, $\tilde{L}_{\pm 1}$ generate an $SO(3,1)$ subalgebra, which is the analog of the $SO(d+1,1)$ conformal algebra in $d>2$ dimensions. In particular,

$$L_{-1}\cdot\phi(0) = \partial\phi(0)$$
$$\tilde{L}_{-1}\cdot\phi(0) = \bar{\partial}\phi(0),$$

$$L_0\cdot\phi(0) = h\,\phi(0)$$
$$\tilde{L}_0\cdot\phi(0) = \tilde{h}\,\phi(0)$$

with $\Delta = h+\tilde{h}$, $j = h-\tilde{h}$ $(s=|j|)$ the scaling dimension and spin of ϕ.

An $SO(3,1)$-primary is annihilated by L_1 and $\tilde{L}_1$.

The full 2D "conformal algebra", a.k.a. the "Virasoro algebra", is ∞ - dimensional and involves the central charges $c, \tilde{c}$.

The space of local operators in a 2D CFT, or equivalently (by state/operator map) states of the CFT on S^1, can be organized according to (irreducible) representations of the Virasoro algebra.

- A Virasoro primary ϕ is an operator (or state $|\phi\rangle$) that obeys

$$L_n |\phi\rangle = 0, \quad \forall\, n \geq 1$$

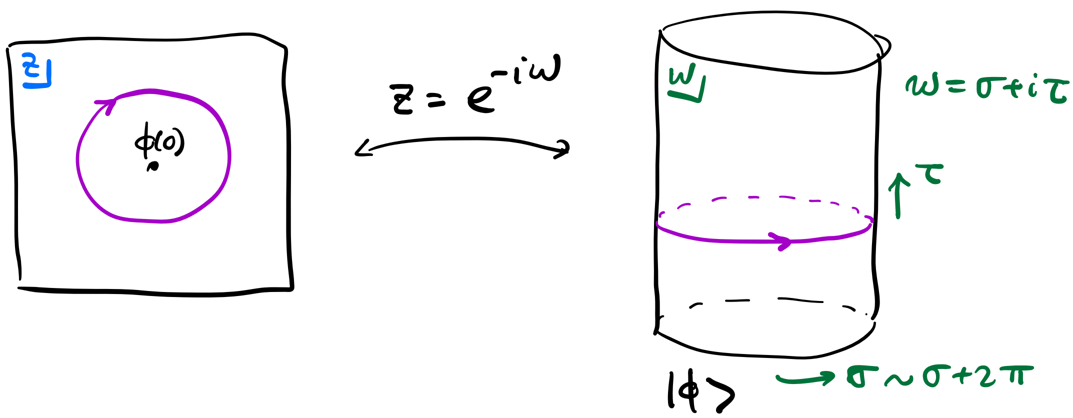
$$L_0 |\phi\rangle = h |\phi\rangle$$

and likewise

$$\tilde{L}_n |\phi\rangle = 0 \quad \forall\, n \geq 1$$

$$\tilde{L}_0 |\phi\rangle = \tilde{h} |\phi\rangle.$$

- An irreducible rep. of Virasoro algebra can be built by acting on a primary $|\phi\rangle$ with $L_{-n}, \tilde{L}_{-n}$, for $n \geq 1$.



$$L_n \longleftrightarrow \text{CKV} \quad z^{n+1}\partial_z$$
$$= i\, e^{-niw}\partial_w$$
$$\xrightarrow{\text{Wick rot}^{n}} i\, e^{-nix^+}\partial_+ \quad \text{on Lorentzian cylinder}$$

with respect to inner product on $\mathcal{H}_{S^1}$,

$L_n^\dagger = L_{-n}$, $\tilde{L}_n^\dagger = \tilde{L}_{-n}$

(generalizing $K_\mu^\dagger = P_\mu$ in $d > 2$)

- unitarity representations of Virasoro alg.

Virasoro primary $|\phi\rangle$ of weight $(h, \tilde{h})$

- level 1 descendant $L_{-1}|\phi\rangle$

$$\| L_{-1}|\phi\rangle \|^2 = \langle\phi| L_1 L_{-1} |\phi\rangle$$
$$= \langle\phi| [L_1, L_{-1}] |\phi\rangle = 2h \qquad (\langle\phi|\phi\rangle = 1)$$

$\Rightarrow \quad h \geq 0$.

similarly $\tilde{h} \geq 0$.

- level 2 descendants $L_{-1}^2|\phi\rangle$, $L_{-2}|\phi\rangle$

$$0 \preceq \mathcal{M}^{(2)} = \begin{pmatrix} \langle\phi|L_1^2 \\ \\ \langle\phi|L_2 \end{pmatrix} \begin{pmatrix} L_{-1}^2|\phi\rangle & L_{-2}|\phi\rangle \end{pmatrix}$$

pos. semidefinite

repeatedly applying Virasoro algebra

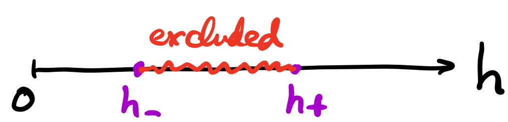
$$= \begin{pmatrix} \overbrace{8h^2+4h}^{\text{already } \geq 0} & 6h \\ 6h & 4h+\frac{c}{2} \end{pmatrix}$$

Need $\det \mathcal{M}^{(2)} \geq 0$

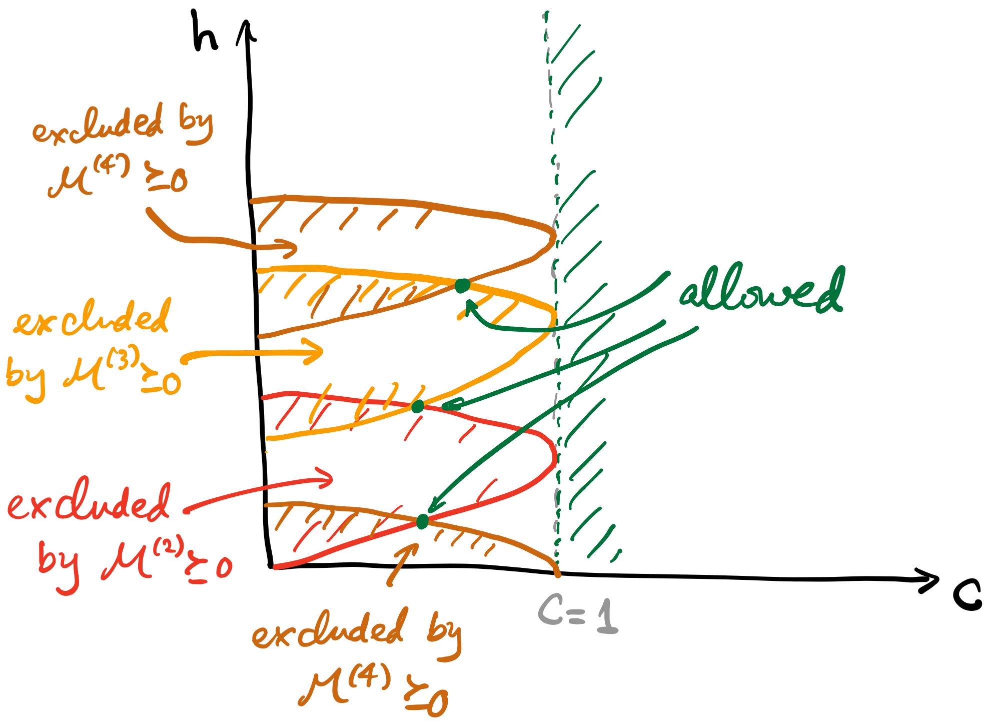
$$\det \mathcal{M}^{(2)} = 32h(h-h_+)(h-h_-)$$

$$h_\pm = \frac{5-c \pm \sqrt{(1-c)(25-c)}}{16} .$$

- if $c \geq 25$, $h_\pm \leq 0$, $\det \mathcal{M}^{(2)} \geq 0$ for $h \geq 0$. ✓
- if $1 < c < 25$, $h_\pm$ complex, $\det \mathcal{M}^{(2)} \geq 0$ ✓
- if $0 < c \leq 1$, $h_\pm$ real, ≥ 0.



More unitarity bounds



Taking into account $\mathcal{M}^{(k)} \geq 0$ for all levels k, eventually all (h, c) with $c < 1$ are ruled out except for a discrete set of values

$$c = 1 - \frac{6}{m(m+1)}, \qquad m \geq 3$$

$$h_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \qquad \begin{array}{l} 1 \leq r \leq m-1 \\ 1 \leq s \leq r \end{array}$$

There are indeed unitary CFT with such values of $c\ (= \tilde{c})$ and primaries with $(h, \tilde{h}) = (h_{r,s}, h_{r',s'})$, known as "minimal models".

Focus on the case $c = \frac{1}{2}$ $(m=3)$

- $h_{1,1} = 0, \quad h_{2,1} = \frac{1}{2}, \quad h_{1,2} = \frac{1}{16}$
  are the only possible weights of unitary representations of the $c=\frac{1}{2}$ Virasoro alg.

- Suppose we have a scalar primary op. σ of weight $(h,\tilde{h}) = (h_{1,2}, h_{1,2}) = (\frac{1}{16}, \frac{1}{16})$

$$\det \mathcal{M}^{(2)}(c,h) = 0$$

$\Rightarrow$ σ has a (holomorphic) level-2 null descendant, i.e. a linear combination of $L_{-2}|\sigma\rangle$ and $L_{-1}^2|\sigma\rangle$ has vanishing norm, thereby vanishes as a state.

Explicitly, can verify

$$\| (L_{-2} - \tfrac{4}{3} L_{-1}^2)|\sigma\rangle \|^2 = 0$$

$$\Rightarrow \left(L_{-2} - \tfrac{4}{3} L_{-1}^2\right)|\sigma\rangle = 0$$

$$\text{i.e. } L_{-2}\cdot\sigma = \tfrac{4}{3}\partial^2\sigma.$$

Similarly, $\tilde{L}_{-2}\cdot\sigma = \frac{4}{3}\bar{\partial}^2\sigma.$

Let us inspect the 4-point function

$$\langle \sigma(z_1,\bar{z}_1) \cdots \sigma(z_4,\bar{z}_4) \rangle$$

$$= |z_{12}|^{-2\Delta_\sigma} |z_{34}|^{-2\Delta_\sigma} f(z,\bar{z})$$

$$z = \frac{z_{12} z_{34}}{z_{13} z_{24}}$$

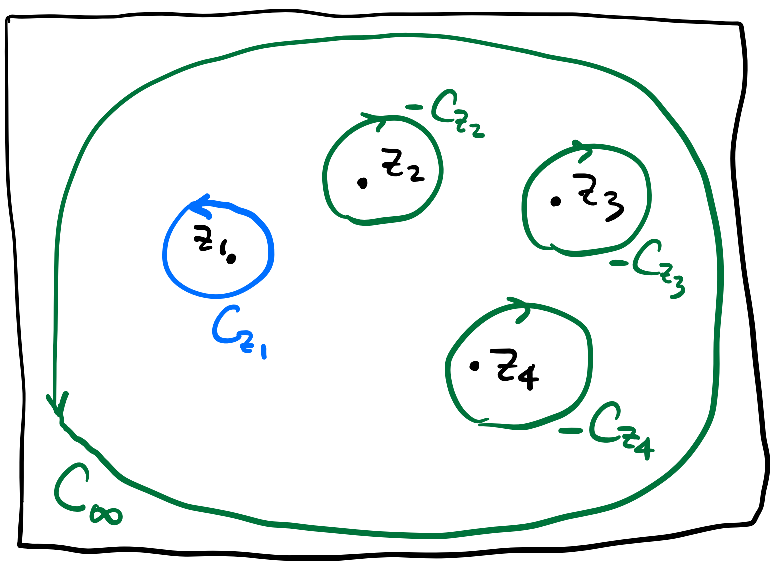
It follows from the null state relation that

$$\frac{4}{3} \partial_{z_1}^2 \langle \sigma(z_1,\bar{z}_1) \cdots \sigma(z_4,\bar{z}_4) \rangle$$

$$= \langle \underbrace{L_{-2} \cdot \sigma(z_1,\bar{z}_1)}_{= \oint_{C_{z_1}} \frac{dw}{2\pi i} \frac{1}{w-z_1} T(w)\, \sigma(z_1,\bar{z}_1)} \sigma(z_2,\bar{z}_2)\, \sigma(z_3,\bar{z}_3)\, \sigma(z_4,\bar{z}_4) \rangle \qquad (\star)$$

can replace

$$C_{z_1} \to -C_{z_2} - C_{z_3} - C_{z_4} + C_\infty$$



$$\langle T(w) \cdots \rangle \sim w^{-4}, \quad w \to \infty$$

$$\Rightarrow \oint_{C_\infty} \frac{dw}{2\pi i} \frac{1}{w-z_1} \langle T(w) \cdots \rangle = 0$$

$$\circledast \quad = -\oint_{C_{z_2}+C_{z_3}+C_{z_4}} \frac{dw}{2\pi i} \frac{1}{w-z_1} \langle T(w)\, \sigma(z_1,\bar{z}_1) \cdots \sigma(z_4,\bar{z}_4) \rangle$$

using $T(w)\,\sigma(z_i,\bar{z}_i) = \frac{h_\sigma}{(w-z_i)^2}\sigma(z_i,\bar{z}_i) + \frac{1}{w-z_i}\partial\sigma(z_i,\bar{z}_i) + \text{regular}$

$$= -\sum_{j=2}^{4} \operatorname*{Res}_{w\to z_j} \frac{1}{w-z_1}\left(\frac{h_\sigma}{(w-z_j)^2} + \frac{1}{w-z_j}\partial_{z_j}\right)\langle \sigma \cdots \sigma \rangle$$

$$= -\sum_{j=2}^{4} \left(-\frac{h_\sigma}{z_{j1}^2} + \frac{1}{z_{j1}}\partial_{z_j}\right)\langle \sigma \cdots \sigma \rangle$$

$$\Rightarrow \left[\frac{4}{3}\partial_{z_1}^2 + \sum_{j=2}^{4}\left(-\frac{1}{16\, z_{j1}^2} + \frac{1}{z_{j1}}\partial_{z_j}\right)\right] \cdot |z_{12}|^{-\frac{1}{4}} |z_{34}|^{-\frac{1}{4}} f(z,\bar{z}) = 0.$$

After simplification,

$$\left[\partial_z^2 + \frac{2-5z}{4z(1-z)}\partial_z - \frac{3}{64(1-z)^2}\right] f(z,\bar{z}) = 0$$

A pair of linearly independent solutions

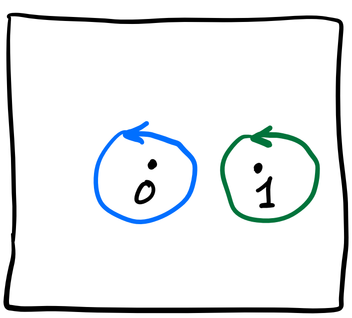
$$f_\pm(z) = (1-z)^{-\frac{1}{8}} \sqrt{1 \pm \sqrt{z}}$$

There is a similar anti-holomorphic diff. eqn for $f(z,\bar{z})$ $\Rightarrow$ $f(z,\bar{z})$ must be a linear combination of

$$f_+(z)\overline{f_+(z)}, \quad f_+(z)\overline{f_-(z)},$$

$$f_-(z)\overline{f_+(z)}, \quad f_-(z)\overline{f_-(z)}.$$

Note that $f_\pm(z)$ are not single-valued on complex z-plane, but $f(z,\bar{z})$ must be.



z going around 0 : $\quad \sqrt{z} \to -\sqrt{z}$

$$f_+(z) \longleftrightarrow f_-(z)$$

z going around 1 : $\quad f_+(z) \to e^{-\frac{2\pi i}{8}} f_+(z)$

$$f_-(z) \to -e^{-\frac{2\pi i}{8}} f_-(z)$$

The only possible linear combo of

$f_\pm(z)\,\overline{f_\pm(z)}$ that is invariant under the monodromy around both 0 and 1 is proportional to

$$|f_+(z)|^2 + |f_-(z)|^2$$

Fix normalization from

$$f(z=\bar{z}=0)=1$$

and $f_\pm(0)=1$

$$\Rightarrow\quad f(z,\bar{z}) = \tfrac{1}{2}|f_+(z)|^2 + \tfrac{1}{2}|f_-(z)|^2$$

$$= \frac{|1+\sqrt{z}| + |1-\sqrt{z}|}{2\,|1-z|^{\frac{1}{4}}}. \qquad ✓$$

- What about the crossing equation?

$z_2 \leftrightarrow z_4$, $z \to 1-z$,

Expect:

$$|1-z|^{\frac{1}{4}} f(z,\bar{z}) \overset{?}{=} |z|^{\frac{1}{4}} f(1-z, 1-\bar{z})$$

From ✓:

$$|1+\sqrt{z}| + |1-\sqrt{z}| \overset{???}{=} |1+\sqrt{1-z}| + |1-\sqrt{1-z}|$$

A miracle!

Compare to expected Virasoro conformal block expansion

$$f(z,\bar{z}) = |G_0(z)|^2 + C_{\sigma\sigma\varepsilon}^2 |G_{\frac{1}{2}}(z)|^2$$

$$G_0(z) = 1 + \mathcal{O}(z), \qquad G_{\frac{1}{2}}(z) = z^{\frac{1}{2}}(1+\mathcal{O}(z)).$$

$$\Rightarrow \quad G_0(z) = \frac{f_+(z) + f_-(z)}{2}$$

$$= 1 + \frac{z^2}{64} + \frac{z^3}{64} + \frac{117z^4}{8192} + \cdots$$

$$G_{\frac{1}{2}}(z) = f_+(z) - f_-(z)$$

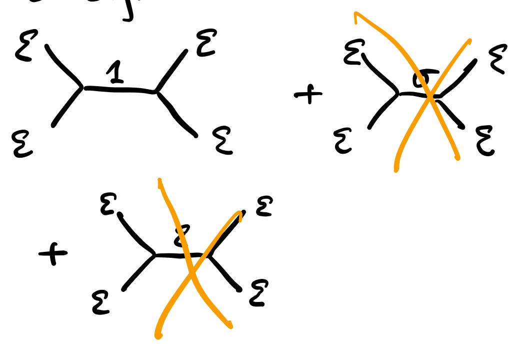
$$= z^{\frac{1}{2}}\left(1 + \frac{z}{4} + \frac{9z^2}{64} + \frac{25z^3}{256} + \cdots\right)$$

$$f(z,\bar{z}) = |G_0(z)|^2 + \frac{1}{4}|G_{\frac{1}{2}}(z)|^2$$

$$\Rightarrow \quad C_{\sigma\sigma\varepsilon} = \frac{1}{2}.$$

Can determine other structure constants from a similar analysis of

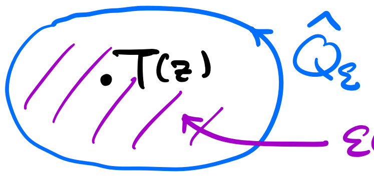
$$\langle \varepsilon\,\varepsilon\,\varepsilon\,\varepsilon \rangle =$$



$$\Rightarrow \quad C_{\varepsilon\varepsilon\sigma} = 0 \qquad [\mathbb{Z}_2\text{-symmetry}]$$

$$C_{\varepsilon\varepsilon\varepsilon} = 0 \qquad [\text{has to do with a “non-invertible symmetry”}]$$

Let us inspect a general infinitesimal conformal transf. of $T(z)$:



$\varepsilon(w)$ should be non-singular in a neighborhood of z

$$\delta_\varepsilon T(z) = -\oint_{C_z} \frac{dw}{2\pi i}\, \varepsilon(w)\, T(w)\, T(z)$$

$$= -\operatorname*{Res}_{w\to z}\, \varepsilon(w)\, \underbrace{T(w)\, T(z)}_{\text{only the singular terms as } w\to z \text{ contributes to Res}}$$

$$= \underbrace{-\frac{c}{12}\partial^3\varepsilon(z)}_{\text{would be absent for } SO(3,1) \text{ global conformal transf.}} - 2\partial\varepsilon(z)\, T(z) - \varepsilon(z)\, \partial T(z).$$

Conformal transf. associated with

$$z \rightsquigarrow z' = z + \varepsilon(z)$$

takes

$$T(z) \rightsquigarrow T'(z) = T(z) + \delta_\varepsilon T(z).$$

Finite form?

under a general finite conformal transf. associated with the coord. map

$$z \rightsquigarrow z' = f(z)$$

any holomorphic function

the stress-energy tensor operator transforms by

$$T(z) \rightsquigarrow T'(z)$$

such that

$$T'(z') = (\partial_z z')^{-2} \Big(T(z) - \frac{c}{12} \underbrace{\{z', z\}_S}_{\text{"Schwarzian derivative"}} \Big)$$

$$\{z', z\}_S \equiv \frac{2(\partial_z^3 z')(\partial_z z') - 3(\partial_z^2 z')^2}{2(\partial_z z')^2}$$

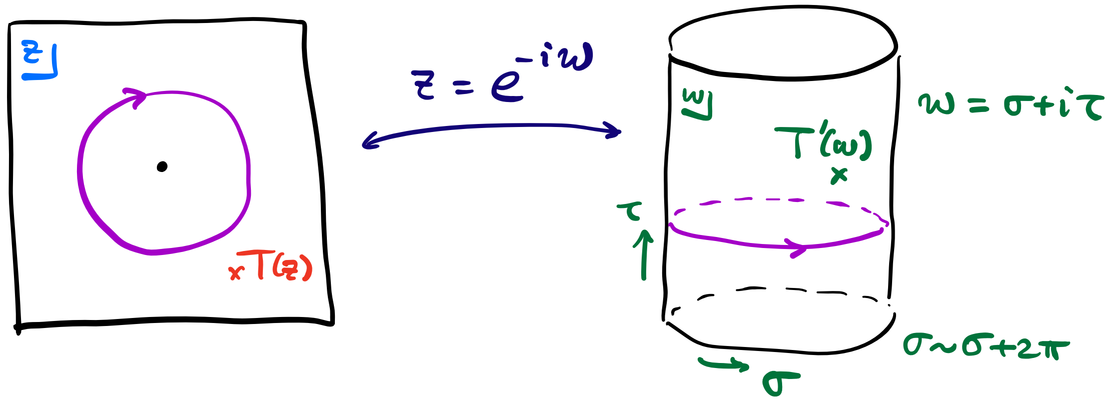
Key properties:

$$\{z'', z\}_S = (\partial_z z')^2 \{z'', z'\}_S + \{z', z\}_S$$

and

$$\{z + \varepsilon(z), z\}_S = \partial_z^3 \varepsilon(z) + \mathcal{O}(\varepsilon^2).$$

A special case of the conformal transf. $z' = i \log z$, maps the z-plane to the cylinder.



$$(*) \quad T'(w) = (\partial_w z)^2\, T(z) + \frac{c}{24}$$

stress tensor on cylinder — stress tensor on plane — from Schwarzian

- Note that the Schwarzian term was absent in our previous discussions of state/operator map in $d>2$, which should also apply to $d=2$ by consideration of $SO(3,1)$ global conformal symmetry only ?!

This is because in $d>2$ we did not have an a priori unambiguous definition of the CFT on $\mathbb{R}\times S^{d-1}$, due to possible couplings to curvature of S^{d-1}. The CFT on S^{d-1} may be viewed as defined via the state/op. map, in such a way that local operators on $\mathbb{R}\times S^{d-1}$ have the same properties as those on $\mathbb{R}^d$ in the short-distance limit.

In contrast, for $d=2$, the cylinder $\mathbb{R}\times S^1$ is flat, thus there is no ambiguity of curvature coupling, and $T'(w)$ is a priori defined unambiguously.

In terms of Virasoro generators,

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}$$

$$\overset{*}{\Longrightarrow} \quad T'(w) = -\sum_{n=-\infty}^{\infty} e^{inw} L_n + \frac{c}{24}.$$

Similarly, $\tilde{T}'(\bar{w}) = -\sum_{n=-\infty}^{\infty} e^{-in\bar{w}} \tilde{L}_n + \frac{\tilde{c}}{24}$

For the CFT on $S^1(\times \mathbb{R}_t)$, the Hamiltonian H and spatial momentum P are related by

$$H = -\int_0^{2\pi} \frac{d\sigma}{2\pi} (T')_{tt} = \int_0^{2\pi} \frac{d\sigma}{2\pi} (T')_{\tau\tau}$$

(different sign & normalization convention from $d>2$)

$$= -\int_0^{2\pi} \frac{d\sigma}{2\pi} \left(T'(w) + \tilde{T}'(\bar{w})\right) \quad \text{(using tracelessness)}$$

$$= \left(L_0 - \frac{c}{24}\right) + \left(\tilde{L}_0 - \frac{\tilde{c}}{24}\right)$$

$$P = \int_0^{2\pi} \frac{d\sigma}{2\pi} (T')_{t\sigma} = i \int_0^{2\pi} \frac{d\sigma}{2\pi} (T')_{\tau\sigma}$$

$$= - \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(T'(w) - \tilde{T}'(\bar{w}) \right)$$

$$= \left(L_0 - \frac{c}{24}\right) - \left(\tilde{L}_0 - \frac{\tilde{c}}{24}\right).$$

The shift of H versus $L_0 + \tilde{L}_0 = \Delta$ by $-\frac{c}{24} - \frac{\tilde{c}}{24}$ has interpretation as "Casimir energy": if we consider CFT on S^1 of radius R instead of unity,

$$H = \frac{\Delta}{R} + \underbrace{\frac{1}{R}\left(-\frac{c}{24} - \frac{\tilde{c}}{24}\right)}_{}$$

not extensive,
cannot be interpreted
as vacuum energy density!

[rather, the Casimir energy is intrinsically determined if we insist on conformal invariance.]

Key consequence: modular invariance (for now, on the torus)

Consider thermal partition function of CFT on S^1:

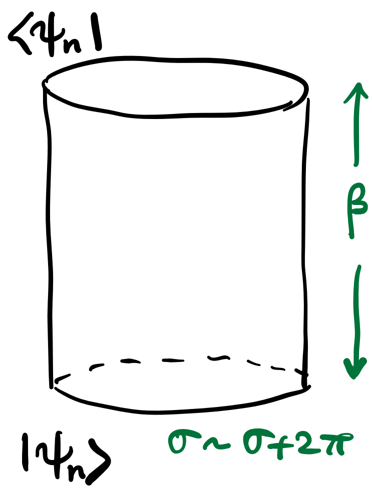
$$Z(\beta) = \mathrm{Tr}_{\mathcal{H}_{S^1}} e^{-\beta H}$$

$$= \mathrm{Tr}_{\mathcal{H}_{S^1}} e^{-\beta (L_0 + \tilde{L}_0 - \frac{c}{24} - \frac{\tilde{c}}{24})}$$

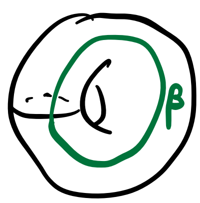
same as space of local op's

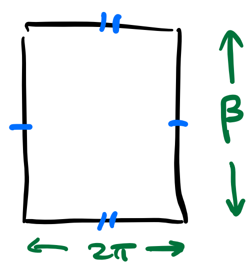
$$= \sum_n$$

$|\psi_n\rangle$ orthonormal basis



(Euclidean cylinder)

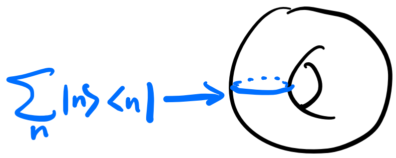
$=$ 

$\equiv$ 

"torus partition function"

(in the sense of Euclidean path integral)

Euclidean symmetry + locality suggest that



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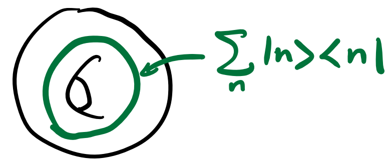


$$\text{Tr}_{\mathcal{H}_{S^1}} e^{-\beta H} \qquad\qquad \text{Tr}_{\mathcal{H}_{S^1(\beta)}} e^{-2\pi H}$$

$$= \text{Tr}_{\mathcal{H}_{S^1}} e^{-\frac{4\pi^2}{\beta} H}$$

i.e
$$Z(\beta) \overset{?}{=} Z\left(\frac{4\pi^2}{\beta}\right)$$

with
$$Z(\beta) = \sum_n e^{-\beta E_n},$$

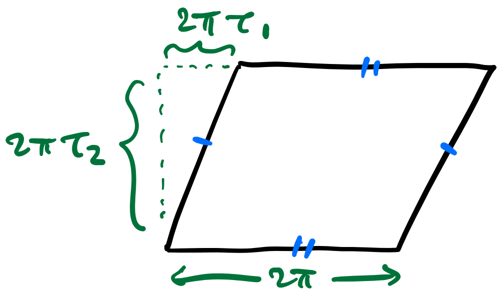
$$E_n = \underbrace{\Delta_n}_{\geq 0} + \left(-\frac{c}{24} - \frac{\tilde{c}}{24}\right)$$

in unitary CFT

Such a relation is only possible when the shift by Casimir energy is taken into account!

[highly nontrivial constraints on spectrum]

More generally, consider Euclidean path integral / partition function on



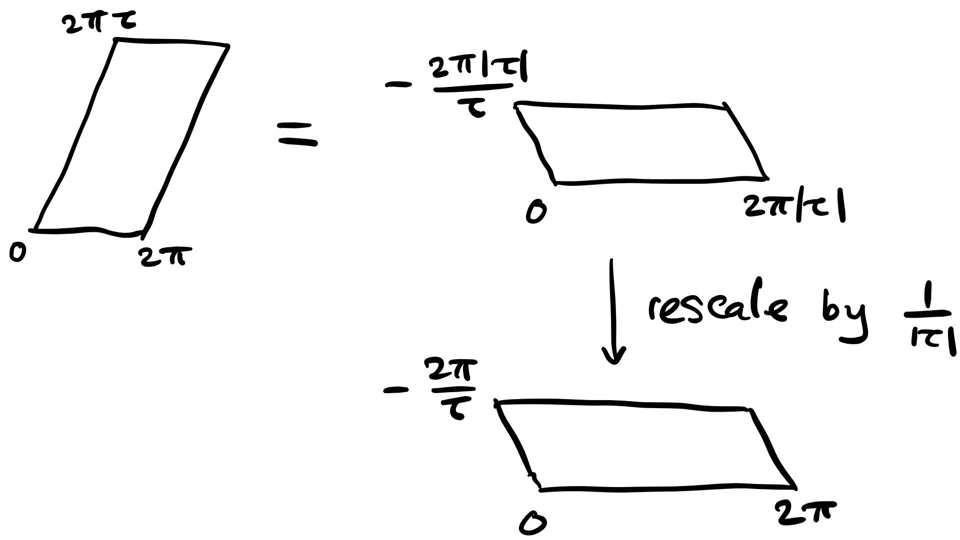
$$= \mathrm{Tr}_{\mathcal{H}_{S^1}}\, e^{-2\pi\tau_2 H + i2\pi\tau_1 P}$$

$$= \mathrm{Tr}_{\mathcal{H}_{S^1}}\, e^{2\pi i\tau\left(L_0 - \frac{c}{24}\right) - 2\pi i\bar{\tau}\left(\tilde{L}_0 - \frac{\tilde{c}}{24}\right)}$$

$$\equiv Z(\tau).$$

$\tau \equiv \tau_1 + i\tau_2$

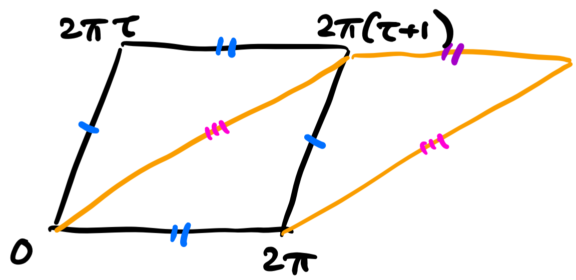
[with the understanding that $\bar{\tau}$ is the complex conjugate of τ]



Expect : $Z(\tau) = Z(-\frac{1}{\tau})$, Ⓢ

for any complex τ with $\mathrm{Im}\,\tau > 0$.

Also :



$$Z(\tau) = Z(\tau+1).$$ Ⓣ

Ⓢ and Ⓣ together implies

$$Z(\tau) = Z\left(\frac{a\tau+b}{c\tau+d}\right)$$ for any integers a, b, c, d that obey

$ad - bc = 1$. ($PSL(2,\mathbb{Z})$-invariant)

- These properties of the partition function Z are a part of the so-called modular invariance of a 2D CFT.

Deformation of background metric

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$$

can be represented by deformation of action

$$\delta S = \frac{1}{4\pi} \int d^2x \sqrt{\det g}\, \delta g_{\mu\nu} T^{\mu\nu},$$

or more precisely in a QFT via insertion of

$$\exp\left[-\frac{1}{4\pi} \int d^2x \sqrt{\det g}\, \delta g_{\mu\nu} \hat{T}^{\mu\nu} \right]$$

in correlation functions.

In 2D, a general metric deformation can be expressed, locally, as

$$\delta g_{\mu\nu} = \underbrace{\nabla_\mu \varepsilon_\nu + \nabla_\nu \varepsilon_\mu}_{\text{can undo with coordinate transf.}} + \underbrace{2\delta\omega\, g_{\mu\nu}}_{\text{Weyl transf.}}$$

A 2D CFT is expected to be Weyl-invariant (modulo an anomaly that will be analyzed shortly), and thus should be canonically defined on any 2D surface.

An anomaly concerning metric deformations arises due to contact terms in the overlapping integrals of the stress-energy tensor insertions of

$$\exp\left[-\frac{1}{4\pi}\int d^2x\sqrt{\det g}\;\delta g_{\mu\nu}\hat{T}^{\mu\nu}\right]$$

To see that, we begin with the TT OPE (in a flat background metric)

$$T_{zz}(z,\bar{z})\,T_{zz}(0)$$

$$= \frac{c}{2z^4} + \frac{2}{z^2}T_{zz}(0) + \frac{1}{z}\partial_z T_{zz}(0)$$

$$+ (\text{holomorphic})$$

The polar terms are not quite holomorphic, e.g.

$$\partial_{\bar{z}}\left(\frac{1}{z}\right) = 2\pi\delta^2(z)$$

convention: $\delta^2(z) = \frac{1}{2}\delta^2(x)$

$\int d^2z\,\delta^2(z) = 1$

[This is so that

$$\int d^2z\,\partial_{\bar{z}}\left(\frac{1}{z}\right) = -i\oint dz\cdot\frac{1}{z} = 2\pi$$
]

It follows that

$$\partial_{\bar z} T_{zz}(z,\bar z)\, T_{zz}(0)$$
$$= 2\pi \left[-\frac{c}{12}\, \partial_z^3 \delta^2(z) - 2\, \partial_z \delta^2(z)\, T_{zz}(0) + \delta^2(z)\, \partial_z T_{zz}(0) \right]$$

compare to contact term associated with the conservation law

$$\left(\partial_{\bar z} T_{zz} + \partial_z T_{\bar z z}\right)(z,\bar z)\, T_{zz}(0)$$
$$= 2\pi \left[\underbrace{-2\, \partial_z \delta^2(z)\, T_{zz}(0) + \delta^2(z)\, \partial_z T_{zz}(0)}_{\text{determined by Poincaré transf. property of } T_{zz}} \right]$$

$$\Rightarrow \cancel{\partial_z} T_{\bar z z}(z,\bar z)\, T_{zz}(0) = 2\pi \cdot \frac{c}{12}\, \cancel{\partial_z}\, \partial_z^2 \delta^2(z)$$

a contact term involving trace of T_{ab}!

i.e.

$$T_{\bar z z}(z_1,\bar z_1)\, T_{zz}(z_2,\bar z_2) = 2\pi \frac{c}{12}\, \partial_z^2\, \delta^2(z_{12})$$

Acting on both sides with $\partial_{\bar z_2}$, we have

$$T_{\bar z z}(z_1, \bar z_1)\, \partial_{\bar z} T_{zz}(z_2, \bar z_2)$$
$$= -2\pi \frac{c}{12}\, \partial_z^2 \partial_{\bar z}\, \delta^2(z_{12}).$$

Compare to

$$T_{\bar z z}(z_1, \bar z_1)\, \left(\partial_{\bar z} T_{zz} + \partial_z T_{\bar z z}\right)(z_2, \bar z_2)$$
$$= (\text{infinitesimal Poincaré transf. of } T_{\bar z z}(z_1, \bar z_1))$$
$$= 0 \quad \text{as a local operator},$$

we find

$$T_{\bar z z}(z_1, \bar z_1)\, \partial_z T_{\bar z z}(z_2, \bar z_2)$$
$$= 2\pi \frac{c}{12}\, \partial_z^2 \partial_{\bar z}\, \delta^2(z_{12}).$$

Finally removing ∂_{z_2} from both sides,

$$\boxed{T_{z\bar z}(z_1, \bar z_1)\, T_{z\bar z}(z_2, \bar z_2) = -2\pi \frac{c}{12}\, \partial \bar\partial\, \delta^2(z_{12})}$$

Note that a similar manipulation with $z \leftrightarrow \bar z$ would result in an identical expression except $c \rightsquigarrow \tilde c$. Consistency requires $c = \tilde c$ (absence of gravitational anomaly).

under a deformation of background metric of the form (Weyl transf.)

$$\delta g_{\mu\nu} = 2\delta\omega\, g_{\mu\nu}\,,$$

$\delta_{\mu\nu}$ to begin with

which amounts to inserting

$$\exp\left(-\frac{1}{4\pi}\int d^2x\; 2\delta\omega\, \hat{T}^{\mu}{}_{\mu}\right)$$

in correlation functions,

$$\left\langle \cdots \hat{T}^{\mu}{}_{\mu}(x) \right\rangle_{g+\delta g}$$

$$= \left\langle \cdots \hat{T}^{\mu}{}_{\mu}(x) \left[1 - \frac{1}{2\pi}\int d^2x'\, \delta\omega\, \hat{T}^{\nu}{}_{\nu}(x')\right] \right\rangle_g$$

contact term

The contact term effectively replaces $\hat{T}^{\mu}{}_{\mu}(x)$ with $\hat{T}^{\mu}{}_{\mu}(x) + \delta_W \hat{T}^{\mu}{}_{\mu}(x)$,

$$\delta_W \hat{T}_{z\bar{z}}(z_1,\bar{z}_1) = -\frac{1}{\pi}\int d^2 z_2\; \delta\omega(z_2,\bar{z}_2)$$

$$\times\, T_{z\bar{z}}(z_1,\bar{z}_1)\, T_{z\bar{z}}(z_2,\bar{z}_2)$$

$$= \frac{c}{6}\,\partial\bar{\partial}\,\delta\omega(z_1,\bar{z}_1)\,.$$

We can express this relation in a manifestly covariant manner, using the Ricci scalar curvature

$$\delta R = -2\nabla^2 \delta\omega \quad \text{for} \quad g_{\mu\nu} = e^{2\delta\omega}\delta_{\mu\nu}$$

and so

$$\delta_W \hat{T}^\mu{}_\mu(x) = -\frac{c}{12}\delta_W R(x).$$

This analysis can be iterated for consecutive Weyl transformations, leading to the operator equation in a general background metric

$$\hat{T}^\mu{}_\mu = -\frac{c}{12}R.$$

"Weyl anomaly"

For $g'_{\mu\nu} = e^{2\omega} g_{\mu\nu} \; (\equiv g^\omega_{\mu\nu})$

$$R[g'] = e^{-2\omega}(R - 2\nabla^2\omega)$$

defined wrt g

partition function of the CFT

$$Z[g] \to Z[g']$$

$$= Z[g]\, e^{-\frac{1}{2\pi}\int d^2x \sqrt{\det g} \int_0^{\omega(x)} \delta\tilde{\omega}\, e^{2\tilde{\omega}} [\hat{T}^\mu{}_\mu]_{g^{\tilde{\omega}}}}$$

using $\frac{1}{2\pi}\int d^2x\sqrt{\det g}\int_0^{\omega(x)} d\tilde{\omega}\; e^{2\tilde{\omega}}\left[\hat{T}^{\mu}{}_{\mu}\right]_{g^{\tilde{\omega}}}$

$$= \frac{1}{2\pi}\int d^2x\sqrt{\det g}\int_0^{\omega(x)} d\tilde{\omega}\; e^{2\tilde{\omega}}\left(-\frac{c}{12} R[g^{\tilde{\omega}}]\right)$$

$$= -\frac{c}{24\pi}\int d^2x\sqrt{\det g}\int_0^{\omega(x)} d\tilde{\omega}\,(R - 2\nabla^2\tilde{\omega})$$

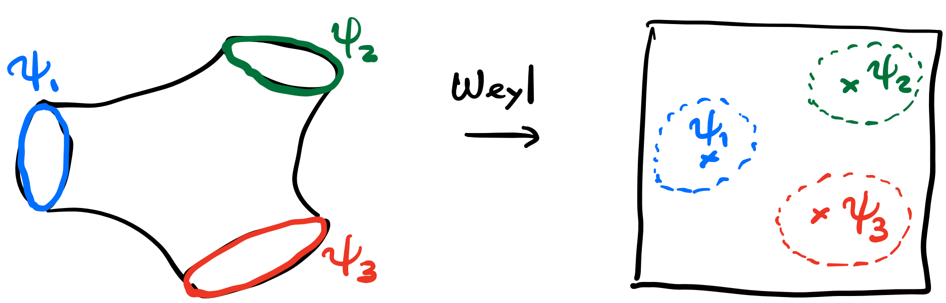
$$= -\frac{c}{24\pi}\int d^2x\sqrt{\det g}\,(R\,\omega + g^{\mu\nu}\partial_\mu\omega\partial_\nu\omega),$$

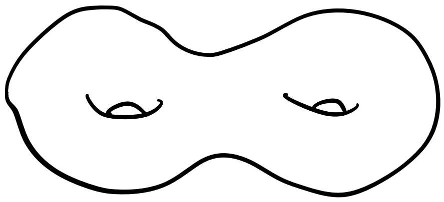
$$\equiv S_W[\omega]$$ "linear dilaton action"

we arrive at

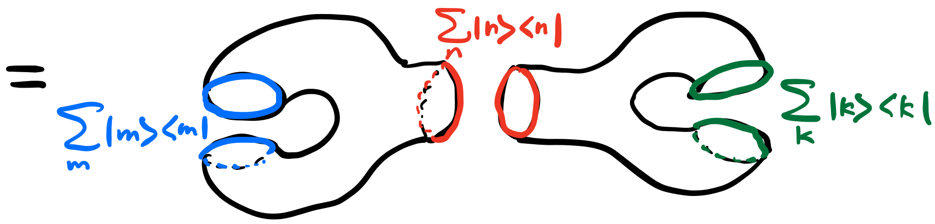
$$Z[g'] = e^{-S_W[\omega]}\cdot Z[g]$$

Equipped with this understanding, we can determine the correlation functions of a 2D CFT on any 2D Riemannian manifold in terms of conformal data on the plane.



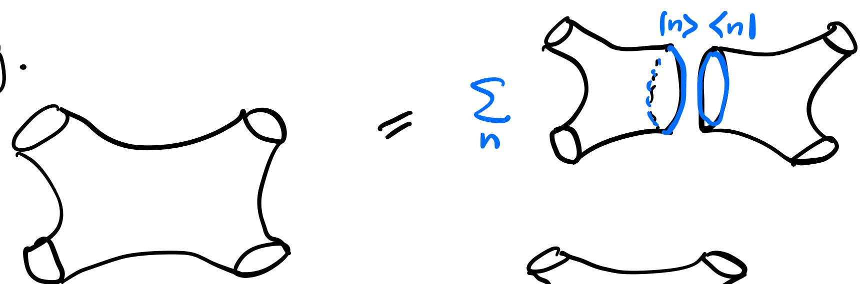


(partition function)



modular invariance is the statement that the partition function, or more generally correlation functions, on any surface, can be determined by such pair-of-pants decompositions, and that the result is independent of the choice of decomposition.

e.g.



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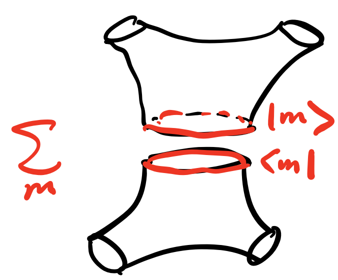
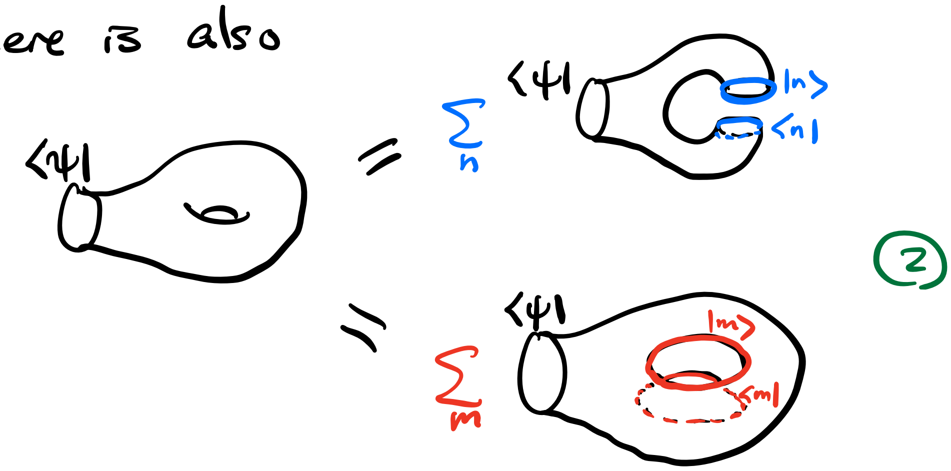


①

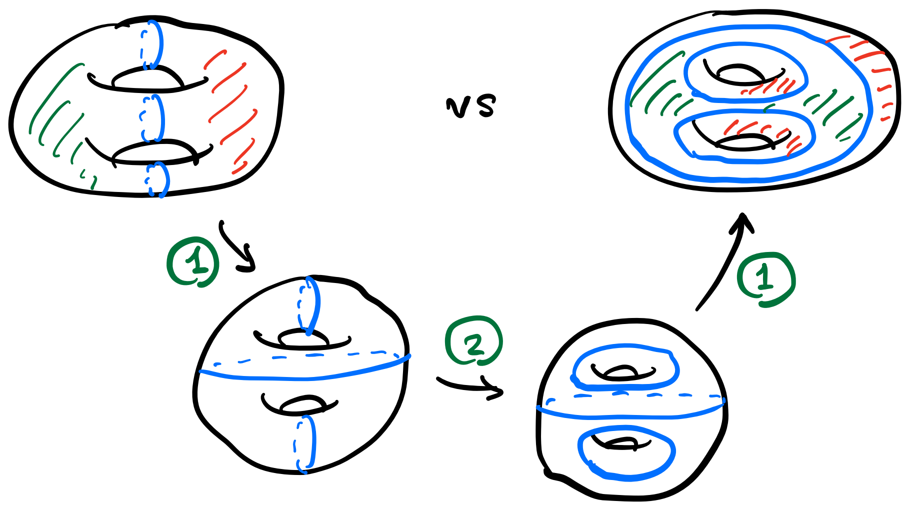
This is nothing but the crossing equation for 4-pt functions.

There is also



which includes the modular invariance of the torus partition function as a special case ($|\psi\rangle = |1\rangle$)

It turns out that any two different pair-of-pants decomp. of a surface can be related by a sequence of basic crossing moves 1 and 2, e.g.



And so 1 + 2 (for all ψ) are equivalent to the full modular invariance.

4D gauge theories w/ massless matter

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$
$$-\bar{\Psi}_I\gamma^\mu D_\mu\Psi_I - (D^\mu\Phi_A)^* D_\mu\Phi_A + \cdots$$

convention: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$

$A_\mu = \sum_a A_{a\mu}\, t^a, \qquad [t^a, t^b] = i f^{ab}{}_c\, t^c,$

$\mathrm{tr}(t^a t^b) \equiv d^{ab} = \frac{1}{2}\delta^{ab}.$

e.g. gauge group $G = SU(2)$, $t^a = \frac{1}{2}\sigma^a$.

tr = ordinary trace of 2×2 matrices

$I = 1, \cdots, N_f$, Ψ_I in representation R_f

$A = 1, \cdots, N_b$, Φ_A in representation R_b.

renormalized gauge coupling $g(\mu)$ defined through 1PI eff. action

$$\Gamma[\tilde{A}] = \int d^4x \left[-\frac{1}{2g^2(\mu)}\,\mathrm{tr}(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}) + \cdots \right]$$

bkgrd field

IR scale of bkgrd field config.

In dim reg + MS scheme $(D = 4-\epsilon)$,

$\lambda^2(\mu) \equiv \mu^\epsilon g^2(\mu)$, with

$$\frac{1}{g^2} = \mu^{-\epsilon}\left[\frac{1}{\lambda^2(\mu)} + \frac{b}{2\pi^2}\frac{1}{\epsilon} + \xcancel{\text{finite}}^{\;0 \text{ in } MS} + \text{higher order in } \lambda(\mu)\right]$$

bare coupling

$$\frac{d\lambda(\mu)}{d\log\mu} = \beta(\lambda(\mu)).$$

$$\beta(\lambda) = -\frac{b}{2\pi^2}\lambda^3 + \mathcal{O}(\lambda^5)$$

For $G = SU(N)$, complex reps R_f & R_b,

$$b = \frac{11}{12}N - \frac{N_f}{3}C(R_f) - \frac{N_b}{12}C(R_b)$$

Special case: $R_f = R_b = \text{adj}$, $C(\text{adj}) = N$,

real representation

$$b = \frac{N}{12}\left(11 - 4\cdot\tfrac{1}{2}N_f - \tfrac{1}{2}N_b\right)$$

$$= 0 \quad \text{for } N_f = 4,\ N_b = 6.$$

Merely a coincidence? There are other marginal couplings, e.g. $\phi\psi^2$ or ϕ^4. Must take their RG running into account as well! Not a scale/conformal invariant theory in general.

The "$\mathcal{N}=4$" Super-Yang-Mills theory

- $SU(N)$ gauge field $A_\mu = \sum_a A_{a\mu} t^a$
- 4 Weyl fermions in adjoint rep.

$$\lambda_{I\alpha} = \sum_a \lambda_{Ia\alpha} t^a$$

- 6 scalars in adjoint rep.

$$\Phi^{IJ} = \sum_a \Phi^{IJ}_a t^a$$

subject to
$$\Phi^{IJ} \equiv -\Phi^{JI} \equiv (\Phi_{IJ})^\dagger \equiv \frac{1}{2}\epsilon^{IJKL}\Phi_{KL}$$

- $SO(6) \simeq SU(4)$ global "R-symmetry" under which $\lambda_{I\alpha}$ transforms as fundamental rep of $SU(4)$ or equivalently chiral spinor rep of $SO(6)$, while Φ^{IJ} transforms as vector of $SO(6)$.

$$\mathcal{L} = \frac{1}{g^2}\mathrm{tr}\Big(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2\bar{\lambda}_I \gamma^\mu D_\mu \lambda^I - D_\mu \Phi_{IJ} D^\mu \Phi^{IJ} \underbrace{- 2\bar{\lambda}^I[\Phi_{IJ}, \bar{\lambda}^J] - 2\lambda_I[\Phi^{IJ}, \lambda_J]}_{\text{"Yukawa coupling"}} \underbrace{+ \frac{1}{2}[\Phi^{IJ}, \Phi^{KL}][\Phi_{IJ}, \Phi_{KL}]}_{\text{scalar potential}}\Big) + \frac{\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma}).$$

Spinor convention:

Weyl spinor $\lambda = \frac{1+\gamma_5}{2}\zeta$

$$\gamma^0 = -i\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \vec{\gamma} = -i\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix},$$

$$(\gamma^\mu)^* = B\gamma^\mu B^{-1}, \qquad B = \gamma_2$$

$$\gamma_5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

- $B^*\lambda^*$ obeys

$$\frac{1+\gamma_5}{2}B^*\lambda^* = B^*\frac{1-\gamma_5}{2}\lambda^* = B^*\left(\frac{1-\gamma_5}{2}\lambda\right)^* = 0$$

- $\zeta = \lambda + B^*\lambda^*$ obeys

$$\zeta^* = B\zeta \quad \text{(Majorana condition)}$$

- $\bar{\psi} \equiv \psi^\dagger\beta, \qquad \beta = i\gamma^0$

- For a pair of Weyl spinors λ and χ,

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda\chi \equiv \overline{(B^*\lambda^*)}\,\chi$$

$$= \lambda^T \underbrace{\gamma_2^T i\gamma^0}_{= \left(\begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & 0 \\ \hline & 0 & * \end{array}\right)} \chi = \underbrace{\epsilon^{\alpha\beta}\lambda_\beta}_{\equiv \lambda^\alpha}\chi_\alpha$$

(This convention is used in writing e.g. the Yukawa term $\lambda^I[\Phi_{IJ},\lambda^J]$ in the Lagrangian of $\mathcal{N}=4$ SYM)

Key feature of $\mathcal{N}=4$ SYM:

- Global "super-" symmetry generators Q^I_α, $\bar{Q}_{I\dot\beta}$, that obey

$$\{Q^I_\alpha, \bar{Q}_{J\dot\beta}\} = -2\,\delta^I_J\,\gamma^\mu_{\alpha\dot\beta}\,P_\mu$$

and

$$[P_\mu, Q^I_\alpha] = 0$$

$$[J_{\mu\nu}, Q^I_\alpha] = (S_{\mu\nu})_\alpha{}^\beta\, Q^I{}_\beta\,,$$

$$S_{\mu\nu} \equiv -\frac{i}{4}[\gamma_\mu, \gamma_\nu]$$

- At classical level,

$$Q^I_\alpha \cdot \Phi_{JK} \sim \delta^I_{[J}\,\lambda_{K]\alpha}$$

$$Q^I_\alpha \cdot \lambda_{J\beta} \sim \delta^I_J(\gamma^{\mu\nu})_{\alpha\beta}F_{\mu\nu} + \epsilon_{\alpha\beta}[\Phi^{IK}, \Phi_{JK}]$$

$$Q^I_\alpha \cdot \bar\lambda^J_{\dot\beta} \sim \gamma^\mu_{\alpha\dot\beta}\,D_\mu\Phi^{IJ}$$

$$Q^I_\alpha \cdot A_\mu \sim (\gamma_\mu)_{\alpha\dot\beta}\,\bar\lambda^{I\dot\beta}$$

This fixes the relative coefficients of Yukawa coupling and scalar potential in the Lagrangian, thereby tying their β-fns to that of the gauge coupling g!

- The β-fn for the gauge coupling g is in fact exactly zero.

  - a non-perturbative argument can be made based on $\mathcal{N}=1$ supersymmetry constraints on the Wilsonian effective action

[Novikov, Shifman, Vainshtein, Zakharov '83
Seiberg '93 Arkani-Hamed, Murayama '97]

Roughly speaking, the idea is that one can promote
$$\tau \equiv \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2}$$
to a dynamical complex scalar field that is part of a "chiral superfield", in such a way that $\mathcal{N}=1$ (i.e. a single spinor worth) of supersymmetry is preserved, which in turn constrains how τ can appear in the Wilsonian effective action.

- scale-invariance $\rightsquigarrow$ dilatation sym D

  $T^\mu{}_\mu$ = total derivative.

  - expect $T^\mu{}_\mu = 0$ i.e. conformal symmetry
  - K_μ's in addition to D, P_μ, $J_{\mu\nu}$

- More symmetries :

Since $[P_\mu, K_\nu] = 2i(\eta_{\mu\nu} D - J_{\mu\nu})$

and $\{Q, Q\} \sim P$,

K_μ cannot commute with Q^I_α

$$[K^\mu, Q^I_\alpha] = \gamma^\mu_{\alpha\dot\beta}\, \bar S^{I\dot\beta}$$

$$[K^\mu, \bar Q_{I\dot\beta}] = \gamma^\mu_{\alpha\dot\beta}\, S^\alpha_I$$

S^α_I and $\bar S^{I\dot\beta}$ are called "special supercharges"

It follows from Jacobi identity that S's further obey

$$[P^\mu, S_{I\alpha}] = -\gamma^\mu_{\alpha\dot\beta}\, \bar Q^{\dot\beta}_I$$

$$[P^\mu, \bar S^I_{\dot\beta}] = -\gamma^\mu_{\alpha\dot\beta}\, Q^{I\alpha}$$

$$\{S_{I\alpha}, \bar S^J_{\dot\beta}\} = -2\delta^J_I \gamma^\mu_{\alpha\dot\beta} K_\mu$$

and

$$\{Q^I_\alpha, S_{J\beta}\} = 2i\delta^I_J \gamma^{\mu\nu}_{\alpha\beta} J_{\mu\nu} - i\delta^I_J \epsilon_{\alpha\beta} D - 2\epsilon_{\alpha\beta} R^I{}_J .$$

$$\{Q, \bar S\} = 0 = \{\bar Q, S\}$$

where $R^I{}_J$ are generators of the SU(4) global "R-symmetry", with

$$R^I{}_I \equiv 0,$$

$$[R^I{}_J, R^K{}_L] = \delta^K_J R^I{}_L - \delta^I_L R^K{}_J,$$

$$[R^I{}_J, Q^K_\alpha] = \delta^K_J Q^I_\alpha - \frac{1}{4}\delta^I_J Q^K_\alpha,$$

etc.

- The $\mathcal{N}=4$ superconformal algebra may look complicated, but can be represented very simply in terms of super-spinor helicity variables

$$(\lambda_\alpha, \tilde{\lambda}_{\dot\alpha}, \eta^I)$$

$$-\cancel{P}_{\alpha\dot\beta} \simeq \lambda_\alpha \tilde{\lambda}_{\dot\beta}, \quad -\cancel{K}_{\alpha\dot\beta} \simeq \frac{\partial}{\partial\lambda^\alpha}\frac{\partial}{\partial\tilde{\lambda}^{\dot\beta}}$$

$$Q^I_\alpha \simeq \sqrt{2}\,\lambda_\alpha \eta^I, \quad S_{I\alpha} \simeq \sqrt{2}\,\frac{\partial}{\partial\lambda^\alpha}\frac{\partial}{\partial\eta_I}$$

$$\bar{Q}_{I\dot\alpha} \simeq \sqrt{2}\,\tilde{\lambda}_{\dot\alpha}\frac{\partial}{\partial\eta^I}, \quad \bar{S}^I{}_{\dot\alpha} \simeq \sqrt{2}\,\frac{\partial}{\partial\tilde{\lambda}^{\dot\alpha}}\eta^I, \quad \text{etc.}$$

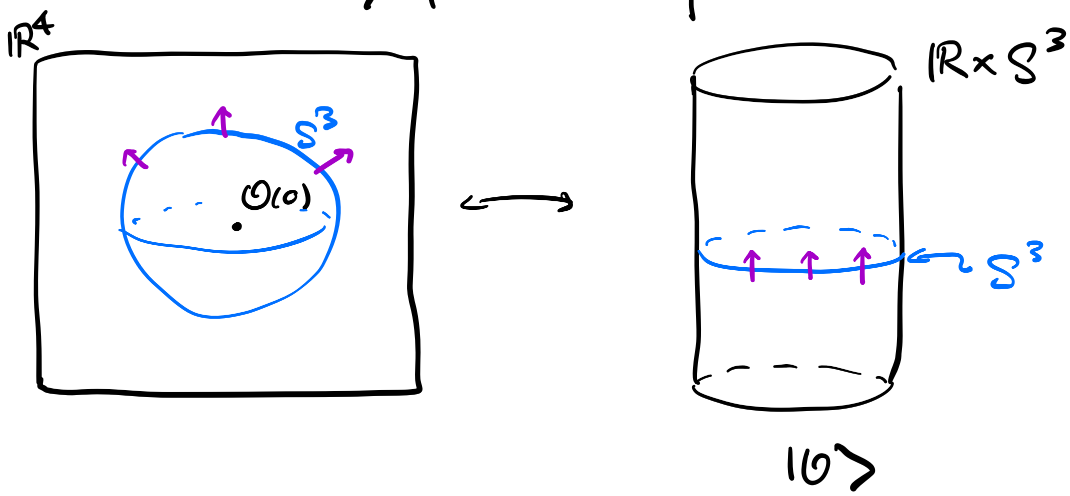
with $\lambda^\alpha \equiv \epsilon^{\alpha\beta}\lambda_\beta$, $\tilde{\lambda}^{\dot\alpha} \equiv \epsilon^{\dot\alpha\dot\beta}\tilde{\lambda}_{\dot\beta}$,

Together they generate the Lie super-algebra psu(2,2|4) on the subspace

$$C \equiv 2 + \lambda^\alpha\frac{\partial}{\partial\lambda^\alpha} - \tilde{\lambda}^{\dot\alpha}\frac{\partial}{\partial\tilde{\lambda}^{\dot\alpha}} - \eta^I\frac{\partial}{\partial\eta^I} = 0.$$

[would be su(2,2|4) without restriction on C]

State/operator map



After Wick rotation to Euclidean signature, P_μ and K_μ as operators acting on $\mathcal{H}_{S^3}$ are Hermitian conjugate to one another.

In a similar manner, Q and $\overline{Q}$ in Euclidean signature are no longer related by complex conjugation. Rather, S and Q are related by Hermitian conjugation as operators on $\mathcal{H}_{S^3}$, as do $\overline{S}$ and $\overline{Q}$. We will still use the barred notation $\overline{Q}$ and $\overline{S}$ to denote the Wick rotated version of the corresponding supercharges, but now viewed as operators on $\mathcal{H}_{S^3}$, they obey

$$S_I{}^\alpha = (Q^I{}_\alpha)^\dagger, \quad \overline{S}^{I\dot\alpha} = (\overline{Q}_{I\dot\alpha})^\dagger.$$

We can organize the (gauge-invariant) operators of $\mathcal{N}=4$ SYM according to representations of the $psu(2,2|4)$ superconformal algebra.

Q, P raise scaling dim Δ,

S, K lower Δ.

Superconformal primary $|\phi\rangle$ obeys

$$S^{\alpha}_{I}|\phi\rangle = \bar{S}^{I\dot{\beta}}|\phi\rangle = 0$$

$$(\Rightarrow K_\mu |\phi\rangle = 0 \text{ also})$$

Superconformal descendants are obtained by acting on $|\phi\rangle$ repeatedly with Q's.

$S \sim Q^\dagger$.

$$\langle\phi| \underbrace{\{Q, S\}}_{\Delta + J + R} |\phi\rangle \geq 0$$

leads to unitarity bounds on Δ in terms of the $\underbrace{SO(4)_{\text{rotation}}}_{\cong\, SU(2)_L \oplus SU(2)_R} \oplus \underbrace{SO(6)_R}_{\cong\, SU(4)}$ rep. of ϕ.

$SU(2)_L \oplus SU(2)_R$ rep. labeled by $(j, \bar{j})$, $j, \bar{j} = 0, \frac{1}{2}, 1, \ldots$

$SU(4)$ rep. labeled by highest weights or equivalently Dynkin labels

Representations of $SU(4)$:

1. Pick Cartan generators $H_1, H_2, H_3 \in su(4)$

$[H_i, H_j] = 0$,

e.g.

$$H_1 = \begin{pmatrix} \frac{1}{2} & & & 0 \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ 0 & & & -\frac{1}{2} \end{pmatrix}$$

$$H_2 = \begin{pmatrix} \frac{1}{2} & & & 0 \\ & -\frac{1}{2} & & \\ & & \frac{1}{2} & \\ 0 & & & -\frac{1}{2} \end{pmatrix}$$

$$H_3 = \begin{pmatrix} \frac{1}{2} & & & 0 \\ & -\frac{1}{2} & & \\ & & -\frac{1}{2} & \\ 0 & & & \frac{1}{2} \end{pmatrix}$$

2. Given a rep. R, H_i are represented as linear maps $\rho(H_i) : R \to R$ that commute w/ one another; can find a basis v_I of R that diagonalize $\rho(H_i)$ simultaneously:

$$\rho(H_i) \cdot v_I = (w_I)_i \, v_I .$$

$w_I \equiv ((w_I)_1, (w_I)_2, (w_I)_3)$ is called the eight of v_I.

3. The remaing non-Cartan generators of $SU(4)$ can also be organized into basis elmts that change weights by a definite amount, e.g. a (complex) element E_α

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

$\alpha \equiv (\alpha_1, \alpha_2, \alpha_3)$ is called a root.

e.g. for $E_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

$[H_1, E_\alpha] = 0$, $[H_2, E_\alpha] = [H_3, E_\alpha] = E_\alpha$

$\Rightarrow \alpha = (0, 1, 1)$

$E_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\beta = (1, -1, 0)$

$E_\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\gamma = (0, 1, -1)$

all other roots are of the form $k_1 \alpha + k_2 \beta + k_3 \gamma$, where k_i's are either all ≥ 0 or all ≤ 0.

$\{\alpha, \beta, \gamma\}$ are called simple roots.

4. For a given irrep R, all weights differ by some integer combo of simple roots.

"highest weight" w is such that for any weight μ associated with a vector in R, $w-\mu$ is a ≥ 0 linear combo of simple roots.

Theorem: (for semi-simple Lie algebra) an irrep has a unique highest weight.

e.g. for $su(4)$,

- fundamental rep "4 of $su(4)$" has highest weight $w_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, realized by the vector $v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

  $(\rho_4(H_i) = H_i, \ \rho_4(E_\alpha) = E_\alpha)$

- anti-fund rep "$\bar{4}$ of $su(4)$" has highest weight $w_3 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ realized by $v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

  $(\rho_{\bar{4}}(H_i) = -H_i, \ \rho_{\bar{4}}(E_\alpha) = -E_\alpha^T)$

- $\mathbf{6}$ of $su(4)$ (anti-sym tensor $\Lambda^2 4 \simeq \Lambda^2 \bar{4}$) has highest weight $\omega_2 = (1, 0, 0)$.

A general irrep of $su(4)$ has highest weight

$$\omega = r_1 \underset{\substack{\uparrow \\ 4}}{\omega_1} + r_2 \underset{\substack{\uparrow \\ 6}}{\omega_2} + r_3 \underset{\substack{\uparrow \\ \bar{4}}}{\omega_3}$$

where r_i are ≥ 0 integers.

eg. adjoint rep has highest weight

$$\begin{aligned} \omega &= \omega_1 + \omega_3 \\ &= (1, 1, 0) \\ &= \alpha + \beta + \gamma \end{aligned}$$

realized by $E_{\alpha+\beta+\gamma} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$[r_1, r_2, r_3]$ are called the Dynkin labels of the representation.

BPS bound on a superconformal primary ϕ of $PSU(2,2|4)$

labeled by $(j, \bar{j}\,;\, r_1, r_2, r_3)$

Either $\Delta \geq 2 + j + \frac{3r_1 + 2r_2 + r_3}{2}$, $j \geq 0$

or $\Delta = \frac{3r_1 + 2r_2 + r_3}{2}$, $j = 0$

and likewise either

$$\Delta \geq 2 + \bar{j} + \frac{r_1 + 2r_2 + 3r_3}{2} \,, \quad \bar{j} \geq 0$$

or $\Delta = \frac{r_1 + 2r_2 + r_3}{2}$, $\bar{j} = 0$

- BPS (or shortened) primaries/representations

A representation of the superconformal algebra is shortened if there are nontrivial linear relations among Q, $\bar{Q}$ - descendants of the s.c. primary $|\phi\rangle$ (i.e. null states)

The null states / shortening condition typically imply that a certain BPS bound is saturated.

For a complete classification of rep. of $PSU(2,2|4)$, see Dolan, Osborn, hep-th/0209056

Cordova, Dumitrescu, Intriligator, 1612.00809

Example: $\Phi_i \equiv (\Gamma_i)^{IJ}\Phi_{IJ}$, $i=1,\dots,6$

$\uparrow$ 6D gamma matrices

$$\mathcal{O}_{ij} = \mathrm{Tr}(\Phi_i \Phi_j) - \tfrac{1}{6}\delta_{ij}\mathrm{Tr}(\Phi_k\Phi_k).$$

$\in$ rk 2 sym. traceless rep of $SO(6)_R$

Dynkin label $[0,2,0]$

vs $\tilde{\mathcal{O}} = \mathrm{Tr}(\Phi_k\Phi_k)$ $\in$ singlet of $SO(6)_R$

Both $\mathcal{O}$ and $\tilde{\mathcal{O}}$ must be annihiled by $S, \bar{S}$ as there are no candidate lower weight op.'s, and must be superconformal primaries.

$$Q^I_\alpha \cdot \Phi_i \sim (\Gamma_i)^{IJ}\lambda_{J\alpha}$$

$$\Rightarrow \quad Q^I_\alpha \cdot \tilde{\mathcal{O}} \sim (\Gamma^k)^{IJ}\,\underbrace{\mathrm{Tr}(\lambda_{J\alpha}\Phi_k)}_{\in}$$

proj.

$4 \oplus 20 = \bar{4} \otimes 6$ of $SO(6)_R$

$$\text{“}\bar{4}\text{''} \otimes \text{“}6\text{''} = V_{[0,0,1]} \otimes V_{[0,1,0]} = V_{[0,1,1]} \oplus V_{[1,0,0]}$$

$\uparrow$ “20” $\quad\quad \uparrow$ “4”

Young tableaux notation

no null states among $Q^I_\alpha \cdot \tilde{\mathcal{O}}$

$\rightsquigarrow$ a non-BPS “long” representation.

On the other hand,

$$Q^I_\alpha \cdot \mathcal{O}_{ij} \sim (\Gamma_i)^{IJ}\, \mathrm{Tr}(\underbrace{\lambda_{J\alpha} \phi_j}) \Big|_{\substack{\text{sym, traceless} \\ \text{wrt } (ij)}}$$

$4 \otimes 20' = 60 \oplus 20$ $\qquad \downarrow$ proj.

$20 \oplus 4 = \bar{4} \otimes 6$

$V_{[1,0,0]} \otimes V_{[0,2,0]}$

$= V_{[1,2,0]} \oplus V_{[0,1,1]}$

$\uparrow$ "60" $\qquad \uparrow$ "20"

The 60-dim'l rep with Dynkin label $[1,2,0]$ appearing in $Q^I_\alpha \cdot \mathcal{O}_{ij}$ are null states!

$\rightarrow$ a shorten representation

The classical or zero coupling scaling dim. of $\mathcal{O}_{ij}$, $\Delta_{\mathcal{O}} = r_2 = 2$, holds in the quantum $\mathcal{N}=4$ SYM at any coupling due to the shortening condition.

This is in contrast with $\tilde{\mathcal{O}}$, whose scaling dimension $\Delta_{\tilde{\mathcal{O}}}$ is a nontrivial function of Yang-Mills coupling g.

- The Lagrangian density $\mathcal{L} = \frac{1}{2g^2} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu} + \dots)$ is a superconformal descendant of $\mathcal{O}_{ij}$!

$$\mathcal{L} \sim (Q^4)^{ij}\, \mathcal{O}_{ij}$$

implies that $\mathcal{L}$ has exact scaling dimension 4,
$\Rightarrow$ vanishing of β-function for g!

The 1/N expansion

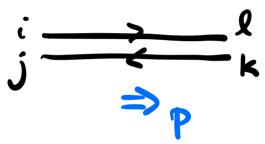
consider a typical $N \times N$ matrix-valued field Φ, with Lagrangian of the form

$$\mathcal{L} = \frac{1}{g^2} \operatorname{Tr}\left(\frac{1}{2} (\partial_\mu \Phi)^2 + V(\Phi) + \cdots \right)$$

invariant under $\Phi \to U^\dagger \Phi U$, $U \in U(N)$

In perturbation theory, the (Euclidean) propagator for $\langle \Phi_i{}^j \, \Phi_k{}^l \rangle$ takes the form

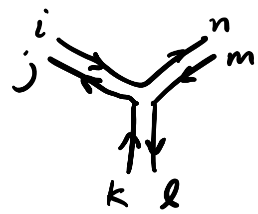
$\Phi_i{}^j$ $\Phi_k{}^l$



$$= \delta_i^l \, \delta_k^j \, \frac{g^2}{p^2 + m^2}$$

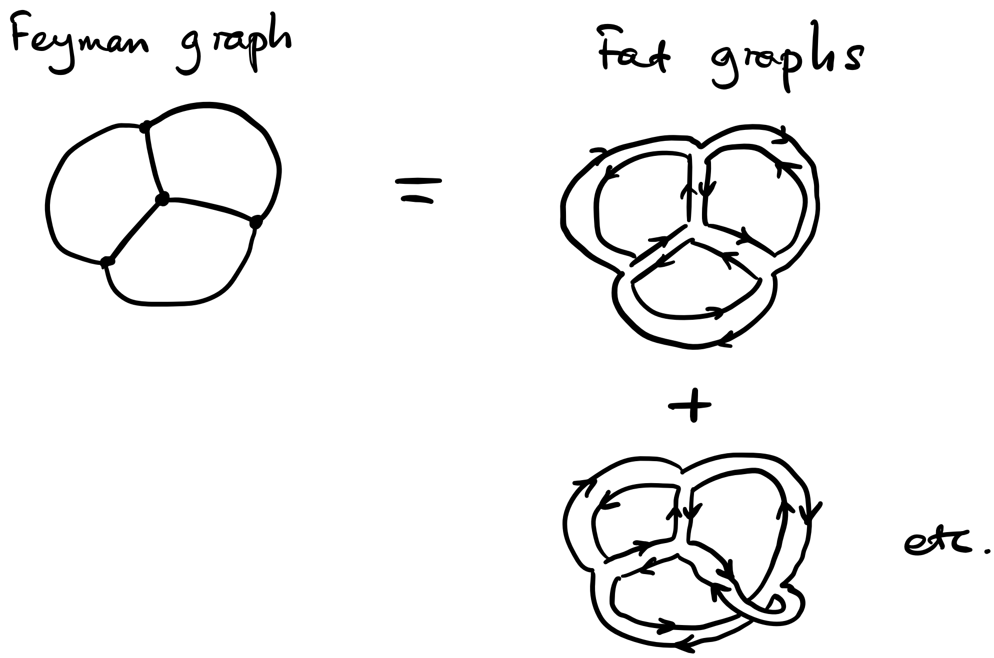
Suppose $V(\Phi) \supset \Phi^3$

$\frac{1}{g^2} \operatorname{Tr} \Phi^3 \to$ vertex



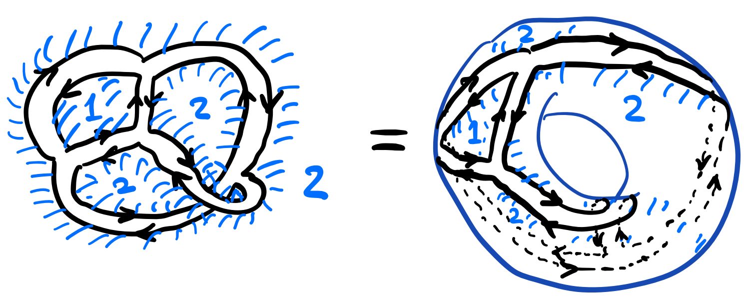
$$\sim \frac{1}{g^2} \delta^j_k \, \delta^l_m \, \delta^n_i$$

The index structure in Feynman diagrams can be represented through "fat graphs", e.g.



By filling in index loops, we turn each fat graph into a closed oriented surface, tiled by polygons along the edges/propagators, e.g.

planar



non-planar

A fat graph with

- V vertices
- E edges / propagators
- L index loops / faces

(not to be confused with the # of loops in the Feynman diagram sense)

comes with g and N dependence

$$\left(\frac{1}{g^2}\right)^V (g^2)^E N^L$$

$$= (g^2 N)^{E-V} \cdot N^{\overbrace{V-E+L}^{\chi}}$$

χ Euler characteristic of the underlying surface

for cubic vertices, $2E = 3V$

generally, $E \geq \frac{3}{2} V$

$g^2 N \equiv \lambda$ "'t Hooft coupling"

partition function

$$Z = \sum_{\text{fat graphs}} A_{n,\chi} \lambda^n N^\chi$$

$n = E - V \left(\geq \frac{V}{2}\right)$

$\chi = V - E + L$

$= 2 - 2h$

$h =$ genus of underlying surface

expansion in λ and $\frac{1}{N}$

" 't Hooft limit ": $N \to \infty$, $\lambda \equiv g^2 N$ fixed

Z dominated by planar (i.e. $h=0$) fat graphs.

In 4D $SU(N)$ Yang-Mills theory, the RG equation takes the form

$$\frac{d\, g(\mu)}{d \log \mu} = -\frac{b}{4\pi^2} \left(g(\mu)\right)^3 + \mathcal{O}(g^5)$$

$b = \frac{11}{12} N$

renormalized 't Hooft coupling

$$\lambda(\mu) \equiv g(\mu) N .$$

2-loop & higher planar graphs

$$\frac{d\lambda(\mu)}{d\log\mu} = -\frac{11}{24\pi^2}\left(\lambda(\mu)\right)^2 + \mathcal{O}(\lambda^4) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

non-planar graphs

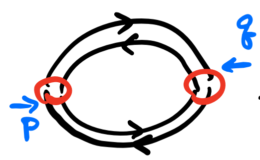
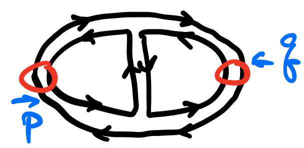
$\lambda(\Lambda_{QCD}) = \infty$ at (from 1-loop RGE)

$$\Lambda_{QCD} = \mu_0\, e^{-\frac{24\pi^2}{11\lambda(\mu_0)}}$$

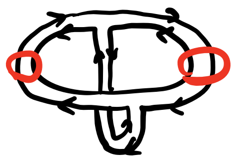
Expect glueball mass $M \sim \Lambda_{QCD}$

flux string tension $T \sim \Lambda_{QCD}^2$.

$$\underbrace{\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})}_{\equiv\ \mathcal{O}(x)} |\Omega\rangle \supset |1\text{-glueball}\rangle$$

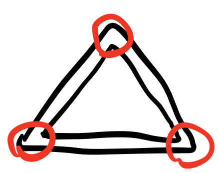
$\langle \tilde{\mathcal{O}}(p)\, \tilde{\mathcal{O}}(q) \rangle =$  $+$  $+ \cdots$

$\sim N^2 \cdot g^4 = \lambda^2$ $\qquad$ $\sim N^3 \cdot g^6 = \lambda^3$

$+$  $+ \cdots$

$\sim N \cdot g^6 = \dfrac{\lambda^3}{N^2}$

$$= \underset{\text{planar}}{\underset{\uparrow}{f_1(\lambda)}} + \underset{\text{non-planar}}{\underset{\uparrow}{\mathcal{O}(N^{-2})}}$$

$\langle \mathcal{O}\, \mathcal{O}\, \mathcal{O} \rangle =$  $+ \cdots$

$\sim N^2 g^6 \sim \dfrac{\lambda^3}{N}$

$$= \underset{\text{planar}}{\underset{\uparrow}{\frac{f_2(\lambda)}{N}}} + \underset{\text{non-planar}}{\underset{\uparrow}{\mathcal{O}(N^{-3})}}$$

via LSZ, $\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle$ determines cubic effective coupling of glueballs $\sim \frac{1}{N}$ in the 't Hooft limit.

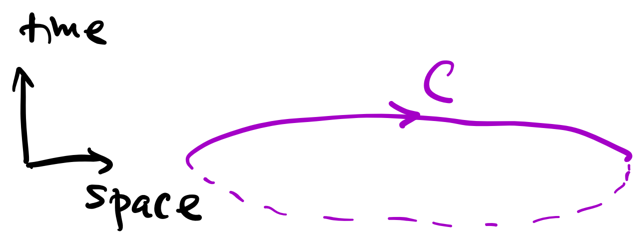
$\Rightarrow$ the SU(N) Yang-Mills theory at large N describes weakly-interacting glueballs.

Flux strings are created by Wilson line operators e.g.

$$W_R(C) = \mathrm{Tr}\left(P\, e^{i\int_C A_{a\mu}(x)\, t^a_R\, dx^\mu}\right)$$

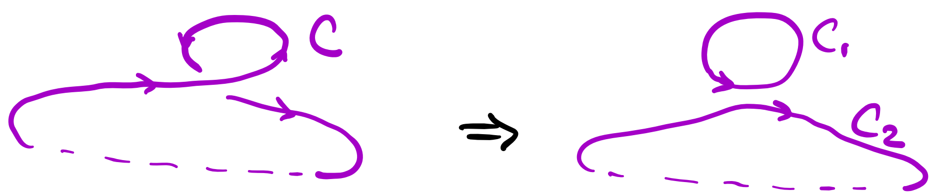
for R = fundamental rep of SU(N).
C a closed path in spacetime

e.g.



Amplitude of splitting (or joining) of flux strings can be extracted from correlators of Wilson lines e.g.

$$\langle \Omega | (W(C_1) W(C_2))^\dagger W(C) | \Omega \rangle \sim \frac{1}{N}$$



Conclusion: SU(N) Yang-Mills theory at large N is a theory of weakly interacting (flux) strings!

Note: the "ripples" on a flux string may be strongly interacting among themselves, but do not leave the string (e.g. as a glueball) at $N = \infty$.

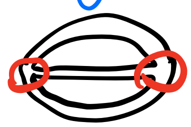
Back to 4D SU(N) $\mathcal{N}=4$ SYM.

In the 't Hooft limit, $N \to \infty$, $\lambda \equiv g^2 N$ fixed, we can organize local operators of finite scaling dim Δ (i.e. not diverging in $N\to\infty$ limit) according to single-trace and multi-trace op's.

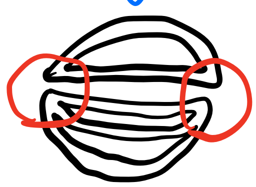
Single-trace: $\mathcal{O}_1 = \mathrm{Tr}(\phi^i F_{\mu\nu} \cdots)$

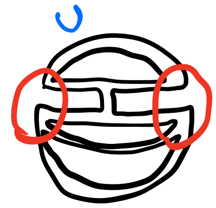
double-trace: $\mathcal{O}_2 = \mathrm{Tr}(\phi^i \cdots)\,\mathrm{Tr}(\phi^j \cdots)$

$$\langle \mathcal{O}_1 | \mathcal{O}_1 \rangle = f_1(\lambda) + \mathcal{O}(N^{-2})$$

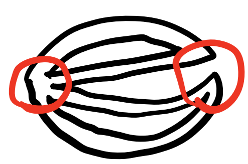


$$\langle \mathcal{O}_2 | \mathcal{O}_2 \rangle = f_2(\lambda) + \mathcal{O}(N^{-2})$$





$$\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle = \frac{1}{N} f_{12}(\lambda) + \mathcal{O}(N^{-3})$$



In the 't Hooft / planar limit, the overlaps between single-trace and multi-trace op's are suppressed. Correlators are dominated by factorized 2-point functions of single-trace operators.

To understand the spectrum of local operators with Δ that remain finite in the limit of $N \to \infty$, fixed λ, it suffices to focus on single-trace operators.

Notation:

$$X^1 = \Phi_1 + i\Phi_2, \quad X^2 = \Phi_3 + i\Phi_4, \quad X^3 = \Phi_5 + i\Phi_6$$

$$X^A \in \text{fundamental rep of } U(3) \subset SO(6)_R$$

can pick an $\mathcal{N}=1$ subalgebra of the supersymmetry algebra, generated by Q_α, $\overline{Q}_{\dot\alpha}$, etc. such that

$$Q_\alpha \cdot X^A = 0, \qquad Q_\alpha \cdot \overline{X}_A = \lambda_{A\alpha},$$

$$Q_\alpha \cdot \lambda_{A\beta} = \epsilon_{ABC}[X^B, X^C],$$

$$Q_\alpha \cdot \chi_\beta = \sigma^{\mu\nu}_{\alpha\beta} F_{\mu\nu}, \quad \text{etc.}$$

- $\mathrm{Tr}(X^{(A_1} \cdots X^{A_n)})$ is a BPS s.c. primary (annhilated by Q_α and all $S, \overline{S}$'s)

- More generally, $\mathrm{Tr}(X^{A_1}\cdots X^{A_n})$ are not BPS primaries, but can only mix among themselves in perturbation theory, as they are the only operators carrying $SO(6)_R$-charge

$$H_1 + H_2 + H_3 = n,$$

w/ $\Delta = n +$ (quantum corrections, a series in λ in 't Hooft limit)

- We will restrict ourselves to the slightly simpler "SU(2) sector", consisting of operators of the form

$$\mathrm{Tr}(X^{A_1}\cdots X^{A_n}), \quad A_i = 1 \text{ or } 3$$

and write $X^1 \equiv X, \quad X^3 \equiv Z.$

- $\mathrm{Tr}(Z^L)$ is a BPS primary

$$\Delta = L \text{ at any } \lambda$$

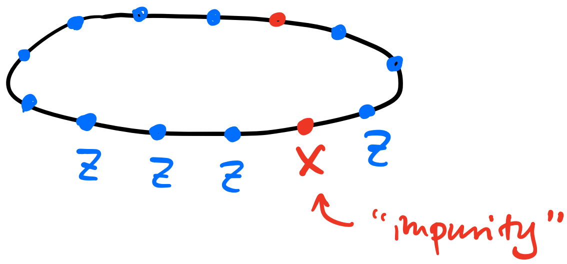
$$\mathrm{Tr}(XZ^{L-1}) = \frac{1}{L}\left[\mathrm{Tr}(XZ^{L-1}) + \mathrm{Tr}(ZXZ^{L-2}) + \cdots\right]$$

is also a BPS primary.

So is $\displaystyle\sum_{\ell=0}^{L-2} \mathrm{Tr}(XZ^\ell X Z^{L-2-\ell})$.

symmetrized within the trace

However, $\mathrm{Tr}(X Z^{\ell} X Z^{L-2-\ell})$ is itself not a BPS primary, and generally not an operator with definite scaling dimension.



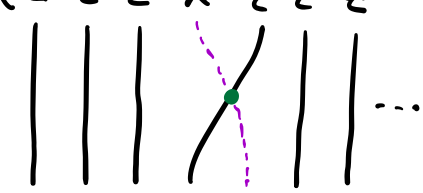
How does the dilatation operator $D = i\Delta$ act on the space of single-trace operators of a total L of Z's and X's ?

Calculate anomalous dimension from 2-point function

$$\langle \overline{\mathcal{O}'(x)}\, \mathcal{O}(0) \rangle$$

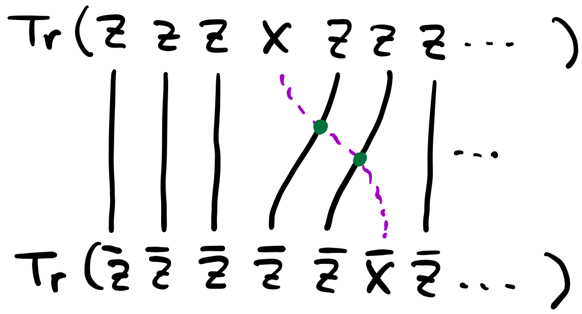
order λ contribution:

$$\mathrm{Tr}(Z\, Z\, Z\, X\, Z\, Z\, Z \cdots)$$



$$\mathrm{Tr}(\bar{Z}\, \bar{Z}\, \bar{Z}\, \bar{Z}\, \bar{X}\, \bar{Z}\, \bar{Z} \ldots)$$

order λ^2 contribution:

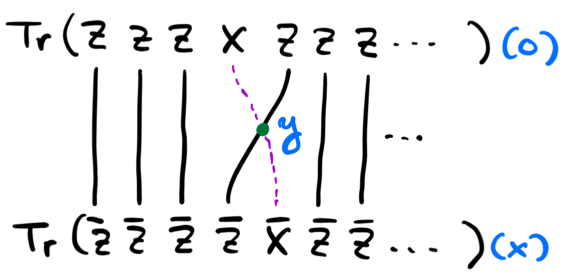


- Expect $\Delta(\lambda)$ to act as a sort of "spin-chain-Hamiltonian" that involves n-th nearest neighbor interactions at order λ^n.

- We normalize $\mathcal{O} = \#\,\mathrm{Tr}(ZZZXZZ\cdots)$ such that $\langle \overline{\mathcal{O}}(x)\,\mathcal{O}(0)\rangle\big|_{\lambda=0} = \dfrac{1}{|x|^{2L}}$.

Euclidean propagator for a massless scalar in position space $= \dfrac{1}{4\pi^2 x^2}$.

The diagram



which contains the vertex

$$\mathrm{Tr}(XZ\bar{X}\bar{Z}) \subset \mathrm{Tr}\,[Z,X][\bar{Z},\bar{X}]$$

from the scalar potential of $\mathcal{N}=4$ SYM gives the 1-loop contribution

$$\langle \overline{\mathcal{O}'(x)\, \mathcal{O}(0)} \rangle \Big|_{1\text{-loop}}$$

$$= \frac{1}{|x|^{2(L-2)}} \frac{\lambda}{(4\pi^2)^2} \underbrace{\int d^4y \frac{1}{(y^2)^2((x-y)^2)^2}}_{\substack{\parallel \\ \frac{4\pi^2}{|x|^4} \log(\Lambda |x|) + \text{finite}}}$$

Λ = UV cutoff

$$= \frac{\lambda}{4\pi^2 |x|^{2L}} \left[\log(\Lambda |x|) + \text{finite} \right]$$

Contributes to Δ at order λ

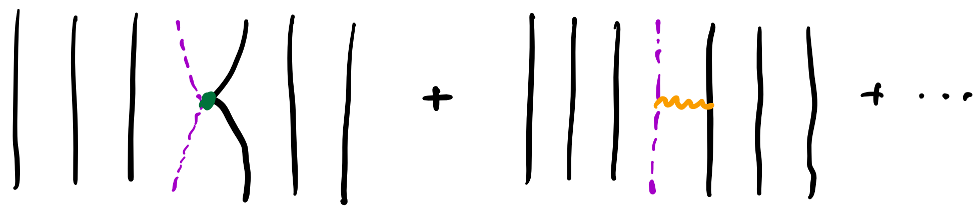
$$\Delta = \Delta_0 + \lambda \Delta_1 + \cdots$$

$$\langle \overline{\mathcal{O}'(x)\, \mathcal{O}(0)} \rangle = \langle \mathcal{O}' | \frac{1}{|x|^{2(\Delta_0 + \lambda \Delta_1 + \cdots)}} | \mathcal{O} \rangle$$

$$= \langle \mathcal{O}' | \frac{1}{|x|^{2\Delta_0}} \left(1 - 2\lambda \Delta_1 \log|x| + \cdots \right) | \mathcal{O} \rangle$$

$$\Rightarrow \quad \langle \mathcal{O}' | \Delta_1 | \mathcal{O} \rangle = -\frac{\lambda}{8\pi^2}.$$

There are other 1-loop diagrams that do not "move" X in the chain, e.g.



their contribution can be determined from the BPS condition, e.g. when $\mathcal{O}$ is a BPS operator, $\Delta_{\mathcal{O}} = L$ without λ-dependent corrections.

Result:

$$\Delta_1 = \frac{1}{8\pi^2} \sum_{i=1}^{L} \left(1 - P_{i,i+1}\right)$$

permuting i^{th} and $(i+1)^{st}$-sites of the chain

An impurity X inserted at the i^{th}-site can be viewed as a "magnon" of the spin chain. Interactions are short range between impurities. Energy eigenstates of a single impurity, far from the others, are momentum eigenstates:

$$|p\rangle \sim \sum_{\ell} \mathrm{Tr}(\cdots Z\,Z\,\underset{\uparrow}{X}\,Z\cdots)\, e^{ip\ell}$$

ℓ-th site

$p = \frac{2\pi n}{L}$, $n \in \mathbb{Z}$

$$\Delta_1 |p\rangle = \frac{1}{8\pi^2}\left(1 - e^{ip} - e^{-ip}\right)|p\rangle$$

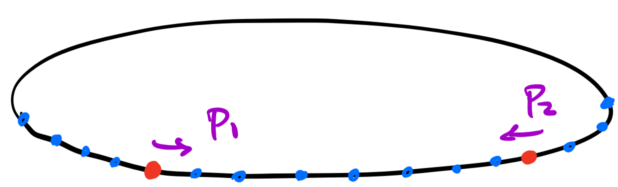
$$= \frac{1}{2\pi^2} \sin^2\left(\frac{p}{2}\right) |p\rangle$$

dispersion relation for a single impurity

$$\Delta(\lambda) = 1 + \frac{\lambda}{2\pi^2}\sin^2\left(\frac{p}{2}\right) + \mathcal{O}(\lambda^2)$$

Turn out the exact dispersion relation can be determined from an enhanced centrally-extended $SU(2|2)$ symmetry of the $L=\infty$ chain, to be

$$\Delta(\lambda) = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}$$



Conjecturally, scattering of impurities is integrable, and governed by a (now exactly known factorized S-matrix), which can be used to determine $\Delta(\lambda)$ on the entire space of single trace operators (at $N=\infty$).