Quantum theory with relativistic locality

- Poincaré symmetry

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\nu$$

$$\underset{\nwarrow \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}}{\eta_{\mu\nu}} dx'^\mu dx'^\nu = \eta_{\mu\nu} dx^\mu dx^\nu$$

In QM, represented as a unitary operator $U(a, \Lambda)$ acting on the Hilbert Space $\mathcal{H}$ of states.

- Infinitesimal form:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu .$$

$$\omega_{\mu\nu} \equiv \eta_{\mu\rho}\, \omega^\rho{}_\nu \quad \text{anti-sym in } [\mu\nu].$$

$$U(\epsilon^\mu, \Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu)$$

$$= 1 - i\epsilon^\mu \underset{\nearrow}{P_\mu} + \frac{i}{2}\omega_{\mu\nu} \underset{\uparrow}{J^{\mu\nu}} + \text{higher order.}$$

energy-momentum   angular momentum & Lorentz boost

P_μ and $J^{\mu\nu}$ are self-adjoint operators

$$P^\mu = (P^0, \vec{P})$$

Assumptions :

- energy $P^0 \geq 0$
- invariant mass squared

$$-P^2 \equiv (P^0)^2 - \sum_{i=1}^{3} P^i P^i$$

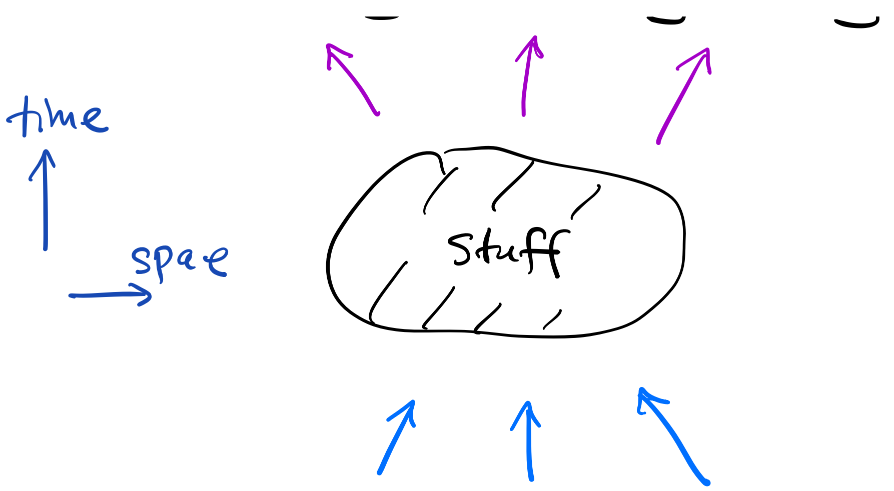
$$\geq 0.$$

- Poincaré-invt vacuum state $|0\rangle$

$$P^\mu |0\rangle = 0 = J^{\mu\nu} |0\rangle.$$

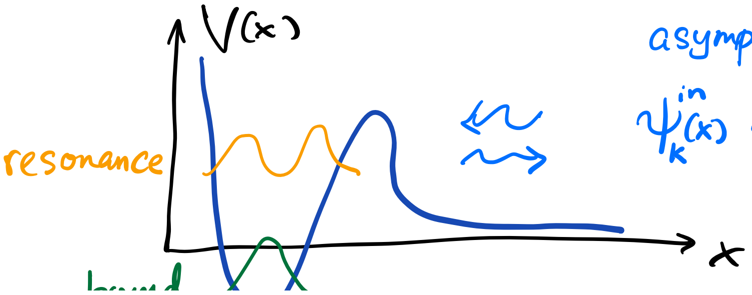
- "particles" – what are they ?

Assume: any state under time evolution decays into wave packets of particles that are weakly interacting at long distances.



- This notion of particles refers to only stable particles
  - "unstable particles" are resonances that will be seen as singularities in the analytic continuation of scattering amplitudes.

- Some intuition from a non-relativistic QM model for 2 interacting particles separated by distance x, $H = \frac{p^2}{2m} + V(x)$.



asymptotic region $x \to \infty$

$$\psi_k^{in}(x) \sim e^{-ikx} + \underbrace{S(k)}_{\text{scattering amplitude}} e^{ikx}$$

bound state

→ particle

$$\psi^{out}_k(x) = (S(k))^{-1}\psi^{in}_k(x)$$

$$\sim (S(k))^{-1}e^{-ikx} + e^{ikx}$$

- bound state $E = -\frac{\alpha^2}{2m} + V_\infty$.

$$\psi(x) \sim e^{-\alpha x}, \quad x\to\infty \quad , \alpha>0.$$

If we do not impose normalizability of wave function, can analytically continue ψ^{out}_k to the complex k-plane.

$k \to i\alpha$, ψ^{out}_k becomes

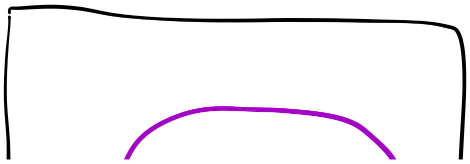
$$\psi(x) \sim \underbrace{(S(i\alpha))^{-1}}_{\text{need this to vanish}} e^{\alpha x} + e^{-\alpha x}$$

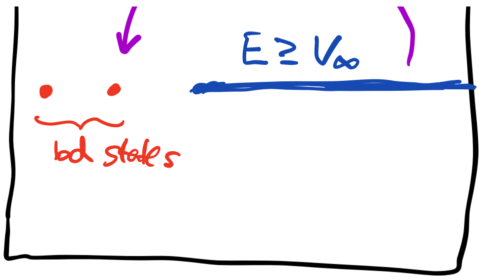
bound state ⟷ pole of $S(k)$

at $k = i\alpha$, $\alpha>0$.

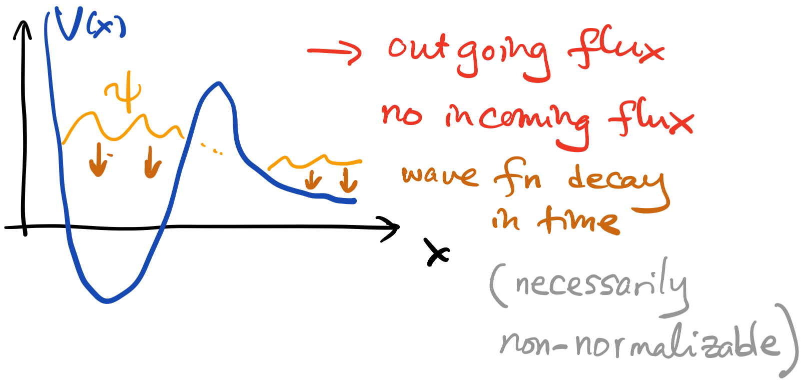
$$E = \frac{k^2}{2m} + V_\infty$$

cplx E-plane





. resonance



restore time-dependence

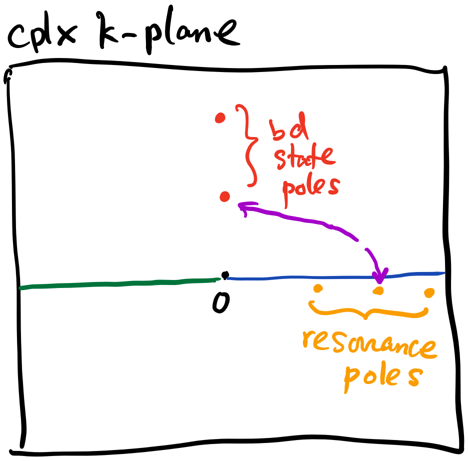
$$\psi^{out}_k(x,t) \sim \underset{0}{(\cancel{S(k)})^{-1}}\, e^{-ikx-iEt} + e^{ikx-iEt}, \quad x\to\infty.$$

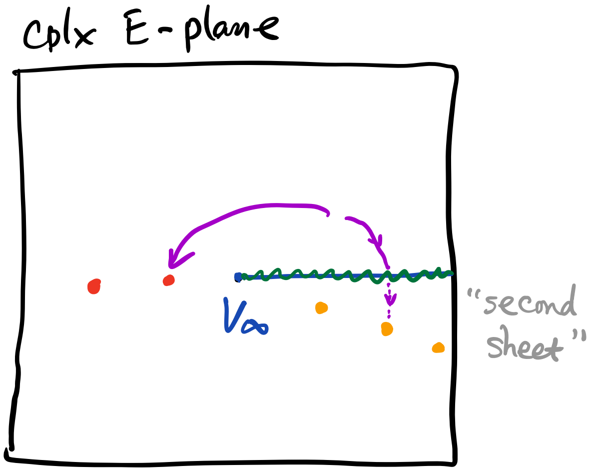
$S(k)$ has pole at $k=k_*$,

where $E = \frac{k_*^2}{2m} + V_\infty$

$= \omega - i\gamma$, $\gamma > 0$ decay width

$$k_* = \sqrt{2m(\underbrace{\omega - V_\infty}_{\overset{\vee}{0}} - i\gamma)}$$





Back to relativistic theory:

The quantum states of a particle form a representation of Poincaré sym.

$$|p^\mu, \sigma\rangle$$

$\uparrow$ internal d.o.f.

$$P^2 \equiv P^\mu P_\mu$$ commutes w/ $U(\Lambda)$

$$= -M^2.$$ M = invariant mass.

Can label the state as

$$|\vec{p}, \sigma\rangle.$$ with the understanding $P^0 = \sqrt{\vec{p}^{\,2} + M^2}$

for some fixed $M \geq 0$.

Q: How does $U(\Lambda)$ act on $|\vec{p}, \sigma\rangle$?

Expect $U(\Lambda)\,|\vec{p}, \sigma\rangle$

$$= \sum_{\sigma'} C_{\sigma\sigma'}(\Lambda, \vec{p})\, |\overrightarrow{\Lambda p}, \sigma\rangle$$

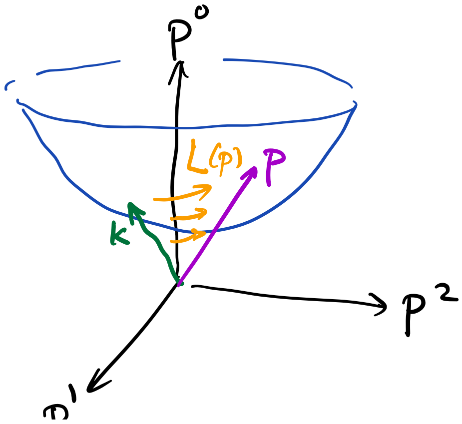
$\uparrow$ spatial component of $\Lambda^\mu{}_\nu p^\nu$.

To proceed, pick a "reference momentum"

$$k^\mu, \qquad k^2 \equiv k^\mu k_\mu = -M^2.$$

For every possible momentum vector p^μ of the particle, choose a "standard boost" $L^\mu{}_\nu(\vec{p})$, with

$$p^\mu = L^\mu{}_\nu(\vec{p})\, k^\nu.$$



Define

$$|\vec{p},\sigma\rangle := \underbrace{N(\vec{p})}_{\substack{\text{normalization}\\ \text{to be chosen later}}}\, U(L(\vec{p}))\,|\vec{k},\sigma\rangle$$

It follows that

$$U(\Lambda)\,|\vec{p},\sigma\rangle = N(\vec{p})\,U(\Lambda)\,U(L(\vec{p}))\,|\vec{k},\sigma\rangle$$

$$= N(\vec{p})\,U(\Lambda L(\vec{p}))\,|\vec{k},\sigma\rangle$$

$$= N(\vec{p})\,U(L(\overrightarrow{\Lambda p}))$$

$$\underbrace{\cdot\, U(\underbrace{L^{-1}(\overrightarrow{\Lambda p})\,\Lambda\, L(\vec{p})}_{\substack{\equiv \\ W\,:\, k \to Lk = p \to \Lambda p \to k}})\,|\vec{k},\sigma\rangle}_{U(W)\,|\vec{k},\sigma\rangle = \sum_{\sigma'} D_{\sigma\sigma'}(W)\,|\vec{k},\sigma'\rangle}$$

Thus, we find

$$U(\Lambda)|\vec{p},\sigma\rangle = \frac{N(\vec{p})}{N(\overrightarrow{\Lambda p})}\sum_{\sigma'} D_{\sigma\sigma'}(W)\,|\overrightarrow{\Lambda p},\sigma'\rangle$$

- The set of W's that leave k invariant

forms the "little group" G

$D_{\sigma\sigma'}(W)$ gives a representation of G.

$$\begin{tikzcd} D: & G \arrow[r] & \mathrm{End}(V) \\ & W \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & (D_{\sigma\sigma'}(W)) \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$\nwarrow$ spanned by $|\vec{k},\sigma\rangle$

$$U(W_1 W_2) = U(W_1)\cdot U(W_2)$$

$$\Rightarrow \quad D_{\sigma\sigma'}(W_1 W_2) = \sum_{\sigma''} D_{\sigma\sigma''}(W_2)\, D_{\sigma''\sigma'}(W_1)$$

"

Two cases:

(1) $M>0$. WLOG take

$$k^\mu = (M, \vec{0})$$

(2) $M=0$, WLOG take

$$k^\mu = (E, 0, 0, E).$$

massive case, little group $G = SO(3)$.

The states $|\vec{0},\sigma\rangle$ form a representation of $SO(3)$. or more generally,

of its covering group $\widetilde{SO(3)} = Spin(3)$
$(\simeq SU(2))$

- irreducible unitary rep of $\widetilde{SO(3)}$ labeled by spin $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. of dimension $2j+1$.

i.e. $D_{\sigma\sigma'}(W)$ are $(2j+1)\times(2j+1)$ matrices

- $j=0$ case, massive scalar particle

$$D(W) = 1.$$

$$U(\Lambda)\,|\vec{p}\rangle = \frac{N(\vec{p})}{N(\overrightarrow{\Lambda p})}\,|\overrightarrow{\Lambda p}\rangle.$$

normalization convention:

$$\langle\vec{p}\,|\,\vec{p}\,'\rangle = \delta^3(\vec{p}-\vec{p}\,')$$

$\Rightarrow$ $|N(\vec{p})|^{-2}\,\delta^3(\vec{p}-\vec{p}\,')$ invariant under $p \to \Lambda p$, $p' \to \Lambda p'$.

Compare with

$$2p^0\,\delta^3(\vec{p}-\vec{p}\,') = \frac{\delta^4(p^\mu - p'^\mu)}{\delta(p^2 - p'^2)}$$

$\nearrow$
manifestly Lorentz invt

$$\Rightarrow \quad N(\vec{p}) = \sqrt{\frac{k^0}{p^0}} . \qquad \checkmark .$$

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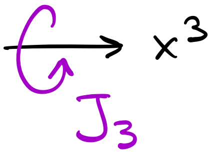
massless case

$$k^\mu = (E, 0, 0, E)$$

$W \in SO(3,1)$ leaving k fixed ?



$$(J_3)^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix}$$

$$R^\mu{}_\nu(\theta) \equiv (e^{i\theta J_3})^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}$$

Also : $A = K_1 + J_2 , \qquad B = K_2 - J_1$

K_i = Lorentz boost generator in x^i-direction

$$A^\mu{}_\nu = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \qquad B^\mu{}_\nu = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$[A, B] = 0 .$$

$$S^{\mu}{}_{\nu}(\alpha,\beta) = \left(e^{i\alpha A + i\beta B}\right)^{\mu}{}_{\nu} = \begin{pmatrix} 1+\zeta & \alpha & \beta & -\zeta \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \zeta & \alpha & \beta & 1-\zeta \end{pmatrix}$$

$$\zeta = \frac{\alpha^2+\beta^2}{2} .$$

General little group element

$$W(\theta,\alpha,\beta) = S(\alpha,\beta)\, R(\theta) .$$

can diagonalize $\hat{A}$, $\hat{B}$

with a basis of 1-particle states

$$|\vec{k}, \underset{\uparrow}{a}, \underset{\uparrow}{b}\rangle$$

eigenvalues wrt $\hat{A}$, $\hat{B}$.

$$[J_3, A] = iB ,$$
$$[J_3, B] = iA$$

$\Rightarrow$ $U(R(\theta))\,|\vec{k}, a, b\rangle$

&

$|\vec{k}, a\cos\theta + b\sin\theta, a\sin\theta + b\cos\theta\rangle$

⇝ a continuous family of orthogonal

1-particle states with fixed momentum k^μ.

- seems pathological.
- set $a = b = 0$. A, B act trivially

Can diagonalize J_3 with helicity basis:

$$J_3 \, |\vec{k}, h\rangle = h \, |\vec{k}, h\rangle$$

$$U(R(\theta)) \, |\vec{k}, h\rangle = e^{ik\theta} \, |\vec{k}, h\rangle$$

no "preferred" phase for 1-particle state of nonzero h.

..

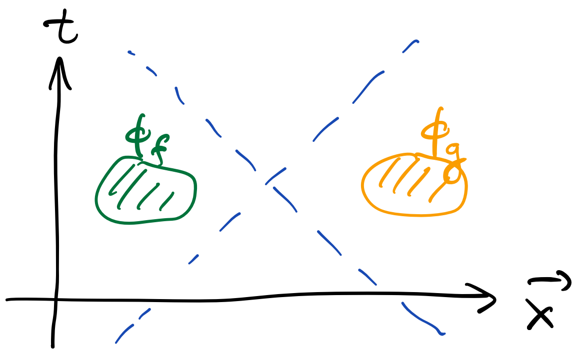
Local ("field") operators.

- Assume the existence of local operator $\phi(x)$, such that

$$\phi_f = \int dx \, \phi(x) f(x)$$

for suitable function $f(x)$ on $\mathbb{R}^{1,d-1}$ is well-defined as a linear operator acting on "most" physical states.

Such that for $\mathrm{Supp}(f)$ and $\mathrm{Supp}(g)$ space-like separated, $[\phi_f, \phi_g] = 0$.



$\phi(x)$ should transform under Poincaré symmetry according to

$$U(a,\Lambda)\phi(x)(U(a,\Lambda))^{-1} = \phi(\Lambda x + a).$$

Example: free massive scalar particles.

annihilation and creation operators

$a_{\vec{k}}$, $a_{\vec{k}}^{\dagger}$,

$$[a, a] = 0 = [a^{\dagger}, a^{\dagger}]$$

$$[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = \delta^{(d-1)}(\vec{k}-\vec{k}') .$$

$$\phi(x) = \int \frac{d^{d-1}\vec{k}}{\sqrt{(2\pi)^{d-1}2\omega_k}} \left(a_{\vec{k}} e^{ik\cdot x} + a_{\vec{k}}^{\dagger} e^{-ik\cdot x} \right)$$

$k^0 = \omega_k = \sqrt{\vec{k}^2 + m^2}$.

$$[\phi(x), \phi(y)] = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}2\omega_k} \left[e^{ik\cdot(x-y)} - e^{-ik\cdot(x-y)} \right]$$

$$= \int \underbrace{\frac{d^{d}k}{(2\pi)^{d-1}} \theta(k^0)\, \delta(k^2+m^2)}_{\text{invariant under } x\to\Lambda x,\ y\to\Lambda y .} \left[e^{ik\cdot(x-y)} - e^{-ik\cdot(x-y)} \right]$$

function of $(x-y)^2$ only.

- If x, y spacelike-separated, WLOG take $(x-y)^0 = 0$, integrand odd under $\vec{k} \to -\vec{k}$.

$$\Rightarrow [\phi(x), \phi(y)] = 0$$

- NOT commuting if $(x-y)^2 < 0$!

Is $\phi(x)$ a well-defined linear operator ?

e.g. $$\phi(x)|0\rangle = \int \frac{d^{d-1}\vec{k}}{\sqrt{(2\pi)^{d-1} 2\omega_k}} e^{-ik\cdot x} |\vec{k}\rangle$$

$$\| \phi(x)|0\rangle \|^2 = \langle 0| \phi(x) \phi(x) |0\rangle$$

$$= \infty .$$

In fact, $\phi(x)$ is ill-defined acting on any state.

Fix: $$\phi_f = \int dx\, f(x)\, \phi(x)$$

e.g. $$\phi_f |0\rangle = \int \frac{d^{d-1}\vec{k}}{\sqrt{(2\pi)^{d-1} 2\omega_k}} \underbrace{\int dx\, f(x) e^{-ik\cdot x}}_{\tilde{f}(k)} |\vec{k}\rangle$$

$$\| \phi_f |0\rangle \|^2 = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1} 2\omega_k} |\tilde{f}(k)|^2 \underset{\nearrow}{<} \infty .$$

restriction on $f(x)$?

- Will consider $f(x) \in C^\infty(\mathbb{R}^d)$ rapidly decreasing

$$\sup_{x \in \mathbb{R}^d} \left| x_1^{\alpha_1} \cdots x_d^{\alpha_d} \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d} f \right| < \infty,$$

$$\forall \alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}$$

- "Schwartz function"

denote the space of such function by $\mathcal{S}$

Axiom 1 : ϕ_f is a well-defined linear operator on a dense subset of $\mathcal{H}$, $\forall f \in \mathcal{S}$

Furthermore, $\phi_{f_1} \cdots \phi_{f_n} |0\rangle$ well-defined

[often span $\mathcal{D} \subset \mathcal{H}$ dense subspace]

Axiom 2: Poincaré transf.

$$(U(a,\Lambda))^{-1} \phi_f \, U(a,\Lambda) = \phi_{f^U}$$

$$f^U(x) := f(\Lambda x + a)$$

[equivalent to

$$U(a,\Lambda)\,\phi(x)\,(U(a,\Lambda))^{-1} = \phi(\Lambda x+a)$$

but better defined.

More generally, non-scalar field cp. $\psi_\alpha(x)$

$$U\psi_\alpha(x)U^{-1} = \underbrace{(R(\Lambda))_\alpha{}^\beta}_{} \psi_\beta(\Lambda x+a)$$

not unitary representation in general.]

Axiom 3: microcausality (locality)

If Supp(f) and Supp(g) spacelike separated,

$$[\phi_f,\ \phi_g] = 0.$$

- Wight "function"

$$\langle 0|\,\phi(x_1)\cdots\phi(x_n)\,|0\rangle$$

is a tempered distribution,

i.e. given $f_1, \dots, f_n \in \mathcal{S}$,

$$\langle 0| \phi_{f_1} \cdots \phi_{f_n} |0\rangle \overset{\text{formally}}{=} \int dx_1 \cdots dx_n \, f(x_1) \cdots f(x_n) \langle 0| \phi(x_1) \cdots \phi(x_n) |0\rangle$$

is well-defined and continuous w.r.t. $f_1, \dots, f_n$.

Example: 2-pt function

$$\langle 0| \phi(x) \phi(y) |0\rangle$$

$$= \int d\alpha \, \langle 0| \phi(x) |\alpha\rangle \langle \alpha| \phi(y) |0\rangle.$$

a basis of P^μ-eigenstates

$$\int d\alpha \, |\alpha\rangle\langle\alpha| = \mathbb{1}.$$

$$\langle 0| \phi(x) |\alpha\rangle = \langle 0| U(x,0) \phi(0) (U(x,0))^{-1} |\alpha\rangle$$

$$= e^{i P_\alpha \cdot x} \langle 0| \phi(0) |\alpha\rangle$$

$$= \int d\alpha \; e^{i P_\alpha \cdot (x-y)} \, |\langle 0 | \phi(0) | \alpha \rangle|^2$$

(assuming $\phi(x)$ Hermitian)

$$= \int d^d p \; e^{i p \cdot (x-y)} \underbrace{\int d\alpha \; \delta^d(p - P_\alpha) \, |\langle 0 | \phi(0) | \alpha \rangle|^2 .}_{/\!/ \ \text{Lorentz inv't}}$$

$$\frac{\theta(p^0)}{(2\pi)^{d-1}} \, \rho(-p^2)$$

for some $\underbrace{\rho(-p^2)}$ supported at $-p^2 \geq 0$.

"Spectral function", real, ≥ 0.

$$= \int_0^\infty d\mu^2 \; \rho(\mu^2) \underbrace{\int \frac{d^d p}{(2\pi)^{d-1}} \, \theta(p^0) \, \delta(p^2 + \mu^2) \, e^{i p \cdot (x-y)}}_{\substack{/\!/\!/ \\ \Delta_+(x-y;\, \mu^2)}}$$

we have already seen this,

2-pt function of <u>free</u> field of mass μ.

$$\Delta_+(x;\mu^2) = \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{1}{2\sqrt{\vec{p}^2+\mu^2}} e^{i\vec{p}\cdot\vec{x} - i\sqrt{\vec{p}^2+\mu^2}\,x^0}$$

$$\Delta_+(x;\mu^2) = \Delta_+(-x;\mu^2) \quad \text{if } x^2 > 0$$

(spacelike)

$$\neq \quad \text{if } x^2 < 0$$

(time like)

In contrast, the time-ordered 2-pt fn is

$$\langle 0| T \phi(x)\phi(y) |0\rangle$$

$$\equiv \theta(x^0-y^0) \langle 0| \phi(x)\phi(y) |0\rangle$$

$$+ \theta(y^0-x^0) \langle 0| \phi(y)\phi(0) |0\rangle$$

$$= \int_0^\infty d\mu^2 \rho(\mu^2) \Big[\theta(x^0-y^0)\Delta_+(x-y;\mu^2)$$

$$+ \theta(y^0-x^0)\Delta_+(y-x;\mu^2) \Big]$$

// check using residue

$$-i\int\frac{d^dp}{(2\pi)^d}\,\frac{e^{ip.(x-y)}}{p^2+\mu^2-i\epsilon}\equiv\Delta_F(x-y;\mu^2)$$

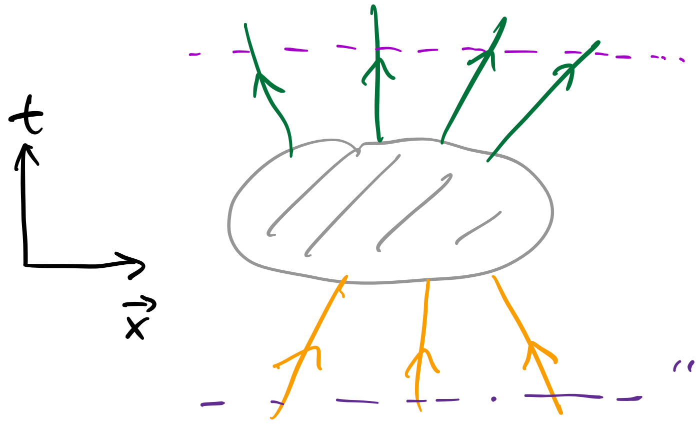
"Feynman propagator"

- What does $\rho(\mu^2)$ look like ?

$$\rho(\mu^2)=(2\pi)^{d-1}\int d\alpha\;\delta^d(p-\underline{P}_\alpha)\,|\langle\alpha|\underbrace{\phi(0)|0\rangle}|^2$$

choose any p^μ such that $p^2=-\mu^2,\quad p^0>0$

What is the content of $\phi(x)\,|0\rangle$?



"out-states"

$|0\rangle$, 1-particle, 2-particles, etc.

two bases related by a unitary transf. (S-matrix)

"in-states"

Assume that there is only 1 species of particle of mass m.

1-particle state

$$p^\mu , \qquad -p^2 = m^2 .$$

2-particle in- or out- state

$$-p^2 \geq (2m)^2 .$$

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can shift $\phi(x)$ by a constant such that $\langle 0 | \phi(x) | 0 \rangle = 0$,

$$\phi(x) |0\rangle = \underbrace{|1\text{-particle}\rangle}_{\text{determined by}} + |2\text{-particles}\rangle^{in} + \cdots$$

$$\langle \vec{k} | \phi(x) | 0 \rangle = e^{-ik\cdot x} \langle \vec{k} | \phi(0) | 0 \rangle$$

$$\langle \vec{k} | \phi(0) | 0 \rangle = \langle \vec{k} | U^{-1}(\Lambda) \, \phi(0) \, U(\Lambda) | 0 \rangle$$

$$= \left(\frac{N(\vec{k})}{N(\vec{\Lambda k})}\right)^* \langle \vec{\Lambda k} \,|\, \phi(0) \,|\, 0 \rangle$$

$$N(\vec{k}) \propto \frac{1}{\sqrt{k^0}}$$

$$\Rightarrow \quad \langle \vec{k} | \phi(0) | 0 \rangle \propto \frac{1}{\sqrt{k^0}},$$

Can write

$$\langle \vec{k} \,|\, \phi(0) \,|\, 0 \rangle \equiv \left[\frac{Z}{(2\pi)^{d-1}} \, \frac{1}{2\sqrt{\vec{k}^2 + m^2}} \right]^{\frac{1}{2}}$$

Constant $Z \geq 0$

[Z is often called "field renormalization constant", sometimes said to be unphysical. The latter is false: Z is intrinsic to the local field operator ϕ, and is unambiguously defined whenever ϕ is defined]

Recall spectral function

$$\rho(\mu^2) = (2\pi)^{d-1} \int d\alpha \; \delta^d(p - P_\alpha) \, \left| \langle \alpha | \phi(0) | 0 \rangle \right|^2$$

$$= \int d^{d-1}\vec{k} \; \frac{Z}{\quad} \; \delta^d(p - k) \, \Big|$$

$$\underbrace{\int \qquad \frac{\quad}{2\sqrt{\vec{k}^2+m^2}}}\qquad \Big|_{\substack{p^2=-\mu^2 \\ k^0=\sqrt{\vec{k}^2+m^2}}}$$

$$+ \left(\text{contribution from multi-particle } |\alpha\rangle\right)$$

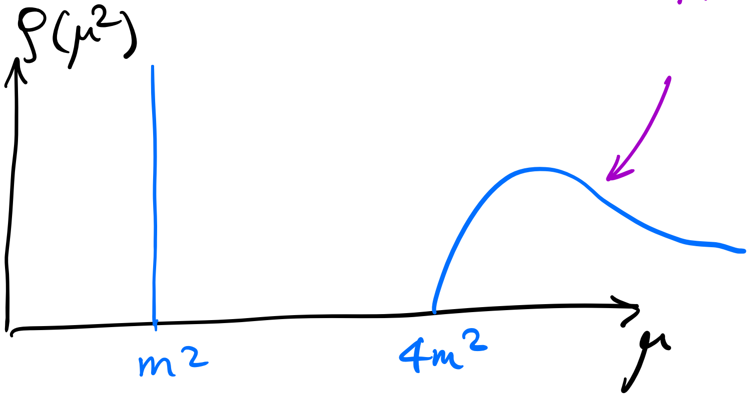
$$Z \cdot \frac{1}{2\sqrt{\vec{p}^2+m^2}}\,\delta\left(p^0-\sqrt{\vec{p}^2+m^2}\right) = Z\,\delta(p^2+m^2)$$

$$= Z\,\delta(\mu^2-m^2)$$

$$\Rightarrow \quad \rho(\mu^2) = Z\,\delta(\mu^2-m^2) + \sigma(\mu^2).$$

↑ multi-particle contribution

supp. @ $\mu > 2m$.



- Lorentzian $\rightleftarrows$ Euclidean

Wightman fn $\longleftrightarrow$ Schwinger fn.

$\uparrow$

$$\langle 0| \underbrace{\phi(x_1) \cdots\cdots \phi(x_n)}_{\text{ordering matters !}} |0\rangle . \qquad x_i \in \mathbb{R}^{1,d-1}$$

$$x_i = (x_i^0 , \vec{x}_i)$$

$\Big\downarrow$ analytic continuation

$$x_i^0 = -i\tau_i , \qquad \tau_i = \text{“Euclidean time”}$$

Q: Is such analytic continuation well-defined for the Wightman function?

$$\langle 0| \phi(x_1)\, \phi(x_2) \cdots \phi(x_n) |0\rangle$$

$$= \int d\alpha_1\, d\alpha_2 \cdots \langle 0| \phi(x_1) |\alpha_1\rangle \langle \alpha_1 | \phi(x_2) |\alpha_2\rangle \cdots\cdots \langle \alpha_{n-1} | \phi(x_n) |0\rangle$$

$$= \int$$

$$- \int d\alpha_1\, d\alpha_2 \cdots \langle 0|\phi(0)|\alpha_1\rangle\, e^{\underbrace{iP_{\alpha_1}\cdot x_{12}}}$$

$$\cdot \langle \alpha_1|\phi(0)|\alpha_2\rangle\, e^{iP_{\alpha_2}\cdot x_{23}} \cdots\cdots$$

$$e^{i\vec{P}_{\alpha_1}\cdot\vec{x}_{12} - iP^0_{\alpha_1} x^0_{12}} , \qquad P^0_{\alpha_1} \geq 0 .$$

$\int$ over α_1 converges if $\mathrm{Im}\, x^0_{12} < 0$

$\rightsquigarrow$ analytic in x^0_i

- More rigorously: Paley–Wiener theorem

$$f(t) = \int d\omega\, \tilde{f}(\omega)\, e^{i\omega t} ,$$

if $\tilde{f}(\omega)$ is supported at $\omega \geq 0$,

then $f(t)$ is analytic on the upper half complex t-plane.

Conclusion: can analytically continue

from (Lorentzian) Wightman function
to Euclidean Schwinger function"
(a.k.a. Euclidean correlator)

provided $\operatorname{Im} x^0_{i,i+1} < 0$.

For $x_i^E \equiv (\tau_i, \vec{x}_i) \in \mathbb{R}^d$

$$\langle \phi(x_1^E) \cdots \phi(x_n^E) \rangle$$

$:=$ analytic continuation of

$$\langle 0 | \phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

to $x_i^0 = -i\tau_i$,

subject to the condition

$$\tau_1 > \tau_2 > \tau_3 > \cdots$$

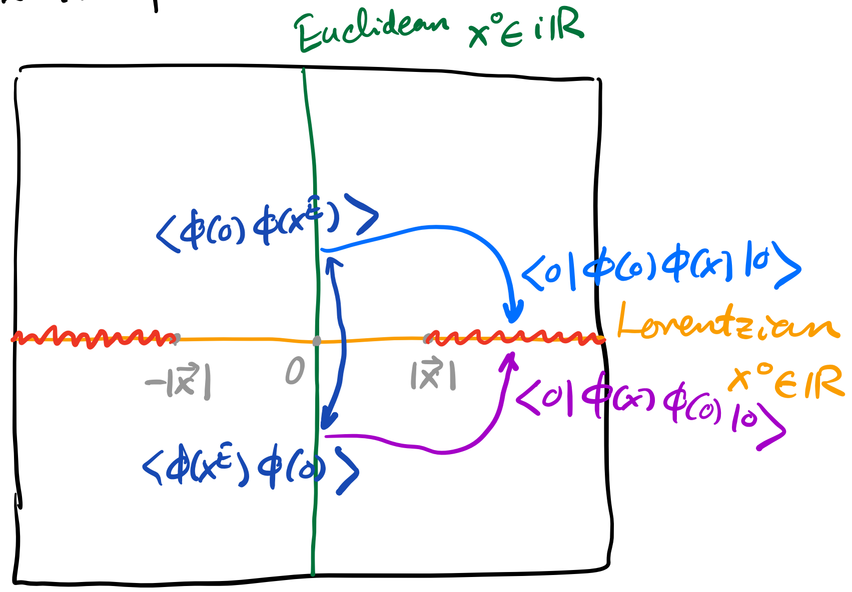
e.g.

$$\langle 0 | \phi(x) \phi(0) | 0 \rangle \xleftrightarrow{\operatorname{Im} x^0 < 0} \langle \phi(x^E) \phi(0) \rangle$$

$\#$ if $x^2 < 0$
x timelike

$$\langle 0|\phi(0)\,\phi(x)|0\rangle \longleftarrow \mathrm{Im}\, x^0 > 0$$

complex x^0-plane



It follows from microcausality that Euclidean correlator

$$\langle \phi(x_1^E)\,\phi(x_2^E)\cdots\rangle_{\tau_1>\tau_2>\tau_3>\cdots}$$

is related to the Euclidean correlator with a different ordering of τ_i, say

$$\langle \phi(x_2^E)\,\phi(x_1^E)\cdots\rangle$$

$$\tau_2>\tau_1>\tau_3>\cdots$$

by analytic continuation.

In this sense, we can forget about the ordering of operators in a Euclid. corr. and simply view $\langle \phi(x_1^E)\, \phi(x_2^E) \cdots \rangle$ as a single analytic function in the x_i^E's.

- In & Out -States.

Recall

$$\phi(x)|0\rangle = \int d^{d-1}\vec{k}\, e^{-i\vec{k}\cdot\vec{x} + i\omega_k x^0} \cdot \left[\frac{Z}{(2\pi)^{d-1}} \frac{1}{2\omega_k}\right]^{\frac{1}{2}} |\vec{k}\rangle$$

$$+ \underbrace{\int d\alpha}_{\text{multi-particle}}\; e^{-iP_\alpha\cdot x} \langle \alpha | \phi(0) | 0\rangle \cdot |\alpha\rangle$$

$(\omega_k \equiv \sqrt{\vec{k}^2 + m^2})$

$\phi_f |0\rangle$

$$= \int d^{d-1}\vec{k}\ \tilde{f}(\omega_k, \vec{k}) \left[\frac{Z}{(2\pi)^{d-1}} \frac{1}{2\omega_k} \right]^{\frac{1}{2}} |\vec{k}\rangle$$

$$+ \int_{\text{m.p.}} d\alpha\ \tilde{f}(p_\alpha) \langle \alpha | \phi(0) | 0 \rangle \cdot | \alpha \rangle .$$

where $\tilde{f}(k) = \int d^d_x\, f(x)\, e^{-ik\cdot x}$

$$f(x) = \int \frac{d^d k}{(2\pi)^d} \tilde{f}(k)\, e^{ik\cdot x}$$

Choose $\tilde{f}(k) \in S$ such that

$$\text{Supp}\,\tilde{f} \subset \{ k : k^0 > 0, |k^2 + m^2| < \varepsilon \}.$$

choose ε to be small enough

so that $|p_\alpha^2 + m^2| = \mu_\alpha^2 - m^2 > \varepsilon$

$\forall$ m. p. state $|\alpha\rangle$.

$\phi_f |0\rangle$ contains only 1-particle state.

Note: $\phi_f^\dagger |0\rangle = \phi_{f^*} |0\rangle = 0$

since $\text{Supp}\,\tilde{f^*} \subset \{ k : k^0 < 0 \}$.

Now consider transformation $f \mapsto f^{(t)}$

$$\tilde{f}^{(t)}(k) := e^{i(k^0 - \omega_k)t}\, \tilde{f}(k).$$

$$\phi_{f^{(t)}}|0\rangle = \phi_f|0\rangle. \qquad (\text{1-particle only})$$

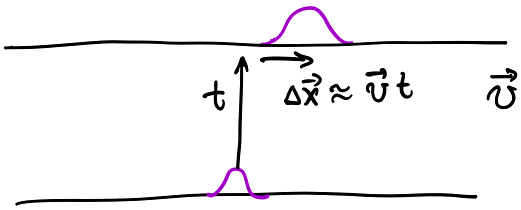
- what does $f^{(t)}(x)$ look like?

$$f^{(t)}(x) = \int \frac{d^dk}{(2\pi)^d}\, e^{ik\cdot x + i(k^0 - \omega_k)t}\, \tilde{f}(k)$$

$$= \int d^dy \int \frac{d^dk}{(2\pi)^d}\, e^{ik\cdot(x-y) + i(k^0-\omega_k)t}\, f(y)$$

$$= \int d^dy \cdot \delta(x^0 - y^0 - t) \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y}) - i\omega_{\vec{k}}t}\, f(y)$$

$$= \int d^{d-1}\vec{y}\; K(\vec{x}-\vec{y}; t)\, f(x^0 - t; \vec{y})$$



$\vec{v} = \frac{d\omega_k}{d\vec{k}}$

More explicitly, consider 1+1 D, $m=0$ case

choose $\tilde{f}(k^0, k^1)$ to be peaked at $k^0 = k^1 = p_* > 0$.

[not quite supported exactly near 1-particle shell, but overlap w/ multi-particle domain is small]

$$\tilde{f}(k^0, k^1) = e^{-\frac{\alpha}{2}(k^0 - p_*)^2 - \frac{\alpha}{2}(k^1 - p_*)^2 - i k^1 \cdot x_*}$$

$$\Big\downarrow \text{Fourier}$$

$$f(x^0, x^1) = \frac{1}{2\pi\alpha} e^{-\frac{1}{2\alpha}(x^0)^2 - \frac{1}{2\alpha}(x^1 - x_*)^2 + i p_*(x^1 - x_*)}$$

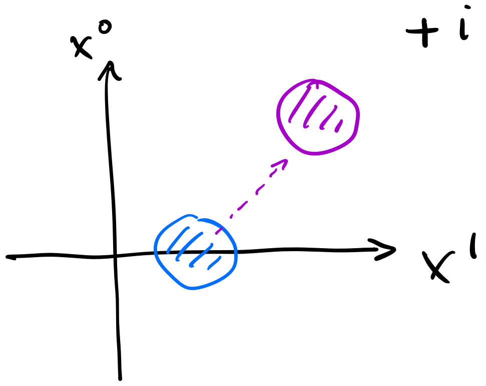
$$\tilde{f}^{(t)}(k^0, k^1) = \exp\Big[-\frac{\alpha}{2}(k^0 - p_*)^2 - \frac{\alpha}{2}(k^1 - p_*)^2$$

$$+ i k^0 t - i k^1 x_* - i|k^1| t\Big]$$

$$\Big\downarrow \text{Fourier, dominated by } k^1 \text{ near } p_*, \ |k^1| \rightsquigarrow k^1$$

$$f^{(t)}(x^0, x^1) \approx \frac{1}{2\pi\alpha} \exp\Big[-\frac{1}{2\alpha}(x^0 - t)^2 - \frac{1}{2\alpha}(x^1 - x_* - t)^2$$

$$+ i p_*(x^1 - x_* - t)\Big]$$



Yet, $\phi_{f^{(t)}}|0\rangle \approx \phi_f|0\rangle$

To formulate scattering theory, we need to further assume:

- Cluster property

Given constant vector a^μ

define $f^{(a)}(x) := f(x+a)$.

$$\lim_{\substack{a^\mu \text{ spacelike} \\ a^2 \to \infty}} \langle 0 | \phi_{f_1} \cdots \phi_{f_n} \phi_{f^{(a)}_{n+1}} \cdots \phi_{f^{(a)}_{n+m}} | 0 \rangle$$

$$= \langle 0 | \phi_{f_1} \cdots \phi_{f_n} | 0 \rangle \langle 0 | \phi_{f_{n+1}} \cdots \phi_{f_{n+m}} | 0 \rangle$$

[Note: in a QFT with degenerate vacua, the cluster property is expected to hold if we choose $|0\rangle$ to be one of the specific basis vacuum states $|\alpha_1\rangle, |\alpha_2\rangle, \cdots$ but would be violated if we had taken to be a generic linear superposition of the $|\alpha_i\rangle$'s.]

Given Wightman functions

$$W_n(x_1, \cdots, x_n) = \langle 0 | \phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

we define "connected" (or "truncated") Wightman functions via

$$W_n = \sum_{\substack{\sqcup S_I = \{1, \cdots, n\} \\ \text{(partition)}}} \prod_I \left(W^{\text{conn}}_{|S_I|}(\{x_\alpha : \alpha \in I\}) \right)$$

e.g. $W_1(x) = W_1^{\text{conn}}(x)$

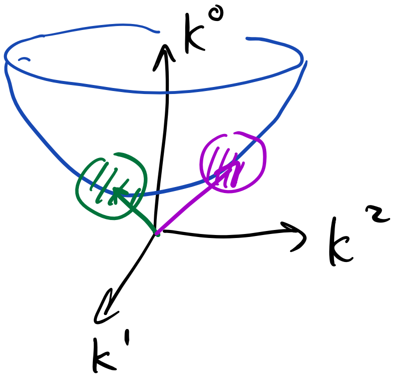
$$W_2(x,y) = W_1(x) W_1(y) + W_2^{\text{conn}}(x,y)$$

etc.

Cluster property

$\Rightarrow$ $W_n^{\text{conn}}(x_1, \cdots, x_n)$ decreases rapidly as $x_1, \cdots, x_n$ are separated by large distance in spacelike direction.

Suppose $\operatorname{Supp} \tilde{f}_1$ and $\operatorname{Supp} \tilde{f}_2$ do not overlap



Consider

$$\phi_{f_1^{(t)}} \phi_{f_2^{(t)}} |0\rangle$$

"regions of concentration" of $f_{1(x)}^{(t)}, f_{2(x)}^{(t)}$ move apart as $t \to \pm\infty$.

(Hepp 1965)

Claim: $\left\| \frac{d}{dt} \phi_{f_1^{(t)}} \phi_{f_2^{(t)}} |0\rangle \right\| \longrightarrow 0$

as $t \to \pm\infty$,

faster than any inverse power of t.

The limit $\lim\limits_{t \to \pm\infty} \phi_{f_1^{(t)}} \phi_{f_2^{(t)}} |0\rangle$

is a well-defined state in $\mathcal{H}$.

$$:= |\phi_{f_1}, \phi_{f_2}\rangle^{out/in}$$

Assume Supp $\tilde{f}_i$ do not overlap $i=1,\dots,n$

$$\lim_{t\to\pm\infty} \phi_{f_1}(t)\cdots\phi_{f_n}(t)|0\rangle = |\phi_{f_1},\dots,\phi_{f_n}\rangle^{out/in}$$

$$= \int \prod_{j=1}^{n} d^{d-1}\vec{k}_j\, \tilde{f}_j(\omega_{k_j},\vec{k}_j)\left[\frac{Z}{(2\pi)^{d-1}}\frac{1}{2\omega_{k_j}}\right]^{\frac{1}{2}} \cdot |\vec{k}_1,\dots,\vec{k}_n\rangle^{out/in}.$$

$$\boxed{\begin{array}{l} {}^{out}\langle \vec{k}_1,\dots,\vec{k}_n|\vec{k}'_1,\dots,\vec{k}'_m\rangle^{in} := S(\vec{k}_1,\dots|\vec{k}'_1,\dots) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{S-matrix.} \end{array}}$$

$${}^{out}\langle \phi_{f_1}\cdots|\phi_{g_1}\cdots\rangle^{in} = \int d\vec{k}_1\, \tilde{f}^*(\omega_{k_1},\vec{k}_1)\left[\frac{Z}{(2\pi)^{d-1}}\frac{1}{2\omega_{k_1}}\right]^{\frac{1}{2}}$$

$$\cdots$$

$$\times\, d\vec{k}'_1\, \tilde{g}(\omega_{k'_1},\vec{k}'_1)\left[\frac{Z}{(2\pi)^{d-1}}\frac{1}{2\omega_{k'_1}}\right]^{\frac{1}{2}}\cdots$$

$$\times\, S(\vec{k}_1\cdots|\vec{k}'_1\cdots)$$

$$\Vert$$

$$\lim_{t\to\infty}\langle 0|\phi^*_{f_n}(t)\cdots\phi^*_{f_1}(t)\,\phi_{g_1}(-t)\cdots\phi_{g_m}(-t)|0\rangle$$

$$= \lim_{t\to\infty}\langle 0|T\,\phi^*_{f_1}(t)\cdots\phi_{g_1}(-t)\cdots|0\rangle$$

$$= \lim_{t\to\infty} \int dx_1\, f_1^{(+t)*}(x_1) \cdots dy_1\, g_1^{(-t)}(y_1) \cdots$$

$$\cdot \langle 0 | T\, \phi(x_1) \cdots \phi(y_1) \cdots | 0 \rangle$$

$$= \lim_{t\to\infty} \int \frac{d^d k_1}{(2\pi)^d}\, \tilde f_1^*(-k_1)\, e^{-i(-k_1^0 - \omega_{k_1})t} \cdots$$

$$\cdot \frac{d^d k_1'}{(2\pi)^d}\, \tilde g_1(k_1')\, e^{-i(k_1'^0 - \omega_{k_1'})t} \cdots$$

$$\cdot \underbrace{\tilde G(k_1, \dots, k_1', \dots)}_{} . \longrightarrow \int dx_1\, e^{-ik_1\cdot x_1} \cdots dy_1\, e^{-ik_1'\cdot y_1} \cdots \langle 0 | T \phi(x_1) \cdots \phi(y_1) \cdots | 0 \rangle$$

vanishes unless

$\tilde G$ is singular at

$k_i^0 = -\omega_{k_i}$ and $k_j'^0 = \omega_{k_j}$

Consider:

$$\int \frac{d^d k}{(2\pi)^d} \frac{\tilde g(k)\, e^{-i(k^0 - \omega_k)t}}{k^2 + m^2 - i\epsilon} \qquad (t > 0)$$

$\tilde g(k^0, \vec k)$ analytic for $\mathrm{Im}\, k^0 < 0$

provided $g(x^0, \vec x)$ supported at $x^0 < 0$.

$$= -2\pi i \int \frac{d^{d-1} \vec k}{(2\pi)^d} \frac{\tilde g(\omega_k, \vec k)}{-2\omega_k}$$

and

$$\int \frac{d^dk}{(2\pi)^d} \frac{\tilde{f}^*(-k)\, e^{i(k^0+\omega_k)t}}{k^2+m^2-i\epsilon}$$

$$= 2\pi i \int \frac{d^{d-1}\vec{k}}{(2\pi)^d} \frac{\tilde{f}^*(\omega_k, -\vec{k})}{2\omega_k}$$

Conclusion:

$$S(-\vec{k}_1, \dots, -\vec{k}_n | \vec{k}_1', \dots, \vec{k}_m')$$

$$= \prod_{i=1}^{n} \left[Z (2\pi)^{d-1} 2\omega_{k_i} \right]^{-\frac{1}{2}} \prod_{j=1}^{m} \left[Z (2\pi)^{d-1} 2\omega_{k_j'} \right]^{-\frac{1}{2}}$$

$$\times \lim_{\substack{k_i^0 \to -\omega_{k_i} \\ k_j'^0 \to \omega_{k_j}'}} \prod_i i(k_i^2+m^2) \prod_j i(k_j'^2+m^2)$$

$$\times\ G(k_1, \dots, k_1', \dots)$$

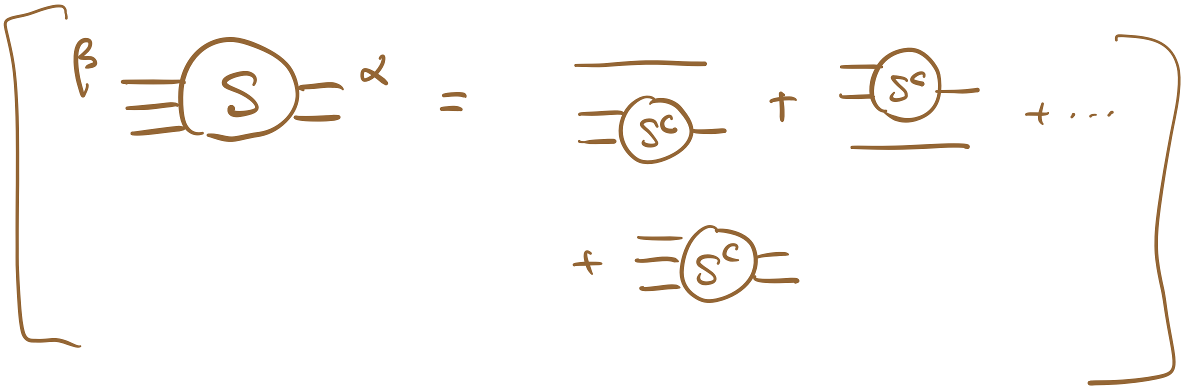
Lehmann - Symanzik - Zimmermann reduction

- The S -matrix

$$S(\beta|\alpha) = {}^{out}\langle \beta | \alpha \rangle^{in}$$

obeys cluster property

$$S(\beta|\alpha) = \sum_{\substack{\alpha = \amalg \alpha_I \\ \beta = \amalg \beta_I}} \prod_I S^{\text{conn}}(\beta_I|\alpha_I)$$



each <u>connected</u> S-matrix elmt

$$S^{\text{conn}}(\vec{k}_1,\dots,\vec{k}_n|\vec{k}_1',\dots,\vec{k}_m')$$

$$= (2\pi)^d\,\delta^d(k_1+\dots+k_n-k_1'-\dots-k_m')$$

$$\cdot\, M\underbrace{(\vec{k}_1,\dots,\vec{k}_n|\vec{k}_1',\dots,\vec{k}_m')}_{\substack{\text{subject to } k_i^2 = k_j'^2 = -m^2 \\ \text{and } k_1+\dots+k_n = k_1'+\dots+k_m'}}$$

Due to the normalization convention of

1-particle states, it is convenient to write

$$\mathcal{M}(\vec{k}_1, \dots, \vec{k}_n \,|\, \vec{k}'_1, \dots, \vec{k}'_m)$$

$$\equiv \prod_{i=1}^{n} \frac{1}{(2\pi)^{\frac{d-1}{2}} \sqrt{2\omega_{k_i}}} \prod_{j=1}^{m} \frac{1}{(2\pi)^{d-1} \sqrt{2\omega_{k'_j}}}$$

$$\times \mathcal{A}(k_1, \dots, k_n \,|\, k'_1, \dots, k'_m)$$

$\mathcal{A}$ is expected to be an analytic function of $k_1, \dots k_n, k'_1, \dots, k'_m$ subject to the kinematic constraints.

Example: 1+1 D, particle mass m.

2 → 2 S-matrix elmt

$k_i^\mu = (E_i, p_i)$.

$k_j'^\mu = (E'_j, p'_j)$.

$$S(p_1, p_2 \,|\, p'_1, p'_2) = (2\pi)^2 \delta^2(k_1 + k_2 - k'_1 - k'_2)$$

$$\times \frac{1}{(2\pi)^2} \frac{1}{\sqrt{2E_1 \cdot 2E_2 \, 2E'_1 \, 2E'_2}} \cdot \mathcal{A}(k_1, k_2 \,|\, k'_1, k'_2)$$

↑ [illegible] disconnected

- include disconnected part here

A special feature of 1+1 D kinematics is that $2 \to 2$ elastic scattering preserve momenta, i.e. we have

$$\text{either} \quad p_1 = p_1', \quad p_2 = p_2' \quad (\text{Transmission})$$

$$\text{or} \quad p_1 = p_2', \quad p_2 = p_1' \quad (\text{Reflection})$$

For identical bosons,

$$|p_1, p_2\rangle^{out} = |p_2, p_1\rangle^{out},$$

we do not need to distinguish T vs R.

and can wrie $\mathcal{A} = \mathcal{A}(s)$,

$$\text{where} \quad s \equiv (\text{invariant mass})^2$$

$$= -(k_1+k_2)^2 = -(k_1'+k_2')^2 .$$

Alternatively, we can write

$$|p_1', p_2'\rangle^{in} = S(s)\, |p_1', p_2'\rangle^{out} + \underbrace{\cdots\cdots}_{\text{more particles}}$$

and so

$$S(p_1, p_2 | p_1', p_2') = \underbrace{{}^{out}\langle p_1, p_2 | p_1', p_2'\rangle^{out}}_{..} S(s)$$

〃

$$\delta(p_1-p_1')\,\delta(p_2-p_2') + \delta(p_1-p_2')\,\delta(p_2-p_1')$$

$$= \frac{\sqrt{s(s-4m^2)}}{2E_1E_2}\,\delta^2(k_1+k_2-k_1'-k_2')$$

$$\Rightarrow \quad A(s) = 2\sqrt{s(s-4m^2)}\;S(s).$$

unitarity

$\Rightarrow \quad |S(s)|^2 \leq 1$ (= if no inelastic 2→n process possible)

Recall

$$\langle 0|T\,\phi(x)\,\phi(y)|0\rangle$$

$$= \int_0^\infty d\mu^2\,\rho(\mu^2)\,(-i)\int\frac{d^dp}{(2\pi)^d}\,\frac{e^{ip\cdot(x-y)}}{p^2+\mu^2-i\epsilon}$$

or equivalently,

$$\tilde{G}(k) := \int d^dx\; e^{-ik\cdot x}\langle 0|T\,\phi(x)\,\phi(0)|0\rangle$$

$$= \int_0^{\infty} d\mu^2 \, \rho(\mu^2) \cdot \frac{-i}{k^2+\mu^2-i\epsilon}$$

1- particle contribution to $\rho(\mu^2)$

$\rightsquigarrow$ pole of $\tilde{G}(k)$ at $k^2=-m^2$.

The derivation only made use of Poincaré - transf property, and not locality, of $\phi(x)$, and can be straightfowardly generalized to higher - point time-order correlation functions.

e.g.

$$\tilde{G}(k_1,k_2,k_3) := \int d^dx_1 \, d^dx_2 \, d^dx_3 \, e^{-i\sum_{j=1}^{3} k_j \cdot x_j}$$

$$\langle 0 | T \, \phi(x_1) \, \phi(x_2) \underset{\int d\alpha \, |\alpha\rangle\langle\alpha|}{\wedge} \phi(x_3) \, \phi(0) | 0 \rangle$$

has a pole in (k_1+k_2) at

$$(k_1+k_2)^2 = -m^2$$

or more generally, $-m_*^2$
if another particle of
mass m_* exists

By LSZ reduction, the 2→2 amplitude
$\mathcal{A}(k_1, k_2 ; k_1', k_2')$ also has a pole
at $(k_1+k_2)^2 = -m_*^2$.

This is a generalization of the observation we made in the context of scattering in non-relativistic QM w/ 1 degree of freedom,

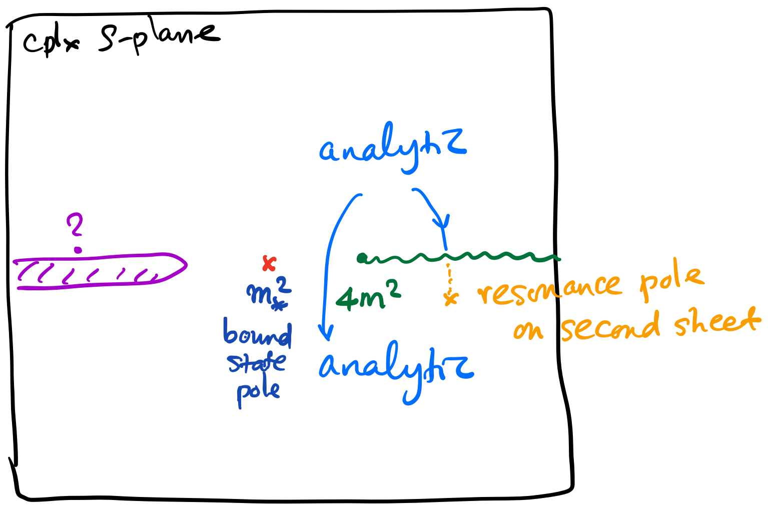
namely the S-matrix (connected part, factor out δ^d for momentum conservation)

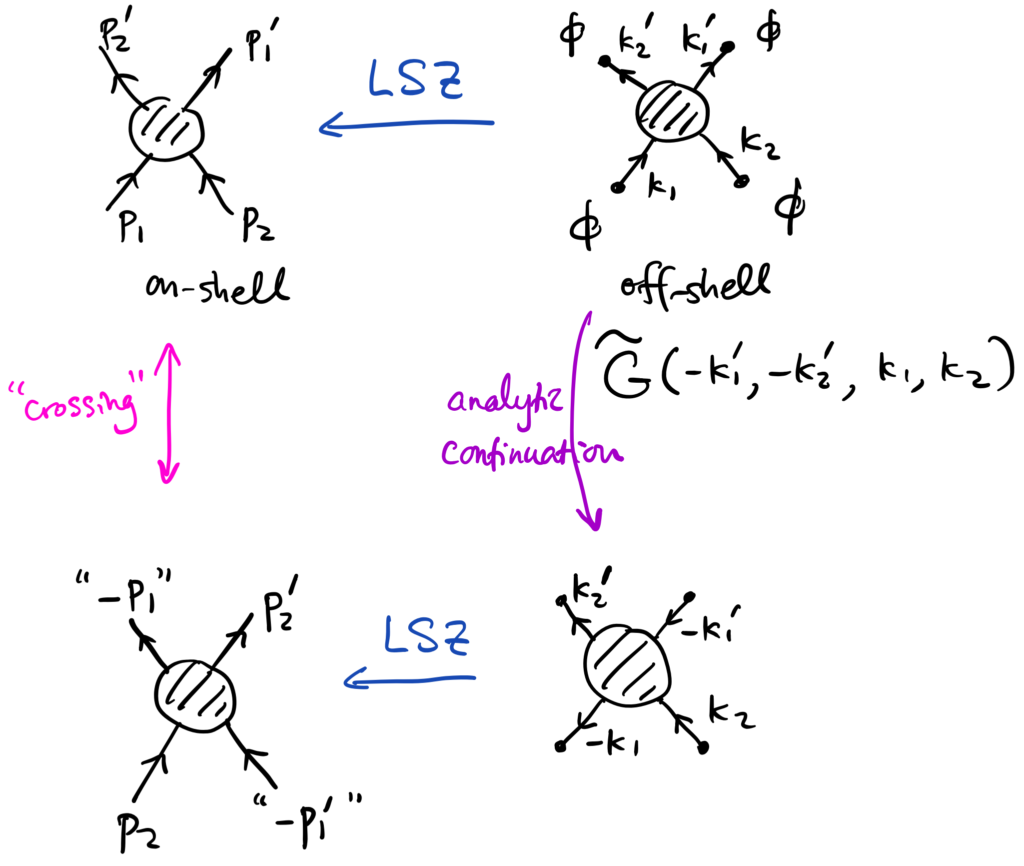
has a pole in C.O.M. energy (invt mass)

$$E = m_*$$

corresponding to every bound state (particle)
of mass m_*

Expected property of $S(s)$



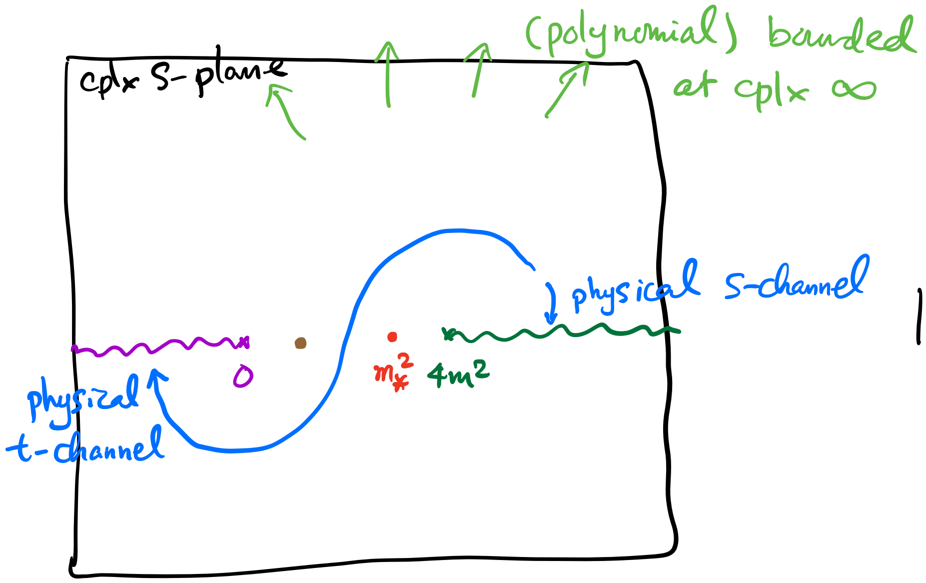


Crossing "symmetry":

$$S\left(s=-(k_1+k_2)^2\right) = S\left(\tilde{s}=-(k_2-k_1')^2\right)$$

$$\underset{t=4m^2-s}{\parallel}$$

$$s > 4m^2 \longleftrightarrow t < 0.$$



$$|S(s)| \leq 1$$
$$s \geq 4m^2$$

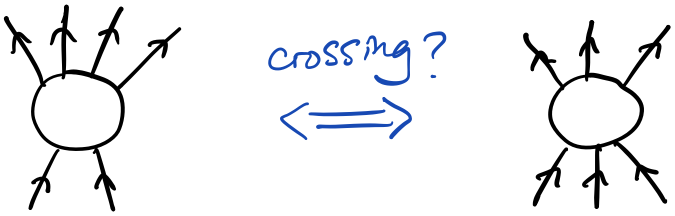
An example:

$$S_{SG}(s) = \frac{\sqrt{s(4m^2-s)} + m_* \sqrt{4m^2-m_*^2}}{\sqrt{s(4m^2-s)} - m_* \sqrt{4m^2-m_*^2}}$$

Here $|S_{SG}(s)| = 1$ for $s \geq 4m^2$.

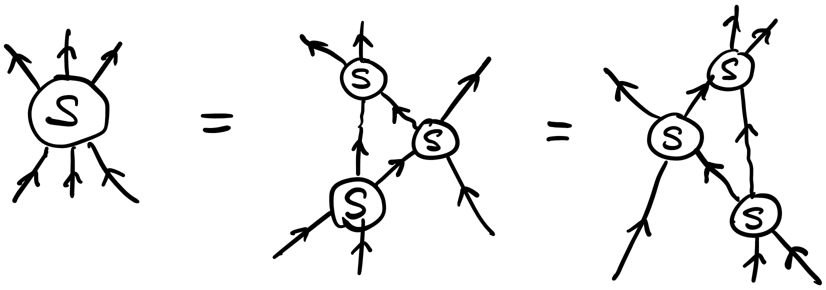
2→2 scattering purely elastic, no particle production

"sine-Gordon breather"



Absence of particle production is possible when connected scattering amplitudes are non-analytic.

Integrable QFT (in 1+1 D)

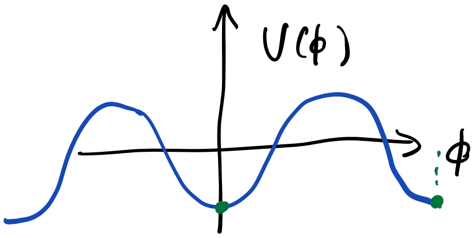


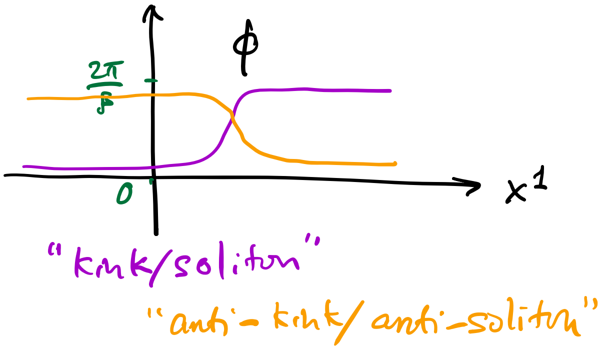
etc.

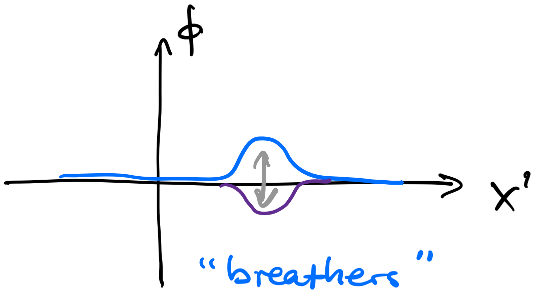
An example is sine-Gordon model

$$S = \int dx^0 dx^1 \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial x^0}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x^1}\right)^2 + \underbrace{\frac{m_0^2}{\beta^2}\cos(\beta\phi)}_{\substack{\shortparallel \\ \frac{m_0^2}{\beta^2} - \frac{1}{2}m_0^2\phi^2 + \frac{1}{24}m_0^2\beta^2\phi^4 + \cdots}} \right]$$

Quantization of modes of $\phi(x^0, x^1)$







particle spectrum :

soliton / anti-soliton

mass m_s

breather masses

$$m_n = 2m \sin \frac{n\gamma}{16},$$

$$\gamma = \frac{\beta^2}{\sqrt{1-\frac{\beta^2}{8\pi}}}.$$

$$n = 1, 2, \cdots < \frac{8\pi}{\gamma}.$$

The 2→2 S-matrix elmt of a pair of $n=1$ breathers is known to be

$$S(s) = \frac{\sqrt{s(4m_1^2 - s)} + m_2\sqrt{4m_1^2 - m_2^2}}{\sqrt{s(4m_1^2 - s)} - m_2\sqrt{4m_1^2 - m_2^2}}$$

- Form factors

we have already seen

$$\langle 0|\phi(0)|\vec{k}\rangle = \frac{Z^{\frac{1}{2}}}{(2\pi)^{\frac{d-1}{2}}\sqrt{2\omega_k}}.$$

Generalization to matrix elmt w/ arbitrary in-states.

$$\langle 0|\phi(0)\,|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\rangle^{in}$$

$$= \prod_{j=1}^{n} \frac{1}{(2\pi)^{\frac{d-1}{2}}\sqrt{2\omega_{k_j}}}\ \underbrace{F(k_1, k_2, \dots, k_n)}_{\substack{\text{Lorentz} \\ \text{invariant fn of } k_i\text{'s} \\ \text{subject to } k_i^2 = -m^2, \\ (\text{but no restriction on } k_1+\dots+k_n)}}$$

- We will focus on 1+1D case

- for analyzing 1+1D kinematics, convenient to work with "rapidity" variable θ for the particle,

$$p = m\sinh\theta, \quad E = m\cosh\theta.$$

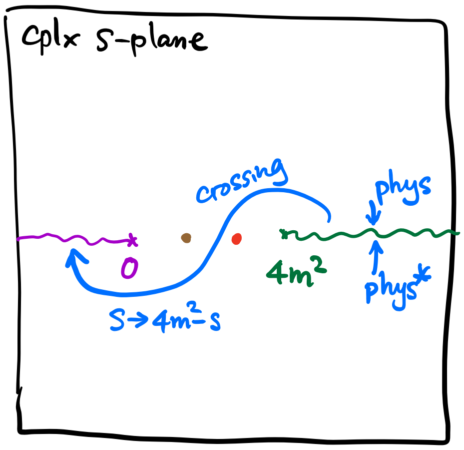
under Lorentz boost, $\theta \to \theta + \text{const}$

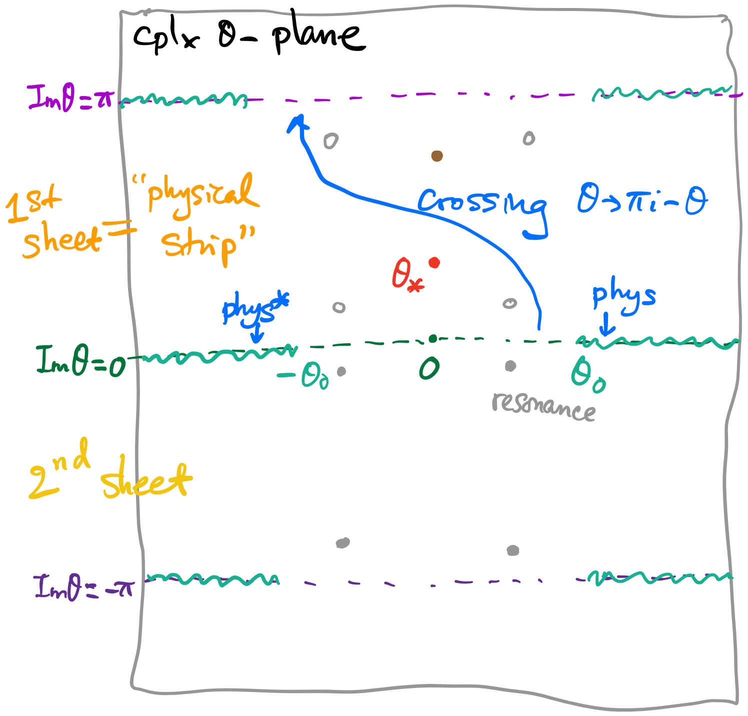
$2 \to 2$ scattering. $p_1, p_2 \rightsquigarrow \theta_1, \theta_2$.

$$s = -(k_1+k_2)^2 = 4m^2 \cosh^2 \frac{\theta_{12}}{2}.$$

We will write $S(s)$ as $S(\theta)$

$\theta \equiv \theta_1 - \theta_2$.





Consider the example

$$S_{SG}(s) = \frac{\sqrt{s(4m^2-s)} + m_* \sqrt{4m^2 - m_*^2}}{\sqrt{s(4m^2-s)} - m_* \sqrt{4m^2 - m_*^2}}$$

$$= \frac{\sinh\theta + \sinh\theta_*}{\sinh\theta - \sinh\theta_*}$$

$$m_* = 2m \cosh\frac{\theta_*}{2} .$$

for $0 < m_* < 2m$, $\quad \theta_* \in i\mathbb{R}$

$0 < \mathrm{Im}\,\theta_* < \pi$

Note the absence of branch cuts in θ:

$S_{SG}(\theta)$ is analytic on the entire θ-plane!

- If there are inelastic processes e.g. $2 \to 3$,

  $S(\theta)$ will have branch cuts at $\theta \in [\theta_0, \infty)$

  $\theta_0 =$ inelastic threshold

  e.g. for $2 \to 3$. $\quad 2m \cosh\frac{\theta_0}{2} = 3m$

- unitarity: $\quad |S(\theta)|^2 \leq 1$, for $\theta \in \mathbb{R}$.

- real analyticity $\quad (S(\theta))^* = S(-\theta^*)$

- unitarity in elastic domain

$$1 = S(\theta)(S(\theta))^*$$

$$= S(\theta)\, S(-\theta) \quad \text{for } \theta \in (0, \theta_0)$$

by analytic continuation,

$1 = S(\theta)\, S(-\theta)$ holds on entire θ-plane.

- crossing

$$S(\theta) = S(\pi i - \theta)$$

It follows that

$$S(\theta) = \frac{1}{S(-\theta)} = \frac{1}{S(\pi i+\theta)}$$

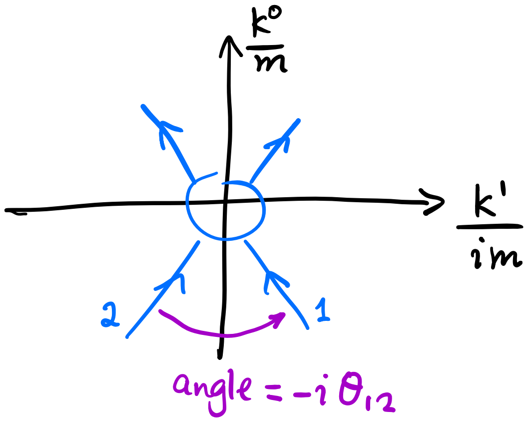
$$= S(\theta + 2\pi i)$$

Recall

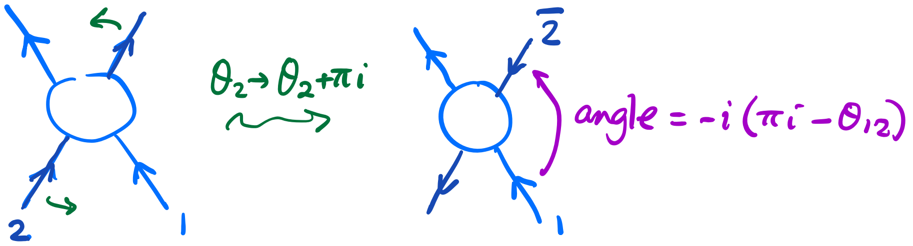
$$(k^0, k^1) = (m\cosh\theta, m\sinh\theta)$$

we can equivalently write

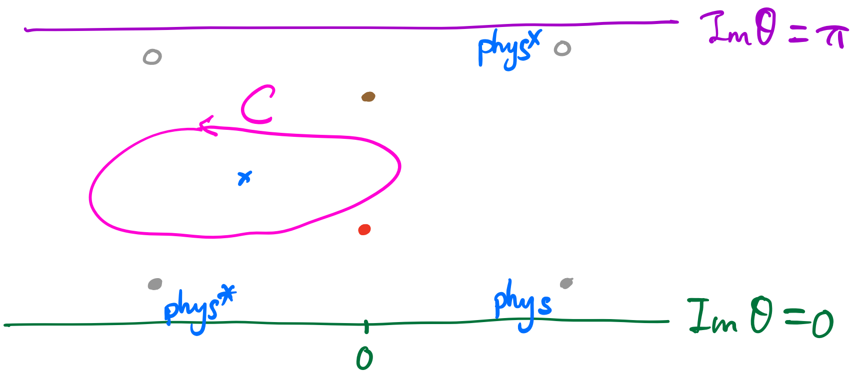
$$(ik^0, k^1) = im\left(\cos(-i\theta), \sin(-i\theta)\right).$$



crossing:



- Resonance poles on 2^{nd} sheet $(-\pi < \operatorname{Im}\theta < 0)$ correspond to zeros of $S(\theta)$ on 1^{st} sheet due to $S(\theta)\,S(-\theta) = 1$.



write

$$S(\theta) = \underbrace{S_{CDD}(\theta)}_{\text{“CDD factor”}} \; \underbrace{S^{min}(\theta)}_{\substack{\text{no zeros nor} \\ \text{poles on the strip}}}$$

$$\mathbin{/\!/} \quad \pm \prod_j \frac{\sinh\theta + i\sin\alpha_j}{\sinh\theta - i\sin\alpha_j}$$

Assuming $|S^{min}(\theta)|$ does not grow as fast as $e^{e^{\theta}}$ as $\operatorname{Re}\theta \to \pm\infty$,

$$\ln S^{min}(\theta) = \oint_{C} \frac{dz}{2\pi i} \; \frac{\ln S^{min}(z)}{\sinh(z-\theta)}$$

$$= \left(\int_{\mathbb{R}+i\epsilon} - \int_{\mathbb{R}+i(\pi-\epsilon)} \right) \frac{dz}{2\pi i} \, \frac{\ln S^{min}(z)}{\sinh(z-\theta)}$$

$$= \int_{-\infty}^{\infty} \frac{dz}{2\pi i} \, \frac{1}{\sinh(z-\theta)} \, \ln \underbrace{S^{min}(z+i\epsilon)\, S^{min}(z+i(\pi-\epsilon))}_{\substack{\shortparallel \\ |S(z+i\epsilon)|^2 \\ \equiv f(z) \\ \text{elastic scattering} \\ \text{probability}}}$$

for $0 < \operatorname{Im}\theta < \pi$.

i.e., $S(\theta)$ is determined, up to overall sign, by the locations of its (bound state) poles and (resonance) zeros on the strip $0 < \operatorname{Im}\theta < \pi$, together with total elastic scattering probability.

- For integrable QFT, $f(z) = 1 \quad (z \in \mathbb{R})$

$$\Rightarrow \quad S^{min}(\theta) = 1.$$

[a sign ambiguity, due to branch of above log, is absorbed into S_{CDD}]

$$S(\theta) = S_{CDD}(\theta).$$

- Form factors in 1+1 D.

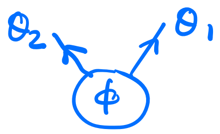
we will denote

$$|p_1, \cdots, p_n\rangle^{in/out} := \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}\cdot\sqrt{2E_j}} \; |\theta_1, \cdots, \theta_n\rangle^{in/out}$$

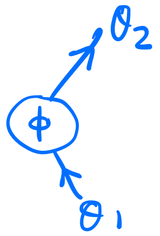
crossing relations among 2-particle form factors

$$\langle 0|\phi(0)|\theta_1, \theta_2\rangle^{in} = F(\theta_{12})$$

$$\langle 0|\phi(0)|\theta_1, \theta_2\rangle^{out} = F(-\theta_{12})$$



$$\langle\theta_1|\phi(0)|\theta_2\rangle = F(\pi i - \theta_{12})$$



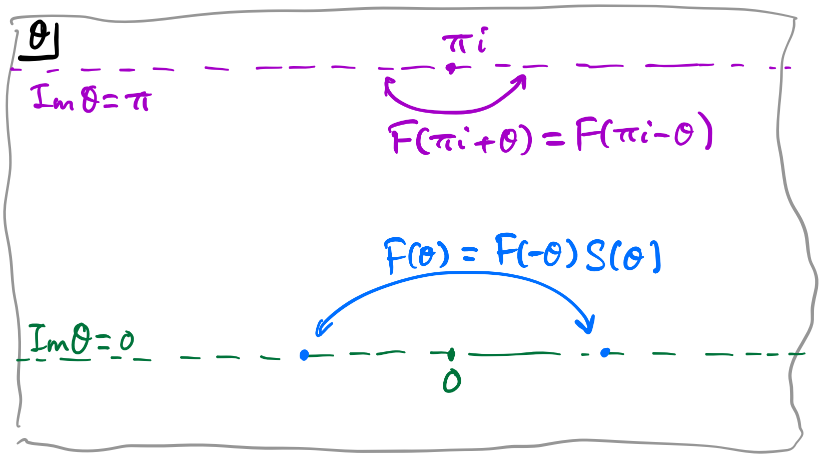
In elastic regime, or for integrable theories,

$$|\theta_1, \theta_2\rangle^{in} = |\theta_1, \theta_2\rangle^{out} \cdot S(\theta_{12})$$

$$\Rightarrow \quad F(\theta) = F(-\theta)\, S(\theta)$$

Also: $\langle\theta_1|\phi(0)|\theta_2\rangle = \langle\theta_2|\phi(0)|\theta_1\rangle$

$$\Rightarrow \quad F(\pi i - \theta) = F(\pi i + \theta).$$



"Watson's equations"

[Karowski, Weisz 1978]

If $F(\theta)$ is meromorphic on the strip $0 \leq \operatorname{Im}\theta \leq \pi$, and $|F(\theta)|$ grows slower than $e^{e^{|\theta|}}$ as $\operatorname{Re}\theta \to \pm\infty$.

then

$$F(\theta) = K(\theta) \cdot F^{min}(\theta)$$

$$K(\theta) = \text{Const} \cdot \prod_{k} \left[\sinh\frac{\theta - i\alpha_k}{2} \sinh\frac{\theta + i\alpha_k}{2} \right]^{\sigma_k}$$

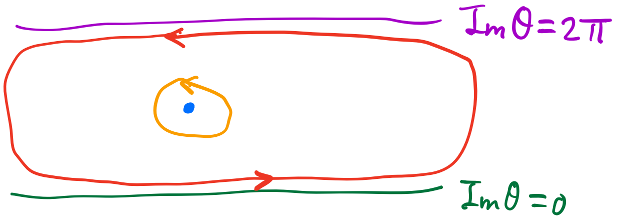
$$\sigma_k = \begin{cases} +1, & \text{zero at } \theta = i\alpha_k \\ -1, & \text{pole.} \end{cases}$$

$F^{min}(\theta)$ "minimal solution"

has no poles nor zeros on the strip,

i.e. $\ln F^{min}(\theta)$ analytic on the strip

$$\frac{d}{d\theta}\ln F^{min}(\theta) = \oint_{C} \frac{dz}{2\pi i}\, \frac{\ln F^{min}(z)}{4\sinh^2\frac{z-\theta}{2}}$$



$$= \int_{-\infty}^{\infty}\frac{dz}{2\pi i}\, \frac{1}{4\sinh^2\frac{z-\theta}{2}}\, \ln \underset{\substack{\| \\ S(z)}}{\boxed{\frac{F^{min}(z)}{F^{min}(z+2\pi i)}}}$$

- If we can write $S(\theta)$ in the form

$$S(\theta) = e^{\int_0^\infty dx\, f(x)\, \sinh\frac{\theta x}{\pi i}}$$

then integrating the above relation gives

$$F^{min}(\theta) = e^{\int_0^\infty dx\, f(x)\, \frac{\sin^2\left(\frac{x}{2\pi}(\pi i-\theta)\right)}{\sinh x}}$$

- Example: sine - Gorden breather

$$S_{SG}(\theta) = \frac{\sinh\theta + \sinh\theta_*}{\sinh\theta - \sinh\theta_*}$$

$$= - e^{\int_0^\infty dx\, f_{SG}(x)\ \sinh\frac{\theta x}{\pi i}}.$$

$$f_{SG}(x) = \frac{2}{x} \cdot \frac{\cosh\left(x\left(\frac{1}{2} - \frac{\theta_*}{\pi i}\right)\right)}{\cosh\frac{x}{2}}.$$

$$F^{min}_{SG}(\theta) = \left(\cosh\frac{\pi i - \theta}{2}\right) \cdot e^{\int_0^\infty dx\ f_{SG}(x)\ \frac{\sin^2\left(\frac{x}{2\pi}(\pi i - \theta)\right)}{\sinh x}}$$

Actual form factor of $\mathcal{O} = :\phi^2:$ in sine-Gordon model

$$\langle 0 | \mathcal{O}(0) | \theta_1, \theta_2 \rangle^{in} = F^{\mathcal{O}}_{SG}(\theta_{12}),$$

$$F^{\mathcal{O}}_{SG}(\theta) = \frac{Z^{(2)}(\theta_*)}{\sinh\frac{\theta - \theta_*}{2} \sinh\frac{\theta + \theta_*}{2}} \cdot F^{min}_{SG}(\theta)$$

$$= \text{const.} \exp\left\{ \int_0^\infty \frac{dx}{x \cdot \sinh x} \left[\frac{\cosh\left(x\left(\frac{1}{2} + \frac{\theta_*}{\pi i}\right)\right)}{\cosh\frac{x}{2}} - 1 \right] \cosh\left(x\left(1 - \frac{\theta}{\pi i}\right)\right) \right\}$$

Karowski, Weisz 1978 (NPB139, 455)

also, Babujian, Karowski, hep-th/0204097

- A simpler example: $S(\theta) = -1$

Interpretation: free massive fermion

- asymptotic 2-particle wave function ($p_1 > p_2$)

$$\psi(x_1, x_2) \approx e^{ip_1x_1+ip_2x_2} + \underbrace{S(p_1,p_2)}_{\overset{\shortparallel}{-1}} e^{ip_2x_1+ip_1x_2} \qquad x_1 \ll x_2$$

||| "identical bosons"

$\psi(x_2, x_1)$

$\updownarrow$ not the same plane wave, interacting boson!

$$\Rightarrow \quad \psi(x_1, x_2) \approx e^{ip_1x_2+ip_2x_1} - e^{ip_1x_1+ip_2x_2} \qquad x_1 \gg x_2$$

Define

$$\psi'(x_1,x_2) = \psi(x_1,x_2), \quad x_1 \ll x_2$$
$$= -\psi(x_1,x_2), \quad x_1 \gg x_2$$

now

$$\psi'(x_1,x_2) = e^{ip_1x_1+ip_2x_2} - e^{ip_2x_1+ip_1x_2}$$

plane wave for all x_1, x_2. $\Rightarrow$ "free particles"

$\psi'(x_2,x_1) = -\psi'(x_1,x_2)$. fermion!

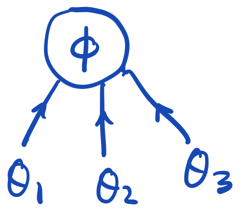
Consider a possible scalar field operator that is odd w.r.t. the $\mathbb{Z}_2$-symmetry $(-)^F$.

$$\langle 0|\,\phi(0)\,|\theta_1,\cdots,\theta_n\rangle^{in} = F^{(n)}(\theta_1,\cdots,\theta_n)$$

non-zero for $n=1,3,5,\cdots$

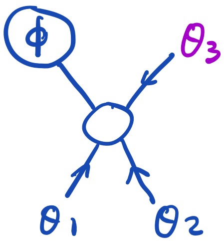
$F^{(1)}(\theta) = Z$ set to 1 by choosing normalization of ϕ.

$$F^{(3)}(\theta_1,\theta_2,\theta_3) = K(\theta_{12},\theta_{23}) \prod_{1\le i<j\le 3} \sinh\frac{\theta_{ij}}{2}.$$



K inv't under $\theta_i \leftrightarrow \theta_j$.
as well as $\theta_j \to \theta_j + 2\pi i$.

pole in θ_3 due to 1-particle exchange



$\theta_3 \to \theta_1 + \pi i$

$$\longrightarrow \frac{-i}{(k_1+k_2+k_3)^2+m^2}\cdot 2\sqrt{s_{12}(s_{12}^2-4m^2)}\cdot\Big(\overbrace{S(\theta_{12})}^{\overset{-1}{\shortparallel}}-1\Big)$$

$$\longrightarrow \frac{-2i}{\theta_3 - \theta_1 - \pi i} \cdot (-2)$$

simplest possibility:

$$K = \frac{2i}{\prod_{1 \le i < j \le 3} \cosh \frac{\theta_{ij}}{2}} .$$

$$\Rightarrow \quad F^{(3)} = 2i \prod_{1 \le i < j \le 3} \tanh \frac{\theta_{ij}}{2} .$$

Generalization to n-body form factor

(n odd)

$$F^{(n)} = (2i)^{\frac{n-1}{2}} \prod_{1 \le i < j \le n} \tanh \frac{\theta_{ij}}{2} .$$

[Berg, Karowski, Weisz. 1978]

$\Rightarrow$

$$\langle 0 | \phi(x) \, \phi(0) | 0 \rangle$$

$k_j = (m \cosh \theta_j, m \sinh \theta_j)$

$$= \sum_{\text{odd } n \ge 1} \frac{1}{n!} \int \frac{d\theta_1}{4\pi} \cdots \frac{d\theta_n}{4\pi} \, e^{i \sum_{j=1}^{n} \overset{\downarrow}{k_j} \cdot x}$$

$$\cdot \left| F^{(n)}(\theta_1,\cdots,\theta_n) \right|^2$$

$$= \frac{1}{2}\sum_{\text{odd } n\geq 1} \frac{(2\pi)^{-n}}{n!}\int d\theta_1\cdots d\theta_n\, e^{i\sum_{j=1}^{n} k_j\cdot x}$$

$$\cdot \prod_{1\leq i<j\leq n} \tanh^2\frac{\theta_{ij}}{2} .$$

To analyze this result, consider

$$G(\lambda;\underset{\substack{\uparrow \\ \text{Lorentz} \\ \text{invariance.}}}{x^2}) \equiv \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\int d\theta_1\cdots d\theta_n\, e^{i\sum_j k_j\cdot x}$$

$$\cdot \prod_{1\leq i<j\leq n} \tanh^2\frac{\theta_{ij}}{2} .$$

$$= \exp\left[\sum_{n=1}^{\infty} \frac{\lambda^n}{n!}\int d\theta_1\cdots d\theta_n\, e^{i\sum_j k_j\cdot x} h_n(\theta_1,\cdots,\theta_n) \right]$$

where h_n are defined by

$$\prod_{1\leq i<j\leq n} \tanh^2\frac{\theta_{ij}}{2} = \sum_{\sqcup I_\alpha = \{1,\cdots,n\}} \prod_\alpha h_{|I_\alpha|}(\{\theta_i\}_{i\in I_\alpha})$$

e.g. $h_1(\theta) = 1$

$$h_2(\theta_1,\theta_2) = \tanh^2\frac{\theta_{12}}{2} - 1 = -\frac{1}{\cosh^2\frac{\theta_{12}}{2}}$$

$$h_3(\theta_1,\theta_2,\theta_3) = \tanh^2\frac{\theta_{12}}{2}\tanh^2\frac{\theta_{13}}{2}\tanh^2\frac{\theta_{23}}{2} - \tanh^2\frac{\theta_{12}}{2} - \tanh^2\frac{\theta_{13}}{2} - \tanh^2\frac{\theta_{23}}{2} + 2$$

$$= \frac{2}{\prod_{1\le i<j\le 3}\cosh\frac{\theta_{ij}}{2}}.$$

etc.

choose $(x^0, x^1) = (-i\tau, 0)$

$$i\sum_{j=1}^{n} k_j\cdot x = -m\tau\sum_{j=1}^{n}\cosh\theta_j$$

$$\theta_j' \equiv \theta_j - \theta_n$$

$$= -m\tau\sum_{j=1}^{n}\cosh(\theta_j'+\theta_n)$$

$$= -m\tau\Big[\cosh\theta_n\Big(\overbrace{\sum_{j=1}^{n-1}\cosh\theta_j' + 1}^{y^0}\Big) + \sinh\theta_n\overbrace{\sum_{j=1}^{n-1}\sinh\theta_j'}^{y^1}\Big]$$

$\Rightarrow$

$$G(\lambda;\tau^2) = \exp\Big[\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\int d\theta_1'\cdots d\theta_{n-1}'\, h_n(\theta_1',\cdots,\theta_{n-1}',0) \cdot \underbrace{\int d\theta_n\, e^{-m\tau(\cosh\theta_n\cdot y^0 + \sinh\theta_n\cdot y^1)}}_{}\Big]$$

$$\parallel$$

$$2K_0\left(m\tau\sqrt{(y^0)^2-(y^1)^2}\right)$$

(Bessel fn)

- short distance limit $\tau \to 0$

$$K_0(m\tau\xi) = -\ln(m\tau\xi) + \ln 2 - \gamma_E + \mathcal{O}(\tau^2 \log\tau).$$

$$G(\lambda;\tau^2) \approx C(\lambda)\exp\left[\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\underbrace{\int d\theta_1' \cdots d\theta_{n-1}'\, h_n(\theta_1',\cdots,\theta_{n-1}',0)}_{\equiv\, I_n} \cdot 2(-\ln\tau)\right]$$

$$= C(\lambda)\cdot \tau^{-2\Delta(\lambda)}$$

$$I_1 = \frac{1}{2\pi}$$

$$I_2 = -\frac{1}{\pi^2}$$

$$I_n = (n-2)^2 I_{n-2},$$

$n \geq 3$.

$$\Delta(\lambda) = \sum_{n=1}^{\infty}\frac{\lambda^n}{n!} I_n$$

$$= \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{\lambda^{2k+1}}{(2k+1)!}\left(\frac{(2k)!}{2^k k!}\right)^2$$

$$-\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{\lambda^{2k}}{(2k)!}\left(2^{k-1}(k-1)!\right)^2$$

$$= \frac{1}{2\pi}\arcsin(\lambda)\cdot\left(\pi - \arcsin\lambda\right).$$

$$\langle 0 | \phi(x)\, \phi(0) | 0 \rangle = \frac{1}{4}\left[G(1; x^2) - G(-1; x^2) \right]$$

$$\longrightarrow \text{Const} \cdot |x|^{-2\Delta} , \qquad \begin{matrix} x \to 0 \\ x^2 > 0 . \end{matrix}$$

$$\Delta = \Delta(\lambda = 1) = \frac{1}{8} .$$

- Interpretation: the 1+1D QFT of a free massive fermion admits a scalar field operator $\phi(x)$ that creates/annihilates an odd number of fermions, and behaves in the short distance limit as an operator of effective mass dimension $\Delta = \frac{1}{8}$.

- The short distance limit of this QFT is captured by a scale-invariant QFT known as the (2D) critical Ising model.

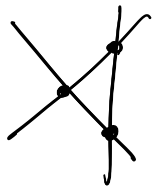
- free massive fermion theory.

Hilbert space $\mathcal{H}$ spanned by

$$|\theta_1, \theta_2, \dots, \theta_n\rangle^{\text{in}}$$

or

$$|\theta_1, \dots, \theta_n\rangle^{\text{out}} = (-)^{\frac{n(n-1)}{2}} |\theta_1, \dots, \theta_n\rangle^{\text{in}} .$$



already constructed form factors of $\phi(x)$

rename $\rightsquigarrow$ $\sigma(x)$ "spin field"

$$\langle 0 | \sigma(0) | \theta_1, \dots, \theta_n\rangle^{\text{in}} = \begin{cases} Z_\sigma \cdot (2i)^{\frac{n-1}{2}} \prod_{1\leq i<j\leq n} \tanh \frac{\theta_{ij}}{2} , & n \text{ odd} \\ 0 , & n \text{ even.} \end{cases}$$

$${}^{\text{out}}\langle \theta_1', \dots, \theta_m' | \sigma(0) | \theta_1, \dots, \theta_n\rangle^{\text{in}}$$

$$\underline{\underline{\text{crossing}}} \quad Z_\sigma \cdot (2i)^{\frac{n+m-1}{2}} \prod_{1\leq i<j\leq n} \tanh \frac{\theta_{ij}}{2} \prod_{1\leq k<l\leq m} \tanh \frac{\theta_{kl}'}{2}$$

$$\cdot \prod_{\substack{1\leq i\leq n \\ 1\leq k\leq m}} \coth \frac{\theta_i - \theta_k'}{2} , \quad (n+m \text{ odd})$$

$$\left[\text{Note: } \tanh\frac{\theta_{ij}}{2} = \frac{p_i - p_j}{E_i + E_j} . \right]$$

This determines all matrix elements of $\sigma(x)$ thereby completely specifies $\sigma(x)$ as a linear operator (-valued distribution)

- Note the matrix elmts involving
$$|p_1, \cdots, p_n\rangle^{in}$$
vanish in the limit $p_i \to p_j$.
consistent with Pauli exclusion.

- Define fermion annihilation and creation operators $b(p)$, $b^{+}(p)$ by
$$|p_1, \cdots, p_n\rangle^{in} = b^{+}(p_1) \cdots b^{+}(p_n)|0\rangle.$$
$$b(p)|0\rangle = 0.$$
$$\{b^{+}(p), b^{+}(p')\} = 0 = \{b(p), b(p')\}.$$
$$\{b(p), b^{+}(p')\} = \delta(p-p').$$

- Some other local field operators in the free massive fermion theory.

- "free fermion field"

$$\psi(t,x) := \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\omega_p}} \sqrt{\omega_p - p} \Big(b(p) e^{ipx - i\omega_p t} + b^\dagger(p) e^{-ipx + i\omega_p t} \Big)$$

$$\tilde{\psi}(t,x) := \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\omega_p}} \sqrt{\omega_p + p} \Big(i b(p) e^{ipx - i\omega_p t} - i b^\dagger(p) e^{-ipx + i\omega_p t} \Big)$$

They obey

$$\partial_t \psi - \partial_x \psi = - m \tilde{\psi}$$

$$\partial_t \tilde{\psi} + \partial_x \tilde{\psi} = m \psi .$$

and anti-commutators at <u>equal</u> time

$$\{ \psi(t,x) , \tilde{\psi}(t,x') \} = 0$$

$$\{ \psi(t,x) , \psi(t,x') \} = 2\pi \delta(x-x')$$

$$= \{ \tilde{\psi}(t,x) , \tilde{\psi}(t,x') \} .$$

Furthermore,

$$\{ \psi(t,x), \psi(t',x') \} = 0$$

$$\text{for } (x-x')^2 - (t-t')^2 > 0.$$

$\psi(t,x)$ is a local "fermionic operator",

but not a local operator in the (bosonic) sense introduced previously.

e.g. $[\psi(t,x), \psi(t',x')] \neq 0$

for spacelike seperated (t,x), (t',x').

Nonetheless, we can build local bosonic field operators using ψ, $\tilde{\psi}$.

e.g.

$$\varepsilon(x) \equiv \frac{-i}{2} : \psi(x) \tilde{\psi}(x) :$$

and

$$T(x) = \frac{-i}{2} : \psi(x) (\partial_{x^0} + \partial_{x^1}) \psi(x) :$$

$$\tilde{T}(x) = \frac{-i}{2} : \tilde{\psi}(x) (\partial_{x^0} - \partial_{x^1}) \tilde{\psi}(x) :$$

Note: the spin field $\sigma(x)$ cannot be expressed as normal ordered product of $\psi(x)$, $\tilde{\psi}(x)$, and their derivatives

. The Hamiltonian is

$$H_0 = \int_{-\infty}^{\infty} dp \;\; \omega_p \, b^{\dagger}(p)\, b(p)$$

$$= \int \frac{dx'}{2\pi} \left[\frac{1}{2} T(x) + \frac{1}{2} \tilde{T}(x) + m\, \varepsilon(x) \right]$$

- Consider a deformation of the Hamiltonian

$$H = H_0 + \underset{\substack{\uparrow \\ \text{a ``coupling'' constant.}}}{h} \int dx' \, \sigma(x)$$

"Ising field theory"

- interacting massive particles, not integrable for $h \neq 0$.

- at short distances, m negligible

$\sigma(x)$ has effective mass dimension $\Delta = \frac{1}{8}$.

$\Rightarrow$ h has mass dimension $\frac{15}{8}$

Define dimensionless coupling parameter

$$\eta \equiv \frac{m}{|h|^{\frac{8}{15}}}.$$

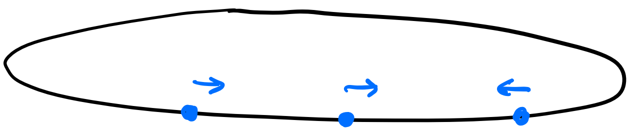
$\rightsquigarrow$ A family of QFTs parameterized by η.

- What are the (stable) particles of IFT?

$$m_1 = m \cdot f_1(\eta)$$
$$m_2 = m \cdot f_2(\eta)$$
$$\vdots$$
$$?$$

- can determine the answer by diagonalizing H of the system on a spatial circle $x^1 \sim x^1 + 2\pi R$, and then take $R \to \infty$ limit.

- Free fermions on a circle



Viewed as identical bosons with pairwise scattering phase $S(\theta) = -1$,

wave function $e^{ip(x+2\pi R)} = (-)^{N-1} e^{ipx}$

$N =$ total number of particles.

Two cases:

- $N =$ even, $p_i \in \frac{1}{R}(\mathbb{Z} + \frac{1}{2})$, $i = 1, \dots, N$

  "Neveu-Schwarz sector" (NS)

- $N =$ odd, $p_i \in \frac{1}{R}\mathbb{Z}$, $i = 1, \cdots, N$.

  "Ramond sector" (R)

Hilbert space

$$\mathcal{H} = \mathcal{H}_{NS} \oplus \mathcal{H}_R .$$

$\sigma(x)$ only has nonzero matrix elmts between $\mathcal{H}_{NS}$ and $\mathcal{H}_R$ states.

For $R \gg \frac{1}{m}$, can approximate matrix elmts of σ using the form factors in uncompactified theory, label states w/ discrete momenta $P_n = \frac{n}{R}$

$$\delta(P_n - P_m) \rightsquigarrow R\,\delta_{nm}$$

$$ {}_{NS}\langle \underbrace{k_1, \dots, k_m}_{\overset{\in}{\mathbb{Z}+\frac{1}{2}}} | \, \sigma(0) \, | \underbrace{\ell_1, \dots, \ell_n}_{\overset{\in}{\mathbb{Z}}} \rangle_R $$

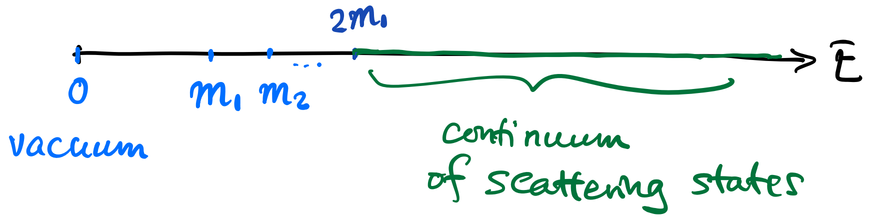
$$ \overset{\text{large } R}{\approx} Z \cdot (-)^{\frac{m(m-1)}{2}} \prod_{i=1}^{m} \frac{1}{\sqrt{4\pi R E_{k_i}}} \prod_{j=1}^{n} \frac{1}{\sqrt{4\pi R E_{\ell_j}}} $$

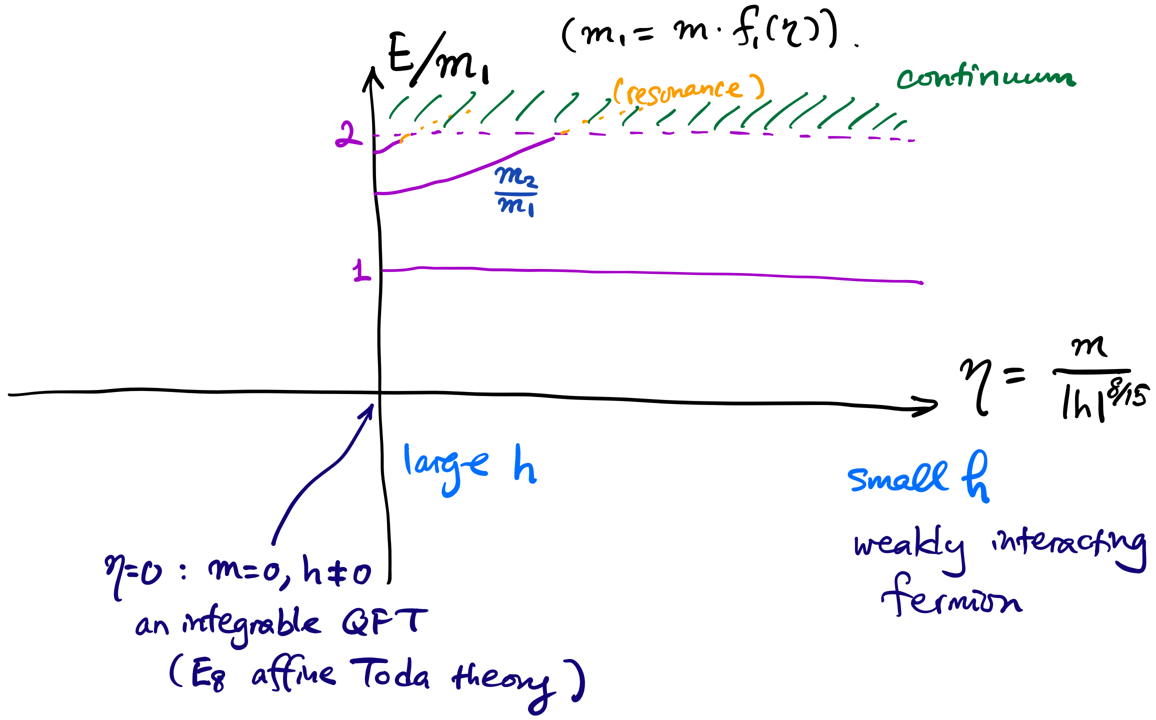
$$ \cdot (2i)^{\frac{m+n-1}{2}} \prod_{1 \le i < j \le m} \frac{P_{k_i} - P_{k_j}}{E_{k_i} + E_{k_j}} \prod_{1 \le i' < j' \le n} \frac{P_{\ell_{i'}} - P_{\ell_{j'}}}{E_{\ell_{i'}} + E_{\ell_{j'}}} $$

$$ \times \prod_{\substack{1 \le i \le m \\ 1 \le j' \le n}} \frac{E_{k_i} + E_{\ell_{j'}}}{P_{k_i} - P_{\ell_{j'}}} . $$

Homework: diagonalize H on a truncated basis numerically to find the energy spectrum.

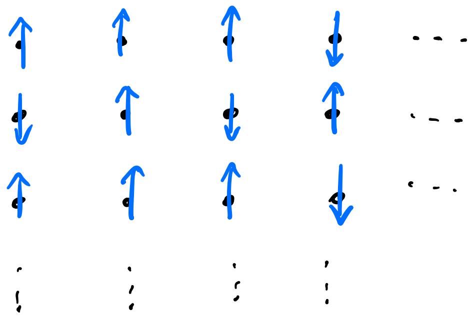
Expect:





- The (statistical) Ising model

2D case: square lattice Λ of spins



$$H = -\sum_{\langle x x' \rangle} S_x S_{x'}$$

$x = (i,j)$
$x' = (i \pm 1, j)$ or $(i, j \pm 1)$.

Hilbert space $\mathcal{H}$ (in the sense of stat-mech,
not to be confused with
that of IFT)

$$\mathcal{H} = \bigotimes_{x\in\Lambda} V_x .$$

$$V_x \simeq \mathbb{C}^2 .$$

$$= \mathrm{Span}\{|\uparrow\rangle, |\downarrow\rangle\}$$

$S_x = +1 , -1 .$ resp.

- thermal partition function

$$Z = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H}$$

$$= \sum_{\substack{S_x = \pm 1 \\ x\in\Lambda}} e^{\beta \sum_{\langle xx'\rangle} S_x S_{x'}}$$

- thermal expectation value

$$\langle \hat{O} \rangle = \frac{1}{Z} \mathrm{Tr}_{\mathcal{H}}\, O\, e^{-\beta H}$$

e.g. spin correlation function

$$\langle S_{x_1} S_{x_2} \cdots S_{x_n} \rangle = \frac{1}{Z} \mathrm{Tr}_{\mathcal{H}}\, S_{x_1} \cdots S_{x_n}\, e^{-\beta H} .$$

- $T = 1/\beta \to 0$, dominated by ground state,

  $S_x = +1$ for all x

  or $S_x = -1$ for all x.

  "spontaneous magnetization"

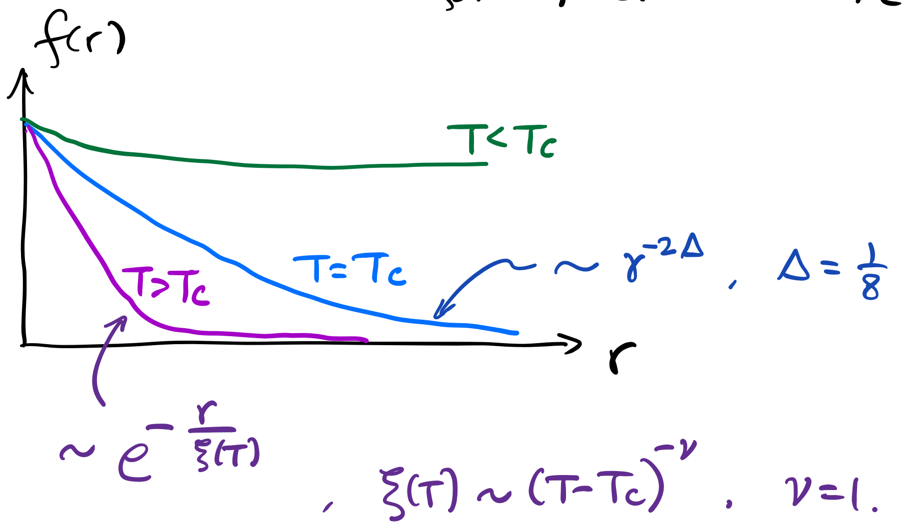
- as T increases, thermal fluctuations tend to wash out correlation

  spont. mag. disappears at Curie temp. $T = T_c$.

$$\langle S_{0,0}\, S_{N,M} \rangle \equiv f_{N,M} \approx f(r)$$

$r = \sqrt{N^2+M^2}$.

for T close to T_c.



$\sim r^{-2\Delta}$, $\Delta = \frac{1}{8}$

$\sim e^{-\frac{r}{\xi(T)}}$, $\xi(T) \sim (T-T_c)^{-\nu}$, $\nu = 1$.

- Deform the model by turning on "magnetic field"

$$H = -\sum_{\langle xx' \rangle} S_x S_{x'} + \tilde{h} \sum_x S_x .$$

Claim: In the limit $T \to T_c$, $\tilde{h} \to 0$ spin correlators at large distances are captured by Euclidean correlator of $\sigma(x)$ in Ising field theory.

$$H_{IFT} = \int dx' \left[\frac{1}{4\pi} T(x) + \frac{1}{4\pi} \tilde{T}(x) + \frac{m}{2\pi} \varepsilon(x) + h\,\sigma(x) \right]$$

$$m \propto (T - T_c) \sim L^{-1}$$

$$h \propto \tilde{h} \sim L^{-\frac{15}{8}} , \qquad L \to \infty .$$

$$\langle S_{x_1} S_{x_2} \cdots S_{x_n} \rangle \qquad |x_{ij}| \sim L$$

$$\longrightarrow \langle \sigma(x_1)\, \sigma(x_2) \cdots \sigma(x_n) \rangle$$

up to normalization of σ

- Why does it work?

"universality": details of Ising lattice and Hamiltonian not relevant at long distances, except for two parameters, $T - T_c$ and $\tilde{h}$.

This is related to the fact that IFT at $m = h = 0$ is scale invariant, possible deformations of H that respect Poincaré symmetry and locality are of the form $\int dx' \lambda_{\mathcal{O}} \mathcal{O}(x)$

$\mathcal{O}$ has effective scaling dimension Δ.

$\lambda_{\mathcal{O}}$ has mass dim $2 - \Delta$.

only those with $\Delta \leq 2$ are important in the long distance limit.

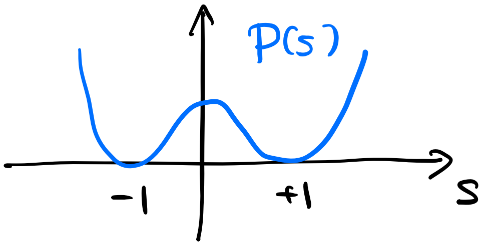
$$\mathcal{O} = \sigma \text{ and } \varepsilon.$$

$$(\Delta_\sigma = \tfrac{1}{8}) \qquad (\Delta_\varepsilon = 1)$$

- Might as well have considered a generalization of Ising model:

replace $$Z = \sum_{S_x = \pm 1} e^{-\beta H[S_x]}$$

with $$\tilde{Z} = \int \prod_x \left(ds_x \, e^{-P(s_x)} \right) \cdot \, e^{-\beta H[s_x]}$$



$$= \int \prod_x ds_x \, e^{-S[s_x]}$$

$\uparrow$ lattice version of path integral

$$S[s_x] = \beta H[s_x] + \sum_x P(s_x)$$

$$= -\beta \sum_{\langle x x' \rangle} s_x s_{x'} + \sum_x \left(P(s_x) + \beta \tilde{h} s_x \right)$$

$$= \frac{\beta}{2} \sum_{\langle x x' \rangle} (s_x - s_{x'})^2 + \sum_x \left(P(s_x) + \beta \tilde{h} s_x - \beta d \, s_x^2 \right)$$

$\uparrow$ $d = 2$ dim. of lattice

$$s_x \rightsquigarrow \phi(x), \quad x \in \mathbb{R}^2 .$$

$S[s_x]$ is the lattice-discretized version of

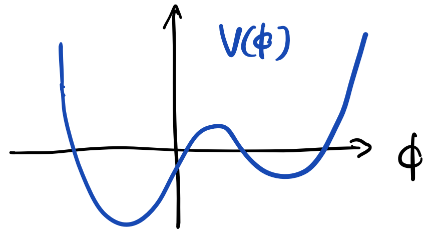
$$S[\phi] = \int d^2x \left[\frac{\beta}{2} (\partial \phi)^2 + \frac{1}{a^2} V(\phi) \right]$$

$\uparrow$ $V(\phi) = P(\phi) + \beta \tilde{h} \phi$

a = lattice spacing

$-2\beta\phi^2.$

$V(\phi)$ looks like



IFT is reached by tuning

$$(\beta-\beta_c) \sim \delta\,, \quad \tilde{h} \sim \delta^{\frac{15}{8}}$$

and simultaneously taking continuum limit
(same as long distance on the lattice)

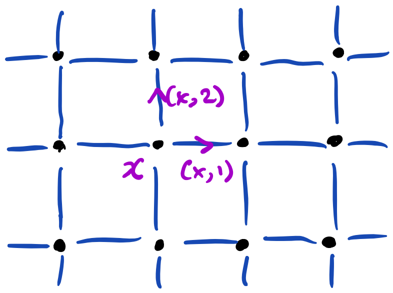
$$a \sim \delta\,, \quad \delta \to 0.$$

- Starting with a lattice path integral and taking a suitable critical & long-distance limit can be used to define a large class of QFTs (through Euclidean correlators of local op's)

Example: Yang-Mills theory

Consider a d-dimensional lattice

e.g. d=2 case



unlike in Ising case, here our variables are associated with links rather than sites.

$$(x,\mu) \rightsquigarrow U_{x,\mu} \in G$$

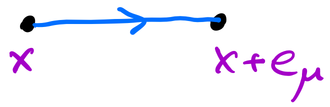
↑ "gauge" group

e.g. $G = SU(N)$

We will consider an action $S[U_{x,\mu}]$ that is invariant under "gauge transformation"

$$U_{x,\mu} \longmapsto g_x U_{x,\mu} g^{-1}_{x+e_\mu}$$

$g_x \in G$



and define the lattice path integral via

$$Z = \int \prod_{x,\mu} [dU_{x,\mu}] \; e^{-S}$$

↗ Haar measure on G

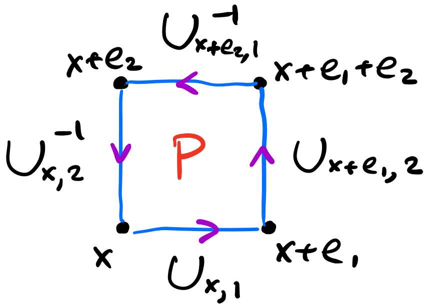
$[dU]$ invariant under
$U \mapsto g_L U g_R^{-1}$,
$\forall\ g_L, g_R \in G$

correlation function

$$\langle O_1 O_2 \cdots \rangle = \frac{1}{Z}\int \prod_{x,\mu}[dU_{x,\mu}]\, e^{-S} O_1 O_2 \cdots$$

Simplest choice of S

$$S[U_{x,\mu}] = -\frac{2}{g}\sum_{\substack{P \\ \uparrow \\ \text{“plaquette”}}} \mathrm{Re}\,\mathrm{tr}\, U_P$$



$$U_P \equiv U_{x,2}^{-1}\, U_{x+e_2,1}^{-1}\, U_{x+e_1,2}\, U_{x,1}$$

$\mathrm{tr}\, U_P$ gauge-invariant

- Continuum limit

write
$$U_{x,\mu} = e^{i\sum_a t^a \tilde{A}^a_\mu(x)}$$

t^a generators of Lie algebra of G

$$\tilde{A}_\mu(x) \equiv \sum_a t^a \tilde{A}^a_\mu(x).$$

$\mathrm{Re}\,\mathrm{tr}\,U_p$

$$= \mathrm{Re}\,\mathrm{tr}\left(e^{-i\tilde{A}_2(x)} e^{-i\tilde{A}_1(x+e_2)} e^{i\tilde{A}_2(x+e_1)} e^{i\tilde{A}_1(x)}\right)$$

$$\approx \mathrm{Re}\,\mathrm{tr}\left(e^{-i\tilde{A}_2} e^{-i\tilde{A}_1 - ia\partial_2\tilde{A}_1} e^{i\tilde{A}_2 + ia\partial_1\tilde{A}_2} e^{i\tilde{A}_1}\right)$$

$$= \mathrm{tr}\Big(-\frac{a^2}{2}(\partial_2\tilde{A}_1)^2 - \frac{a^2}{2}(\partial_1\tilde{A}_2)^2 + a^2\,\partial_2\tilde{A}_1\,\partial_1\tilde{A}_2$$

$$+ \cdots\Big)$$

$$= -\frac{a^2}{2}\,\mathrm{tr}(\tilde{F}_{12}^2) + \cdots$$

$$\uparrow \quad \tilde{F}_{12} = \partial_1\tilde{A}_2 - \partial_2\tilde{A}_1 + \frac{i}{a}[\tilde{A}_1, \tilde{A}_2].$$

under gauge transformation

$$U_{x,\mu} \mapsto g_x U_{x,\mu} g^{-1}_{x+e_\mu}$$

$$g_x \rightsquigarrow g(x) = e^{i\zeta(x)}, \quad \zeta(x) = \sum_a t^a \zeta^a(x)$$

$$\tilde{A}_\mu(x) \rightsquigarrow -a\partial_\mu\zeta(x) + i[\zeta(x), \tilde{A}_\mu(x)]$$

$\mathrm{tr}(\tilde{F}_{12}^2)$ is gauge-invariant.

$$S = -\frac{2}{\;}\sum \mathrm{Re}\,\mathrm{tr}\,U_p$$

$$\rightsquigarrow \frac{1}{g}\int d^2x\, \mathrm{tr}\, \tilde{F}_{12}^2 + \cdots$$

rescale $\tilde{A}_\mu \equiv a A_\mu$.

$$\tilde{F}_{12} = a F_{12},$$

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 + i[A_1, A_2].$$

$$S = \frac{a^2}{g}\int d^2x\, \mathrm{tr}\, F_{12}^2 + \cdots$$

For d-dim'l lattice, same derivation gives

$$S = \frac{a^{4-d}}{g}\int d^dx \underbrace{\sum_{1\le\mu<\nu\le d} \mathrm{tr}\, F_{\mu\nu}^2}_{\text{Yang-Mills action.}} + \cdots$$

As a QFT, the quantum Yang-Mills theory is defined by a limit of the lattice path $\int$

$$Z = \int \prod_{x,\mu}[dU_{x,\mu}]\, e^{-S[U]}$$

simultaneously $g \to g_c$, $a \to 0$.

- Exactly how g approaches g_c in the continuum limit is a question concerning dynamics.

- perturbative RG calculation

$\Rightarrow$ $g_c = 0$ (asymptotic freedom)

$$g \sim \frac{1}{b \cdot \log \frac{1}{Ma}} \quad \text{as} \quad a \to 0.$$

$b = \frac{1}{2\pi^2} \cdot \frac{11}{12} N$, (1-loop β-fn coeff.)

M = a mass scale ("Λ_{QCD}")

- $$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \prod_{x,\mu} [dU_{x,\mu}] \, e^{-S} \mathcal{O}$$

in this limit give Euclidean correlators of a QFT

e.g. $\mathcal{O} = \mathrm{tr}\, U_{P(x_1,\mu_1,\nu_1)} \mathrm{tr}\, U_{P(x_2,\mu_2,\nu_2)} \cdots$

$$\langle \phi(x) \phi(0) \rangle \rightsquigarrow \langle 0 | \phi(x) \phi(0) | 0 \rangle$$

Euclidean $\nearrow$ Lorentzian

from spectral decomp.

$\langle 0|\phi(x)|\alpha\rangle\langle\alpha|\phi(0)|0\rangle$

can in principle read off

particle spectrum :

glueballs of mass $\sim M$.

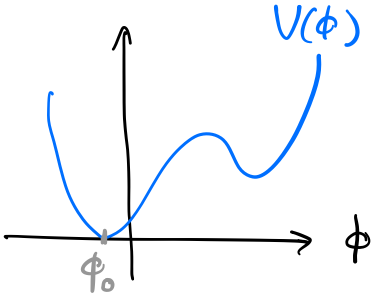
- Some comments on perturbation theory and its limitation

lattice path integral

$$Z = \int \prod_x d\phi_x \; e^{-\frac{1}{\hbar}S[\phi]}$$

Simple model of 1 site , 1D integral

$$Z = \int_{-\infty}^{\infty} d\phi \; e^{-\frac{1}{g}V(\phi)}$$



- $g \to 0^+$, $\int$ dominated by contribution near $\phi = \phi_0$, can expand

$$V(\phi) = V(\phi_0) + \frac{1}{2} V''(\phi_0) (\phi-\phi_0)^2 + \frac{1}{3!} V'''(\phi_0) (\phi-\phi_0)^3 + \cdots$$

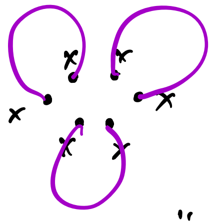
$$Z = e^{-\frac{1}{g}V(\phi_0)} \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2g} V''(\phi_0) x^2 - \frac{1}{6g} V'''(\phi_0) x^3 + \cdots}$$

$$= e^{-\frac{1}{g}V(\phi_0)} \sqrt{\frac{2\pi g}{V''(\phi_0)}} \left\langle e^{-\frac{1}{g} \sum_{n=3}^{\infty} \frac{1}{n!} V^{(n)}(\phi_0) x^n} \right\rangle_0$$

$$\langle x^N \rangle_0 = \frac{\int dx \, e^{-\frac{1}{2g} V''(\phi_0) x^2} x^N}{\int dx \, e^{-\frac{1}{2g} V''(\phi_0) x^2}}$$

$$= \begin{cases} \frac{N!}{(\frac{N}{2})! \, 2^{\frac{N}{2}}} \left(\langle x^2 \rangle_0\right)^{\frac{N}{2}}, & N \text{ even} \\ 0, & N \text{ odd} \end{cases}$$

group N x's into $\frac{N}{2}$ pairs



"propagator" $= \langle x^2 \rangle_0$

$$= \frac{g}{V''(\phi_0)}.$$

Formally, we can write the expansion

$$Z \sim e^{-\frac{1}{g}V(\phi_0)} \sum_{n=0}^{\infty} a_n g^{n+\frac{1}{2}}$$

- problem: the series expansion in g has radius of convergence $R=0$.

e.g. set $V''(\phi_0)=1$,

$$\left\langle e^{-\frac{1}{g}x^4} \right\rangle_0 = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-1}{g}\right)^k \cdot \langle x^{4k}\rangle_0$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-1}{g}\right)^k \frac{(4k)!}{(2k)!\, 2^{2k}} \cdot g^{2k}$$

$$\Rightarrow \quad a_k = \sqrt{2\pi} \cdot (-)^k \cdot \frac{(4k)!}{k!\,(2k)!\,2^{2k}}$$

$$|a_k| \sim \frac{2^{4k}}{\sqrt{\pi}\, k} k!$$

- perturbative series does not converge.

Q: Is it nonetheless possible to recover the exact integral Z from the a_n's?

A: Depends.

WLOG set $V(\phi_0)=0$ from now,

Formal series

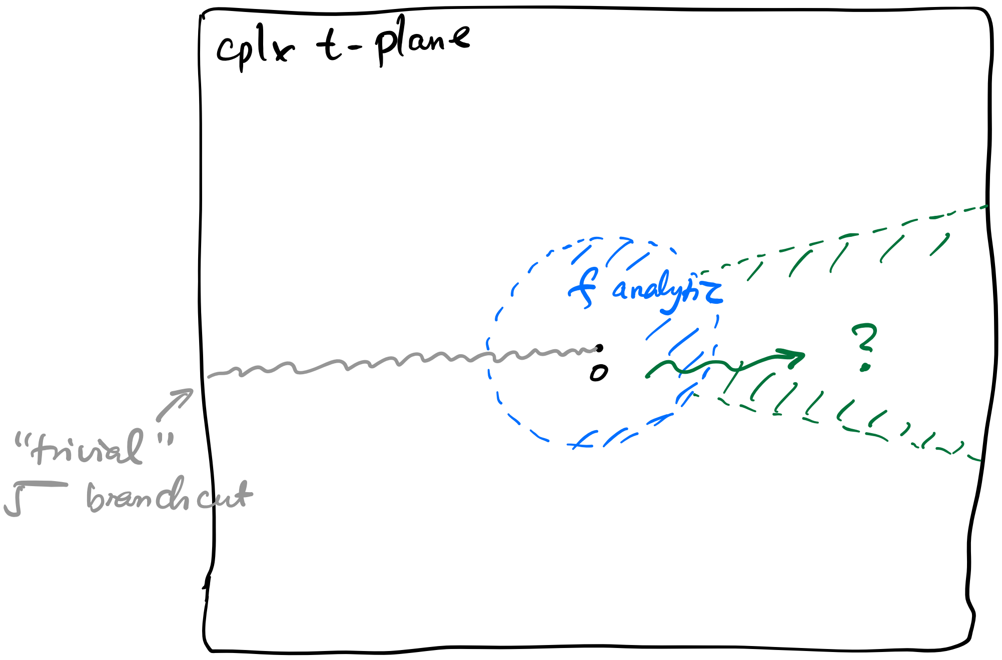
$$Z \sim \sum_{n=0}^{\infty} a_n g^{n+\frac{1}{2}} .$$

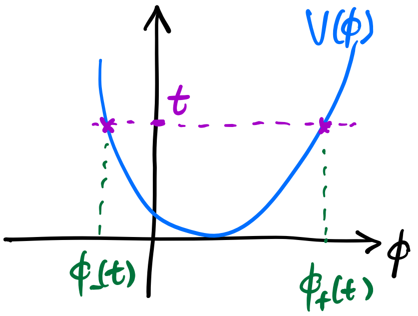
Consider "Borel transformation"

$$\sum_{n=0}^{\infty} a_n g^{n+\frac{1}{2}} \rightsquigarrow \underbrace{\sum_{n=0}^{\infty} a_n \frac{t^{n-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})}}_{\text{finite radius of convergence in } t} := f(t)$$

?

formally, $\int_0^{\infty} dt\, e^{-\frac{t}{g}} f(t)$





$$Z = \int_{-\infty}^{\infty} d\phi\, e^{-\frac{1}{g}V(\phi)}$$

$$= \int_0^{\infty} dt\, e^{-\frac{t}{g}} \underbrace{\int_{-\infty}^{\infty} d\phi\, \delta(t - V(\phi))}_{\substack{\equiv \\ F(t)}}$$

$$F(t) \overset{t\in(0,\delta)}{=} \sum a_n \frac{t^{n-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})}$$

[Sanity check:

$$F(t) = \int_{-\infty}^{\infty} d\phi\, \delta(t - V(\phi)) = \frac{1}{V'(\phi_+(t))} - \frac{1}{V'(\phi_-(t))}$$

for small $t > 0$, $t \approx \frac{1}{2} V''(\phi_0)(\phi_\pm - \phi_0)^2$.

$$V'(\phi_\pm) \approx V''(\phi_0)(\phi_\pm - \phi_0)$$

$$\approx \pm\sqrt{2V''(\phi_0)\, t} .$$

$$F(t) \approx \frac{2}{\sqrt{2V''(\phi_0)\, t}} = \frac{a_0}{\Gamma(\frac{1}{2})} t^{-\frac{1}{2}}$$]

- If $V(\phi)$ real analytic, has a single local minimum at $\phi = \phi_0$, $V'(\phi) \neq 0$ elsewhere,

then
$$F(t) \equiv \int_{-\infty}^{\infty} d\phi \, \delta(t - V(\phi))$$

is an analytic function for $t \in \mathbb{R}_+$, and hence can be analytically continued to a neighborhood of $\mathbb{R}_+$ on the cplx t-plane. Furthermore, in a disc $|t| < R$,

$$F(t) = \sum_{n \geqslant 0} a_n \frac{t^{-\frac{1}{2}}}{\Gamma(n + \frac{1}{2})} .$$

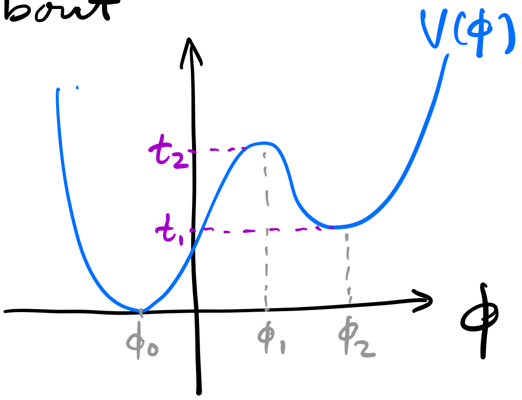
Said equivalently, we can define $f(t)$ to be the analytic continuation of the Taylor series $\sum_{n \geqslant 0} a_n \frac{t^{-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})}$ from $|t| < R$ to a neighborhood of $\mathbb{R}_+$,

and recover Z by

$$Z = \int_0^{\infty} dt \, e^{-\frac{t}{g}} f(t) .$$

"Borel resummation"

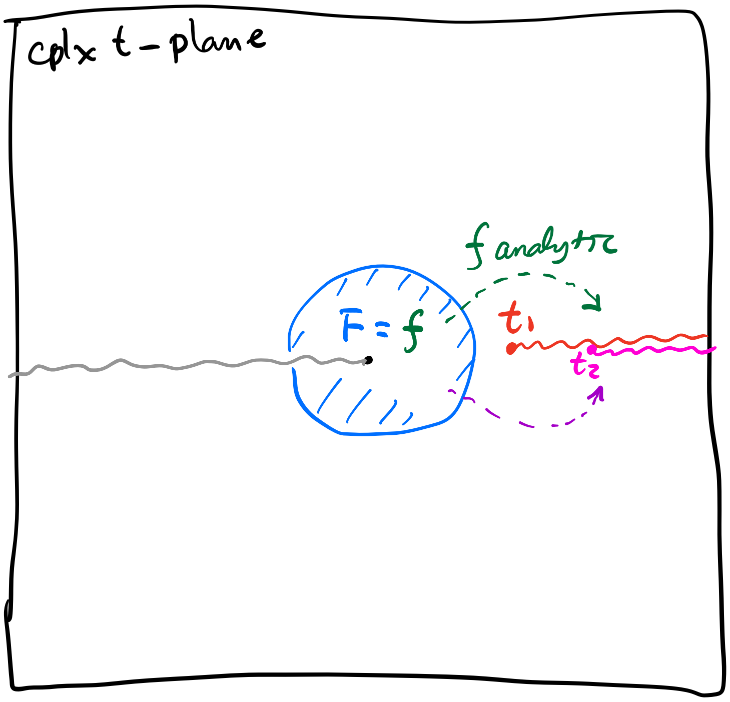
- What about



?

$$F(t) \equiv \int_{-\infty}^{\infty} d\phi \, \delta(t - V(\phi)) \qquad t \in \mathbb{R}_+$$

is NOT analytic at $t = t_1, t_2$.



By construction, we still have

$$Z = \int_0^\infty dt \, e^{-\frac{t}{g}} F(t)$$

But $F(t) \neq f(t)$ for $t > t_1$

$\uparrow$ defined by analytic continuation of $\sum a_n \frac{t^{n-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})}$ from $|t| < R$

Q: how do we recover $F(t)$ from $f(t)$?

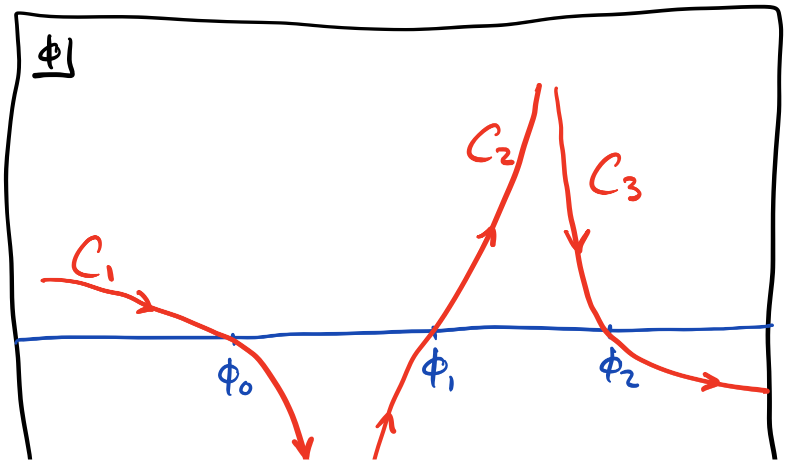
- Deform the model by

$$V(\phi) \to \tilde{V}(\phi) = e^{i\theta} V(\phi)$$

small $\theta > 0$.

$$\tilde{Z} = \int_{\mathbb{R}} d\phi \; e^{-\frac{1}{g}\tilde{V}(\phi)} \xrightarrow{\theta \to 0} Z$$

Since $\tilde{V}(\phi)$ is analytic on the entire cplx ϕ-plane, we can replace the real contour $\mathbb{R}$ with a deformed contour C, chosen as follows.



$$C = C_1 + C_2 + C_3 .$$

Each C_i is a steepest descent path followed by a steepest ascent path with respect to the "height function" $\mathrm{Re}\,\tilde{V}(\phi)$ that passes through the saddle point ϕ_i.

i.e. $\phi = \phi(s)$ along the path C_i, $s \in \mathbb{R}$.

such that
$$\frac{d\phi}{ds} = \pm \frac{\partial}{\partial \phi^*} \mathrm{Re}\,\tilde{V}(\phi)$$
$$= \pm \frac{1}{2} \left(\tilde{V}'(\phi)\right)^* .$$

$$\Rightarrow \quad \frac{d}{ds} \mathrm{Im}\,\tilde{V}(\phi) = \mathrm{Im}\left(\frac{d\phi}{ds} \tilde{V}'(\phi)\right)$$
$$= \mathrm{Im}\left(\pm \frac{1}{2} |\tilde{V}'(\phi)|^2\right) = 0 .$$

i.e. $\mathrm{Im}\,\tilde{V}(\phi) = \text{const}$ along C_i.

$$\int_{C_i} d\phi \, e^{-\frac{1}{g}\tilde{V}(\phi)}$$

$$= e^{-\frac{1}{g}\tilde{V}(\phi_i)} \int_0^\infty dt \, e^{-\frac{t}{g}} \underbrace{\int_{C_i} d\phi \, \delta\Big(t - \overbrace{(V(\phi) - V(\phi_i))}^{\overset{\text{for } \phi \in C_i}{\mathbb{R}_+ \ni}}\Big)}_{\overset{\cong}{f_i(t)}}$$

$f_i(t)$ = analytic continuation of $\sum a_n^{(i)} \frac{t^{n-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})}$

$\sum a_n^{(i)} g^{n+\frac{1}{2}}$ is the perturbative series for $\tilde{Z}$ around $\phi = \phi_i$.

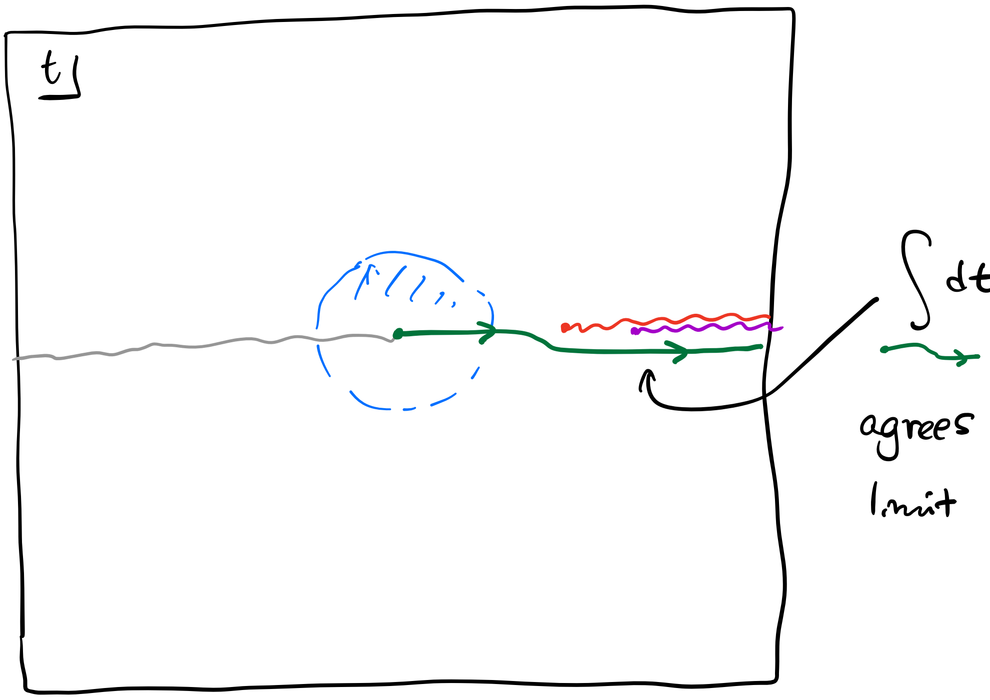
Conclusion:

$$\tilde{Z} = \tilde{Z}_1 + \overbrace{\tilde{Z}_2 + \tilde{Z}_3}^{\text{"instanton corrections"}}$$

each $\tilde{Z}_i$ is given by the Borel-resummation of the pert. series around the saddle $\phi = \phi_i$.

Note: in relation to $f(t)$ defined previously as Borel transform of pert. series for $Z = \int_{\mathbb{R}} d\phi \, e^{-\frac{1}{g} V(\phi)}$ around $\phi = \phi_0$,

$$f_1(t) = f(e^{-i\theta} t) \quad ,$$



$\int dt \, e^{-\frac{t}{g}} f(t)$

agrees with $\theta \to 0^+$ limit of $\tilde{Z}_1$.

Generalization to multi-dim'l integrals

$$Z = \int_{C_{\mathbb{R}}} d^N\phi \, e^{-\frac{1}{g} V[\phi]} \; .$$

- deform $V[\phi]$ with cplx coefficients

  so that
$$Z = \int_C d^N\phi \, e^{-\frac{1}{g} V[\phi]}$$

$C = \sum_i n_i C_i$, each C_i is a "stable submanifold" that is the union of all steepest ascent paths with respect to height function Re $V[\phi]$.

that start at the saddle point $\phi = \phi_i$

"Lefschetz thimble"

$n_i \in \mathbb{Z}$, Let $C_i^\perp$ be the "unstable submanifold" formed by all steepest descent paths from $\phi = \phi_i$. $\#(C_i^\perp \cap C_j) = \delta_{ij}$

$n_\perp$ = (algebraic) intersection number of $C_i^\perp$ with the real contour $C_{\mathbb{R}}$