Physics 253c Problem set 1

Due Tuesday September 26, 2023

In this course we adopt the mostly positive signature convention for the Minkowskian metric, $\eta_{\mu\nu} = \text{diag}\{-1, 1, \cdots, 1\}$, the convention for Dirac gamma matrices γ_μ such that $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$, and the four-dimensional spinor conjugation convention $\bar{\psi} \equiv \psi^\dagger i\gamma^0$.

Problem 1. In each of the following classical field theories in $D = 4$ spacetime dimensions, find the expression for the stress-energy tensor $T_{\mu\nu}$ via Noether's procedure:

(a) the massless scalar field ϕ, with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi. \tag{1}$$

(b) the massless Dirac fermion field ψ, with Lagrangian density

$$\mathcal{L} = -\bar{\psi}\gamma^\mu\partial_\mu\psi. \tag{2}$$

(c) the free Maxwell theory, with Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3}$$

where the field strength $F_{\mu\nu}$ is related to the vector potential A_μ as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

In each case, is $T_{\mu\nu}$ traceless modulo the equation of motion? Is the theory (classically) conformally invariant?

Problem 2. Consider a free massless scalar field ϕ propagating in D-dimensional spacetime with background metric $g_{\mu\nu}$, described by the covariant action

$$S[\phi; g] = -\frac{1}{2}\int d^D x\sqrt{-\det g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + aR\phi^2\right). \tag{4}$$

Show that the action is invariant under the Weyl transformation

$$g_{\mu\nu}(x) \mapsto e^{2\omega(x)}g_{\mu\nu(x)}, \qquad \phi(x) \mapsto e^{\lambda\omega(x)}\phi(x), \tag{5}$$

where $\omega(x)$ is an arbitrary function, for a unique choice of the constants λ and a. By varying the action with respect to the background metric, find the expression of the stress-energy tensor $T_{\mu\nu}(x)$ in the Minkowskian spacetime that is both conserved and traceless (it may differ from your answer to problem 1(a) by an "improvement term").

Problem 3. In a free massless *quantum* scalar field theory described by the Lagrangian density (1), construct the conserved *traceless* stress-energy tensor operator $\hat{T}_{\mu\nu}(x)$ as a normal-ordered version of the classical expression you found in problem 2. The charge operator associated with a conformal Killing vector field $\varepsilon^\mu(x)$, as defined in class, is

$$\hat{Q}_\varepsilon = \int d^{D-1}\vec{x}\,\hat{T}^{0\mu}(x)\varepsilon_\mu(x). \tag{6}$$

In particular, the energy-momentum vector $\hat{P}_\mu$, the angular momentum/Lorentz boost generator $\hat{J}_{\mu\nu}$, the dilatation operator $\hat{D}$, and the special conformal generator $\hat{K}_\mu$, are associated with the following CKVs (for convenience we adopt the differential notation for vector fields, $\varepsilon^\mu\partial_\mu$, in each case)

$$\partial_\mu, \qquad x_\mu\partial_\nu - x_\nu\partial_\mu, \qquad x^\mu\partial_\mu, \qquad x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu. \tag{7}$$

By the expanding the fields in terms of creation and annihilation operators, or equivalently using the equal-time canonical commutator relations, show that the following commutation relation holds:

$$\begin{aligned}
&[\hat{D}, \hat{P}_\mu] = i\hat{P}_\mu, \\
&[\hat{D}, \hat{K}_\mu] = -i\hat{K}_\mu, \\
&[\hat{P}_\mu, \hat{K}_\nu] = 2i(\eta_{\mu\nu}\hat{D} - \hat{J}_{\mu\nu}),
\end{aligned} \tag{8}$$

in addition to the familiar Poincaré commutator relations among $\hat{P}_\mu$ and $\hat{J}_{\mu\nu}$.

Problem 4. Consider the conformally coupled free scalar field theory in $D = 4$ dimensions, whose action is given by (4) (where the coefficient a was determined in problem 2), in the spacetime $\mathbb{R} \times S^3$ where S^3 stands for the 3-sphere of unit radius. Via canonical quantization, find the energy spectrum of single particle states, and thereby constructing an energy eigenbasis of the full Hilbert space in the form of multi-particle Fock states.

Next, consider the free massless quantum scalar field theory in $D = 4$ flat spacetime, and find all linearly independent local operators (at one point) of scaling dimension $\Delta \leq 4$. Show that these are in 1-1 correspondence with the first few energy eigenstates of the aforementioned conformally coupled free scalar field theory on S^3, whose excitation energy is proportional to Δ. *Bonus: extend this correspondence to a complete basis of local operators.*