Physics 253c Problem set 3

Due Tuesday October 24, 2023

The 3D $O(N)$ model is a quantum field theory defined by the Euclidean action

$$S[\phi_i, \sigma] = \int d^3x \left[\frac{1}{2} \sum_{i=1}^{N} (\partial^\mu \phi_i)(\partial_\mu \phi_i) + \frac{i}{2} \sigma \left(\sum_{i=1}^{N} \phi_i^2 - \frac{N}{g} \right) \right], \tag{1}$$

where $\phi_1, \cdots, \phi_N$ and σ are real scalar field variables, and g a (dimensionful) bare coupling parameter.[1]

(a) Deduce an effective action $S_{\text{eff}}[\sigma]$ by performing the (Gaussian) functional integration over the fields $\phi_i(x)$,

$$e^{-S_{\text{eff}}[\sigma]} = \int [D\phi_i]\, e^{-S[\phi_i, \sigma]}, \tag{2}$$

defined with dimensional regularization scheme. You should find that $S_{\text{eff}}[\sigma]$ is proportional to N, and thus in the large N limit the path integral $Z = \int [D\sigma]\, e^{-S_{\text{eff}}[\sigma]}$ can be evaluated by a saddle point approximation. Assuming that the dominant saddle point is translation-invariant i.e. $\sigma(x) = \overline{\sigma}$ a constant independent of spacetime coordinates x^μ, determine $\overline{\sigma}$ in terms of g.

(b) One can generalize (2) by introducing source terms for ϕ_i and evaluate the Gaussian functional integral

$$e^{-S_{\text{eff}}[\sigma, J_i]} = \int [D\phi_i]\, e^{-S[\phi_i, \sigma] + \int d^3x \sum_{i=1}^{N} J_i \phi_i}, \tag{3}$$

where $J_i(x)$ are arbitrary functions, and from the generating functional

$$e^{-W[J_i]} \equiv \int [D\sigma] e^{-S_{\text{eff}}[\sigma, J_i]} \tag{4}$$

extract correlation functions of ϕ_i in a $1/N$ expansion. Calculate the two-point function $\langle \phi_i(x) \phi_j(0) \rangle$ at the leading order in the large N limit. For what value of $\bar{\sigma}$ or g is the theory gapless?

[1] The appearance of the factor i in the second term may seem unusual, but in fact the underlying quantum field theory is unitary (reflection positive); indeed, σ plays the role of a Lagrangian multiplier, and the effect of performing the Gaussian functional integral over $\sigma(x)$ restricts $(\phi_1, \cdots, \phi_N)$ to take value on an $(N-1)$-sphere of radius $\sqrt{N/g}$.

(c) The *critical* $O(N)$ model is defined by (1) at the special value of g such that the theory is gapless.[2] In this theory, calculate the two-point function $\langle \sigma'(x)\sigma'(0)\rangle$, where $\sigma(x) \equiv \langle\sigma\rangle + \sigma'(x)$, at the leading order in the large N limit. In particular, determine the scaling dimensions of the operators ϕ_i and σ' as $N \to \infty$.

(d) In the critical $O(N)$ model, calculate the three-point functions

$$\langle \phi_i(x_1)\phi_j(x_2)\sigma'(x_3)\rangle \quad \text{and} \quad \langle \sigma'(x_1)\sigma'(x_2)\sigma'(x_3)\rangle \tag{5}$$

at the leading nontrivial order in the large N limit. In particular, show that the result is compatible with the expectations from conformal invariance.

(e) Extend your calculation of $\langle \phi_i(x)\phi_j(0)\rangle$ and $\langle \sigma'(x)\sigma'(0)\rangle$ to the next-to-leading order in the $1/N$ expansion, and extract from your result the scaling dimensions of the operators ϕ_i and σ' in the critical $O(N)$ model at order N^{-1}.

[2]Note that this critical value of g is generally regularization scheme dependent.