Physics 253c Problem set 4

Due Tuesday November 14, 2023

Problem 1. Calculate the four-point function of a 4-dimensional free massless scalar field $\phi(x)$, and express the result explicitly as a sum over conformal blocks labeled by internal primary weight Δ and spin s in the form

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = |x_{12}|^{-2\Delta_\phi}|x_{34}|^{-2\Delta_\phi}\sum_{\Delta,s}\lambda^2_{\Delta,s}G_{\Delta,s}(u,v), \tag{1}$$

and read off all of the coefficients $\lambda^2_{\Delta,s}$ for Δ up to 4. (Hopefully, you will confirm that $\lambda^2_{\Delta,s}$ are non-negative.)

Problem 2. Consider the 2-dimensional free massless scalar field theory described by the Euclidean action

$$S = \frac{1}{2}\int d^2x\, \partial^\mu\phi\partial_\mu\phi \equiv \int d^2z \partial\phi\bar{\partial}\phi, \tag{2}$$

where we have adopted the convention $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$, $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$, and $d^2z \equiv 2dx^1dx^2$.

(a) The stress-energy tensor operators $T(z)$ and $\widetilde{T}(\bar{z})$ are of the form[1]

$$T = \gamma : \partial\phi\partial\phi :, \quad \widetilde{T} = \widetilde{\gamma} : \bar{\partial}\phi\bar{\partial}\phi : . \tag{4}$$

Determine the coefficients γ and $\widetilde{\gamma}$, and verify explicitly that the TT OPE takes the form

$$T(z)T(0) = \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) + \sum_{n\geq 0} z^n\Psi_n(0), \tag{5}$$

and that c is equal to 1.

(b) Consider local operators built out of derivatives of ϕ of the form

$$\mathcal{O}_{n_1,\cdots,n_k;\widetilde{n}_1,\cdots,\widetilde{n}_\ell} =: \partial^{n_1}\phi\cdots\partial^{n_k}\bar{\partial}^{\widetilde{n}_1}\phi\cdots\bar{\partial}^{\widetilde{n}_\ell}\phi :, \tag{6}$$

[1] Note that the normal ordered product, usually defined via the expansion of free fields in terms of creation and annihilation operators and changing the order so that the creation operators are to the left of annihilation operators, is equivalent in this case to

$$: \partial\phi\partial\phi(z) := \lim_{w\to z}\left[\partial\phi(w)\partial\phi(z) - \langle\partial\phi(w)\partial\phi(z)\rangle\right]. \tag{3}$$

where $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$ and $\widetilde{n}_1 \geq \widetilde{n}_2 \geq \cdots \geq \widetilde{n}_\ell \geq 1$. Find all linear combinations of such operators of scaling dimension $\Delta \leq 4$ that are primaries with respect to the $so(3,1)$ conformal algebra.

(c) Find all linear combinations of operators of the form (6) of scaling dimension $\Delta \leq 4$ that are primaries with respect to the Virasoro algebra.

(d) Repeat **(c)** but now including also operators with ϕ itself in the normal order product, namely

$$: \phi^m \partial^{n_1}\phi \cdots \partial^{n_k} \bar{\partial}^{\widetilde{n}_1}\phi \cdots \bar{\partial}^{\widetilde{n}_\ell}\phi : . \tag{7}$$

with arbitrary $m \geq 0$.