Physics 253c Problem set 5

Due Tuesday November 28, 2023

Problem 1. Consider the theory of a free massless scalar field ϕ in 2D, defined by the Euclidean action

$$S = \frac{1}{2}\int d^2x\, \partial^\mu\phi\partial_\mu\phi \equiv \int d^2z \partial\phi\bar{\partial}\phi, \tag{1}$$

and its partition function

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}} \exp\left[2\pi i\tau(L_0 - \frac{c}{24}) - 2\pi i\bar{\tau}(\widetilde{L}_0 - \frac{\widetilde{c}}{24})\right] \tag{2}$$

for $\mathcal{H}$ the space of local operators at the origin. Modular invariance would require

$$Z(\tau) = Z(\tau+1) = Z(-1/\tau). \tag{3}$$

(a) Calculate the contribution to Z from local operators built out of derivatives of ϕ (you have studied as in PSet 4) of the form

$$\mathcal{O}_{n_1,\cdots,n_k;\widetilde{n}_1,\cdots,\widetilde{n}_\ell} =: \partial^{n_1}\phi\cdots\partial^{n_k}\bar{\partial}^{\widetilde{n}_1}\phi\cdots\bar{\partial}^{\widetilde{n}_\ell}\phi : . \tag{4}$$

Show that the resulting function of $\tau, \bar{\tau}$ is *not* modular invariant.

(b) Now consider the theory of a *compact* free scalar, where ϕ is understood to be periodic, namely ϕ and $\phi + 2\pi R$ are viewed as equivalent field configurations. Said differently, the discrete shift symmetry $\phi \to \phi + 2\pi R$ is now regarded as a *gauge redundancy*. The set of local operators of the form

$$\mathcal{O}_{m|n_1,\cdots,n_k;\widetilde{n}_1,\cdots,\widetilde{n}_\ell} =: e^{im\phi/R}\partial^{n_1}\phi\cdots\partial^{n_k}\bar{\partial}^{\widetilde{n}_1}\phi\cdots\bar{\partial}^{\widetilde{n}_\ell}\phi : \tag{5}$$

are well-defined for any integer m. Calculate the contribution of operators of the form (5) to the partition function Z. Is the result modular invariant?

(c) There is in fact a larger family of admissible local operators in the compact free scalar theory, of the form

$$\mathcal{O}_{m,m'|n_1,\cdots,n_k;\widetilde{n}_1,\cdots,\widetilde{n}_\ell} =: e^{im\phi/R + im'2\pi R(\phi_+ - \phi_-)}\partial^{n_1}\phi\cdots\partial^{n_k}\bar{\partial}^{\widetilde{n}_1}\phi\cdots\bar{\partial}^{\widetilde{n}_\ell}\phi :, \tag{6}$$

where ϕ_+ and ϕ_- are formally the holomorphic and anti-holomorphic components of ϕ, with the (non-singled valued) OPE

$$\begin{aligned} \phi(z,\bar{z})\phi_+(0) &=: \phi(z,\bar{z})\phi_+(0) : -\frac{1}{4\pi}\log z, \\ \phi(z,\bar{z})\phi_-(0) &=: \phi(z,\bar{z})\phi_-(0) : -\frac{1}{4\pi}\log \bar{z}. \end{aligned} \tag{7}$$

Here the normal-ordered expression on the RHS has a regular Taylor series around $z = 0$. Show that the OPE between $\phi(z, \bar{z})$ and $\mathcal{O}_{m,m'|n_1,\cdots,n_k;\widetilde{n}_1,\cdots,\widetilde{n}_\ell}(0)$ is single-valued for integer m, m' *up to shifts* of ϕ by integer multiples of $2\pi R$ (and hence is well-defined in theory where the field ϕ and $\phi + 2\pi R$ are viewed as equivalent).

(d) Calculate the contribution of all operators of the form (6) to the partition function (2), and verify that the result is modular invariant.

Problem 2. A free Majorana fermion field in 2D consists of components $(\psi_+, \psi_-) \equiv (\psi, \widetilde{\psi})$, that obey OPE

$$\begin{aligned} \psi(z)\psi(0) &=: \psi(z)\psi(0) : + \frac{1}{z}, \\ \widetilde{\psi}(\bar{z})\widetilde{\psi}(0) &=: \widetilde{\psi}(\bar{z})\widetilde{\psi}(0) : + \frac{1}{\bar{z}}, \\ \psi(z)\widetilde{\psi}(0) &=: \psi(z)\widetilde{\psi}(0) :, \end{aligned} \tag{8}$$

where the normal-ordered expressions have a regular Taylor series around $z = 0$. Furthermore, the fermion fields are anti-commuting in the OPE, e.g.

$$\psi(z_1)\psi(z_2) = -\psi(z_2)\psi(z_1), \tag{9}$$

which in particular implies that

$$: \psi(z)\psi(0) := \sum_{n=1}^{\infty} \frac{z^n}{n!} : \partial^n\psi(0)\psi(0) :, \tag{10}$$

where the $n = 0$ term is absent.

(a) Calculate the partition function $Z_1(\tau)$ of all operators made out of normal-ordered products of the fermion fields and their derivatives, of the form

$$\mathcal{O}_{n_1,\cdots,n_k;\widetilde{n}_1,\cdots,\widetilde{n}_\ell} =: \partial^{n_1}\psi \cdots \partial^{n_k}\psi \bar{\partial}^{\widetilde{n}_1}\widetilde{\psi} \cdots \bar{\partial}^{\widetilde{n}_\ell}\widetilde{\psi} :, \tag{11}$$

where we may take, without loss of generality, $n_1 > n_2 > \cdots > n_k \geq 0$, and similarly for $\widetilde{n}_i$'s. Verify that $Z_1(\tau)$ is invariant under $\tau \to -1/\tau$, but not under $\tau \to \tau + 1$.

(b) Define

$$Z_2(\tau) \equiv Z_1(\tau + 1), \tag{12}$$

and

$$Z_3(\tau) \equiv Z_1(-\frac{1}{\tau} + 1). \tag{13}$$

Verify that Z_3 is invariant under $\tau \to \tau + 1$.

(c) Show that

$$Z(\tau) \equiv \frac{Z_1(\tau) + Z_2(\tau) + Z_3(\tau)}{2} \tag{14}$$

is modular invariant, and that if we interpret $Z(\tau)$ as the partition function of a CFT, the spectrum of Virasoro primaries of such a putative CFT consists of precisely those of conformal weights $(0, 0)$, $(\frac{1}{16}, \frac{1}{16})$, and $(\frac{1}{2}, \frac{1}{2})$.