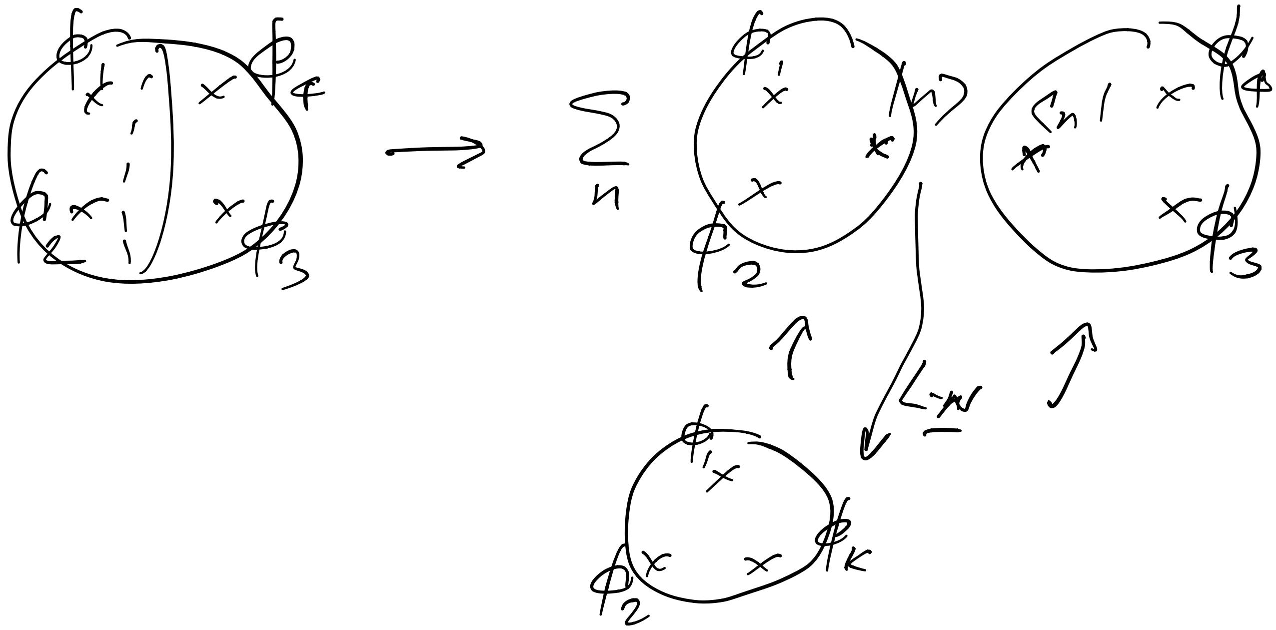

section 3.8 & 3.9

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string worldsheet

Polyakov

$$\Big\downarrow \text{conformal gauge.}$$

$$\underbrace{g_{ab} = \hat{g}_{ab}(t^k)}$$ $k = 1, \ldots, \#\text{moduli}$

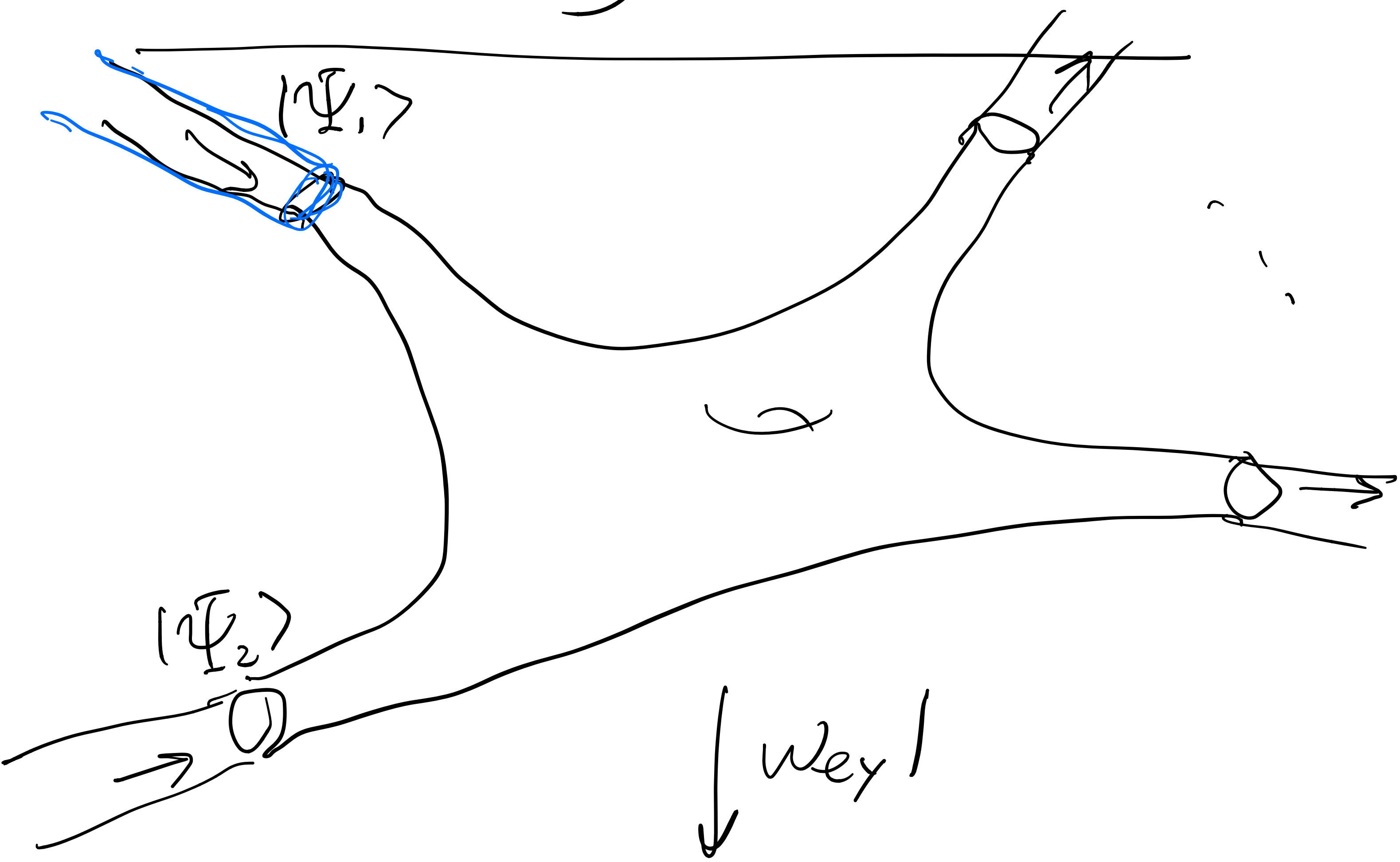
WS CFT

$$\underbrace{X^\mu}_{\mu = 0, \ldots, D-1} \oplus \text{bc-ghost}$$

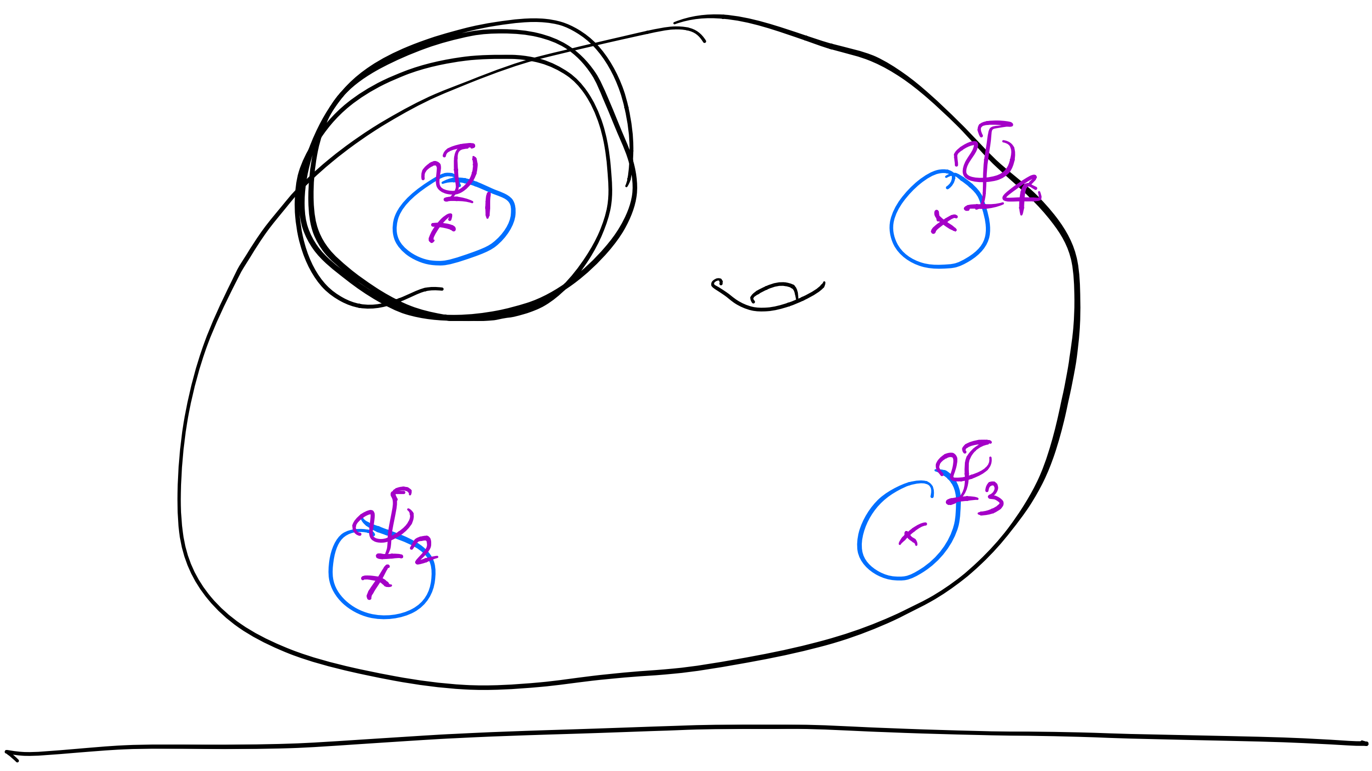
$$c = D \qquad\qquad c = -26.$$

$$0 = c^{\text{total}} = D - 26.$$

~~Weyl anomaly~~



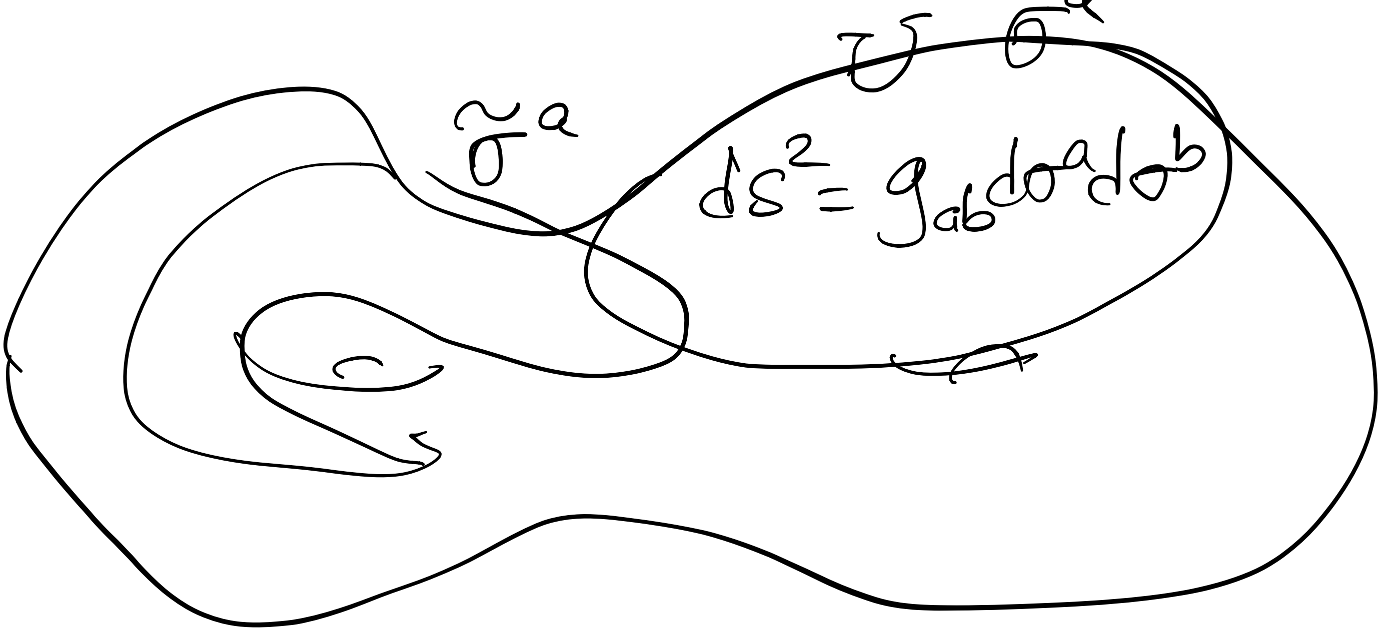
Weyl



$$g_{ab} \to e^{2\omega} g_{ab}$$

What ~~are~~ the space of metrics on an oriented 2D surface (compact) modulo Weyl transf ?

– it is the space of <u>Riemann Surfaces</u>



Locally, on a patch U, $z = \sigma_1 + i\sigma_2$ can find a coord. system (σ_1, σ_2) in which $ds^2 = e^{2\omega}(d\sigma_1^2 + d\sigma_2^2)$

$$= 2g^{(z,\bar{z})}_{z\bar{z}}\, dz\, d\bar{z}$$

$$(g_{zz} = 0 = g_{\bar{z}\bar{z}})$$

agree on $U \cap U'$

on another patch U',

can find $(z', \bar{z}')$

$$ds^2 = 2g_{z'\bar{z}'}\, dz'\, d\bar{z}'.$$

$$z' = f(z,\bar{z}), \quad \bar{z}' = \bar{f}(z,\bar{z})$$

$$ds^2 = 2g_{z'\bar{z}'}\left(\partial_z f\, dz + \partial_{\bar{z}} f\, d\bar{z}\right) \cdot \left(\partial_z \bar{f}\, dz + \partial_{\bar{z}} \bar{f}\, d\bar{z}\right)$$

$$\Rightarrow \bar{\partial} f = 0, \qquad \partial \bar{f} = 0,$$

$$\boxed{z' = f(z)}$$

BRST

F-P ansatz.

$$\int D\phi \, e^{-S[\phi]}$$

$\phi \rightsquigarrow \phi^{\xi}$

gauge transf.

want to fix gauge by imposing $F^A[\phi] = 0$.

$\det(\delta_\alpha F^A(\phi))$

$$Z = \int D\phi \; \underbrace{\delta(F^A[\phi])}_{} \cdot \underbrace{\Delta_{FP}}_{} \cdot e^{-S[\phi]}$$

Lag. multiplier (bosonic)

$$= \int D\phi \, Db_A \, Dc^\alpha \, DB_A$$

$$\times \, e^{-S[\phi] + iB_A F^A[\phi] - b_A c^\alpha \delta_\alpha F^A[\phi]}$$

BRST symmetry

- a fermionic global symmetry "δ_B"

$$* \quad \delta_B \phi = -i c^\alpha \delta_\alpha \phi$$

$$\longrightarrow \delta_B B_A = 0.$$

$$* \quad \delta_B b_A = B_A$$

$$* \quad \delta_B c^\alpha = \frac{i}{2} f_{\beta\gamma}{}^\alpha c^\beta c^\gamma$$

$$\left([\delta_\alpha, \delta_\beta] = f_{\alpha\beta}{}^\gamma \delta_\gamma \right)$$

$$\delta_B S[\phi] = 0.$$

$$* \quad \delta_B^2 = 0 \qquad \text{(exercise)}$$

$$\boxed{* \quad i B_A F^A[\phi] - b_A c^\alpha \delta_\alpha F^A[\phi] = \delta_B \left(i b_A F^A[\phi] \right)}$$

BRST sym

$\Rightarrow$ Noether current j_B

$$= \left((j_B)_z , (j_B)_{\bar{z}} \right).$$

$\Rightarrow$ conserved charge. $Q_B = \oint \frac{dz}{2\pi i} (j_B)_z - \oint \frac{d\bar{z}}{2\pi i} (j_B)_{\bar{z}}$

(BRST charge)

classically, $\underline{\delta_B} = i \{ Q_B , \cdot \}_{\text{Poisson}}$.

QM: Q_B is a (Hermitian) quantum operator

$\delta_B^2 = 0 \rightsquigarrow Q_B^2 = 0$.

$|\Psi\rangle \in \mathcal{H}_{\phi, bc}$.

$\underline{\text{Def}^n}$ $|\Psi\rangle$ is "physical"

if $Q_B |\Psi\rangle = 0$.

in path $\int$ language,

$|\Psi\rangle \longleftrightarrow$ wave functional $\Psi[\phi]$

$$\mathcal{O} = Q_B \Psi[\phi]$$
$$= -i\,\delta_B \Psi[\phi]$$

* transition amplitudes between "physical" states are independent of the choice of $F^A[\phi]$

e.g. $F^A \longrightarrow F^A + \underset{\sim}{\delta' F^A}$

$$\delta' \langle \Psi_f | U_{fi} | \Psi_i \rangle$$

$$= \int D\phi \;\ldots\ldots \Big|_{\phi_i}^{\phi_f}$$

$S[\phi] + S(\phi, F^A)$

$$\Psi_f^*[\phi_f]\, e^{-S[\phi] + \delta_B(b_A F^A)}\, \Psi_i[\phi_i]$$

$$\times\ \delta_B(b_A \delta' F^A)$$

$$\delta_B \Psi_i = \delta_B \Psi_f = 0$$ by assumption

$$\int D\phi \cdots\ \delta_B(\ \cdots\ ) = 0$$

Furthermore,

if $|\Psi\rangle$ is physical, i.e. $Q_B|\Psi\rangle = 0$.

$$|\Psi'\rangle = |\Psi\rangle + Q_B|\chi\rangle .$$

any.

$$Q_B|\Psi'\rangle = 0 \Leftarrow Q_B^2 = 0.$$

$|\Psi\rangle$, $|\Psi'\rangle$ same overlap

$|\Psi\rangle$, $|\Psi'\rangle$ same overlap

w/ any other physical state

$$\langle\tilde{\Psi}|\Psi'\rangle = \langle\tilde{\Psi}|\Psi\rangle + \cancel{\langle\tilde{\Psi}|Q_B|\chi\rangle}^{\,0}.$$

$|\Psi\rangle \sim |\Psi'\rangle$ represents the same phys. state

$$\mathcal{H}_{\text{physical}} \equiv \frac{\{\Psi\in\mathcal{H},\ Q_B|\Psi\rangle=0\}}{\sim}$$

"BRST cohomology"

WS CFT

$$S = S_P[g_{ab}, X^\mu]$$

$$+ S_{bc}[g_{ab}, b_{ab}, c^a]$$

$$\cancel{+ i \int B^{ab}(g_{ab} - \hat{g}_{ab})}$$

BRST variation

Q_B $.X^\mu$ $\rightarrow$

$$\delta_B X^\mu = i c^a \partial_a X^\mu$$

Q_B $.b$ $\rightarrow$

$$\underline{\delta_B b_{ab}} = i \underline{T_{ab}}$$

by E.o.M. wrt δg_{ab} stress-energy tensor of $X^\mu \oplus bc$.

Q_B $.c$ $\rightarrow$

$$\delta_B c^a = i c^b \nabla_b c^a$$

$(z, \bar{z})$ $\qquad b_{ab}, \; c^a$ $\qquad$ central charge $= -26$

weight $(h, \tilde{h}) = (2, 0)$

$$b(z) \equiv b_{zz}(z) \qquad c^z(z) \equiv c(z) \quad (-1, 0)$$

$$\tilde{b}(\bar{z}) \equiv b_{\bar{z}\bar{z}}(\bar{z}) \qquad c^{\bar{z}}(\bar{z}) \equiv \tilde{c}(\bar{z})$$

$(0, 2)$ $\qquad\qquad (0, -1)$

OPE

$$b(z)\, c(0) \sim \frac{1}{z}$$

$$c(z)\, b(0) \sim \frac{1}{z}$$

$$\left[\begin{aligned} b(z)\, c(0) &= \frac{1}{z} + :b(z)c(0): \\ &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^n}{n!} :\partial^n b \cdot c: \end{aligned}\right]$$

$$\boxed{T^{gh} = -:\partial b\, c: - 2:b\,\partial c:}$$

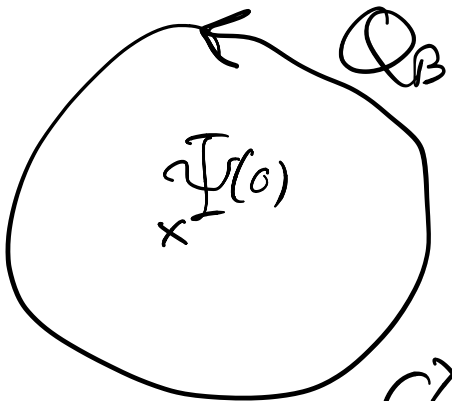
BRST current

$$(j_B)_z(z) \equiv j_B(z)$$

$$(j_B)_{\bar z}(\bar z) \equiv \tilde{j}_B(\bar z).$$

$$\underbrace{\bar\partial (j_B)_z} + \underbrace{\partial (j_B)_{\bar z}} = 0.$$

$$Q_B = \oint \frac{dz}{2\pi i}\, j_B(z) - \oint \frac{d\bar{z}}{2\pi i}\, \tilde{j}_B(\bar{z})$$



$$T^X = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu:$$

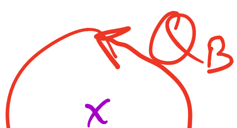
X^μ-CFT stress tensor

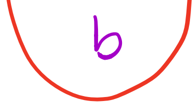
$$\underset{(1,0)\ \text{primary}}{j_B(z)} = \underset{(-1,0)}{c}\,\underset{(2,0)}{T^X} + :\overset{2}{b}\,\overset{-1}{c}\,\overset{1}{\partial}\overset{-1}{c}: + \frac{3}{2}\partial^2 c$$

(1,0) primary ← (only if total central charge $=0$.)

$$\oint j_B(z)\, X^\mu(0) \sim \oint c(z)\, \underbrace{T^X(z)\, X^\mu(0)}_{\frac{1}{z}\partial X^\mu(0)}$$

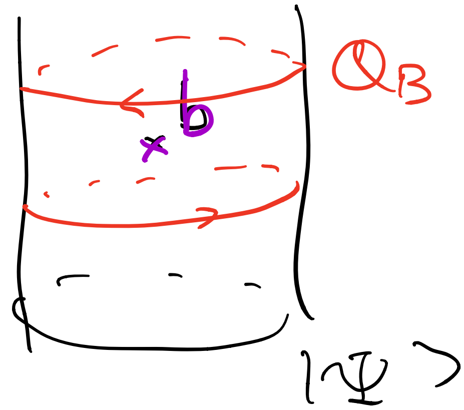
$$= \#\cdot\ c(0)\,\partial X^\mu(0)$$





$$Q_B \cdot b = T \quad (= T^X + T^{gh})$$

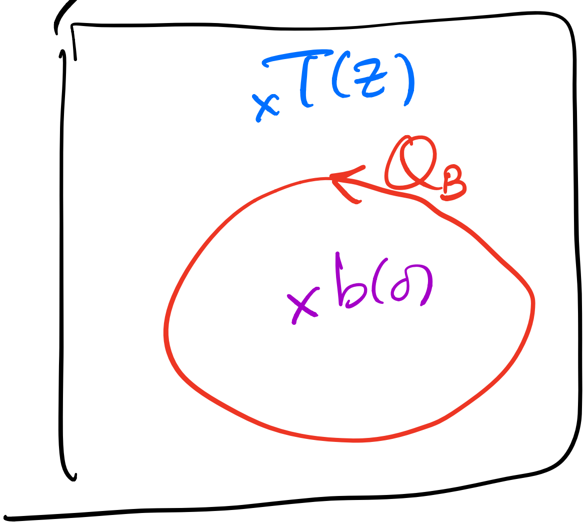
"$\{Q_B, b\}$"



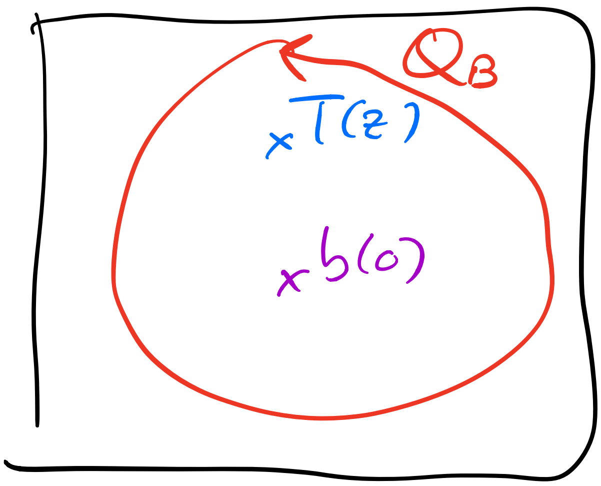
$$T(z)\,T(0) = T(z)\, Q_B \cdot b(0)$$
$$= Q_B \cdot \big(T(z)\, b(0)\big)$$

$\boxed{Q_B^2 = 0,}$

$Q_B \cdot T = 0.$



$\simeq$



$$T(z)\, b(0) \sim \frac{2}{z^2}\, b(0) + \frac{1}{z}\, \partial b(0)$$

no $\frac{1}{z^4}$ term! $\Rightarrow$ total central charge $= 0$.