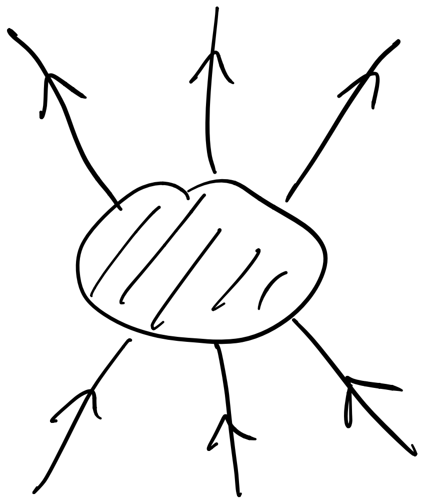
string S-matrix



out-states

$|k_1, k_2, \ldots\rangle^{out}$

in-states

$|k_1', k_2', \ldots\rangle^{in}$

in-states

$$|\{V_i^{in}(k_i)\}_{i=1,\ldots,m}\rangle$$

$$\cdots e^{ik_i\cdot X}$$

$$k_i^2 = -m_i^2$$

$$(k_i^0, \vec{k}_i).$$

$$k_i^0 > 0$$

out-states

$$\left| \{V_j^{out}(k_j)\}_{j=1,\dots,n} \right\rangle$$

$$k_j^0 < 0$$

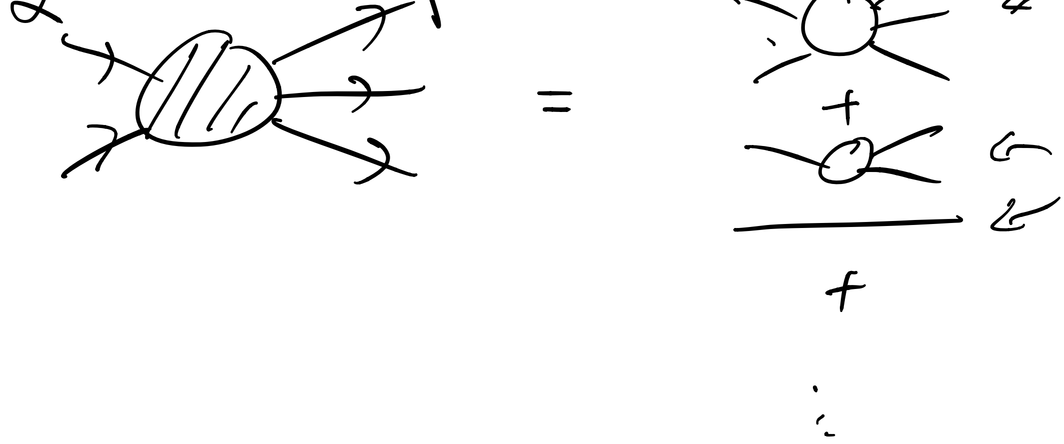
"$S|out\rangle = |in\rangle$"

$$\langle \{V_j^{out}(k_j)\}_{m+1 \le j \le n} | \{V_i^{in}(-k_i)\}_{1 \le i \le m} \rangle$$

$$= \langle \{V_j^{out}(k_j)\} | S | \{V_i^{out}(-k_i)\} \rangle$$

$$\sum_{i=1}^{n} k_i^\mu = 0$$

$$\langle \beta | S | \alpha \rangle = \sum_{\{\alpha_I, \beta_I\}} \prod_I \langle \beta_I | S^{conn} | \alpha_I \rangle$$



$$\left\langle \{V_j(k_j)\}_{m+1\le j\le n} \right| S^{\text{conn}} \left| \{V_i(k_i)\}_{1\le i\le m} \right\rangle$$

"amplitude"

$$\equiv \mathcal{A}[V_1(k_1), \dots, V_n(k_n)]$$

$$= \sum_{g=0}^{\infty} \mathcal{A}_g[\cdots]$$

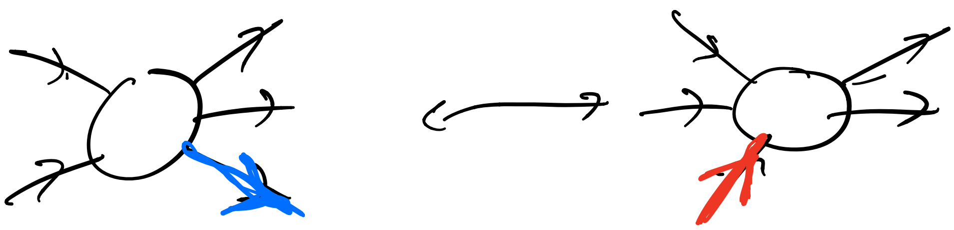
$$i(2\pi)^D \delta^D\left(\sum k_i\right) \cdot \hat{\mathcal{A}}[V_1(k_1), \dots, V_n(k_n)]$$

"reduced amplitude"
(connected)

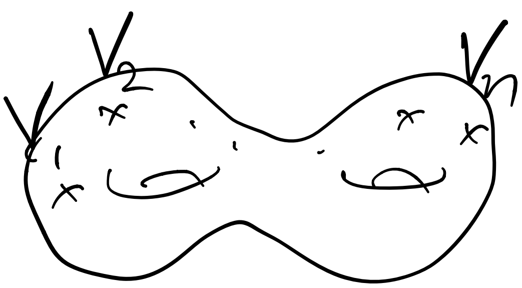
$\hat{\mathcal{A}}$ – analytic function
in momenta

(on-shell, subject to $\sum k_i^\mu = 0$)

- Crossing symmetry
  - w/ suitable analytic continuation, we can exchange momenta of in- vs out- particles.



$$V_i^{out}(k_i) = \overline{V_i^{in}(-k_i)}$$



$\longrightarrow \mathcal{A}_g$

$$\mathcal{A}_g[V_1, \dots, V_n]$$

$$= g_s^{2g-2} \int_{\mathcal{M}_g} \prod_k dt^k$$

$\mu_k{}^{ab} = \dfrac{\partial \hat{g}_{ab}}{\partial t^k}$

$\frac{1}{4\pi}(b, \mu_k)$

$$\times \Big\langle \prod_k \frac{1}{4\pi}\int d^2\sigma \sqrt{\hat{g}}\, \cdot b^{ab}\frac{\partial \hat{g}_{ab}}{\partial t^k} \cdot \prod_{(i,a)\in f} c^a(\hat{\sigma}_i)$$

?

$$\int \prod_{(i,a)\notin f} d\sigma_i^a \cdot \prod_{i=1}^{n} \sqrt{\hat{g}(\sigma_i)} \cdot V_i(\sigma_i) \Big\rangle_{\Sigma, \hat{g}(t)}$$

BRST-inv ✓

BRST invariance ?

$$\int d^2\sigma \sqrt{g(\sigma)}\, V_i(\sigma)$$

primary scalar operator of weight (1,1)

$$Q_B \cdot V_i(\sigma) = \left[\oint \frac{dz}{2\pi i} j_B(z) + c.c. \right] V_i(\sigma)$$

$c(z)T^X(z)$

$\frac{h^{=1}}{z^2} V_i(0) + \frac{1}{z}\partial V_i(0)$

$\partial c(0) V_i(0) + c(0)\,\partial V_i(0)$

$$= \partial(c V_i(0)) + \bar\partial(\tilde c V_i(0))$$

$$= \partial_a(c^a V_i(0))$$

$c\tilde c V_i$

if V_i is X^μ-CFT primary $(1,1)$

Q_B-closed.

(OCQ)

$$Q_B\cdot(b,\mu_k) = (T,\mu_k)$$

$$\equiv \int d^2\sigma \sqrt{\hat{g}} \cdot T^{ab} \cdot \frac{\partial \hat{g}_{ab}(t)}{\partial t^k}$$

$$\left\langle Q_B \cdot (b, \mu_k) \cdots \right\rangle_{\Sigma, \hat{g}(t)}$$

$$= \int d^2\sigma \sqrt{\hat{g}} \cdot \frac{\partial \hat{g}_{ab}}{\partial t^k} \left\langle T^{ab} \cdots \right\rangle_{\Sigma, \hat{g}(t)}$$

$$-\frac{4\pi}{\sqrt{\hat{g}}} \frac{\delta \langle \cdots \rangle}{\delta \hat{g}_{ab}}$$

$$= -4\pi \frac{\partial}{\partial t^k} \langle \cdots \rangle_{\Sigma, \hat{g}(t)}$$

$\mathcal{A}_g[V_1, \ldots, V_n]$ is BRST-invt

(up to possible bdry term on $\mathcal{M}_g$)

$$S = \mathbb{1} + iT$$

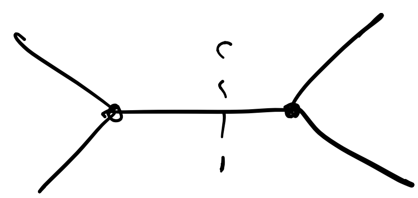
$$SS^\dagger = \mathbb{1} = S^\dagger S$$

$$\Rightarrow \boxed{T - T^\dagger = i T T^\dagger}$$

specialize to tree level.

$$T_{\text{tree}} \left(= -i A_{g=0}\,,\right)$$

real, away from poles



pole $\rightsquigarrow$ nonzero $T - T^\dagger$

$$i T T^\dagger = i T \sum_n |n\rangle\langle n| T^\dagger$$

$$\supset i \int \frac{d^D p}{(2\pi)^{D-1}} \delta(p^2 + M^2)\, T |p\rangle\langle p| T^\dagger$$

$$\boxed{\langle p | p' \rangle = \frac{(2\pi)^{D-1}}{2p^0} \delta^{D-1}(\vec{p} - \vec{p}\,')}$$

$$\frac{1}{p^2+M^2-i\epsilon} - \frac{1}{p^2+M^2+i\epsilon} = 2\pi i\,\delta(p^2+M^2)$$

near $p^2=-M^2$

$$\boxed{\begin{aligned}&\hat{A}_0[V_1(k_1),\dots,V_n(k_n)]\\ &\sim \frac{1}{p^2+M^2-i\epsilon}\,\hat{A}_0[V_1(k_1),\dots,V_m(k_m),\overline{V(p)}]\\ &\qquad\cdot\hat{A}_0[V(p),V_{m+1}(k_{m+1}),\dots V_n(k_n)]\end{aligned}}$$

(tree-level unitarity)

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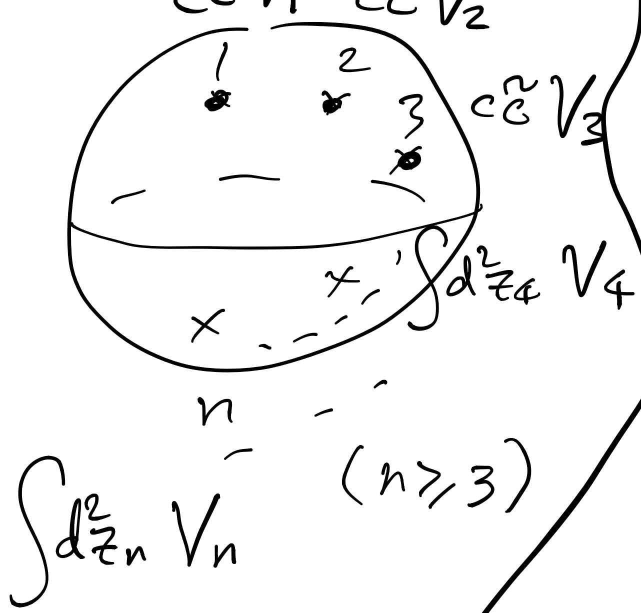
tree-level n-tachyon amplitude

$$A_0(k_1,\dots,k_n) = \int d^2z_4\cdots d^2z_n$$

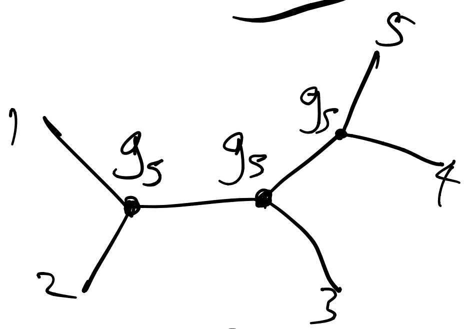


$$\left\langle \prod_{i=1}^{3} c\tilde{c}\, V_i(z_i,\bar{z}_i) \times \prod_{j=4}^{n} V_j(z_j,\bar{z}_j) \right\rangle$$

$$V_i(z,\bar{z}) = g_s\, e^{ik_i\cdot X(z,\bar{z})}$$

$k_i^0 > 0$ in-state

$k_i^0 < 0$, out-state



g_s^{n-2} $\qquad g_s^{-2}$

$$A_0 = g_s^n \cdot \int d^2z_4 \cdots d^2z_n$$

$$\times\, C_{S^2} \cdot \prod_{1\leq a<b\leq 3} |z_{ab}|^2 \cdot i(2\pi)^{26}\delta^{26}\Big(\sum k_i^\mu\Big)$$

$$\times \prod_{1\leq i<j\leq n} |z_{ij}|^{\alpha' k_i\cdot k_j}$$

normalization associated w/ sphere topology

actually

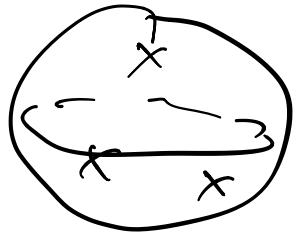
$C \quad k_i^2 = -m_T^2 \left(= \frac{4}{\ldots}\right)$

$$\sum k_i^\mu = 0.$$

actually independent of z_1, z_2, z_3!

$$\underbrace{z_1, z_2, z_3}, \; z_4, \ldots, z_n$$

e.g. $n=3$



$$\prod_{1\le i<j\le 3} |z_{ij}|^{\underbrace{2+\alpha' k_i\cdot k_j}_{=0}}$$

$$k_1\cdot k_2 = \frac{(k_1+k_2)^2 - k_1^2 - k_2^2}{2}$$

$$= \frac{k_3^2 - k_1^2 - k_2^2}{2} = \frac{m_T^2}{2}$$

$$= -\frac{2}{\alpha'}$$

3-tachyon amplitude

$$\mathcal{A}_O(k_1, k_2, k_3) = i(2\pi)^{26}\,\delta^{26}(k_1+k_2+k_3)$$

$$\times\; g_s^3 \cdot C_{S^2}$$

4-tachyon amplitude

$$\mathcal{A}_0(k_1,\dots,k_4) = i\,(2\pi)^{26}\,\delta^{26}(k_1+\dots+k_4)$$

$$\times\, g_s^4 \times C_{S^2} \cdot F(s,t,u)$$

$$\parallel$$

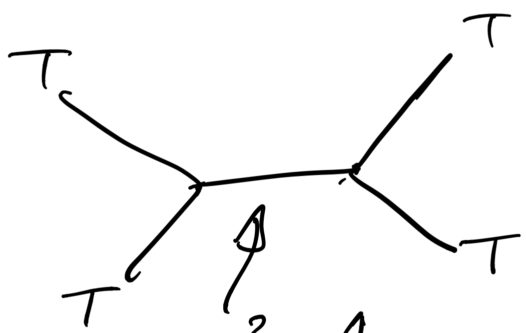
$$\int d^2z \cdots$$

will see that F has poles

at $s \equiv -(k_1+k_2)^2$

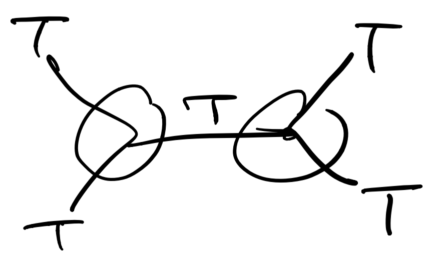
$= \frac{4}{\alpha'} n,$

$n = -1, 0, 1, \dots$



$m^2 = \frac{4}{\alpha'} \cdot n$

$\supset$



will determine C_{S^2}