2D CFT

digression: OPE in QFT

Poincaré inv't QFT
(apply to rel dim CFT on worldsheet)

T_{ab}, $\partial_a T^a{}_b = 0$.

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local $\overset{\text{(field)}}{\text{operator}}$ $\mathcal{O}(x)$

$\langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \rangle$ correlation function

$\Bigg(\quad \mathcal{O}_i(\overset{\leftarrow \text{spacetime}}{x}) \quad \underset{(\vec{x}, t)}{\overset{\shortparallel}{}}$

$\langle 0 | \mathcal{O}_{i_1}(x_1) \cdots\cdots \mathcal{O}_{i_n}(x_n) | 0 \rangle$

microcausality

$$\Big\{ \quad [\mathcal{O}_i(x), \mathcal{O}_j(y)] = 0,$$

$$(x-y)^2 = (\vec{x}-\vec{y})^2 - (x^0 - y^0)^2$$

$> 0.$

analytic continuation

$$x_i \equiv (\vec{x}_i, t_i)$$

$$t_i \to i\tau_i$$

$$\langle \mathcal{O}^E_{i_1}(x^E_1) \cdots \mathcal{O}^E_{i_n}(x^E_n) \rangle$$

$$x^E_i \equiv (\vec{x}_i, \tau_i) \in \mathbb{R}^d$$

$$\langle 0 | \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) | 0 \rangle$$

$$= \sum_\alpha \langle 0 | \mathcal{O}_1(x_1) \cdots \mathcal{O}_{n-1}(x_{n-1}) | \alpha \rangle \underbrace{\langle \alpha | \mathcal{O}_n(x_n) | 0 \rangle}_{e^{-iP_\alpha \cdot x_n} \langle \alpha | \mathcal{O}_n(0) | 0 \rangle}$$

$$P_\alpha = (E_\alpha, \vec{P}_\alpha)$$

bounded from below.

$$e^{-iP_\alpha \cdot x_n} = e^{-i\vec{P}_\alpha \cdot \vec{x}_n + iE_\alpha x^0_n}.$$

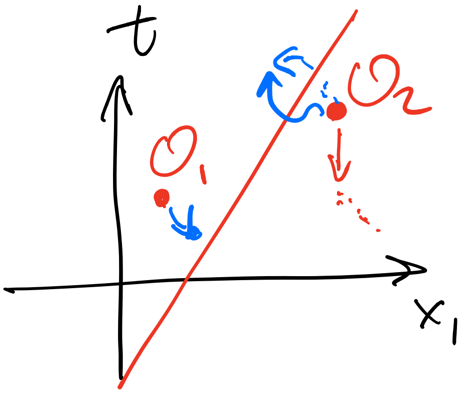
$$x^0_n \to i\tau_n.$$

$$\tau_n > 0.$$

$$\langle \mathcal{O}^E(x_1^E)\, \mathcal{O}^E(x_2^E) \rangle$$

$$\sum_\alpha \langle 0| \mathcal{O}(\vec{x}_1,t_1) |\alpha\rangle\langle\alpha| \mathcal{O}(\vec{x}_2,t_2)|0\rangle$$

$$= \sum_\alpha \langle 0|\mathcal{O}(0)|\alpha\rangle\langle\alpha|\mathcal{O}(0)|0\rangle$$



$$\cdot\ \underbrace{e^{-iP_\alpha\cdot(X_2-X_1)}}$$

$$e^{-i\vec{P}_\alpha\cdot(\vec{x}_2-\vec{x}_1)+iE_\alpha\cdot(t_2-t_1)}$$

$$t_i = i\tau_i$$

$$e^{-E_\alpha(\tau_2-\tau_1)}$$

$$\tau_2 > \tau_1$$

Operator Product Expansion (OPE)

$\mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)$

$$\Big\langle \mathcal{O}_i(x_1)\;\underset{x_3}{\cdot}\;\mathcal{O}_j(x_2)\;\cdots\;\cdot\,\mathcal{O}'(x')\;\cdots \Big\rangle$$

$$\Big\langle \sum_k C_{ij}^{\;k}(x_1,x_2,x_3)\,\mathcal{O}_k(x_3)\;\cdots\;\cdot\,\mathcal{O}'(x')\;\cdots \Big\rangle$$

<u>conformal symmetry</u>.

$T_{ab}(x)$. $\qquad \partial_a T^a{}_b = 0$

T_{ab}

$$J_a{}^{bc} \equiv x^b T_a{}^c - x^c T_a{}^b.$$

$$\partial^a J_a{}^{bc} = 0$$

$T_{ab} \to$ sym, traceless

$T_{ab} \to$ trace

$$T^a{}_a = \sum_I \beta^I \mathcal{O}_I$$

"beta function"

A conformal field theory (CFT)

is a local Poincaré-inv't QFT

whose stress-energy tensor is traceless. $T^a{}_a = 0$.

more conserved currents!

$$\to \quad \partial_a\left(x^b T^a{}_b\right) = \delta_a{}^b T^a{}_b = 0$$

$$\partial_a\left(T^{ab}x^2 - 2T^a{}_c x^b x^c\right)$$

$$= T^{ab}\cancel{2x_a} - 2\cancel{T^a{}_c \delta_a{}^b x^c}$$

$$- 2\underbrace{T^a{}_c x^b \delta_a{}^c}_{=0}$$

$$= 0$$

$$\partial_a J^a = 0, \quad Q = \int_{\mathbb{R}^{d-1}} d\vec{x}\; J^0$$

conserved charge.

$$J_a(x) = T_{ab}(x)\, v^b(x)$$

vector field

$v^a(x)\partial_a$.

generates diffeomorphism

$$x^a \mapsto x'^a = x^a + \epsilon v^a(x).$$

∂_a, $\quad x_a\partial_b - x_b\partial_a$

translation $\quad$ rotation/boosts.

$x^a\partial_a$, $\quad x^2\partial_a - 2x_a x^b\partial_b$

dilatation $\quad$ "special conformal transformation" generate conformal sym algebra.

$\simeq so(d,2)$

$\Big(\overset{\cup}{so(d-1,1)}$ Lorentz group $\Big)$

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specialize to 2D.

(τ, σ)
$\shortparallel$
(σ^0, σ^1)

$\sigma^\pm = \sigma \pm \tau$.

$\sigma^1 \pm \sigma^0$

$\downarrow$ Wick

$z \equiv \sigma^1 + i\sigma^2$

$\bar{z} \equiv \sigma^1 - i\sigma^2$.

T_{++}

T_{--}

$T_{+-} = T_{-+} = 0$ in CFT.

$$\partial_- T_{++} + \cancel{\partial_+ T_{-+}} = 0$$

$$\partial_+ T_{--} + \cancel{\partial_- T_{+-}} = 0.$$

$j_a \leftarrow (j_+ , j_-) = 0.$

$$\partial_- \left(v(\sigma^+) T_{++} \right) = 0.$$

$$\partial_+ \left(\tilde{v}(\sigma^-) T_{--} \right) = 0$$

$(j_+ = 0, j_-)$

$T(z)$

$\parallel$

$$T_{++}(\sigma^+) \rightsquigarrow T_{zz}(z)$$

$$T_{--}(\sigma^-) \rightsquigarrow T_{\bar z \bar z}(\bar z)$$

$\partial \equiv \partial_{\bar z}$

$\partial_{\bar z} T(z) = 0.$ "operator equation" $\qquad \tilde{T}(\bar z)$

$$\langle \bar\partial T(z)\, \mathcal{O}_1(z_1,\bar z_1) \cdots \mathcal{O}_n(z_n,\bar z_n)\rangle = 0$$

provided that z does not coincide w/ the z_i's

$$\langle T(z)\, \mathcal{O}_1(z_1,\bar z_1) \cdots \mathcal{O}_n(z_n,\bar z_n)\rangle$$

is a meromorphic fn in z, possibly w/ poles at $z = z_i$

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state/operator correspondence

$$S[X] \xrightarrow{\text{covariantize}} S[g_{ab}, X]$$

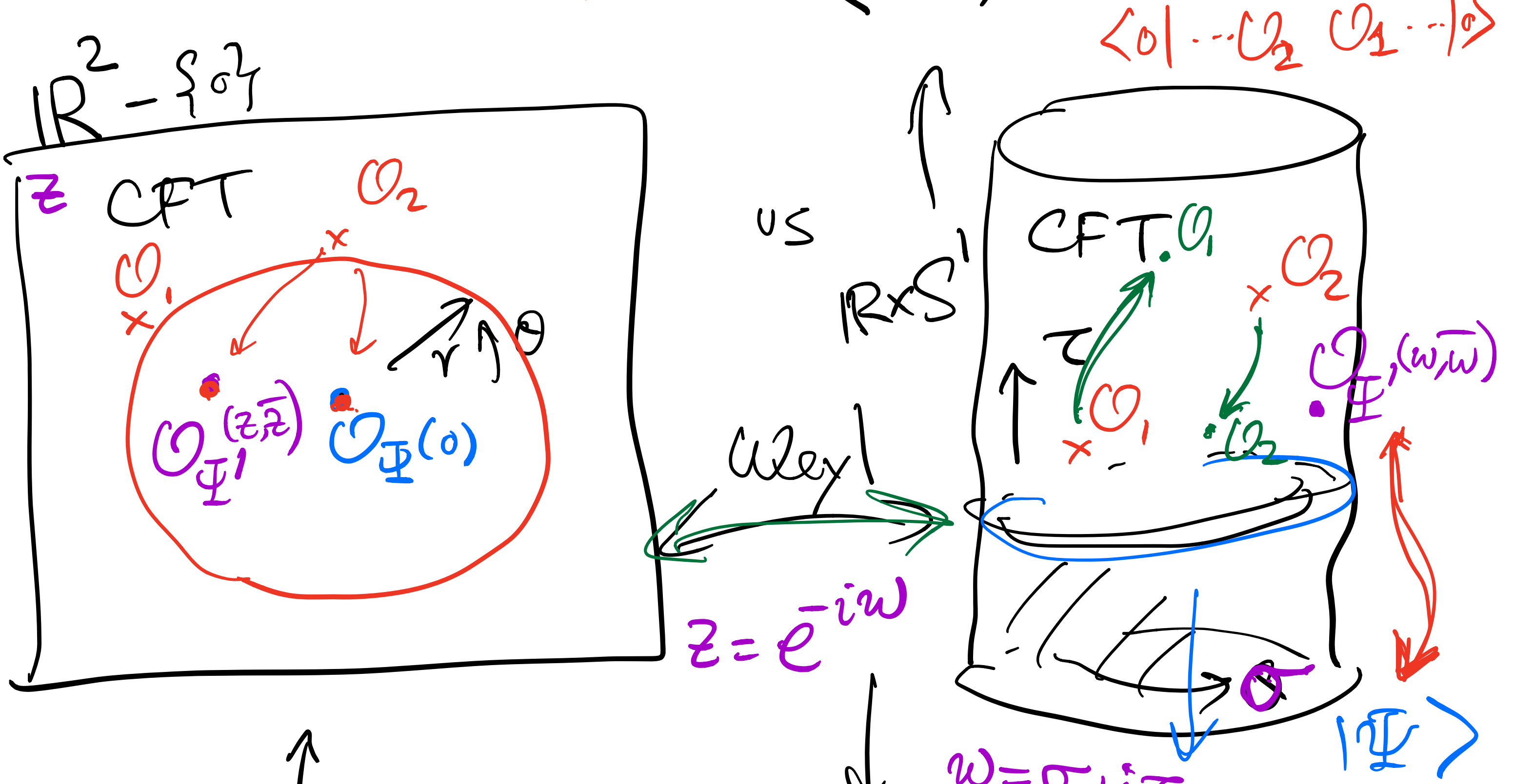
$$T_{ab} = \frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{ab}}$$

$T^a{}_a = 0,$ $\qquad g_{ab} \frac{\delta S}{\delta g_{ab}} = 0,$

S is invt under $g_{ab} \to e^{2\delta\omega} g_{ab}$.

(Weyl transf)

$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \rangle$



$\sim w+2\pi$

$$ds^2 = dr^2 + r^2 d\theta^2$$

$$= r^2\left(\left(\frac{dr}{r}\right)^2 + \theta^2\right) = e^{2\tau}\left(d\tau^2 + d\theta^2\right)$$

$$r \equiv e^{\tau}$$

State/operator:

there is a 1-1 correspondence b/t local operators at a point and states of the CFT on S^1