AlphaGeometry solution to the Olympiad problem 3, 2005

For more information, please read our paper in Nature: "Solving olympiad geometry without human demonstrations" https://doi.org/10.1038/s41586-023-06747-5

Step 1. B, C, M are collinear and AH is perpendicular to BC ⇒ AH is perpendicular to CM.

***Neural Language Model*: construct point D as the midpoint of BH.**

Step 2. B, H, D are collinear and AC is perpendicular to BH ⇒ AC is perpendicular to HD.

Step 3. AC is perpendicular to HD and AH is perpendicular to CM ⇒ ∠HAC = ∠(CM, HD) and ∠AHD = ∠MCA.

Step 4. A, F, H are collinear, B, C, F are collinear, B, C, M are collinear, B, H, D are collinear and ∠HAC = ∠(CM, HD) ⇒ ∠CAF = ∠HBF.

Step 5. A, F, H are collinear, B, C, F are collinear and AH is perpendicular to BC ⇒ ∠AFC = ∠BFH.

Step 6. ∠CAF = ∠HBF and ∠AFC = ∠BFH ⇒ FA/FB = FC/FH.

Step 7. A, F, H are collinear, B, C, F are collinear and AH is perpendicular to BC ⇒ ∠AFB = ∠CFH.

Step 8. ∠AFB = ∠CFH and FA/FB = FC/FH ⇒ ∠BAF = ∠HCF and ∠ABF = ∠CHF.

Step 9. BM = CM and BO = CO ⇒ BC is perpendicular to MO.

Step 10. B, C, M are collinear and BM = CM ⇒ M is the midpoint of BC.

***Neural Language Model*: construct point G as the midpoint of CH.**

Step 11. C, H, G are collinear and CG = HG ⇒ G is the midpoint of CH.

Step 12. M is the midpoint of BC and G is the midpoint of CH ⇒ BH is parallel to MG.

Step 13. B, H, D are collinear and BD = HD ⇒ D is the midpoint of BH.

Step 14. M is the midpoint of BC and D is the midpoint of BH ⇒ CH is parallel to MD.

Step 15. HO2 = O2Q and KO2 = O2Q ⇒ O2 is the circumcenter of HKQ.

Step 16. AQ is perpendicular to HQ and HK is perpendicular to KQ ⇒ ∠AQH = ∠QKH.

Step 17. O2 is the circumcenter of HKQ and ∠AQH = ∠QKH ⇒ AQ is perpendicular to O2Q.

Step 18. HO2 = O2Q, AQ is perpendicular to HQ and AQ is perpendicular to O2Q ⇒ O2 is the midpoint of HQ.

Step 19. O2 is the midpoint of HQ and G is the midpoint of CH ⇒ CQ is parallel to O2G.

Step 20. O2 is the midpoint of HQ and D is the midpoint of BH ⇒ BQ is parallel to O2D.

Step 21. AO = BO, AO = OQ and BO = CO ⇒ A, B, C, Q are cyclic.

Step 22. A, B, C, Q are cyclic ⇒ ∠ABQ = ∠ACQ and ∠ACB = ∠AQB.

Step 23. A, F, H are collinear, B, C, F are collinear, B, C, M are collinear, B, H, D are collinear, ∠BAF = ∠HCF, ∠HAC = ∠(CM, HD), BH is parallel to MG and CH is parallel to MD ⇒ ∠(AB, MD) = ∠(AC, MG).

Step 24. ∠ABQ = ∠ACQ, BQ is parallel to O2D and CQ is parallel to O2G ⇒ ∠(AB, O2D) = ∠(AC, O2G).

Step 25. ∠(AB, MD) = ∠(AC, MG) and ∠(AB, O2D) = ∠(AC, O2G) ⇒ ∠GMD = ∠GO2D.

AB - MD = AC - MG

AB - O2D = AC - O2G

=> O2D - MD = O2G - MG

=> O2D - O2G = MD - MG

=> D O2 G = D M G

Step 26. ∠GMD = ∠GO2D ⇒ M, O2, G, D are cyclic.

Step 27. A, F, H are collinear, B, C, F are collinear and AH is perpendicular to BC ⇒ BF is perpendicular to FH.

Step 28. D is the midpoint of BH and BF is perpendicular to FH ⇒ BD = FD.

Step 29. BD = FD ⇒ ∠BFD = ∠DBF.

Step 30. G is the midpoint of CH and D is the midpoint of BH ⇒ BC is parallel to GD.

Step 31. B, C, F are collinear, B, C, M are collinear, B, H, D are collinear, ∠BFD = ∠DBF, BC is parallel to GD and BH is parallel to MG ⇒ ∠MFD = ∠MGD.

Step 32. ∠MFD = ∠MGD ⇒ F, M, G, D are cyclic.

Step 33. F, M, G, D are cyclic and M, O2, G, D are cyclic ⇒ F, M, O2, G are cyclic.

Step 34. F, M, O2, G are cyclic and F, M, G, D are cyclic ⇒ F, O2, G, D are cyclic.

Step 35. F, O2, G, D are cyclic ⇒ ∠O2FG = ∠O2DG.

Step 36. AC is perpendicular to HD and AQ is perpendicular to HQ ⇒ ∠CAQ = ∠DHQ and ∠(AC, HQ) = ∠(HD, AQ).

Step 37. B, H, D are collinear, ∠CAQ = ∠DHQ, BH is parallel to MG, AQ is perpendicular to HQ and AQ is perpendicular to O2Q ⇒ ∠(AC, MG) = ∠(AQ, HO2).

Step 38. B, C, M are collinear and ∠ACB = ∠AQB ⇒ ∠ACM = ∠AQB.

Step 39. ∠(AC, MG) = ∠(AQ, HO2) and ∠ACM = ∠AQB ⇒ ∠(BQ, HO2) = ∠CMG.

Step 40. AO = BO and BO = CO ⇒ O is the circumcenter of ABC.

Step 41. O is the circumcenter of ABC and M is the midpoint of BC ⇒ ∠(AB, MO) = ∠ACO and ∠BAC = ∠MOC.

Step 42. AO = BO and BO = CO ⇒ AO = CO.

Step 43. AO = CO ⇒ ∠ACO = ∠OAC.

Step 44. B, H, D are collinear, ∠HAC = ∠(CM, HD) and BH is parallel to MG ⇒ ∠(AC, MG) = ∠(AH, CM).

Step 45. $\angle(AB, MO) = \angle ACO$, $\angle ACO = \angle OAC$, AH is perpendicular to BC and BC is perpendicular to MO $\Rightarrow \angle BAH = \angle OAC$.

Step 46. $\angle BAH = \angle OAC$ and $\angle(AC, MG) = \angle(AH, CM) \Rightarrow \angle(AB, CM) = \angle(AO, MG)$.

Step 47. A, F, H are collinear, B, C, F are collinear and AH is perpendicular to BC $\Rightarrow$ CF is perpendicular to FH.

Step 48. G is the midpoint of CH and CF is perpendicular to FH $\Rightarrow$ FG = HG.

Step 49. FG = HG $\Rightarrow \angle FHG = \angle GFH$.

Step 50. B, C, M are collinear, $\angle(BQ, HO2) = \angle CMG$, $\angle O2FG = \angle O2DG$, BC is parallel to GD and BQ is parallel to O2D $\Rightarrow \angle O2FG = \angle(HO2, MG)$.

Step 51. A, F, H are collinear, B, C, F are collinear, B, C, M are collinear, C, H, G are collinear, $\angle(AB, CM) = \angle(AO, MG)$, $\angle ABF = \angle CHF$ and $\angle FHG = \angle GFH \Rightarrow \angle(AH, FG) = \angle(AO, MG)$.

Step 52. $\angle(AH, FG) = \angle(AO, MG)$ and $\angle O2FG = \angle(HO2, MG) \Rightarrow \angle(AH, FO2) = \angle(AO, HO2)$.

Step 53. B, C, M are collinear, B, H, D are collinear, $\angle ACB = \angle AQB$, $\angle AHD = \angle MCA$ and BH is parallel to MG $\Rightarrow \angle(AH, MG) = \angle BQA$.

Step 54. B, H, D are collinear, $\angle(AC, HQ) = \angle(HD, AQ)$, BH is parallel to MG, AQ is perpendicular to HQ and AQ is perpendicular to O2Q $\Rightarrow \angle(AC, MG) = \angle(HO2, AQ)$.

Step 55. $\angle(AC, MG) = \angle(HO2, AQ)$ and $\angle(AH, MG) = \angle BQA \Rightarrow \angle HAC = \angle(BQ, HO2)$.

Step 56. AO = BO, AO = OQ and BO = CO $\Rightarrow$ O is the circumcenter of BCQ.

Step 57. O is the circumcenter of BCQ and M is the midpoint of BC $\Rightarrow \angle(BQ, MO) = \angle QCO$.

Step 58. $\angle HAC = \angle(BQ, HO2)$, $\angle(BQ, MO) = \angle QCO$, CQ is parallel to O2G, AH is perpendicular to BC and BC is perpendicular to MO $\Rightarrow \angle(AC, HO2) = \angle(CO, O2G)$.

Step 59. ∠BAC = ∠MOC, AH is perpendicular to BC and BC is perpendicular to MO ⇒ ∠BAC = ∠(AH, CO).

Step 60. ∠BAC = ∠(AH, CO) and ∠(AC, HO2) = ∠(CO, O2G) ⇒ ∠(AB, HO2) = ∠(AH, O2G).

Step 61. A, F, H are collinear, B, C, F are collinear, B, C, M are collinear, ∠BAF = ∠HCF and CH is parallel to MD ⇒ ∠(AB, MD) = ∠(AH, CM).

Step 62. ∠(AB, HO2) = ∠(AH, O2G) and ∠(AB, MD) = ∠(AH, CM) ⇒ ∠(CM, O2G) = ∠(MD, HO2).

Step 63. M, O2, G, D are cyclic ⇒ ∠O2MD = ∠O2GD.

Step 64. B, C, M are collinear, ∠(CM, O2G) = ∠(MD, HO2), ∠O2MD = ∠O2GD and BC is parallel to GD ⇒ ∠(HO2, MD) = ∠O2MD.

Step 65. ∠(HO2, MD) = ∠O2MD ⇒ HO2 is parallel to MO2.

Step 66. HO2 is parallel to MO2 ⇒ H, M, O2 are collinear.

Step 67. AH is perpendicular to CM and AQ is perpendicular to HQ ⇒ ∠HAQ = ∠(CM, HQ).

Step 68. A, F, H are collinear, B, C, F are collinear, B, C, M are collinear, H, M, O2 are collinear, ∠HAQ = ∠(CM, HQ), AQ is perpendicular to HQ and AQ is perpendicular to O2Q ⇒ ∠AFM = ∠AQM.

Step 69. ∠AFM = ∠AQM ⇒ A, F, M, Q are cyclic.

Step 70. A, F, M, Q are cyclic ⇒ ∠AFQ = ∠AMQ.

Step 71. ∠(AH, FO2) = ∠(AO, HO2), AH is perpendicular to BC, AQ is perpendicular to HQ, AQ is perpendicular to O2Q and BC is perpendicular to MO ⇒ ∠AOM = ∠QO2F.

Step 72. A, F, H are collinear, H, M, O2 are collinear, ∠AFQ = ∠AMQ, AH is perpendicular to BC, AQ is perpendicular to HQ, AQ is perpendicular to O2Q and BC is perpendicular to MO ⇒ ∠AMO = ∠O2QF.

Step 73. ∠AMO = ∠O2QF and ∠AOM = ∠QO2F ⇒ OA/OM = O2F/O2Q.

Step 74. F, M, O2, G are cyclic ⇒ ∠MFO2 = ∠MGO2.

Step 75. AO = KO and AO = OQ ⇒ KO = OQ.

Step 76. KO = OQ and KO2 = O2Q ⇒ O2O is the bisector of ∠KO2Q.

Step 77. KO = OQ and KO2 = O2Q ⇒ KQ is perpendicular to OO2.

Step 78. AO = BO, AO = KO and BO = CO ⇒ A, B, C, K are cyclic.

Step 79. A, B, C, K are cyclic and A, B, C, Q are cyclic ⇒ A, C, K, Q are cyclic.

Step 80. A, C, K, Q are cyclic ⇒ ∠ACK = ∠AQK and ∠CAK = ∠CQK.

Step 81. AC is perpendicular to HD and HK is perpendicular to KQ ⇒ ∠(AC, HK) = ∠(HD, KQ).

Step 82. ∠ACK = ∠AQK, O2O is the bisector of ∠KO2Q, AQ is perpendicular to O2Q and KQ is perpendicular to OO2 ⇒ ∠ACK = ∠OO2K.

Step 83. B, H, D are collinear, ∠(AC, HK) = ∠(HD, KQ), BH is parallel to MG, HK is perpendicular to KQ and KQ is perpendicular to OO2 ⇒ ∠(AC, MG) = ∠(OO2, KQ).

Step 84. ∠(AC, MG) = ∠(OO2, KQ) and ∠ACK = ∠OO2K ⇒ ∠(CK, MG) = ∠O2KQ.

Step 85. A, B, C, K are cyclic ⇒ ∠ACB = ∠AKB.

Step 86. AO = BO, AO = KO and BO = CO ⇒ O is the circumcenter of BCK.

Step 87. O is the circumcenter of BCK and M is the midpoint of BC ⇒ ∠KBO = ∠(CK, MO).

Step 88. AO = BO and AO = KO ⇒ BO = KO.

Step 89. BO = KO ⇒ ∠BKO = ∠OBK.

Step 90. B, C, M are collinear, B, H, D are collinear, ∠ACB = ∠AKB, ∠AHD = ∠MCA and BH is parallel to MG ⇒ ∠(AH, MG) = ∠BKA.

Step 91. ∠KBO = ∠(CK, MO), ∠BKO = ∠OBK, AH is perpendicular to BC and BC is perpendicular to MO ⇒ ∠(AH, CK) = ∠BKO.

Step 92. ∠(AH, CK) = ∠BKO and ∠(AH, MG) = ∠BKA ⇒ ∠AKO = ∠(MG, CK).

Step 93. ∠(CK, MG) = ∠O2KQ and ∠AKO = ∠(MG, CK) ⇒ ∠AKO = ∠QKO2.

Step 94. ∠CAK = ∠CQK and CQ is parallel to O2G ⇒ ∠CAK = ∠(O2G, KQ).

Step 95. ∠CAK = ∠(O2G, KQ) and ∠AKO = ∠QKO2 ⇒ ∠(AC, KO) = ∠GO2K.

Step 96. B, C, F are collinear, B, C, M are collinear, B, H, D are collinear, ∠HAC = ∠(CM, HD), ∠MFO2 = ∠MGO2 and BH is parallel to MG ⇒ ∠CAH = ∠GO2F.

Step 97. ∠CAH = ∠GO2F and ∠(AC, KO) = ∠GO2K ⇒ ∠(AH, KO) = ∠FO2K.

Step 98. AO = KO, KO2 = O2Q and OA/OM = O2F/O2Q ⇒ O2F/O2K = OK/OM.

Step 99. ∠(AH, KO) = ∠FO2K, AH is perpendicular to BC and BC is perpendicular to MO ⇒ ∠FO2K = ∠MOK.

Step 100. ∠FO2K = ∠MOK and O2F/O2K = OK/OM ⇒ ∠FKO2 = ∠OMK.

***Neural Language Model*: construct point E as the midpoint of KM.**

Step 101. KO1= MO1 and KE = ME ⇒ KM is perpendicular to O_2E.

Step 102. FO1= KO1and KO1= MO1⇒ O1is the circumcenter of FKM.

Step 103. K, M, E are collinear and KE = ME ⇒ E is the midpoint of KM.

Step 104. O_1 is the circumcenter of FKM and E is the midpoint of KM ⇒ ∠KFM = ∠KO_2E.

Step 105. B, C, M are collinear, K, M, E are collinear, AH is perpendicular to BC and KM is perpendicular to O_2E ⇒ ∠(AH, CM) = ∠KEO_2.

Step 106. B, C, F are collinear, B, C, M are collinear and $\angle KFM = \angle KO_2E \Rightarrow \angle(CM, FK) = \angle EO_2K$.

Step 107. $\angle(AH, CM) = \angle KEO1$and $\angle(CM, FK) = \angle EO_2K \Rightarrow \angle(AH, FK) = \angle EKO_2$.

Step 108. K, M, E are collinear, $\angle(AH, FK) = \angle EKO_2$, $\angle FKO2 = \angle OMK$, AH is perpendicular to BC and BC is perpendicular to MO $\Rightarrow \angle FKO2 = \angle FKO_2$.

Step 109. $\angle FKO2 = \angle FKO1 \Rightarrow$ KO2 is parallel to KO_2.

Step 110. KO2 is parallel to KO1$\Rightarrow$ K, O2, O1are collinear