Physics 285a Problem Set 3

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This is a longish homework. These problems are, however, clustered about only two basic questions. You can use ChatGPT for any of the problems (in fact, I would encourage it if the problem threatens to take long) – but you are responsible for the answers!

Problem 1. Power Broadening: As shown in lecture, for a two-state atom in a laser field, the probability P_+ of finding the atom in the upper state $|+\rangle$ is

$$P_+ = \frac{\Omega_R^2}{\Omega^2} \sin^2\left(\frac{\Omega t}{2}\right)$$

where $\Omega_R = \mu E_0/\hbar$ is the (angular) Rabi frequency, and $\Omega = \left(\epsilon^2 + \Omega_R^2\right)^{1/2}$ is the generalized Rabi frequency.

(a) Suppose the upper state is weakly damped by spontaneous emission with decay rate γ. Compute the time-average probability $\overline{P_+}$ of finding the atom in the upper state to obtain a power broadened lineshape. *Hint: In order to "sneak" in a small decay, rewrite the eoms found by replacing the usual Hamiltonian,* $H = \begin{pmatrix} 0 & \frac{\Omega_R}{2}e^{-i\epsilon t} \\ \frac{\Omega_R^*}{2}e^{i\epsilon t} & 0 \end{pmatrix}$, *with a non-Hermitian one adding a decay to the diagonal excited state term.*

(b) How does the inclusion of spontaneous emission change the transition linewidth? For simplicity, define the saturation parameter $S=\Omega_R^2/\gamma^2$. Plot $\overline{P_+}$ for saturation parameters of 0.1, 1, and 10.

(c) For a quantized EM field,

$$\gamma = \frac{\omega^3 \mu^2}{3\pi\epsilon_0\hbar c^3}$$

where $\mu = e\langle r\rangle$ is the electric dipole matrix element. Furthermore, the driving field can be expressed as an intenity $I = c\epsilon_0 E_0^2/2$. Using these relations, show that the saturation parameter can be written as I/I_{sat} and obtain an expression for the saturation intensity I_{sat}.

(d) Consider the $3^2S_{1/2}$-$3^2P_{3/2}$ transition in ^{23}Na, also known as the D_2 line. Look up the wavelength and decay time of this transition. What is the saturation intensity? Assume we use a laser beam with a 1 mm diameter spot size. What laser power is necessary to achieve S=1? For this laser power, what is the Rabi frequency?

Problem 2. Transit-Time Broadening: Another phenomenon that alters the transition linewidth is transit-time broadening, which arises when the interaction time between the atom and radiation is small compared to the decay time of the atom. In this problem, it is sufficient to obtain the correct functional forms of the responses i.e. you are not expected to obtain the correct constant prefactors.

(a) Consider an atom traveling through a radiation field with a rectangular field profile. The atom enters the field at time $t = 0$ and exits at time T. For simplicity, consider the atom to be an undamped oscillator $p(t) = p_0 \cos(\omega_0 t)$ that oscillates with constant amplitude between times $t = 0$ and $t = T$. What is the amplitude frequency spectrum $A(\omega)$ of the oscillator's motion (i.e. the Fourier transform of $p(t)$)? What is the power spectrum $|A(\omega)|^2$ and what is the FWHM of its central peak?

(b) In a real lab, lasers have a Gaussan profile rather than a rectangular one. We can write the laser's field as

$$E = E_0 e^{-r^2/a^2} \cos(\omega t)$$

where $E_0 e^{-r^2/a^2}$ is the transverse electric field profile, $r^2 = x^2 + y^2$, and $2a$ is the $1/e$ diameter of the laser field. For this type of field distribution, assume again the linear response of the oscillator (i.e. the oscillation 'tracks' the field strength) and compute the power spectrum (the Fourier transform) of the oscillator's behavior. What is the FWHM? You only need to consider molecules moving with speed v perpendicular to the laser beam and through the center of the Gaussian distribution.

(c) Rotational-vibrational states of molecular iodine $^{127}I_2$ are popular choices for optical frequency standards because of the narrow natural linewidths of these transitions. For instance, at λ=532 nm, there are several transitions with linewidths of 10 kHz. Consider a vapor cell of iodine at room temperature. What is the most probable velocity of the molecules? What beam diameter is needed so that the transit-time broadening is less than the natural linewidth?

Problem 3. Doppler and Recoil Shifts

(a) When an atom with an optical transition $\hbar\omega_0$ absorbs a photon of energy $\hbar\omega$, it also receives a momentum kick. What is the energy of the atom after the absorption? What is the difference in the kinetic energy of the atom before and after the absorption? For simplicity, assume that the problem is one-dimensional, and you can do a classical calculation here. The answer may depend on the initial speed v of the atom and its mass m but NOT on ω_0.

(b) In your answer to (a) you will find a term proportional to kv. This is the first-order Doppler effect. Find yourself a nice type of atom at room temperature with a transistion of your choice (ideally one for which lasers are typically used) using your knowledge, ChatGPT, your lab's spectroscopy manual... What is the velocity distribution of the atoms? What type of linewidth do the atoms now have? What is the FWHM?

(c) You should have also found a term that is inversely proportional to the atom's mass. This is the recoil shift. Estimate the magnitude of the recoil shift for your transition.

Problem 4. Which is largest? For the following experimental setups, which of the broadening mechanisms studied in this problem set is largest: power broadening, transit-time broadening, or Doppler broadening?

(a) Rotational-vibrational transitions at λ=1064 nm in CH_4 have a linewidth of 10 kHz. The molecules are at room temperature in a vapor cell and are illuminated by a 0.5 mW laser beam with a 6 cm diameter.

(b) A cold (77K) atomic beam of Rb atoms are excited by a diode laser whose radiation is directed perpendicular to the beam. The laser's 2 mW output excites the $5^2S_{1/2}$-$5^2P_{3/2}$ transition and is focused to a 0.5 mm diameter.

(c) A 500 mW dye laser at λ=240nm excites the 1S-2S transition in hydrogen via two-photon absorption. The atoms are cold (<4K) and trapped in a MOT. The entire sample (0.5 cm diameter) is illuminated by the laser. [Hint: What needs to happen for two-photon absorption to occur?]

Problem 5. Berry's phase: Understanding Berry's phase may help one better appreciate the adiabatic theorem. We will show how Berry's phase and the associated quantities can arise from the slow evolution limit to the time-dependent Schrodinger equation $H(t)\left|\Psi(t)\right\rangle = i\hbar\partial_t \left|\Psi(t)\right\rangle$. In this slow time dependence limit, the evolution is said to be *adiabatic*.

(a) We will take a snapshot of the evolution at time $\tau = \epsilon t$ and write down the instantaneous eigenvalues $\omega_n(\tau)$ and the associated eigenstates $|n(\tau)\rangle$. Note that it is the eigenkets, and not the quantum numbers, that depend on time. We will also restrict our attention to the non-degenerate case.

Subsitute the general solution $|\Psi\rangle = \sum c_n |n\rangle$ into the Schrodinger equation and show that

$$\dot{c}_n = -i\omega_n c_n - \epsilon \sum_m \langle n| \frac{d}{d\tau} |m\rangle c_m. \tag{1}$$

Show that the zeroth-order solution is $c_n(t) = e^{i\phi_n} c_n(0)$. Find the *dynamical* phase $\phi_n(t)$.

(b) The first-order solution can be obtained by making the change of variable $c_n(t) = e^{i\phi_n}\tilde{c}_n(t)$. Write down the equation of motion for $\tilde{c}_n(t)$. Note that everything we have written so far is exact and there is no higher order correction.

(c) Integrate the equation of motion and observe that any dynamical phase factor oscillates relatively rapidly. Make an appropriate approximation and show that $\tilde{c}_n(t) = e^{i\gamma(t)}\tilde{c}_n(0)$. Find $\gamma(t)$. This is called the *Berry's phase* (also the *geometric phase*).

(d) Now suppose the Hamiltonian depends on parameters $H = H(\mathbf{R})$, where $\mathbf{R} = \mathbf{R}(\tau)$ is a vector containing all parameters. We will adiabatically vary the parameters and come back to the original point and form a closed loop. Use Stokes' rule to show that

$$\gamma = \oint \mathbf{A}(\mathbf{R}) \cdot d\mathbf{R} = \int \mathbf{B} \cdot \mathbf{dS}, \tag{2}$$

where in second integral is performed over a closed surface. Express $\mathbf{A}$ in terms of the eigenstates $|n(\mathbf{R})\rangle$. Find $\mathbf{B}$ in terms of $\mathbf{A}$. This relation resembles that of the gauge field in classical electromagnetism, and $\mathbf{A}$ and $\mathbf{B}$ are called the *Berry connection* and *Berry curvature* respectively.

(e) Find any publication during the last 20 years that shows or uses the Berry phase in the context of AMO physics. In addition to providing us with the reference, give a very brief outline of what they try to do. (This part everybody should do by themselves! I don't want to see same papers by people who worked together. There are tons of papers out there....)