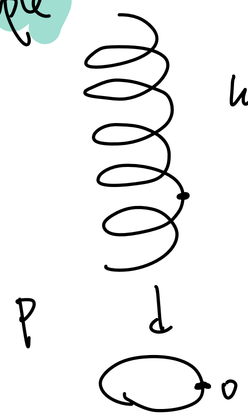
covering spaces.

Example



$\overset{\text{homeo}}{\cong} \mathbb{R} \xrightarrow{\quad} S^1$

$\leftarrow$ has properties.

① lift path
unique lift when source fixed

② $\pi_1(S^1) \leftrightsquigarrow p^{-1}(0)$
$\begin{array}{c} \cong \\ \mathbb{Z} \end{array}$

convention

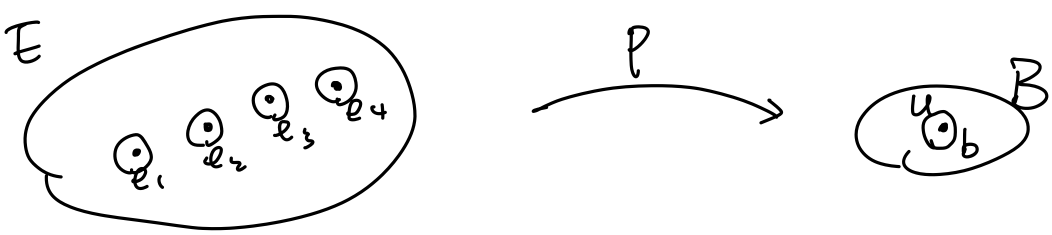
all spaces are path connected.

<u>Def</u> (covering)

$p: E \to B$ is **covering** if

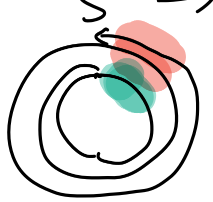
① p is surjective

②. $\forall\, b \in B$, $\exists$ neighborhood U of b, s.t.
each component of $p^{-1}(U)$ maps homeomorphically to U.



<u>Ex</u> : ① $\mathbb{R} \to S^1$

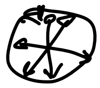
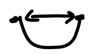
② $S^1 \to S^1$: $t \mapsto e^{2\pi i n t}$ $t \in [0,1]$
$n \in \mathbb{Z} \setminus \{0\}$

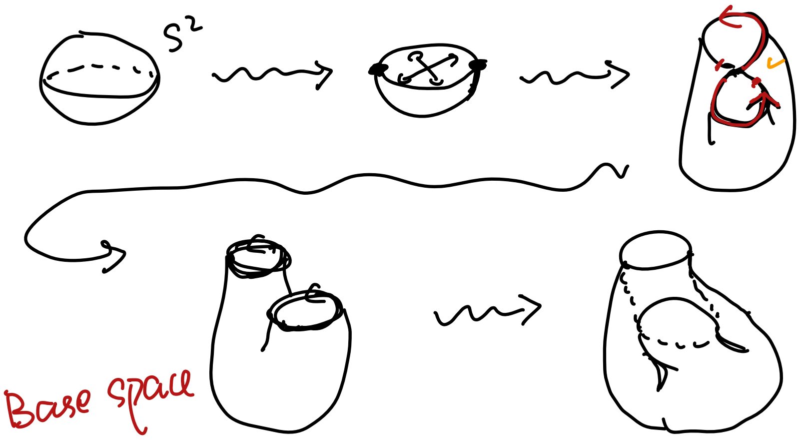


n-times.

③ all identity maps.

④ $S^n \xrightarrow[\text{antipodal pts.}]{\text{identifying}} \mathbb{RP}^n$ $S^n \rightsquigarrow$ glue antipodal points.

$\mathbb{RP}^1$: S^1  $\rightsquigarrow$  $\rightsquigarrow S^1$

$\mathbb{RP}^2$: 

total space ↓, Base space ↓

prop $p : E \longrightarrow B$ is covering. Fix $b \in B$, $e \in p^{-1}(b)$.

Then.

① $\forall f : I \to B$ with $f(0) = b$, $\exists !$ lift of f, denoted by $\tilde{f} : I \to E$ with $\tilde{f}(0) = e$.

$$\begin{tikzcd} & E \ni e \arrow[d] \\ I \arrow[ur] \arrow[r] & B \\ 0 \arrow[r, mapsto] & b \end{tikzcd}$$

② $\forall f \sim f' : I \to B$, then $\tilde{f} \sim \tilde{f}'$, here $\tilde{f}$ & $\tilde{f}'$ are lifts of f & f' with source e.

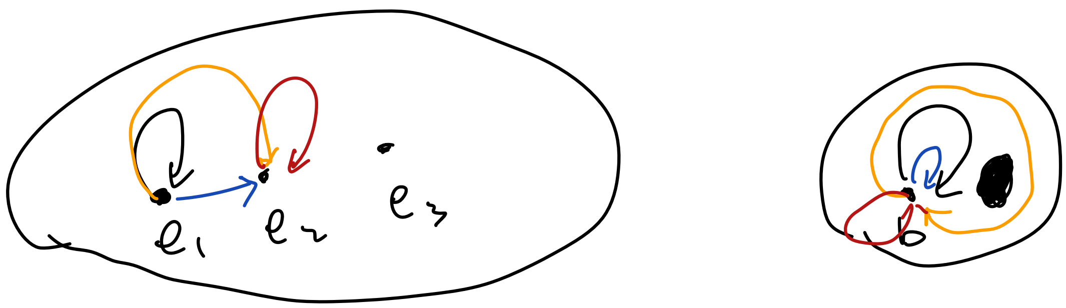
prop ① $p_* : \pi_1(E, e) \to \pi_1(B, b)$ is injective

② choose e & $e' \in p^{-1}(b)$

$p_*(\pi_1(E,e))$ & $p_*(\pi_1(E,e'))$

are conjugate subgrps of $\pi_1(B,b)$

③ when e runs through all points in $p^{-1}(b)$, $p_*(\pi_1(E,e))$ runs through all conjugate subgrps of $p_*(\pi_1(E,e))$ in $\pi_1(B,b)$

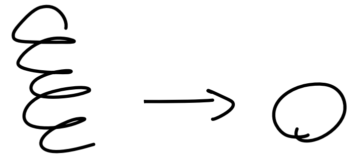


$$
\left.
\begin{array}{l}
p_*(\pi_1(E,e_1)) \subset \\
p_*(\pi_1(E,e_2)) \subset \\
p_*(\pi_1(E,e_3)) \subset
\end{array}
\right. \quad \pi_1(B,b)
$$

<u>Def</u>. $p: E \to B$ is a covering.

- We say p is regular if $p_*(\pi_1(E,e))$ is normal subgrp in $\pi_1(B,b)$.
- We say p is universal if $\pi_1(E,e)$ is trivial.

Ex (universal)

① $\mathbb{R}^1 \to S^1$ 

② $S^n \to \mathbb{R}P^n \quad n \geq 2$

Will see:

for a universal covering

$p^{-1}(b) \leftrightsquigarrow \pi_1(B, b)$.

(In Ex ①, $\mathbb{Z} \leftrightsquigarrow \pi_1(S^1)$)

(In Ex ②, $n \geq 2$, $\mathbb{Z}/2 \cong \pi_1(\mathbb{R}P^n)$)

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fundamental grpoids & coverings.

Defi $\mathcal{E}, \mathcal{B}$ are groupoids.

$p: \mathcal{E} \to \mathcal{B}$ is covering if

① p is surj on objects.

② $p: \{\text{morph with source } e\}$

$\longrightarrow \{\text{morph with source } p(e)\}$

is a bijection for $\forall e$.

Prop If $p: E \to B$ a covering of spaces, then $\Pi(p): \Pi(E) \to \Pi(B)$ is a covering of groupoids.

Recall $\pi_1(X) \simeq \Pi(X)$ equivalence of categories

Notation $\Pi(\mathcal{E}, e_0) := \mathcal{E}(e_0, e_0)$

↑ has group structure

Prop

① $p_*: \Pi(\mathcal{E}, e) \to \Pi(B, b)$ is injective

② choose e & $e' \in p^{-1}(b)$

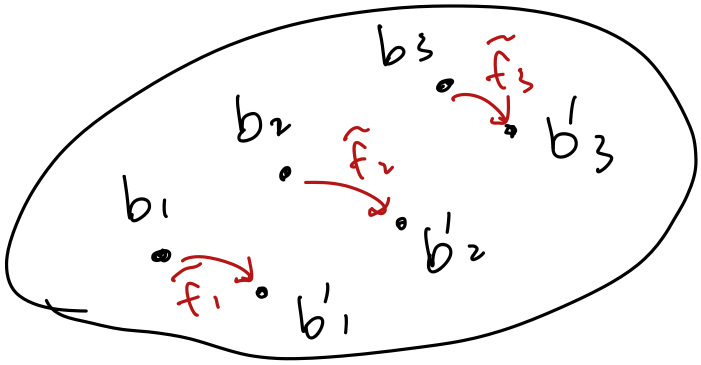
$p_*(\Pi(\mathcal{E}, e))$ & $p_*(\Pi(\mathcal{E}, e'))$

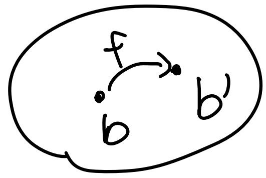
are conjugate subgrps of $\Pi(B, b)$

③ when e runs through all points in $p^{-1}(b)$, $p_*(\Pi(\mathcal{E}, e))$ runs through all conjugate subgrps of $p_*\Pi(\mathcal{E}, e)$. in $\Pi(B, b)$

<u>Notation</u> $F_b = p^{-1}(b)$ fiber over b

relation F_b & $F_{b'}$?





a path $f : b \to b'$

$F_b \cong F_{b'}$ gives $F_b \longrightarrow F_{b'}$

$b_i \longmapsto \tilde{f}_i(1)$

$b = b'$ $\quad f \in \pi(B, b)$ gives

$F_b \longrightarrow F_b \in \underbrace{\mathrm{Aut}(F_b)}_{\parallel}$

all isomorphisms $F_b \to F_b$

$$\rightsquigarrow \quad \overset{G.}{\overset{\shortparallel}{\pi(B,b)}} \longrightarrow \mathrm{Aut}(F_b)$$

a map of groups. (check)

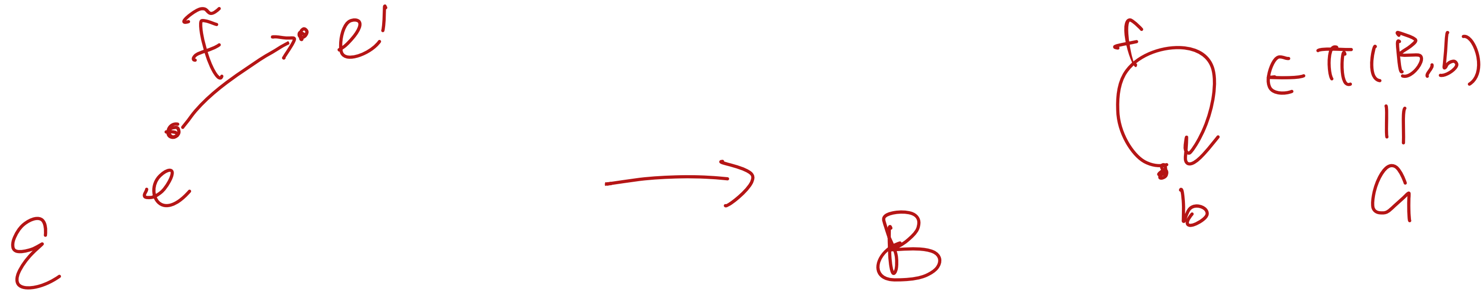
① G has an action on F_b

$G \times F_b \to F_b$

$g \quad e \longmapsto g \circ e = e'$

② G action on F_b is transitive.

$(\forall\, e, e' \in F_b, \ \exists\, g \in G$ s.t. $g \circ e = e')$



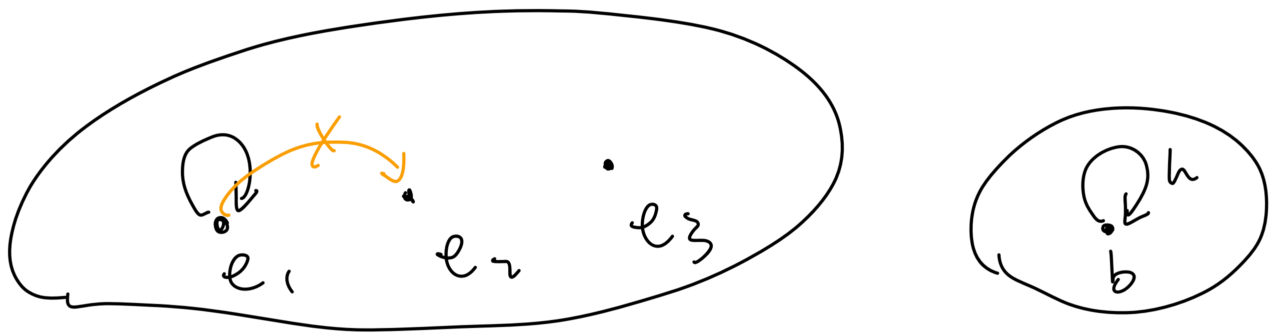
$\rightsquigarrow f \circ e = e'$

① + ② $\Longrightarrow$ $F_b = G/H$

for some subgrp H.

$G := \pi(B,b)$
$H = p_* \pi(\mathcal{E}, e)$.

$h \in H$ $\quad h \curvearrowright F_b$ as id.



$\Longrightarrow$ h lifts to a morph $e_1 \circlearrowleft$.

to an element $\pi(\mathcal{E}, e)$

$\Longrightarrow$ $h \in p_* \pi(\mathcal{E}, e)$

$\Longrightarrow$ $F_b = G/H = \pi(B,b) \,/\, p_* \pi(\mathcal{E}, e)$

When the covering is universal (i.e. $\pi(\mathcal{E}, e) = *$).

$$F_b = \pi(B, b) / e$$

($\pi_1 \mathbb{R}P^n = \mathbb{Z}/2$, $n \geq 2$).

- fix B, want to study all coverings $\mathcal{E} \to B$ & maps between them.

$$\begin{tikzcd} e\ \mathcal{E} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{E}'\ e' \arrow[dl, "p'"] \\ & B \\ & b \end{tikzcd}$$

Q: when does f exist? s.t. $f(e) = e'$

A: Th'm f $\exists$ if and only if

$$p_* \pi(\mathcal{E}, e) \subseteq p'_* \pi(\mathcal{E}', e').$$

Pf

$\Rightarrow$:

$$\begin{tikzcd} \pi(\mathcal{E}, e) \arrow[rr, "f_*"] \arrow[dr, "p_*"'] & & \pi(\mathcal{E}', e') \arrow[dl, "p'_*"] \\ & \pi(B, b) \end{tikzcd}$$

$p_* = p'_* \circ f_*$

$p_*(\pi(\mathcal{E}, e))$
$\quad ||$
$p'_* f_*(\pi(\mathcal{E}, e))$
$\subseteq p'_* \pi(\mathcal{E}', e')$.

$\Leftarrow$: want to construct f.

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{E}' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

e e' b

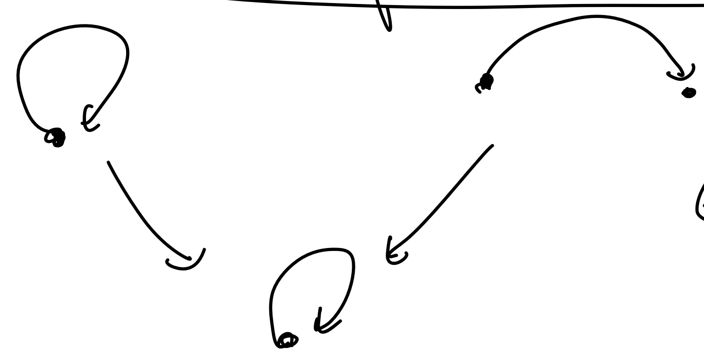
① → ② → ③

$$\begin{tikzcd} ① \; e \arrow[r, bend left] \arrow[dr, bend right, "p"'] & e_1 \; \in \mathcal{E} & ③ \; e' \arrow[r] \arrow[dl, "\text{unique path lifting}"] & e'_1 \; \in \mathcal{E}' \\ & ② \; b \arrow[r, bend left] & b_1 \; \in B & \end{tikzcd}$$

Let $f(e \to e_1) = e' \to e'_1$

need the assumption for well-definedness

Ex:



← don't want this.

The assumption ensures that

$\forall \circlearrowleft \in \pi(\mathcal{E}, e)$, $p_* \circlearrowleft \in \pi(B, b)$

lifts to a loop in $\pi(\mathcal{E}', e')$

$$\begin{array}{c} \mathcal{E} \\ p \downarrow \\ B \end{array}$$

When $p_*(\pi(\mathcal{E}, e)) = 0$.

i.e. $\mathcal{E} \to B$ universal.

$\forall$ $\begin{array}{c} \mathcal{E}' \\ \downarrow \\ B \end{array}$ covering, $\exists$ $\begin{tikzcd} \mathcal{E} \arrow[rr] \arrow[dr] & & \mathcal{E}' \arrow[dl] \\ & B & \end{tikzcd}$

(lemma) $\mathcal{E} \xrightarrow{f} \mathcal{E}'$ is also a covering

$\Longrightarrow$ universal covering covers all other coverings.

subgrp of G $\rightsquigarrow$ covering over $\pi(B, b)$.

$G = \pi(B,b)$

$p_*\pi(\mathcal{E}, e) \subseteq \pi(B, b)$ $\qquad \mathcal{E} \xrightarrow{p} B$

$$\begin{array}{c} p_*(\pi(\mathcal{E}, e)) \\ \downarrow \\ p_*(\pi(\mathcal{E}', e')) \end{array} \qquad \begin{tikzcd} \mathcal{E} \arrow[d] \arrow[dr] & \\ \mathcal{E}' \arrow[r] & B \end{tikzcd}$$

$\mathcal{O}_G$: orbit category of G

obj : G/H orbit of G.

morph : G-maps $G/H \xrightarrow{f} G/K$

$f(g \cdot kH) = g \circ f(kH)$

f is determined by $eH \mapsto rK$

$g \in H$.

$f(gH) = f(eH)$
$= grK = rK \Rightarrow r^{-1}gr \in K$
$\Rightarrow r^{-1}Hr \subset K$

$f(gH) = f(g(eH))$
$= g\, f(eH)$
$= grK$

<u>Th'm</u> There is a functor

$$\mathcal{E} : \mathcal{O}_G \longrightarrow \mathrm{Cov}(B, b).$$

$\mathcal{O}_G$ is the orbit cat of G

$G = \pi(B, b)$

$\mathrm{Cov}(B,b)$ the cat of coverings over B

obj : $\mathcal{E} \longrightarrow B$

morph :

$$\begin{tikzcd} \mathcal{E} \arrow[d] \arrow[r] & B \\ \mathcal{E}' \arrow[ur] & \end{tikzcd}$$

$G/H \rightsquigarrow \mathcal{E} \to B$.

s.t. $p_*\pi(\mathcal{E}, e) = H$

$G/H \to G/K \rightsquigarrow p_*(\pi(\mathcal{E}, e)) \subseteq p_*(\pi(\mathcal{E}', e'))$

iff $rHr^{-1} \subset K$

want to show

$\forall\, G/H \in \mathcal{O}_G$, can construct

$\mathcal{E} \to B$ s.t. $p_* \pi(\mathcal{E}, e) = H$.

①. when $H = \{e\}$. $\leftrightsquigarrow$ $F_b = G/e$.

$\mathcal{E}^{univ}$: ob : $\{$ all path with starting point b $\}$

morph : unique morphism.

$\mathcal{E} \xrightarrow{P} B$: $p(f) = f(1)$.

$F_b = p^{-1}(b)$ = path starting & ending at b

$= \pi(B, b) = G$.

② from universal to $F_b = G/H$.

$\mathcal{E}^{univ}/H$

$\mathcal{E}$ has an action of G.

Graph

Vertices + edges.

$\bullet\ \bullet = \partial I$

Def. A graph is.

1) X^0 a discrete set of pt.

2) a set J of maps. $\{j_r : S^0 \to X^0\}_{r \in J}$

$$X^0 \amalg I \times J \Big/ \begin{array}{l} \{0\} \times r = j_r(0) \\ \{1\} \times r = j_r(1) \end{array}$$