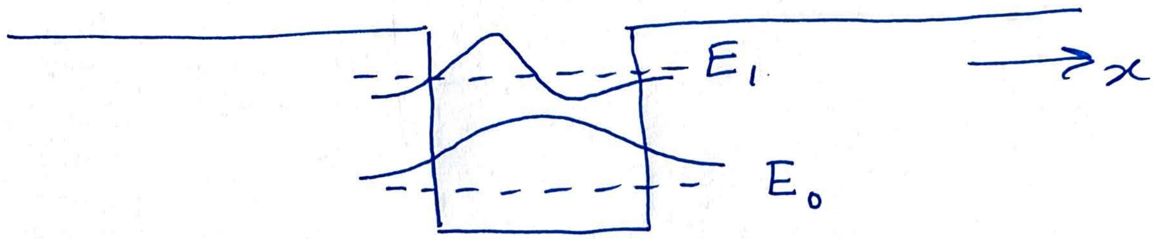
Optical Fibers and Photonic crystals

Analogy with bound state problems in quantum mechanics

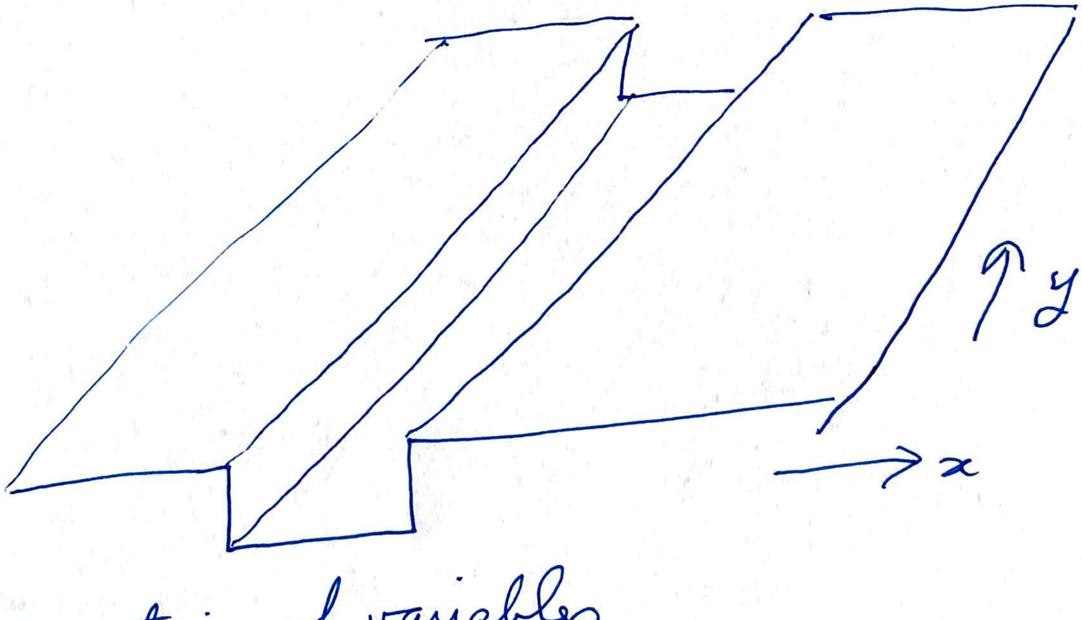
(i) Potential well



$V(x)$

Finite number of bound states E_n, $n = 0, 1, 2 \ldots$

(ii) Potential trench



Separation of variables

$$\psi(x, y) = e^{iky} \phi(x)$$

Time independent Schrodinger equation

$$\left[-\frac{\hbar^2\nabla^2}{2m} + V(x)\right]\psi = E\,\psi$$

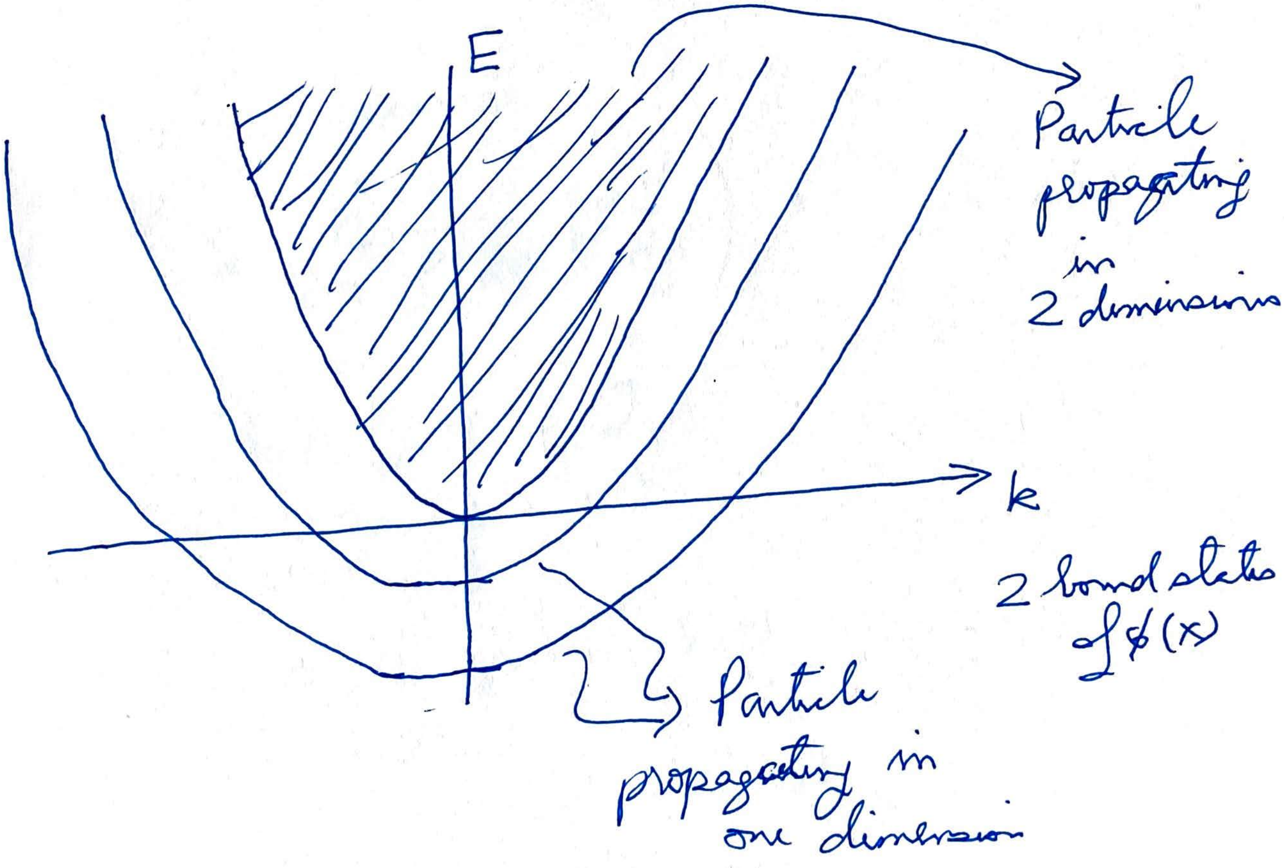
$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2 k^2}{2m} + V(x)\right]\phi(x) = E\,\phi(x)$$

Let $E = \frac{\hbar^2 k^2}{2m} + \epsilon$

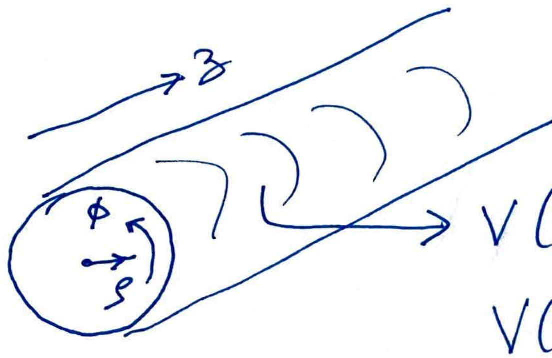
Then $\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\phi(x) = \epsilon\,\phi(x)$

is the one-dimensional problem we have already solved.

& $E_n(k) = \frac{\hbar^2 k^2}{2m} + \epsilon_n$



(iii) Cylindrical potential well



$V(\vec{r}) = -V_0$

$V(\vec{r}) = 0$ outside

Use cylindrical co-ordinates

$$V(\vec{r}) = V(\rho)$$

$$\psi(\vec{r}) = e^{i\bar{m}\phi} e^{ikz} f(\rho) \quad , \quad \bar{m} \text{ integer}$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

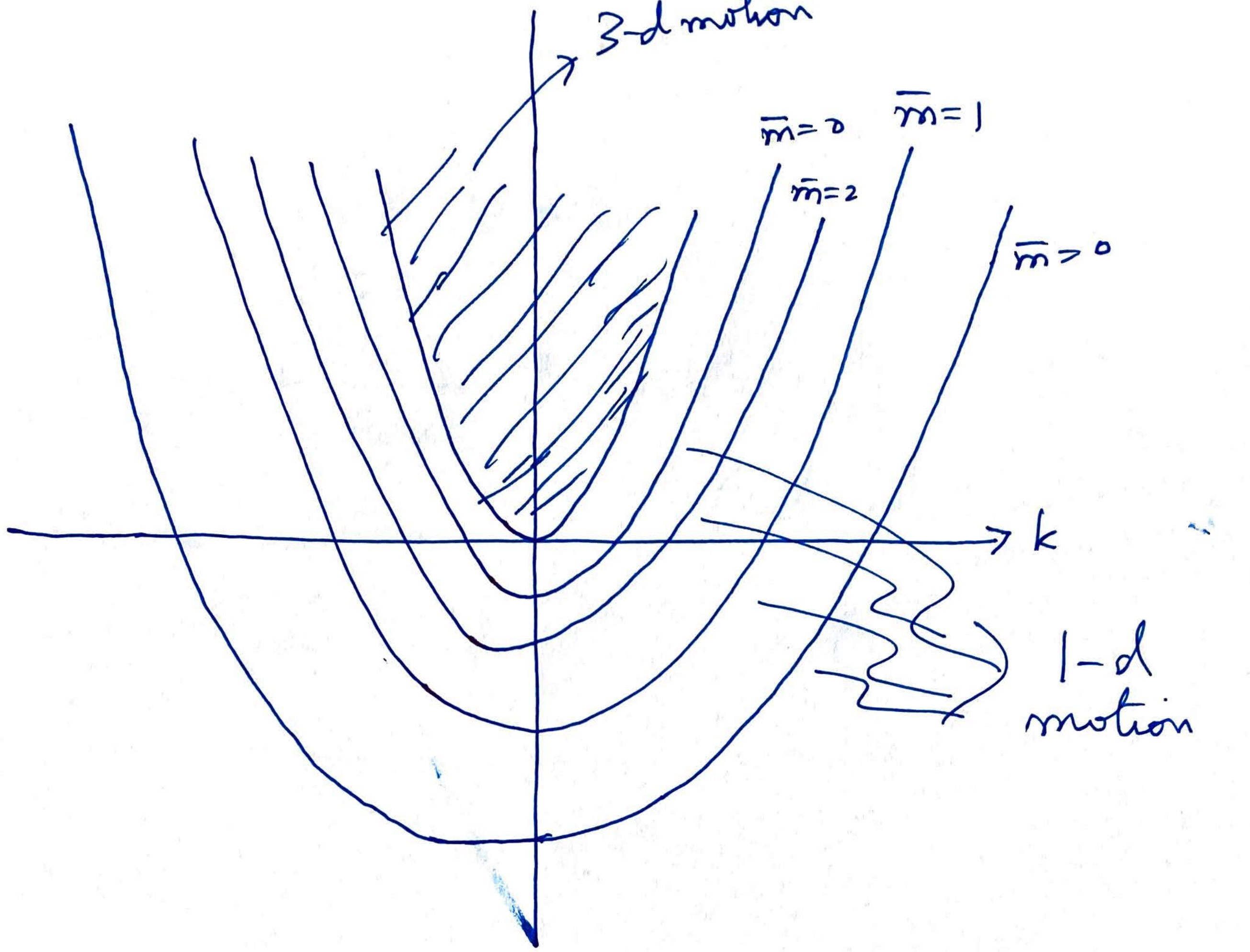
becomes

$$\left[\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2}{2m}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + \frac{\bar{m}^2}{\rho^2} + V(\rho)\right] f = E f$$

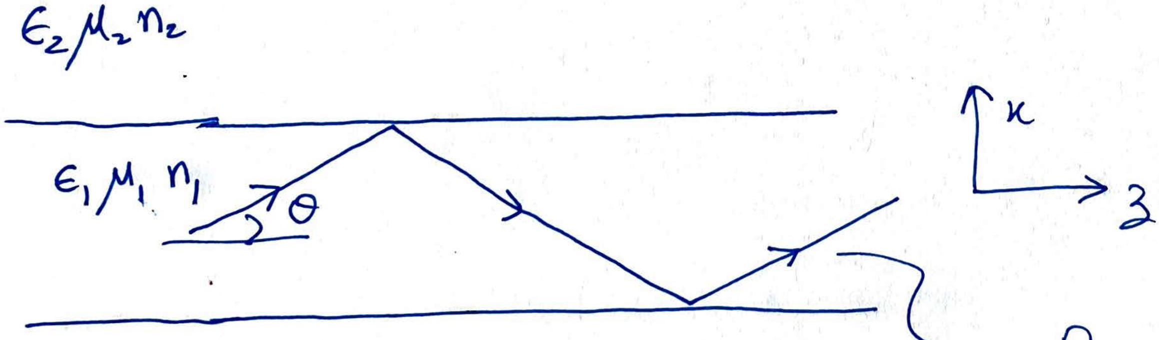
Define $f(\rho) = \dfrac{g(\rho)}{\sqrt{\rho}}$, $E = \epsilon + \hbar^2 k^2/2m$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{d\rho^2} + \frac{\bar{m}^2 - 1/4}{\rho^2} + V(\rho)\right] g(\rho) = \epsilon\, g(\rho)$$

Half-space one-d problem in potential $V(\rho) + \dfrac{\bar{m}^2 - 1/4}{\rho^2}$



Planar Dielectric Waveguide (Zangwill 19.5)



Ray picture

~~Bd~~ Light trapped by total internal reflection

Only possible for $\theta < \theta_{max}$

Analog of finite number of boundstates in wave picture

Choose $\vec{E} = \hat{y}\, E(x)\, e^{i(hz - \omega t)}$

(TE mode)

$$\vec{H} = \frac{1}{i\omega\mu} \vec{\nabla} \times \vec{E}$$

$$= \frac{1}{i\omega\mu} \left(-ih\hat{x} + \hat{z}\frac{d}{dx}\right) E(x) e^{i(hz-\omega t)}$$

(Note: $\vec{H}$ is not <u>purely transverse</u>).

Wave equation

$$\left(\nabla^2 + n^2\frac{\omega^2}{c^2}\right)\vec{E} = 0$$

is $$\boxed{\left[\frac{d^2}{dx^2} + \left(n^2\frac{\omega^2}{c^2} - h^2\right)\right] E(x) = 0.}$$

like 1-d Schroedinger equation.

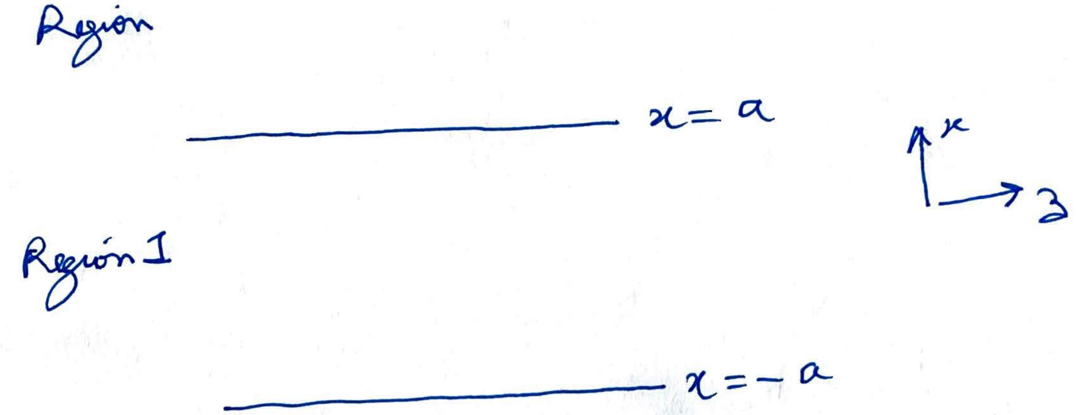
We want $\gamma_1^2 \equiv n_1^2 \dfrac{\omega^2}{c^2} - h^2 > 0$

$$-\gamma_2^2 \equiv n_2^2\frac{\omega^2}{c^2} - h^2 < 0$$

for a wave to be localized inside along the x direction

Requires $n_1 > n_2$ as expected.

Boundary Conditions



(i). $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ and $\vec{E} = \hat{y} E(x)\, e^{i(hz - \omega t)}$

implies $\boxed{\begin{aligned} E_1(a) &= E_2(a) \\ E_1(-a) &= E_2(-a) \end{aligned}}$ and

The $\vec{H}$ field is $\cancel{\vec{B}} = \cancel{\dfrac{hE}{\omega\mu}\hat{x} + \hat{z}}$

$$\vec{H} = \left(-\frac{ihE}{\omega\mu}\hat{x} + \frac{1}{i\omega\mu}\frac{dE}{dx}\hat{z} \right) e^{i(hz - \omega t)}$$

(ii) $\vec{\nabla} \cdot \vec{B} = 0$ implies $B_{1x}(a) = B_{2x}(a)$

and $B_{1x}(-a) = B_{2x}(-a)$.

Already satisfied by (i)

(iii) $\vec{\nabla} \times \vec{H} = \vec{J}_f + \dfrac{\partial \vec{D}}{\partial t}$ implies

$$H_{1z}(\pm a) = H_{2z}(\pm a)$$

or $\boxed{\left.\dfrac{1}{\mu_1}\dfrac{dE_1}{dx}\right|_{x=\pm a} = \left.\dfrac{1}{\mu_2}\dfrac{dE_2}{dx}\right|_{x=\pm a}}$

Even parity solution

$$E_1(x) = \begin{cases} E_i \cos(\gamma_1 x) & |x| < a \\ E_0 e^{-\gamma_2 |x|} & |x| > a \end{cases}$$

Boundary conditions lead to

$$E_i \cos(\gamma_1 a) = E_0 e^{-\gamma_2 a} \tag{1}$$

and $$-\frac{\gamma_1}{\mu_1} E_i \sin(\gamma_1 a) = -\frac{\gamma_2}{\mu_2} E_0 e^{-\gamma_2 a} \tag{2}$$

Dividing (2) by (1) yields

$$\boxed{\tan(\gamma_1 a) = \frac{\mu_1 \gamma_2}{\mu_2 \gamma_1}}$$

where recall $$\boxed{\begin{aligned} \gamma_1^2 &= n_1^2 \frac{\omega^2}{c^2} - h^2 \\ \gamma_2^2 &= h^2 - n_2^2 \frac{\omega^2}{c^2} \end{aligned}}$$

Odd parity solution

$$E_1(x) = \begin{cases} E_i \sin(\gamma_1 x) \\ E_0 \, \mathrm{sgn}(x) \, e^{-\gamma_2 |x|} \end{cases}$$

Similarly we obtain $$\boxed{\cot(\gamma_1 a) = -\frac{\mu_1 \gamma_2}{\mu_2 \gamma_1}}$$

Equations to be solved

(1) $\tan(\gamma_1 a) = \dfrac{\mu_1 \gamma_2}{\mu_2 \gamma_1}$ or $\cot(\gamma_1 a) = -\dfrac{\mu_1 \gamma_2}{\mu_2 \gamma_1}$

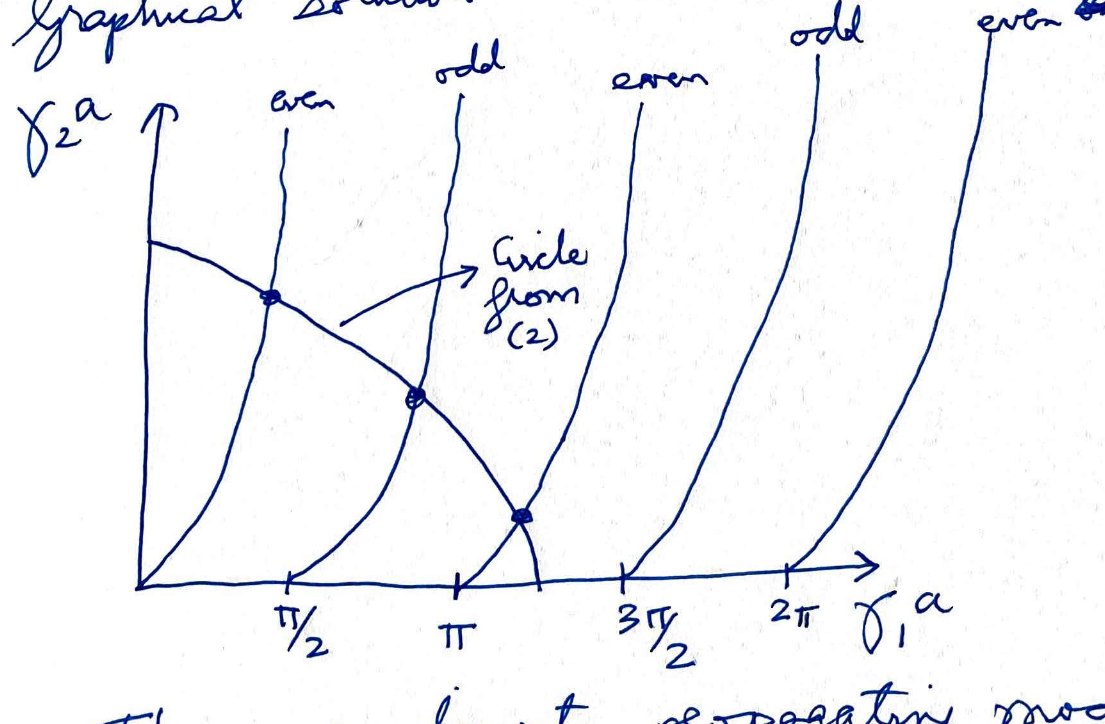
(2) $$\frac{\omega^2}{c^2}(n_1^2 - n_2^2) = \gamma_1^2 + \gamma_2^2$$

(3) $$h^2\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) = \frac{\gamma_2^2}{n_2^2} - \frac{\gamma_1^2}{n_1^2}$$

Unknowns ω, h, γ_1, γ_2.

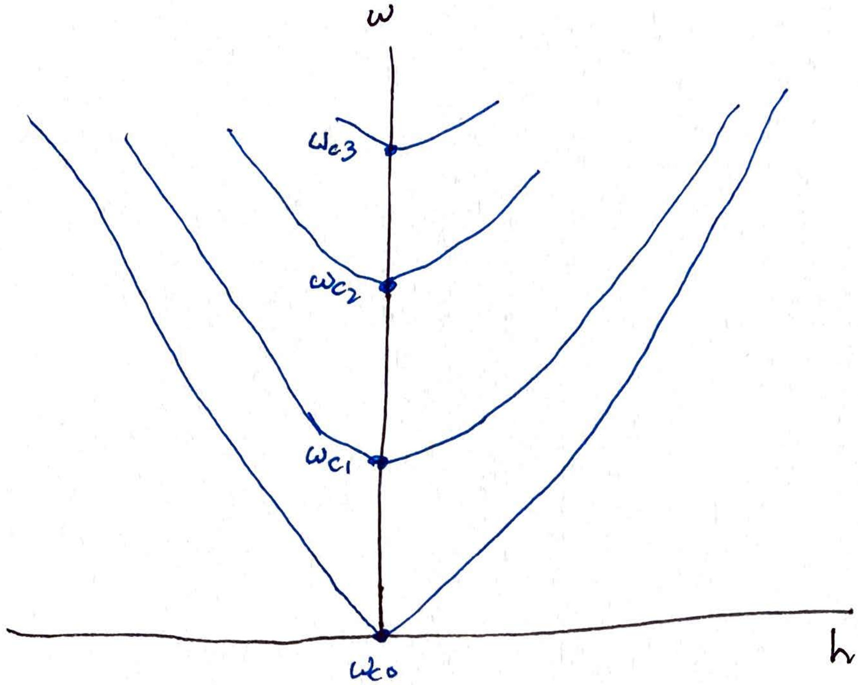
Solve (1), (3) for $\gamma_1(h)$ and $\gamma_2(h)$ and so determine dispersion $\omega = \dfrac{c}{n_1}\sqrt{h^2 + \gamma_1(h)^2}$

Graphical solution



There are discrete propogating modes above cutoff frequencies

$$\omega_{cm} = \frac{m\pi c}{2a\sqrt{n_1^2 - n_2^2}}; \quad \underline{m \to \text{integer}}$$


ω
ωc3
ωc2
ωc1
h
ωc0