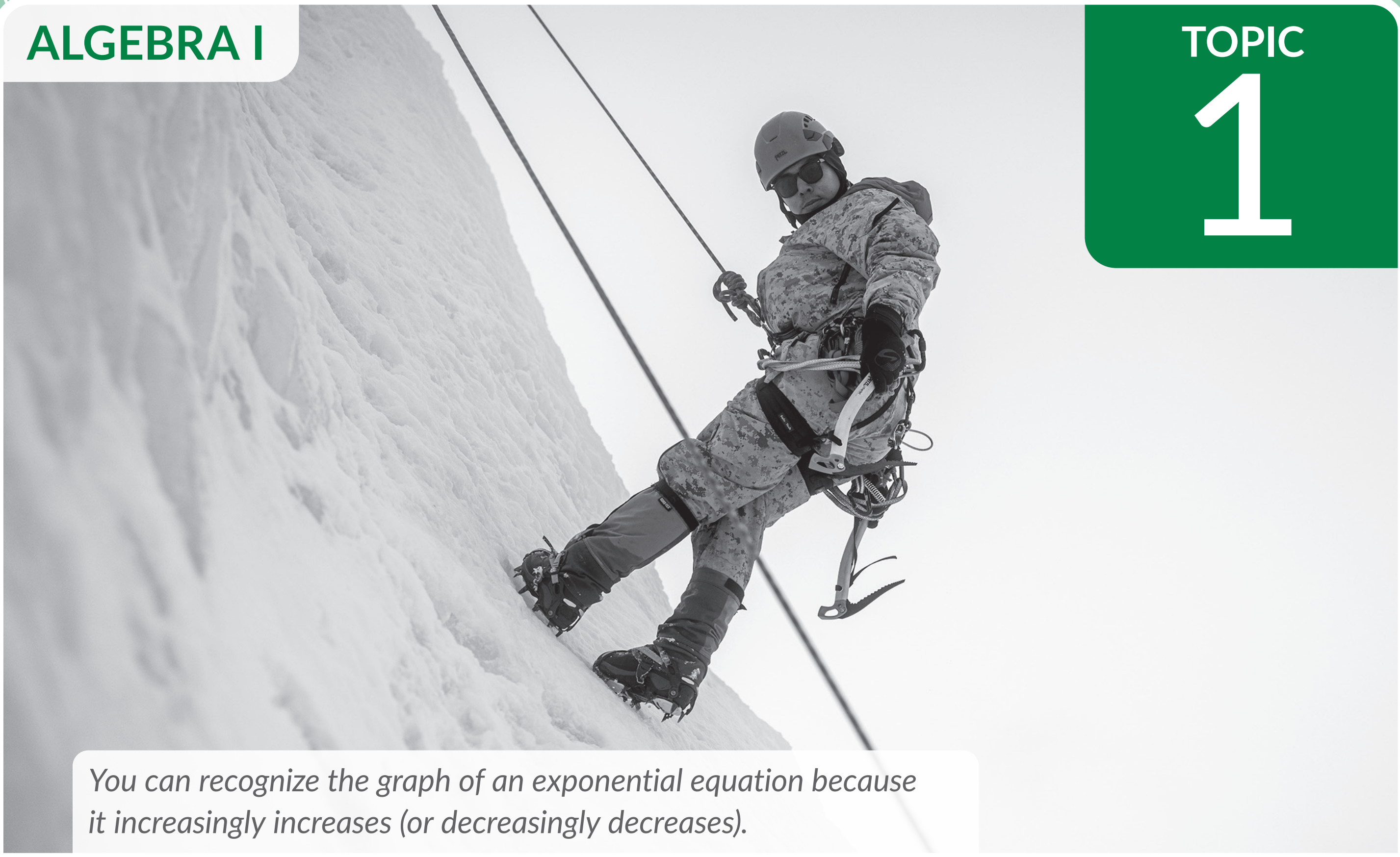
MODULE 4

Investigating Growth and Decay

ALGEBRA I

TOPIC 1



You can recognize the graph of an exponential equation because it increasingly increases (or decreasingly decreases).

Introduction to Exponential Functions

1 Properties of Powers with Integer Exponents

OBJECTIVES

- Expand a power into a product.
- Write a product as a power.
- Simplify numeric and algebraic expressions containing integer exponents.
- Develop rules to simplify a product of powers, a power of a power, and a quotient of powers.
- Apply the properties of integer exponents to create equivalent expressions.

NEW KEY TERMS

- power
- base
- exponent

You have learned how to evaluate numeric expressions involving whole-number exponents.

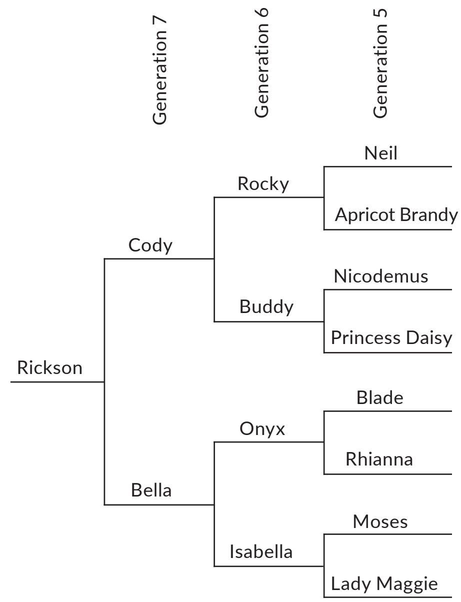
In this lesson, you will develop the properties of integer exponents to generate equivalent numeric and algebraic expressions.

How can you use the properties of integer exponents to generate equivalent numeric expressions?

Getting Started

Three Generations

Lucas adopted an eighth-generation purebred English Mastiff puppy that he named Rickson. The breeder provided documentation that verified a portion of Rickson's lineage going back three generations, as shown.



Ask Yourself . . .
What patterns do you notice?

A dog's lineage is similar to a person's family tree. It shows a dog's parents, grandparents, and great-grandparents.

1. How many parents does Rickson have? What are his parents' names?

2. How many grandparents does Rickson have?

3. How many great-grandparents does Rickson have?

4. What pattern is there in the number of dogs in each generation?

ACTIVITY

1.1 Review of Powers and Exponents

Lucas wants to trace Rickson's lineage back through all eight generations. How many sires (male parents) and dams (female parents) are there in each generation of Rickson's lineage?

1. Complete the second column of the table, Number of Sires and Dams, to show the total number of dogs in each generation.

Generation	Number of Sires and Dams		
Generation 7			
Generation 6			
Generation 5			
Generation 4			
Generation 3			
Generation 2			
Generation 1			

Ask Yourself . . .
Why is the eighth generation not included as part of the table? Which dog is included in the eighth generation?

An expression used to represent the product of a repeated multiplication is a *power*. A **power** has a *base* and an *exponent*. The **base** of a power is the expression that is used as a factor in the repeated multiplication. The **exponent** of a power is the number of times that the base is used as a factor in the repeated multiplication.

$\text{base}^{\text{exponent}}$ ← **power**

WORKED EXAMPLE

You can write a power as a product by writing out the repeated multiplication.

$$2^7 = (2)(2)(2)(2)(2)(2)(2)$$

The power 2^7 can be read as:

- "two to the seventh power."
- "the seventh power of two."
- "two raised to the seventh power."

Think About . . .
How can I write the number of dogs in each generation as a repeated multiplication?

2. Label the third column *Expanded Notation*. Then, write each generation total as a product.

3. Label the fourth column of the table *Power*. Then, write each generation total as a power.

Ask Yourself . . .
How many generations are there before this dog in its lineage?

4. Suppose another dog is a thirteenth-generation purebred. How many dogs are in the first generation? Write your answer as a power and then use a calculator to determine the total number of dogs.

5. How many total sires and dams are there in all three generations shown in Rickson's lineage? Explain your calculation.

ACTIVITY
1.2 Practice with Powers

In this activity, you will investigate the role of parentheses in expressions containing exponents.

1. Identify the base(s) and exponent(s) in each. Then, write each power as a product. Finally, evaluate the power.

   a. 5^3

   b. $(-9)^5$

   c. -11^3

   d. $(4)^5(3)^6$

Think About . . .

When the negative sign is not in parentheses, it's not part of the base. Instead, there is a factor of -1.

2. Write each as a product. Then, calculate the product.

   a. -1^2

   b. -1^3

   c. -1^4

   d. -1^5

   e. $(-1)^2$

   f. $(-1)^3$

   g. $(-1)^4$

   h. $(-1)^5$

3. Consider Questions 2(f) and 2(h). What conclusion can you draw about a negative number raised to an odd power?

4. Consider Questions 2(e) and 2(g). What conclusion can you draw about a negative number raised to an even power?

ACTIVITY 1.3 Multiplying and Dividing Powers

1 gigabyte = 1024 megabytes

1 megabyte = 1024 kilobytes

1 kilobyte = 1024 bytes

File sizes of eBooks, podcasts, and song downloads depend on the complexity of the content and the number of images.

WORKED EXAMPLE

Suppose that a medium-sized eBook contains about 1 megabyte (MB) of information.

Since 1 megabyte is 1024 kilobytes (kB), and 1 kilobyte is 1024 bytes (B), you can multiply to determine the number of bytes in the eBook:

$$1 \text{ MB} = (1024 \text{ kB})\left(\frac{1024 \text{ B}}{1 \text{ kB}}\right) = 1{,}048{,}576 \text{ B}$$

There are 1,048,576 bytes in the eBook.

Remember . . .
Be sure to use units in your calculations.

1. One model of an eBook can store up to 256 MB of data. A USB jump drive can hold 2 GB of storage. Use the method shown in the Worked Example to calculate each.

   a. Calculate the number of bytes the eBook can store.

   256 MB · ________ · _________ = ___________________

   b. A USB jump drive can hold 2 GB of storage. How many bytes can the USB jump drive hold?

   2 GB · _______ · ______ · _________ = ___________________

   c. How many times more storage space does the jump drive have than the eBook? Show your work.

WORKED EXAMPLE

Computers use binary math, or the base-2 system, instead of the base-10 system.

Base 10	Base 2
$10^1 = 10$	$2^1 = 2$
$10^2 = (10)(10) = 100$	$2^2 = (2)(2) = 4$
$10^3 = (10)(10)(10) = 1000$	$2^3 = (2)(2)(2) = 8$

2. Revisit Question 1, parts (a) through (c), by rewriting each factor and either your product or quotient as a power of 2.

   a.

   b.

   c.

3. Analyze your answers to Question 2. What do you notice about all the bases in Question 2?

4. In parts (a) and (b), how does the exponent in each product relate to the exponents in the factors?

5. In part (c), how does the exponent in the quotient relate to the exponents in the numerator and denominator?

ACTIVITY
1.4

Product of Powers

In this activity, you will explore different expressions to develop rules to evaluate powers.

1. Rewrite each expression as a product using expanded notation. Then, identify the base or bases and record the number of times the base is used as a factor.

   a. $2^4 \cdot 2^3$

   b. $(-3)^3(-3)^3$

   c. $(4)(4^5)$

   d. $(5^2)(6^2)(5^3)(6)$

   e. $(9^3)(4^2)(9^2)(4^5)$

2. Rewrite each of your answers from Question 1 as a power or a product of powers.

3. What relationship do you notice between the exponents in the original expression and the number of factors?

4. Write a rule that you can use to multiply powers.

5. Use your rule to write an equivalent expression as a power or product of powers in simplest form.

a. $5^2 \cdot 5^3$

b. $x^4 \cdot x^6$

c. $(-3)^3(-3)^5$

d. $(-a)^2(-a)$

e. $(6)(4^7)(6^8)$

f. $(c^3)(d^9)(c^{12})(d^4)$

A power can also be raised to a power.

WORKED EXAMPLE

The exponential expression $(4^2)^3$ is a power of a power. It can be written as two repeated multiplication expressions using the definition of a power.

$$(4^2)^3 = (4^2)(4^2)(4^2)$$

$$= (4 \cdot 4)\cdot(4 \cdot 4)\cdot(4 \cdot 4)$$

There are 6 factors of 4.

6. Use the definition of a power to write repeated multiplication expressions for each power of a power, as modeled in the Worked Example. Then, record the number of factors.

   a. $(8^2)^3$

   b. $(5^4)^2$

   c. $-(6^1)^6$

   d. $((-6)^2)^2$

7. What relationship do you notice between the exponents in each expression in Question 6 and the number of factors? Write each expression as a single power.

8. Write a rule that you can use to determine a power of a power.

9. Write an equivalent expression in simplest form using the rules that you wrote.

a. $6^4 \cdot 6^3$

b. $y^7 \cdot y^8$

c. $(4^3)^5$

d. $(w^7)^3$

e. $r^5 \cdot r^2 \cdot r$

f. $((2)(3))^4$

10. Mason says that $2^6 = 12$. Luna says that $2^6 = 64$. Who is correct? Explain your reasoning.

11. Nahimana says that $2^2 + 2^3 = 2^5$, and Sebastian says that $2^2 + 2^3 \neq 2^5$. Who is correct? Explain your reasoning.

ACTIVITY
1.5 Quotient of Powers

Now, let's investigate what happens when you divide powers with like bases.

1. Write each numerator and denominator as a product. Then, write an equivalent expression in simplest form using exponents.

   a. $\frac{9^5}{9^2}$

   b. $\frac{5^6}{5^3}$

   c. $\frac{x^8}{x^6}$

   d. $\frac{10^2}{10}$

2. What relationship do you notice between the exponents in the numerator and denominator and the exponents in the simplified expression?

3. Write a rule that you can use to divide with powers.

4. Write an equivalent expression in simplest form using the rule that you wrote for a quotient of powers.

   a. $\frac{6^8}{6^3}$

   b. $-\frac{t^7}{t^5}$

   c. $\frac{2^3}{3^2}$

ACTIVITY 1.6

Powers Equal to 1 and Numbers Less Than 1

You know that any number divided by itself is 1. How can you use that knowledge to develop another rule to evaluate powers?

Consider each representation of 1.

$\frac{4}{4} = 1$ $\quad\quad$ $\frac{9}{9} = 1$ $\quad\quad$ $\frac{25}{25} = 1$ $\quad\quad$ $\frac{x}{x} = 1$

1. Rewrite the numerator and denominator of each fraction as a power. Do not simplify.

2. Next, simplify the fractions you just wrote using the quotient of powers rule. Leave your answer as a power. What do you notice?

An exception is that 0^0 is not equal to 1, because that would mean that using zero as a factor zero times would give you 1, and that's not possible.

3. Write a rule that you can use when raising any base to the zero power.

Let's determine how to use powers to represent numbers that are less than 1.

You know that you can use powers to represent numbers that are greater than or equal to 1.

4. Let's start with 1 and multiply by 10 three times.

   a. Complete the representation. Write each as a power.

	1	=	10^0
Multiply by 10 =	______________	=	______________
Multiply by 10 =	______________	=	______________
Multiply by 10 =	______________	=	______________

b. Describe what happens to the exponents as the number becomes greater.

5. Now, let's start with 1 and divide by 10 three times.

a. Complete the representation. Write the division as a fraction and then rewrite using the definition of powers. Next, apply the quotient of powers rule, and finally, simplify each expression.

$$1 = \frac{10^0}{10^0} = 10^{0-0} = 10^0$$

Divide by 10 = $\frac{1}{10}$ = $\frac{10^0}{10^1}$ = 10^{0-1} = 10^{-1}

Divide by 10 = ________ = ________ = ________ = ________

Divide by 10 = ________ = ________ = ________ = ________

b. Describe what happens to the exponents as the number becomes less.

c. Write each of the powers as a decimal.

6. Rewrite each sequence of numbers using the definition of powers.

a. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8$

b. $\frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27$

c. Describe the exponents in the sequence.

7. Write an equivalent expression in simplest form using the quotient of powers rule. Then, write each as a decimal.

a. $\frac{10^0}{10^3}$

b. $\frac{10^0}{10^5}$

c. $\frac{10^0}{10^4}$

8. Complete the table shown.

Unit	Number of Grams	Number of Grams as an Expression with a Positive Exponent	Number of Grams as an Expression with a Negative Exponent
Milligram	$\frac{1}{1000}$		10^{-3}
Microgram		$\frac{1}{10^6}$	
Nanogram	$\frac{1}{1,000,000,000}$		10^{-9}
Picogram		$\frac{1}{10^{12}}$	

9. Write an equivalent expression in simplest form such that the exponent is positive.

a. 8^{-4}

b. 5^{-6}

c. p^{-5}

d. $(4^{-2})(3^{-3})$

10. Complete the table.

Given Expression	Expression with a Positive Exponent	Value of Expression
$\frac{1}{3^{-2}}$		
$\frac{1}{4^{-2}}$		
$\frac{1}{5^{-2}}$		
$\frac{2^{-2}}{1}$	$\frac{1}{2^2}$	$\frac{1}{4}$
$\frac{3^{-2}}{1}$		
$\frac{5^{-2}}{1}$		

11. Describe how to rewrite any expression with a negative exponent in the numerator.

12. Describe how to rewrite any expression with a negative exponent in the denominator.

Talk the Talk

Simplifying

In this lesson, you have developed rules for operating with powers. A summary of these rules is shown in the table.

Properties of Powers	Words	Rule
Product rule of powers	To multiply powers with the same base, keep the base and add the exponents.	$a^m a^n = a^{m+n}$
Power of a power rule	To simplify a power of a power, keep the base and multiply the exponents.	$(a^m)^n = a^{mn}$
Quotient of powers rule	To divide powers with the same base, keep the base and subtract the exponents.	$\frac{a^m}{a^n} = a^{m-n}$, if $a \neq 0$
Zero power	The zero power of any number, except for 0, is 1.	$a^0 = 1$, if $a \neq 0$
Negative exponents in the numerator	An expression with a negative exponent in the numerator and a 1 in the denominator equals 1 divided by the power with its opposite exponent placed in the denominator.	$a^{-m} = \frac{1}{a^m}$, if $a \neq 0$ and $m > 0$
Negative exponents in the denominator	An expression with a negative exponent in the denominator and a 1 in the numerator equals the power with its opposite exponent.	$\frac{1}{a^{-m}} = a^m$, if $a \neq 0$ and $m > 0$

Write an equivalent expression in simplest form using the properties of powers.

1. $2a^8 \cdot 2a^6$

2. $4b^2 \cdot 8b^9$

3. $-3c \cdot 5c^3 \cdot 2c^9$

4. $(3d^2)^3$

5. $(10ef^3)^5$

6. $\frac{f^8}{f^3}$

7. $\frac{10g^4}{5g^6}$

8. $\frac{30h^8}{15h^2}$

9. $\frac{35i^7j^3}{7i^2j^3}$

10. $\left(\frac{a^2}{a^5}\right)^0$

11. $\frac{2^2}{2^6}$

12. $(4x^2)(3x^5)$

13. $(9^4)(9^{-5})$

14. $(8^0)(8^{-2})$

15. $\frac{3^{-3}}{3^{-3}}$

16. $\frac{4^{-2}}{4^{-3}}$

17. $\frac{(-3)^2}{(-3)^4}$

18. $\frac{h^3}{h^5}$

19. $\frac{x^4}{x^5}$

20. $\frac{m^2p^{-2}}{m^4p^3}$

Lesson 1 Assignment

Write

Use the term *base*, *power*, or *exponent* to complete each sentence.

1. The __________ of a power is the number of times that the factor is repeatedly multiplied.

2. An expression used to represent a factor as repeated multiplication is called a __________.

3. The __________ of a power is the repeated factor in a power.

Remember

Properties of Powers	Words	Rule
Product of powers rule	To multiply powers with the same base, keep the base and add the exponents.	$a^m a^n = a^{m+n}$
Power of a power rule	To simplify a power of a power, keep the base and multiply the exponents.	$(a^m)^n = a^{mn}$
Quotient of powers rule	To divide powers with the same base, keep the base and subtract the exponents.	$\frac{a^m}{a^n} = a^{m-n}$, if $a \neq 0$

Practice

1. As the principal of a middle school, Mr. Park is in charge of notifying his staff about school delays or cancellations due to weather, power outages, or other unexpected events. Mr. Park starts a phone chain by calling three staff members. Each of these staff members then calls three more staff members, who each call three more staff members. This process completes the calling list.

    a. Excluding Mr. Park, how many staff members are there at the middle school? Explain your calculation.

Lesson 1 Assignment

b. Complete the first three columns in the table.

	Round 1	Round 2	Round 3	Round 4
Number of calls made				
Expanded notation				
Power				

c. Dr. Martinez, superintendent of the school district, decides that she should start the phone chain instead of the school principals. She starts the calling list by calling each of the principals of the three schools in her district. The principals continue the phone chain as described previously. Explain why the table in part (b) can be used to represent Dr. Martinez's phone chain.

d. Complete the fourth column in the table to represent the fourth round of calls in Dr. Martinez's phone chain.

e. How many calls are made in the fourth round?

f. Excluding Dr. Martinez, how many principals and staff members are there in the entire school district? Explain your calculation.

Lesson 1 Assignment

2. The *hertz* (Hz) is a unit of frequency that represents the number of complete cycles per second. It is used to measure repeating events, both scientific and general. For instance, a clock ticks at 1 Hz. One scientific application is the electromagnetic spectrum, or the range of all possible frequencies of electromagnetic waves. The spectrum includes frequencies from everyday contexts, such as radio and TV signals, microwaves, light (infrared, visible, and ultraviolet), and X-rays.

Name	Frequency
1 kilohertz (kHz)	1000 Hz
1 megahertz (MHz)	1000 kHz
1 gigahertz (GHz)	1000 MHz
1 terahertz (THz)	1000 GHz

For each question, use powers to write a mathematical expression. Then, evaluate each expression. Express your answer as a power.

a. How many hertz are in 1 gigahertz? 1 terahertz?

b. A television channel has a frequency of 60 megahertz. What is the channel's frequency in hertz?

c. The frequency of a microwave is 30 gigahertz. What is the microwave's frequency in hertz?

Lesson 1 Assignment

d. The frequency of a visible ray of light is 1000 terahertz. A radio station transmits at a frequency of 100 megahertz. How many times greater is the frequency of the light than the frequency of the radio station?

3. Each expression has been simplified incorrectly. Explain the mistake that occurred and then make the correction.

a. $(-2x)^3 = 8x^3$

b. $\frac{16x^5}{4x} = 12x^4$

c. $(x^2y^4)^3 = x^6y^7$

d. $(x^5y^7)(x^2yz) = x^7y^7z$

4. When you take a picture, the camera shutter controls how much light reaches the film or the digital image sensor. The shutter speed is the amount of time, in seconds, that the shutter stays open. Write each shutter speed as a power with a negative exponent.

a. $\frac{1}{4}$ second

b. $\frac{1}{8}$ second

Lesson 1 Assignment

5. True or False: A number raised to a negative power is always a negative number. Give an example to support your answer.

6. Give an example of a number raised to a negative exponent that is a negative number.

7. Write an equivalent expression in simplest form using the properties of powers. Show your work.

a. $\frac{10^2}{10^5}$

b. $\frac{3^{-5}}{3^{-5}}$

c. $(6x^4)(2x^{-2})$

d. $(7^{-6})(7^4)$

e. $(4^0)(4^{-3})$

f. $\frac{5^2}{5^{-2}}$

g. $\frac{4^{-1}}{4^2}$

h. $\frac{15p^4}{3p^9}$

i. $\frac{m^{-2}}{m^{-6}}$

j. $\frac{q^{-2}\,r^{-3}}{q^6\,r^{-4}}$

Lesson 1 Assignment

Prepare

Simplify each expression.

1. $\frac{1}{3^2}$

2. $-(2^5)^2$

3. $\frac{2}{5} \cdot 10$

4. $\frac{32 + 8}{80 \div 2}$

2 Analyzing Properties of Powers

OBJECTIVES

- Review the product of powers property.
- Review the power of a power property.
- Review the quotient of powers property.
- Generate equivalent expressions by applying properties of integer exponents.

You have learned about the properties of powers with integer exponents.

How can you use these properties to justify your reasoning when solving problems?

Getting Started

Are They Equal?

The symbol for "is equal to" is $=$. The symbol for "is not equal to" is $\neq$. Write the appropriate symbol in each box to compare the two expressions. Explain your reasoning.

1. $2^3 \; \boxed{} \; 2^{-3}$

2. $\frac{1}{2^3} \; \boxed{} \; 2^{-3}$

3. $\frac{1}{2^{-3}} \; \boxed{} \; 2^3$

4. $\frac{1}{2^{-3}} \; \boxed{} \; 2^{-3}$

ACTIVITY **2.1**

Using Properties of Powers to Justify Steps in Simplifying

Analyze the Worked Example.

WORKED EXAMPLE

$\left(\frac{g^5}{g^4}\right)^3 =$

$= (g^1)^3$ quotient of powers rule

$= g^3$ power of a power rule

1. Identify the rule that justifies each step to simplify the expression.

a. $2^4 \cdot (-4^1)^3$

$= 2^4 \cdot (-4^3)$ ____________________

$= (16)(-64)$ ____________________

$= -1024$

b. $\frac{(3 \cdot 5^3)^2}{(2 \cdot 5^4)^2}$ ____________________

$= \frac{(3^2 \cdot 5^6)}{(2^2 \cdot 5^8)}$ ____________________

$= \frac{3^2}{(2^2 \cdot 5^2)}$ ____________________

$= \frac{9}{4 \cdot 25}$ ____________________

$= \frac{9}{100}$ ____________________

Ask Yourself . . .
Did you justify your mathematical reasoning?

2. Write an equivalent expression in simplest form using the properties of powers. Express your answers using only positive exponents.

a. $\dfrac{2 \cdot 6^3 \cdot 4^5}{4 \cdot 6^3 \cdot 4^2}$

b. $\dfrac{(4 \cdot 1^2 \cdot 3^3)^4}{(2 \cdot 1^3 \cdot 3^2)^3}$

c. $\dfrac{(-3x^2y^4)^8}{(-3x^2y^4)^8}$

d. $\dfrac{(k^5 \cdot k^5)}{k^6}$

ACTIVITY 2.2

Analyze Errors in Applying Properties of Powers

Determine which student(s) used the properties of powers correctly. Explain why the other expressions are not correct.

1. $\dfrac{n^7m^4}{n^3m^9}$

   Kayla wrote $n^{10}m^{13}$.

   Adriana wrote $\dfrac{n^4}{m^5}$.

   Mei wrote n^4m^5.

   Who is correct?

2. $\dfrac{2(3)^{-4}}{5^{-2}}$

   Kayla wrote $\dfrac{2(5)^2}{3^4}$.

   Adriana wrote $\dfrac{5^2}{2(3)^4}$.

   Mei wrote $\dfrac{2(3)^4}{5^2}$.

   Who is correct?

3. Each expression has been simplified incorrectly. Explain the mistake that occurred, and then make the correction.

   a. $(-2 \cdot 7)^3 = 8 \cdot 7^3$

   b. $\frac{16 \cdot 7^5}{4 \cdot 7} = 12 \cdot 7^4$

   c. $(g^2h^4)^3 = g^6h^7$

   d. $(a^5b^7)(a^2b \cdot c) = a^7b^7c$

Talk the Talk

Organize the Properties

1. Create graphic organizers for each rule:

   Product of powers rule

   Quotient of powers rule

   Power of a power rule

   Zero power rule

   Negative exponent rule

   As you are creating your representations, consider the following:

Definition in your own words:	How would you describe this property to a friend?
Facts/Characteristics:	Are there specific characteristics if the numbers are positive or negative? Does this property work the same for variables and numbers?
Examples:	Include examples with variables and different types of numbers (e.g., positive, negative, and fractions).
General Rule:	Use variables. Be mindful of when a variable cannot be zero.

DEFINITION IN YOUR OWN WORDS

FACTS/ CHARACTERISTICS

PRODUCT OF POWERS RULE

EXAMPLES

GENERAL RULE

DEFINITION IN YOUR OWN WORDS

FACTS/ CHARACTERISTICS

QUOTIENT OF POWERS RULE

EXAMPLES

GENERAL RULE

DEFINITION IN YOUR OWN WORDS

FACTS/ CHARACTERISTICS

POWER OF A POWER RULE

EXAMPLES

GENERAL RULE

DEFINITION
IN YOUR OWN
WORDS

FACTS/
CHARACTERISTICS

ZERO POWER
RULE

EXAMPLES

GENERAL RULE

DEFINITION IN YOUR OWN WORDS

FACTS/ CHARACTERISTICS

NEGATIVE EXPONENT RULE

EXAMPLES

GENERAL RULE

Lesson 2 Assignment

Write

In your own words, write the quotient of powers rule, product of powers rule, and the power of a power rule. Use examples to illustrate your descriptions.

Remember

Negative exponents in the numerator can be moved to the denominator and become positive, $a^{-m} = \frac{1}{a^m}$, if a $\neq$ 0 and m $>$ 0.

The zero power of any number, except for 0, is 1.

Practice

Justify each step to simplify each expression. Choose the properties from the box.

Product of powers rule	Power of a power rule	Negative exponent rule	Quotient of powers rule
Zero power rule	Simplify powers	Identity property of multiplication	Commutative property of multiplication

1. $4x^5 \cdot 6x^2y^6 \cdot xy$

2. $(3a^2b^3)(7ab^5)(a^0b^2)$

3. $(4m^2n^5)^3$

4. $(-3x^7y^3z^0)^5$

Lesson 2 Assignment

5. $\dfrac{27y^8z^5}{-3y^4z^2}$

6. $\dfrac{-96m^9n^2}{8m^2n^6}$

7. $-2x^5y^3 \cdot 8x^2y^{-5} \cdot x^{-9}y^2$

8. $\dfrac{(42m^5n^3)(m^4n^2)}{6m^6n^5}$

Prepare

Use the given explicit formula to generate the first four terms of each geometric sequence.

1. $g_n = 2 \cdot 3^{n-1}$

2. $g_n = 8240 \cdot 1.05^{n-1}$

3. $g_n = 100 \cdot \left(\frac{1}{2}\right)^{n-1}$

4. $g_n = (-2) \cdot 4^{n-1}$

3 Geometric Sequences and Exponential Functions

OBJECTIVES

- Write a geometric sequence as an exponential function in the form $f(x) = ab^x$.
- Identify the constant ratio and y-intercept in different representations of exponential functions.
- Recognize when a relationship is exponential.
- Use algebra to show that, for an exponential function in the form $f(x) = ab^x$, the ratio $\frac{f(x + 1)}{f(x)}$ is constant and equal to b, and y-intercept is represented by the orderedthe pair $(0, a)$.

You have learned about geometric sequences and have briefly explored exponential functions.

How can you use geometric sequences to define exponential functions?

Getting Started

Compare and Contrast

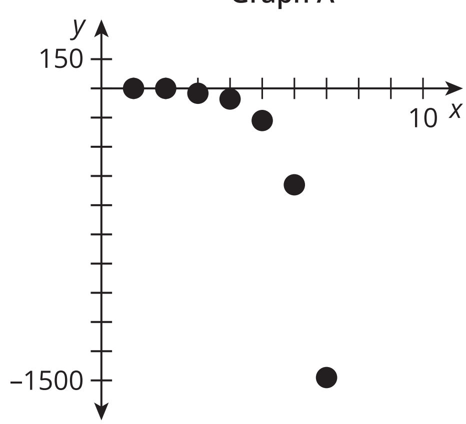
Recall that a *geometric sequence* is a sequence of values in which consecutive terms are separated by a *common ratio*, or constant ratio. For example, the sequence shown is a geometric sequence with a constant ratio of 2.

$$1, \quad 2, \quad 4 \underset{\div 2}{\overset{\times 2}{\rightleftarrows}} 8, \quad 16, \ldots$$

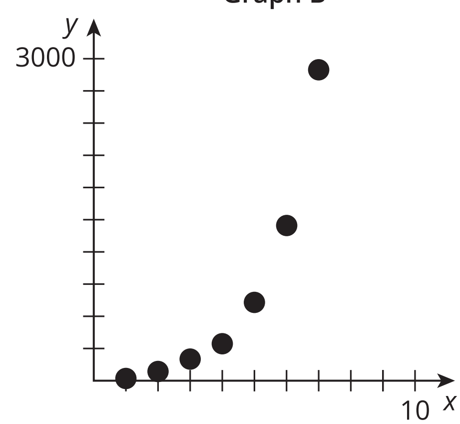
1. Consider a geometric sequence defined by the *recursive formula* $f(n) = \frac{1}{2}(n - 1)$. List the first four terms of the sequence when $f(1) = -5$.

The graphs of six different geometric sequences are shown.

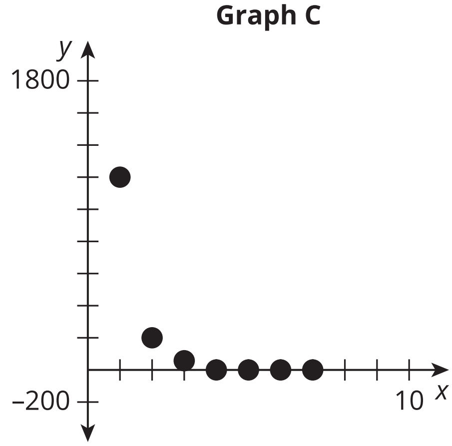
Graph A



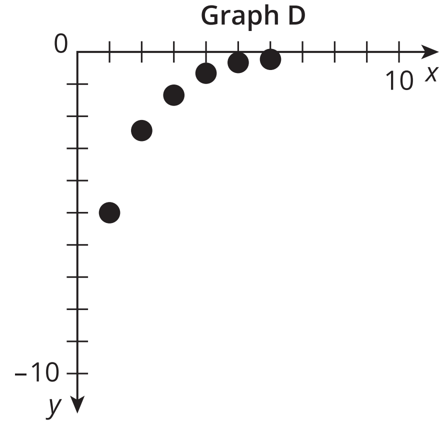
Graph B



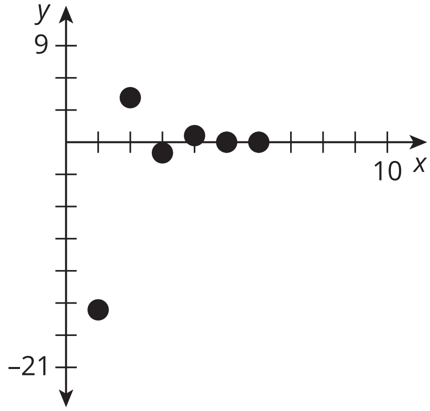
Graph C



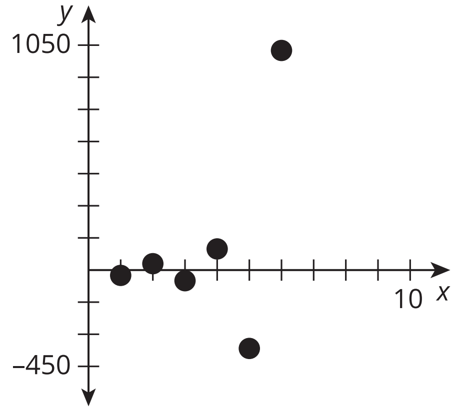
Graph D



Graph E



Graph F



2. Identify similarities and differences among the graphs. What do you notice?

3. Which of the graphs could represent the sequence given in Question 1? Explain your reasoning.

ACTIVITY 3.1 Identifying the Constant Ratio in Geometric Sequences

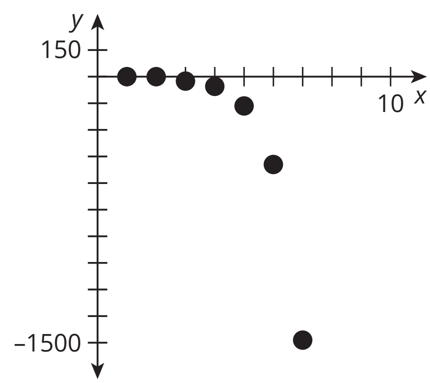
A table of values, a graph, and the explicit formula are given for six geometric sequences.

The *explicit formula* for a geometric sequence is $g_n = g_1(r)^{n-1}$.

1. Identify the constant ratio in each representation.

a. Sequence *A*

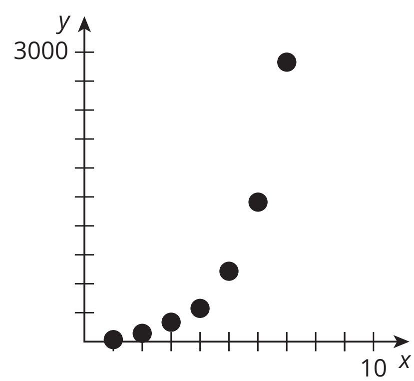
x	y
1	−2
2	−6
3	−18



$y = -2(3)^{x-1}$

b. Sequence *B*

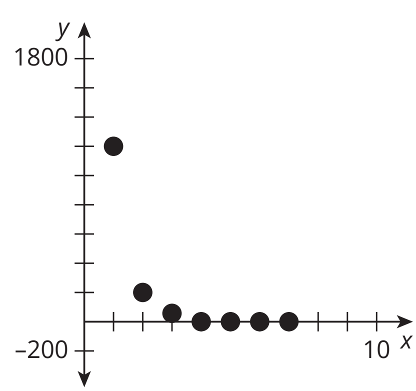
x	y
1	45
2	90
3	180



$y = 45 \cdot 2^{x-1}$

c. Sequence *C*

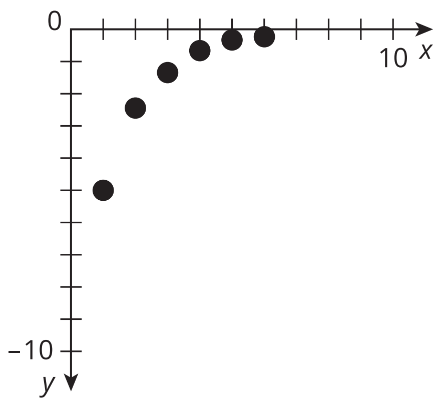
x	y
1	1234
2	123.4
3	12.34



$y = 1234(0.1)^{x-1}$

d. Sequence *D*

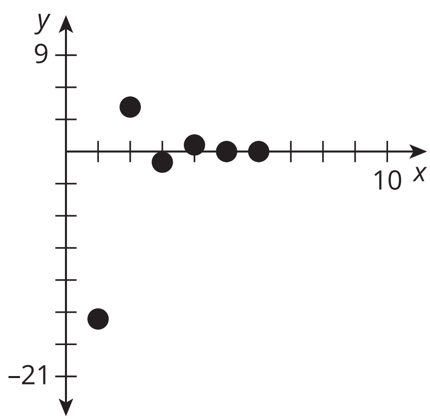
x	y
1	−5
2	−2.5
3	−1.25



$y = -5\left(\frac{1}{2}\right)^{x-1}$

d. Sequence *E*

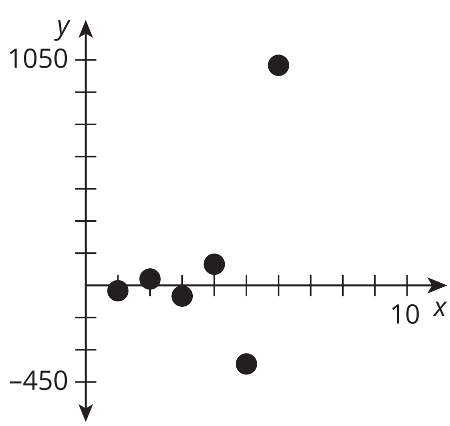
x	y
1	−16
2	4
3	−1



$y = -16\left(-\frac{1}{4}\right)^{x-1}$

f. Sequence *F*

x	y
1	−4
2	12
3	−36



$y = -4(-3)^{x-1}$

2. What strategies did you use to identify the constant ratio for each sequence?

3. Analyze the graphs of the geometric sequences. Do any of the graphs appear to belong to a specific function family? If so, identify the function family. Explain your reasoning.

ACTIVITY
3.2 Exponential Growth

Remember . . .
All arithmetic sequences can be represented as linear functions. Is there a function family that can represent geometric sequences?

A famous legend tells the story of the inventor of the game of chess. When the inventor showed the new game to the emperor of India, the emperor was so astonished, he said to the inventor, “Name your reward!”

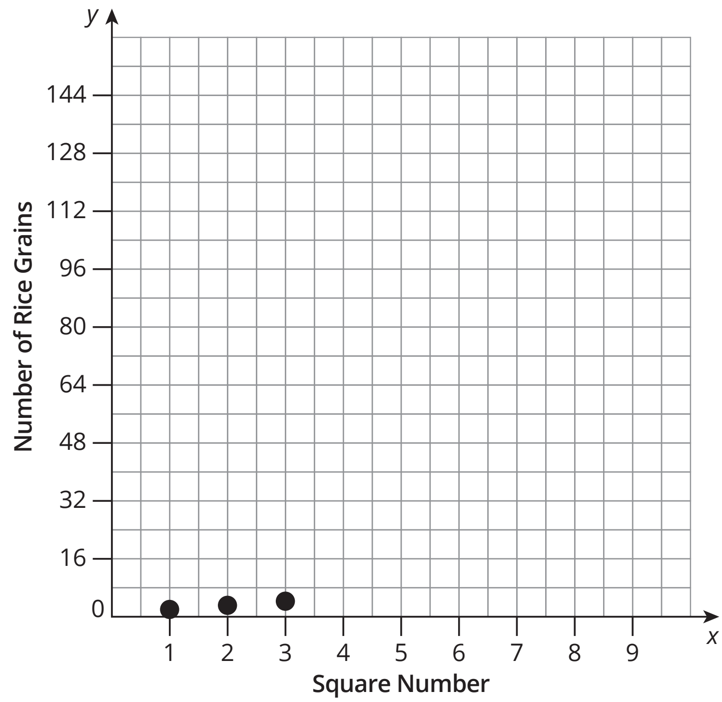
The wise inventor asked the emperor for 1 grain of rice for the first square of the chessboard, 2 grains for the second square, 4 grains for the third square, 8 grains for the fourth square, and so on.

Square Number	Number of Rice Grains	Power
1	1	
2	2	
3	4	
4	8	
5		
6		
7		
8		

1. Determine the number of rice grains on the next 4 squares and include them in the table. Complete the third column by writing each number of rice grains as a power with the same base.

2. What pattern do you notice in the table?

3. Graph the points from your table. The first few points have been plotted. Describe the meaning of the plotted points and then identify the function family represented.



4. Identify the constant ratio in the graph, in the table, and in the situation.

5. Angelina and Gabriel each used different methods to write an exponential function to represent the number of rice grains for any square number on the chessboard.

Angelina

I compared the exponents of the power to the square number in the table. Each exponent is 1 less than the square number.

$$f(s) = 2^{s-1}$$

Gabriel

I know the constant ratio is 2. If I extend the pattern back, I get the y-intercept of $(0, \frac{1}{2})$, so I can rewrite the function as

$$f(s) = \frac{1}{2}(2)^s$$

Use properties of exponents to verify that 2^{s-1} and $\frac{1}{2}(2)^s$ are equivalent.

The function that Gabriel wrote is in exponential form. Recall that an exponential function is a function of the form $f(x) = ab^x$, where a and b are real numbers and b is greater than 0, but is not equal to 1.

6. What do the a-value and b-value represent in terms of the equation and graph?

7. Use the exponential function and a calculator to determine the number of rice grains that would be on the very last square of the chessboard. A chessboard has 64 squares.

You can write the explicit formula for geometric sequences in function notation.

Think About . . .

The product of powers rule of exponents allows you to rewrite the product of two powers with the same base:

$(2)^n(2^{-1}) = 2^{n-1}$

$(6)^x(6^y) = 6^{x+y}$

WORKED EXAMPLE

Represent $g_n = 45(2)^{n-1}$ as a function in the form $f(x) = ab^x$.

$g_n = 45 \cdot 2^{n-1}$ or $45(2)^{n-1}$

$f(n) = 45(2)^{n-1}$

Next, rewrite the expression $45(2)^{n-1}$.

$f(n) = 45(2^n)(2^{-1})$	product of powers rule
$f(n) = 45(2^{-1})(2)^n$	commutative property
$f(n) = 45\left(\frac{1}{2}\right)(2)^n$	definition of negative exponent
$f(n) = \frac{45}{2}(2)^n$	multiply.

So, $g_n = 45(2)^{n-1}$ written in function notation is $f(n) = \frac{45}{2}(2)^n$, or $f(n) = 22.5(2)^n$.

In the previous activity, you identified some of the geometric sequences as exponential functions and some that were not exponential functions.

8. Rewrite each explicit formula of the geometric sequences that are exponential functions in function form. Identify the constant ratio and the y-intercept.

Sequence	Explicit Formula	Exponential Function $f(x) = ab^x$	Constant Ratio	y-Intercept
A	$-2(3)^{x-1}$			
B	$45 \cdot 2^{x-1}$			
C	$1234(0.1)^{x-1}$			
D	$-5\left(\frac{1}{2}\right)^{x-1}$			

Based on the graphs of Sequences E and F, you can tell they do not represent exponential functions.

9. Rewrite each explicit formula in function form and explain why these geometric sequences are not exponential functions.

a. Sequence E: $y = -4(-3)^{x-1}$

b. Sequence F: $y = -16\left(-\frac{1}{4}\right)^{x-1}$

10. You know that all arithmetic sequences are linear functions. What can you say about the relationship between geometric sequences and exponential functions?

11. Complete the table by writing each part of the exponential function that corresponds to each part of the geometric sequence.

Geometric Sequence $g_n = g_1 \cdot r^{n-1}$	Exponential Function $f(x) = ab^x$	Mathematical Meaning
g_n		
$\frac{g_1}{r}$		
r		
n		

ACTIVITY 3.3

Identifying Exponential Functions

As part of a project in health class, Valentina, Logan, and Emily are raising awareness and challenging others to eat a healthy breakfast each morning. Today, they each sent selfies of themselves eating a healthy breakfast to 4 friends and challenged them to do the same the next day. This next day, when others send selfies of themselves eating a healthy breakfast will be considered Day 1 of their results. The following day, only those contacted the previous day will send selfies to 4 friends, and the challenge will continue to spread. Assume everyone contacted completes the challenge and new participants are contacted each day.

The constant ratio of an exponential function must be greater than 0 and not equal to 1.

1. Write an exponential function, $f(x)$, to represent the number of new participants of the challenge as a function of the day number, x.

The results after 4 days of the challenge are shown in the table.

Time (Day)	Number of New Participants
1	12
2	48
3	192
4	768

2. The relationship between time and number of participants is exponential.

   a. Verify the relationship is exponential by identifying the constant ratio.

b. What is the number of new participants for Day 0?
Explain your answer.

c. If the number of new participants for Day 0 is represented by $f(x) = ab^x$, then the number of new participants for Day 1 can be represented by $f(x + 1) = ab^{(x + 1)}$. Complete the table to show the number of new participants as a function of the day in terms of x and $f(x)$.

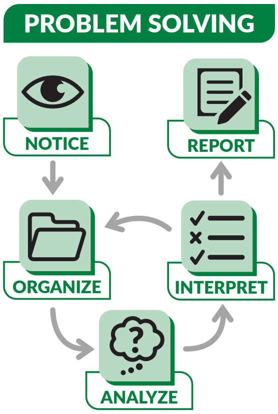
Time (Day)	Number of New Participants		Function Form
x	$f(x)$		$f(x) = ab^x$
0	$f(x)$		$f(x) = ab^x$
1	$f(x + 1)$	12	$f(x + 1) = ab^{(x + 1)}$
2		48	
3		192	
4		768	

d. Use the expressions from the Function Form column of the table and algebra to prove that the table shows a constant ratio between consecutive output values of the function.

Ask Yourself . . .
How can you organize and record your mathematical ideas?

Recall that the quotient of powers rule states that when dividing powers with the same base, you can subtract their exponents.

$\frac{b^{(x + 2)}}{b^{(x + 1)}} = b^{(x + 2) - (x + 1)}$

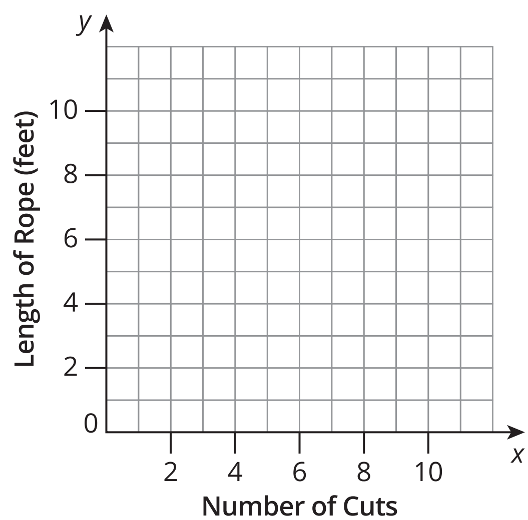


ACTIVITY 3.4 Writing Exponential Functions

A magician is practicing one of his tricks. As part of the trick, he cuts a rope into many pieces and then magically puts the pieces of rope back together. He begins the trick with a 10-foot rope and then cuts it in half. He takes one of the halves and cuts that piece in half. He keeps cutting the pieces in half until he is left with a piece so small he can't cut it anymore.

1. Complete the table to show the length of rope after each of the magician's cuts. Write each length as a whole number, mixed number, or fraction. Then, graph the points from the table.

Number of Cuts	Length of Rope (feet)
0	
1	
2	
3	
4	
5	

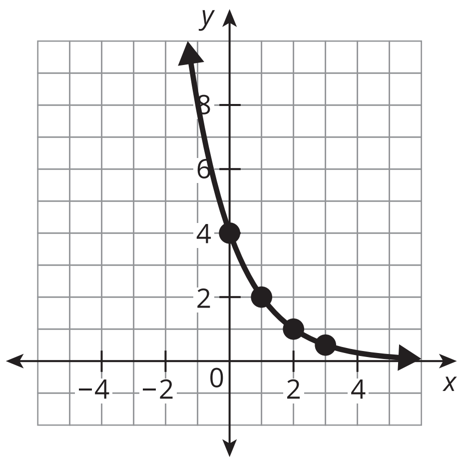


2. Write the function, $L(c)$, to represent the length of the rope as a function of the cut number, c.

3. Use your function to determine the length of the rope after the 7^{th} cut.

4. Write an exponential function of the form $f(x) = ab^x$ for each table and graph, if possible.

a.



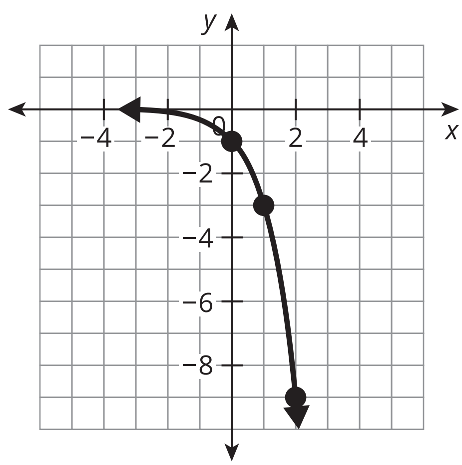
b.

x	y
-2	$-\frac{1}{2}$
-1	-2
0	-8
1	-32

c.

x	y
0	3
1	8
2	13
3	18

d.



Talk the Talk

Did We Mention Constant Ratio?

1. For an exponential function of the form $f(x) = ab^x$, what is the relationship between the base of the power, the expression $\frac{f(x + 1)}{f(x)}$ and the common ratio of the corresponding geometric sequence?

2. How can you decide whether a geometric sequence of the form $g_n = g_1 \cdot b^{n-1}$ represents an exponential function?

Lesson 3 Assignment

Write

Describe the differences between a linear function and an exponential function using your own words.

Remember

All sequences are functions, and some geometric sequences are exponential functions.

The form of an exponential function is $f(x) = ab^x$, where a and b are real numbers and $b > 0$, but $b \neq 1$. The a-value represents the y-intercept, and the b-value represents the constant ratio, or constant multiplier.

Practice

1. Each table shows the population of a city over a three-year period. Write an exponential function to represent each population as a function of time. Round the base to the nearest thousandth.

a.

Blueville	
Time (years)	Population
1	7098
2	7197
3	7298

b.

Yellowville	
Time (years)	Population
1	12,144
2	12,290
3	12,437

c.

Greenville	
Time (years)	Population
1	7860
2	7722
3	7587

2. Consider each situation. If possible, identify a constant ratio and write an exponential function to represent the relationship. Be sure to define your variables.

   a. Diego works in a lab. The number of bacteria over time in a petri dish he is studying is shown in the table.

Bacteria	
Time (years)	Population
0	605
1	2420
2	9680
3	38,720

Lesson 3 Assignment

b. Sarah has been studying the honey bee population. The number of honey bees she documents over time is shown in the table.

Honey Bee Population	
Time (years)	Number of Honey Bees
1	52,910
2	43,069
3	35,058
4	28,537

c. Camilla started depositing money into a savings account. The amount of money over time in the account is shown in the table.

Savings Account	
Time (years)	Value ($)
5	875
10	1200
15	1525
20	1850

Prepare

Write the square root for each perfect square.

1. $\sqrt{16}$
2. $\sqrt{81}$
3. $\sqrt{196}$
4. $\sqrt{225}$

Write the prime factorization of each number.

5. 16
6. 81
7. 196
8. 225

4 Rewriting Square Roots

OBJECTIVES

- Simplify numerical radical expressions involving square roots.
- Multiply and divide with square roots.

NEW KEY TERMS

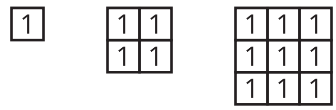
- perfect square
- square root
- radical
- radicand
- product property of radicals
- extracting perfect squares
- index
- quotient property of radicals

You have used operations to compose numeric expressions. How can you operate with numeric radical expressions?

Getting Started

Building Perfect Squares

Each of these models shows a *perfect square*. A **perfect square** is the product of an integer number multiplied by itself.



Remember . . .

A **square root** is one of two equal factors of a given number. For example, 3 is a square root of 9 because $(3)(3) = 9$.

The symbol √ is called a **radical**. The **radicand** is the quantity under a radical.

radical radicand

$\sqrt{25}$

1. Write the area of each perfect square as a power and as a whole number.

2. Write the side length of each perfect square as a radical and as a whole number.

3. Consider the interior squares of each perfect square model.

   a. What is the area of each interior square as a power?

   b. What is the side length of each interior square as a radical?

   c. How could you represent the side length of each perfect square model as a multiplication expression?

4. Can you draw a perfect square model with an area of 20 square units? Why or why not?

Isaiah

Since calculating a square root means to divide by 2, $\sqrt{20} = 10$.

5. Explain why Isaiah is incorrect.

ACTIVITY **4.1**

Rewriting Non-Perfect Squares

Consider this square model. Each interior square has an area of 5 square units.

$$\begin{array}{|c|c|}\hline 5 & 5 \\ \hline 5 & 5 \\ \hline\end{array}$$

1. What is the total area of the square model?

Ask Yourself . . .
Is this a perfect square model?

2. Write the side length of the square model as a radical.

3. Write the side length of the square model as a multiplication expression.

4. Consider the expressions you wrote in Questions 2 and 3. What can you conclude?

5. Write the side length of each square model as a radical and as a multiplication expression.

a. $\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 3 & 3 \\ \hline\end{array}$

b. $\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 2 & 2 & 2 \\ \hline 2 & 2 & 2 \\ \hline\end{array}$

c. $\begin{array}{|c|c|c|}\hline 7 & 7 & 7 \\ \hline 7 & 7 & 7 \\ \hline 7 & 7 & 7 \\ \hline\end{array}$

d. $\begin{array}{|c|c|c|c|}\hline 5 & 5 & 5 & 5 \\ \hline 5 & 5 & 5 & 5 \\ \hline 5 & 5 & 5 & 5 \\ \hline 5 & 5 & 5 & 5 \\ \hline\end{array}$

e. $\begin{array}{|c|c|c|}\hline 4 & 4 & 4 \\ \hline 4 & 4 & 4 \\ \hline 4 & 4 & 4 \\ \hline\end{array}$

f. $\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline\end{array}$

6. Consider your answers to Question 5. How can you tell when a square model represents a perfect square? Explain your thinking.

Avery noticed a pattern and made a conjecture. She said, "I can write $\sqrt{20}$ as $\sqrt{(4 \cdot 5)}$. Since the model shows that $\sqrt{20}$ is equal to $2 \cdot \sqrt{5}$, that means that $\sqrt{(4 \cdot 5)} = \sqrt{4} \cdot \sqrt{5}$. This will work for all radicals!"

7. Do you think Avery's conjecture is correct? Use the models from Question 5 to explain your reasoning.

Parker

I see a connection to division too!

$$\frac{\sqrt{12}}{\sqrt{4}} = \sqrt{\left(\frac{12}{4}\right)}$$

8. Explain Parker's reasoning.

ACTIVITY 4.2 Multiplying and Dividing Radicals

The variable n in the expression $\sqrt[n]{a}$ is called the **index**. For square roots, the index is 2. The index is often not written for square roots.

Avery and Parker discovered two important properties of radicals.

The **product property of radicals** states that $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$, when a and b are greater than 0.

Suppose you have the product $\sqrt{15} \cdot \sqrt{5}$. You can use the product property of radicals to rewrite this radical expression.

WORKED EXAMPLE

$$\begin{aligned}\sqrt{15} \cdot \sqrt{5} &= \sqrt{15 \cdot 5} \\ &= \sqrt{3 \cdot 5 \cdot 5} \\ &= \sqrt{3 \cdot 5^2} \\ &= \sqrt{3} \cdot \sqrt{5^2} \\ &= 5\sqrt{3}\end{aligned}$$

Notice that the interior perfect square, 5^2, was extracted from inside the radicand. The process of removing perfect square numbers from under a radical symbol is **extracting perfect squares.**

1. Carlos calculated that $\sqrt{15 \cdot 5} = 3 \cdot 5$, or 15. What error did Carlos make?

2. Rewrite each radical expression by extracting perfect squares.

   a. $\sqrt{50}$

   b. $\sqrt{24}$

   c. $3\sqrt{20}$

   d. $\sqrt{3} \cdot \sqrt{6}$

   e. $\sqrt{3} \cdot \sqrt{12}$

   f. $\sqrt{8} \cdot \sqrt{12}$

3. Determine each product. Then rewrite the product by extracting perfect squares, if possible.

a. $\sqrt{6} \cdot \sqrt{2}$

b. $\sqrt{10} \cdot \sqrt{3}$

c. $\sqrt{14} \cdot \sqrt{2}$

d. $\sqrt{8} \cdot \sqrt{7}$

e. $\sqrt{10} \cdot \sqrt{15}$

The **quotient property of radicals** states that $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, where $a \geq 0$ and $b > 0$.

4. Rewrite each square root. Extract perfect squares if possible.

a. $\sqrt{\frac{15}{64}}$

b. $\sqrt{\frac{81}{10^2}}$

c. $\sqrt{\frac{48}{25}}$

Jasmine

I can continue to rewrite $\frac{\sqrt{15}}{8}$. Since $\frac{15}{8} = 1.875$, then $\frac{\sqrt{15}}{8} = \sqrt{1.875}$.

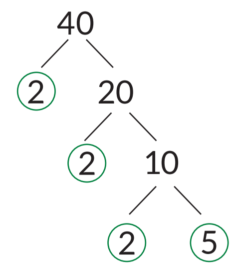
5. Explain how Jasmine reasoned incorrectly.

Another way to rewrite radicals is to use prime factorization.

WORKED EXAMPLE

Rewrite $\sqrt{40}$ in simplest radical form.

Write the prime factorization of 40.



$40 = 2^2 \cdot 2 \cdot 5$

Extract the perfect squares.

$\sqrt{40} = 2 \cdot \sqrt{(2 \cdot 5)}$

Rewrite the radicand.

$\sqrt{40} = 2 \cdot \sqrt{10}$

6. Use prime factorization to rewrite each radical in simplest radical form.

a. $\sqrt{24}$

b. $\sqrt{75}$

c. $\sqrt{35}$

d. $\sqrt{126}$

7. Consider the different strategies you have learned to rewrite radicals. When might you choose each strategy based on the radicand?

Talk the Talk

Strategy Talk

Consider each square root. Determine the strategy you would use to simplify each radical. Then, rewrite each radical in simplest radical form and show your work.

1. $\sqrt{45}$

2. $\sqrt{81}$

3. $\sqrt{600}$

Lesson 4 Assignment

Write

Explain how you can use the area and side length of a square to understand the components of radical form.

Remember

The product property of radicals states that $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$ when a and b are greater than 0.
The quotient property of radicals states that $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, where $a \geq 0$ and $b > 0$.

Rewrite each expression in in simplest radical form.

1. $\sqrt{12}$

2. $\sqrt{30}$

3. $\sqrt{27}$

4. $3\sqrt{75}$

5. $\sqrt{15} \cdot \sqrt{6}$

6. $\sqrt{14} \cdot \sqrt{2}$

7. $\sqrt{8} \cdot \sqrt{7}$

8. $\sqrt{10} \cdot \sqrt{15}$

9. Determine each product. Extract roots if possible.

   a. $\sqrt{5} \cdot \sqrt{20}$

   b. $\sqrt{6} \cdot \sqrt{3}$

   c. $\sqrt{\frac{3}{4}} \cdot \sqrt{\frac{1}{5}}$

   d. $\sqrt{72} \cdot \sqrt{2}$

Lesson 4 Assignment

10. Determine each quotient. Extract roots if possible.

a. $\frac{\sqrt{42}}{\sqrt{7}}$

b. $\frac{\sqrt{26}}{\sqrt{2}}$

c. $\sqrt{\frac{490}{10}}$

d. $\frac{\sqrt{8}}{\sqrt{2}}$

Prepare

Use the properties of exponents to write an equivalent expression in simplest form.

1. $\frac{b^3}{b^0}$

2. $\frac{a \cdot x^5}{a \cdot x^4}$

3. $a \cdot b^2 \cdot a^2 \cdot b^3$

4. $b^{-2} \cdot b^2 \cdot a^0$

5 Rational Exponents and Graphs of Exponential Functions

OBJECTIVES

- Rewrite powers with rational exponents as radical expressions.
- Rewrite radical expressions as powers with rational exponents.
- Use the properties of exponents to interpret output values for non-integer input values in exponential functions.
- Construct exponential functions and identify a common ratio between output values in a graph, a table, and the equation.

NEW KEY TERM

- horizontal asymptote

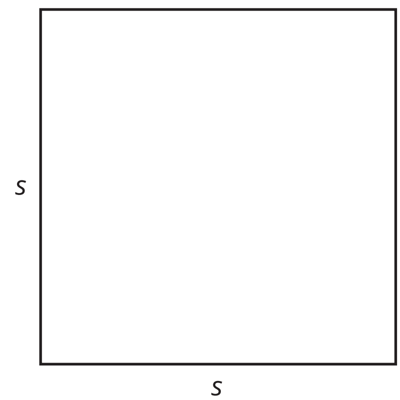
You have determined the constant ratio of exponential functions with integer inputs.

How can you use a constant ratio to determine output values with non-integer inputs?

Getting Started

Squares and Cubes

Write an expression to represent the side length, *s*, of each square given its area. Then, approximate the value of *s*.

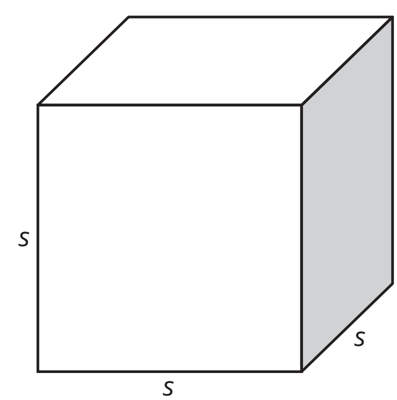


1. Area = 5 ft^2

2. Area = 2 cm^2

3. Area = 81 $in.^2$

Write an expression to represent the side length, *s*, of each cube given its volume. Then, approximate the value of *s*.



4. Volume = 51 ft^3

5. Volume = 343 cm^3

6. Volume = 2 $in.^3$

ACTIVITY 5.1

Characteristics of Exponential Growth

In a laboratory experiment, a certain type of bacteria doubles each hour.

1. Suppose a bacteria population starts with just 1 bacterium.

   a. Complete the table to show the population of bacteria, $f(x)$, over time, x.

x	1	2	3	4
$f(x)$				

   b. Determine the constant ratio and y-intercept. Then, write the exponential function that represents the growth of the bacteria population over time. Show your work.

The constant ratio is a multiplier. To determine the next term of a geometric sequence, you multiply by this value.

   c. How is the constant multiplier evident in the problem situation?

2. Graph the exponential function to show bacteria growth over time on the coordinate plane located at the end of the lesson.

ACTIVITY 5.2 The Square Root Constant Ratio of an Exponential Function

The table shown represents the function $f(x) = 2^x$, which models the laboratory experiment showing that a certain population of bacteria can double each hour.

x	0	1	2	3
$f(x)$	2^0	2^1	2^2	2^3
	1	2	4	8

In the table, the interval between the input values is 1 and the constant multiplier is 2 at the point when the interval changes. What effect, if any, is there on the constant multiplier if the input interval is different?

An exponential function is continuous, meaning that there is a value $f(x)$ for every real number value x.

1. Consider the ratio $\frac{f(2)}{f(0)}$.

   a. Describe the interval of input values. Then, determine the multiplier.

   b. Write two additional ratios that have the same multiplier. Explain your reasoning.

   c. Write a new pair of ratios that have the same multiplier but span a different input interval than the intervals you have already analyzed. Justify your answer.

Nahimana, Lucas, and Luna are interested in the population of bacteria at each $\frac{1}{2}$-hour interval. They have values for the exponential function $f(x)$ when x is an integer. They need the values of the exponential function when x is a rational number between integers.

2. The three students used the idea of the constant multiplier to estimate the value of $f\left(\frac{1}{2}\right)$ for the function $f(x) = 2^x$.

Nahimana

I know the constant multiplier for an interval of 1 is 2. I want to split each interval of 1 into two equal parts, which means I need two equal multipliers.

x	0	$\frac{1}{2}$	1
$f(x)$	2^0	$2^{\frac{1}{2}}$	2^1
	1		2

$1 \cdot r \cdot r = 2$

Multiply by 2.

So, $r^2 = 2$.

Lucas

If r is a constant multiplier for the function as it grows by consecutive integers, it can be split into two equal multipliers of $\sqrt{r}$, because r can be split into two equal factors of $\sqrt{r}$.

$(\sqrt{r})^2 = 2$

Luna

If $f(0) = 2^0 = 1$ and $f(1) = 2^1 = 2$, then $f\left(\frac{1}{2}\right) = 2^{\frac{1}{2}}$ must be equal to 1.5.

a. Use Nahimana's and Lucas's thinking to determine $f\left(\frac{1}{2}\right)$. Write $f\left(\frac{1}{2}\right)$ as a power of 2 and in radical form. Then, enter the values in the table.

x	0	$\frac{1}{2}$	1	2	3
$f(x)$	2^0		2^1	2^2	2^3
	1		2	4	8

b. Use the graph you created in the previous activity to approximate $f\left(\frac{1}{2}\right)$ as a decimal.

c. Explain why Luna's thinking is incorrect.

The number 2 is a rational number because it can be represented as the ratio of two integers. The number $\sqrt{2}$ is an irrational number because it cannot be represented as the ratio of two integers.

Consider the expression $\sqrt{2} = 2^{\frac{1}{2}}$. The square root symbol ($\sqrt{\ }$) is interpreted as the rational exponent $\frac{1}{2}$. All the properties with integer exponents you previously learned continue to apply, even when the exponent is a rational number.

The tables shown represent two different equivalent representations of the constant multiplier, $2^{\frac{1}{2}}$ or $\sqrt{2}$, for the function $f(x) = 2^x$.

	Rational Exponent Representation	
$f(0)$	2^0	
$f\left(\frac{1}{2}\right)$	$2^{\frac{1}{2}}$	$2^0 \cdot 2^{\frac{1}{2}}$
$f(1)$	2^1	$2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$
$f\left(\frac{3}{2}\right)$	$2^{\frac{3}{2}}$	$2^1 \cdot 2^{\frac{1}{2}}$

	Radical Form Representation	
$f(0)$	1	
$f\left(\frac{1}{2}\right)$	$\sqrt{2}$	$1 \cdot \sqrt{2}$
$f(1)$	2	$\sqrt{2} \cdot \sqrt{2}$
$f\left(\frac{3}{2}\right)$	$2\sqrt{2}$	$2 \cdot \sqrt{2}$

3. Use the properties of exponents to justify that $2^1 \cdot 2^{\frac{1}{2}} = 2^{\frac{3}{2}}$. Then, use the graph to estimate $2^{\frac{3}{2}}$ as a decimal.

In the expression $\sqrt[n]{a}$, the n is called the *index*.

A rational exponent can be rewritten in radical form using the definition $a^{\frac{1}{n}} = \sqrt[n]{a}$. When the index is 2, it is usually implied rather than written.

Let's consider the properties of exponents to rewrite expressions in equivalent forms.

Remember . . .
The process of removing perfect square numbers from under a radical symbol is called extracting perfect squares.

WORKED EXAMPLE

Consider the expression $2^{\frac{3}{2}}$.

Using the power of a power rule: $2^{\frac{3}{2}} = \left(2^{\frac{1}{2}}\right)^3$ or $(2^3)^{\frac{1}{2}}$.

You can use the definition of rational exponents to rewrite each expression in radical form.

$$\begin{aligned}\left(2^{\frac{1}{2}}\right)^3 &= (\sqrt{2})^3 \\ &= \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \\ &= 2\sqrt{2}\end{aligned}$$

$$\begin{aligned}(2^3)^{\frac{1}{2}} &= \sqrt{2^3} \\ &= \sqrt{2 \cdot 2 \cdot 2} \\ &= \sqrt{2^2 \cdot 2} \\ &= 2\sqrt{2}\end{aligned}$$

Remember . . .
The product property of radicals states that $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$ when a and b are greater than 0.

You can see that $\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} = \sqrt{2 \cdot 2 \cdot 2}$.

4. Use the Worked Example to explain the product property of radicals.

Now let's consider how to use the product of powers rule to rewrite the expression with rational exponents in radical form.

WORKED EXAMPLE

Consider the expression $2^{\frac{3}{2}}$.

Using the product of powers rule:

$$\begin{aligned} 2^{\frac{3}{2}} &= 2^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} \\ &= \left(2^{\frac{1}{2}}\right)\left(2^{\frac{1}{2}}\right)\left(2^{\frac{1}{2}}\right) \\ &= \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \\ &= \sqrt{2 \cdot 2 \cdot 2} \end{aligned}$$

You can use the definition of *rational exponents* to rewrite each expression in radical form.

5. Analyze the Worked Examples.

a. Explain why $\sqrt{2} \cdot \sqrt{2} = 2$.

You will learn and practice more with rational exponents later in this lesson.

b. Explain why $\sqrt{2^2} = 2$.

6. Mason and Sebastian each calculate the population of bacteria when $t = \frac{5}{2}$ hours.

Mason says that when $t = \frac{5}{2}$ hours, $f\left(\frac{5}{2}\right) = \left(\sqrt{2}\right)^5$ bacteria.

Sebastian says that $f\left(\frac{5}{2}\right) = 4\sqrt{2}$ bacteria.

Who's correct?

Use definitions and rules to justify your reasoning. Then, use the graph to estimate the value of $f\left(\frac{5}{2}\right)$ as a decimal on the graph.

ACTIVITY 5.3 The Cube Root Constant Ratio of an Exponential Function

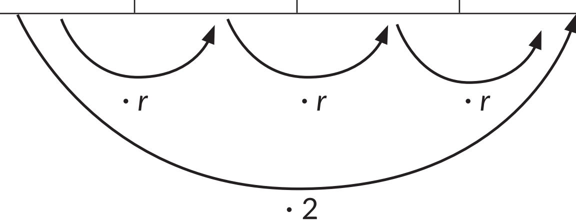
Think About . . .

What is the constant multiplier you can use to build this relationship over intervals of $\frac{1}{3}$?

In the previous activity, you looked at the exponential function $f(x) = 2^x$. When the input interval is 1, the constant ratio is 2^1, and when the input interval is $\frac{1}{2}$, the constant multiplier is $2^{\frac{1}{2}}$.

Now, let's think about the constant multiplier when the input interval is $\frac{1}{3}$.

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$
$f(x)$	2^0			2^1	
	1			2	



In the expression $\sqrt[n]{a}$, only $n = 2$ is implied rather than written. All other index values must be written.

1. Complete the table of values for the exponential function $f(x) = 2^x$. Represent $f(x)$ as a rational exponent and in radical form. Show your work.

2. Write the points represented in the table as ordered pairs. Use the graph you created in the previous activity to estimate each output value as a decimal.

ACTIVITY
5.4

Negative Exponents and Asymptotes

You have explored output values with integer and rational exponents for the exponential function $f(x) = 2^x$. What happens when x is a negative value?

Remember . . .

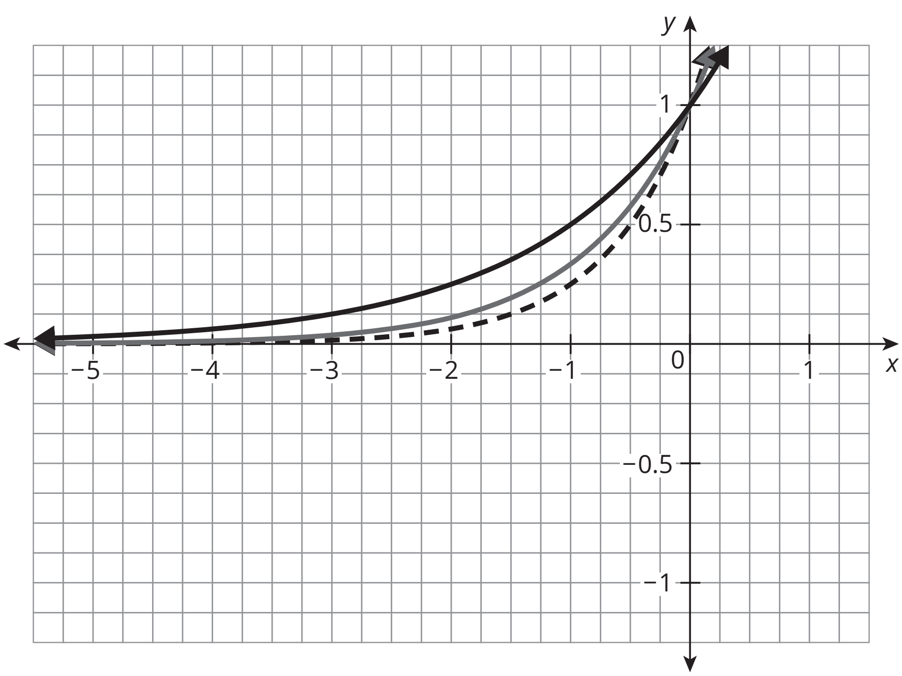
The negative exponent rule states that $a^{-1} = \frac{1}{a}$, $a^{-2} = \frac{1}{a^2}$, and so on.

1. Use what you know about negative exponents to complete the table for the function $f(x) = 2^x$.

x	-4	-3	-2	-1	0
$f(x)$					2^0
					1

2. Consider the table of values you just completed. How does $f(x)$ change as x approaches negative infinity?

An exponential function has a *horizontal asymptote*. For exponential functions, a **horizontal asymptote** is a horizontal line that a function gets closer and closer to, but never intersects.



3. Label the function $f(x) = 2^x$ on the coordinate plane. Identify the horizontal asymptote.

4. The functions $g(x) = 3^x$ and $h(x) = 4^x$ are also shown on the coordinate plane.

   a. Identify each function and explain how you know.

   b. Determine the horizontal asymptotes for the functions. Compare these with the horizontal asymptote of $f(x) = 2^x$.

5. Compare how each of the three functions approaches its horizontal asymptote. What is the same and what is different?

6. Identify the domain and range of each of the three functions. What do you notice?

ACTIVITY 5.5

Rational Exponents

In this lesson, you have been writing powers with rational exponents. You have shown that you can rewrite a rational exponent in radical form, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

1. Rewrite each expression as a power.

   a. $\sqrt[3]{7}$

   b. $\sqrt[5]{x}$

   c. $\sqrt{y}$

Remember . . .

In the expression $\sqrt[n]{a}$, the n is called the *index*.

2. Rewrite each expression in radical form.

   a. $8^{\frac{1}{4}}$

   b. $z^{\frac{1}{5}}$

   c. $m^{\frac{1}{3}}$

3. Use the properties of exponents to rewrite $a^{\frac{m}{n}}$ in radical form.

4. Rewrite each expression in radical form. Simplify if possible.

a. $4^{\frac{3}{2}}$

b. $5^{\frac{3}{4}}$

c. $x^{\frac{4}{5}}$

d. $4^{\frac{2}{3}}$

5. Rewrite each expression as a power with a rational exponent.

a. $\left(\sqrt[4]{2}\right)^3$

b. $\left(\sqrt{5}\right)^4$

c. $\sqrt[5]{x^8}$

d. $\left(\sqrt[5]{y}\right)^{10}$

Ask Yourself . . .
How does representing mathematics in multiple ways help to communicate reasoning?

Let's analyze the product of radicals.

WORKED EXAMPLE

You can rewrite the numeric expression $\left(\sqrt[3]{2}\right)^2\left(\sqrt{2}\right)$ in radical form using the rules of exponents.

$\left(2^{\frac{1}{3}}\right)^2 (2)^{\frac{1}{2}}$	Definition of rational exponents
$\left(2^{\frac{2}{3}}\right)\left(2^{\frac{1}{2}}\right)$	Power of a power rule
$2^{\frac{2}{3}+\frac{1}{2}}$	Product of powers rule
$2^{\frac{7}{6}}$	Add fractions
$\sqrt[6]{2^7}$	Definition of rational exponents

6. Kayla rewrote the expression $\sqrt[6]{2^7}$ in a different way.

$$2^{\frac{7}{6}} = 2^{\frac{6}{6}+\frac{1}{6}} = 2^{\frac{6}{6}} \cdot 2^{\frac{1}{6}} = 2\sqrt[6]{2}$$

Is she correct? Justify your reasoning.

Let's revisit the product property of radical and the process of extracting roots.

Suppose you have the product $\sqrt{15} \cdot \sqrt{5}$. You can use properties of exponents to rewrite this radical expression.

WORKED EXAMPLE

$$\begin{aligned}\sqrt{15} \cdot \sqrt{5} &= 15^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \\ &= (15 \cdot 5)^{\frac{1}{2}} \\ &= (3 \cdot 5 \cdot 5)^{\frac{1}{2}} \\ &= (3 \cdot 5^2)^{\frac{1}{2}} \\ &= 3^{\frac{1}{2}} \cdot 5 \\ &= 5\sqrt{3}\end{aligned}$$

$$\begin{aligned}\sqrt{15} \cdot \sqrt{5} &= \sqrt{15 \cdot 5} \\ &= \sqrt{3 \cdot 5 \cdot 5} \\ &= \sqrt{3 \cdot 5^2} \\ &= 5\sqrt{3}\end{aligned}$$

7. Use either strategy in the Worked Example to rewrite each radical expression by extracting perfect squares.

a. $\sqrt{50}$

b. $\sqrt{24}$

c. $3\sqrt{20}$

d. $\sqrt{3} \cdot \sqrt{6}$

e. $\sqrt{3} \cdot \sqrt{12}$

f. $\sqrt{8} \cdot \sqrt{12}$

8. Explain how the properties of rational exponents extend from the properties of integer exponents.

9. Consider the calculations you made throughout this lesson and the definition of a rational number to answer each question.

   a. Is the product of a nonzero rational number and an irrational number always, sometimes, or never a rational number? Explain your reasoning.

   b. Is the product of an irrational number and an irrational number always, sometimes, or never a rational number? Explain your reasoning.

WORKED EXAMPLE

You can rewrite the numeric expression $\frac{\sqrt[3]{x}\,\sqrt{x}}{\sqrt[6]{x}}$ in radical form using rules of exponents.

$\frac{x^{\frac{1}{3}}x^{\frac{1}{2}}}{x^{\frac{1}{6}}}$	Rewrite using rational exponents.
$\frac{x^{\frac{1}{3}+\frac{1}{2}}}{x^{\frac{1}{6}}}$	Apply the product of powers rule.
$\frac{x^{\frac{5}{6}}}{x^{\frac{1}{6}}}$	Add fractions.
$x^{\frac{4}{6}}$	Apply the quotient of powers rule.
$x^{\frac{2}{3}}$	Rewrite fraction.
$\sqrt[3]{x^2}$	Rewrite in radical form.

10. Rewrite each expression using the rules of exponents. Write responses in radical form.

a. $\left(3^{\frac{3}{2}}\right)^3$

b. $25^{\frac{3}{2}}$

c. $\left(2x^{\frac{1}{2}}\,y^{\frac{1}{3}}\right)\left(3x^{\frac{1}{2}}y\right)$

d. $\left(\frac{24m^{\frac{3}{4}}n^{\frac{5}{2}}}{36m^{\frac{2}{7}}n^{\frac{2}{5}}}\right)^0$

Talk the Talk

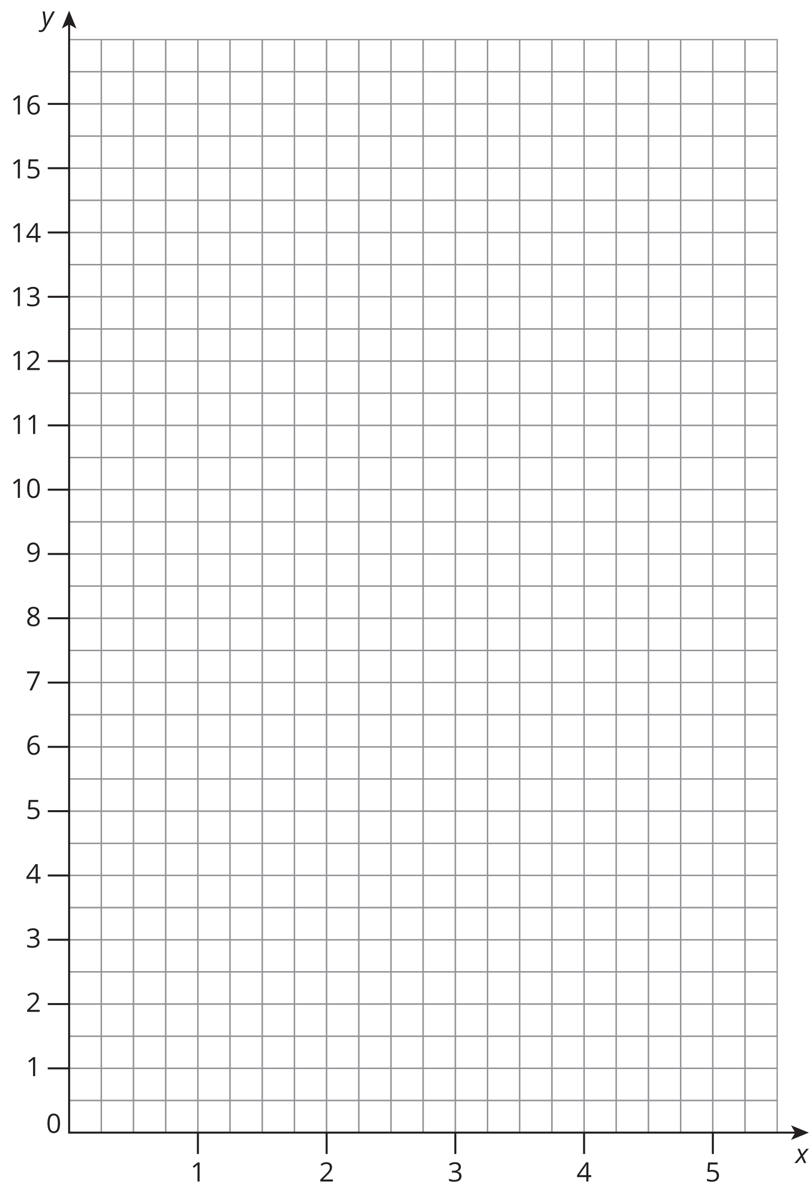
May the Fourths Be With You

1. Consider the exponential function $f(x) = 2^x$. Complete the table. Then, compare the table of fourths to the tables you completed for halves and thirds. What patterns do you notice in the multiplier?

x	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	1
$f(x)$	2^0				2^1
	1				2

2. Match each rational expression with the appropriate radical expression.

Rational Expression	Radical Expression
1. $2^{\frac{5}{10}}$	A. $\sqrt[5]{10^2}$
2. $10^{\frac{5}{2}}$	B. $\sqrt[5]{2^{10}}$
3. $10^{\frac{2}{5}}$	C. $\sqrt[10]{2^5}$
4. $5^{\frac{10}{2}}$	D. $\sqrt{10^5}$
5. $2^{\frac{10}{5}}$	E. $\sqrt{5^{10}}$
6. $5^{\frac{2}{10}}$	F. $\sqrt[10]{5^2}$



y
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
x

Lesson 5 Assignment

Write

Describe how the components of radical form and rational exponent form of an equivalent expression are related.

Remember

If the difference in the input values is the same, an exponential function shows a constant multiplier between output values, no matter how large or how small the gap between input values.

If n is an integer greater than 1, then $\sqrt[n]{a} = a^{\frac{1}{n}}$.

Practice

Rewrite each radical using a rational exponent.

1. $\sqrt[4]{88}$
2. $\sqrt[10]{46}$
3. $\sqrt[6]{x}$
4. $\sqrt{z}$

Rewrite each power in radical form.

5. $9^{\frac{1}{3}}$
6. $5^{\frac{1}{2}}$
7. $20^{\frac{1}{5}}$
8. $41^{\frac{1}{8}}$

Rewrite each power in radical form. Simplify your answer, if possible.

9. $16^{\frac{3}{2}}$
10. $5^{\frac{7}{4}}$
11. $12^{\frac{2}{5}}$
12. $8^{\frac{4}{3}}$

Lesson 5 Assignment

13. $\left(3x^{\frac{1}{3}}y\right)\left(4x^{\frac{2}{3}}y^{\frac{1}{2}}\right)$

14. $(225x^{2}y^{0}z^{3})^{\frac{1}{2}}$

Rewrite each expression using a rational exponent. Simplify your answer, if possible.

15. $\sqrt[5]{10^{4}}$

16. $(\sqrt[4]{t})^{4}$

17. $(\sqrt{w})^{6}$

18. $\sqrt[9]{h^{3}}$

Prepare

Rewrite each expression by combining like terms.

1. $-3x + 4y - 9x - 5y$

2. $2xy^{2} + 5x^{2}y - 7xy + xy^{2}$

3. $6 - m^{2} + 5m^{2}$

4. $-8 - (-4k) + 7 + 1 - 4k$

TOPIC 1 SELF-REFLECTION

Name: ______________________________

Introduction to Exponential Functions

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 **represents I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Introduction to Exponential Functions* topic by:

TOPIC 1: *Introduction to Exponential Functions*	Beginning of Topic	Middle of Topic	End of Topic
using the rules of integer exponents to simplify numeric and algebraic expressions.	◯	◯	◯
justifying the exponent rules, including product of powers, quotient of powers, power of a power, and zero and negative exponent rules.	◯	◯	◯
writing an equation in function notation for an exponential function represented as a table, a graph, a set of ordered pairs, or a scenario.	◯	◯	◯
evaluating an exponential function for any unknown value.	◯	◯	◯
using the equation, graph, or scenario to identify key characteristics of an exponential function, including the y-intercept, the horizontal asymptote, the x-intercept, and intervals of increase or decrease.	◯	◯	◯
describing what the key characteristics of a function indicate about the problem scenario.	◯	◯	◯
creating a graph, including all of the key characteristics, that matches a given scenario.	◯	◯	◯

continued on the next page

TOPIC 1 SELF-REFLECTION *continued*

TOPIC 1: *Introduction to Exponential Functions*	Beginning of Topic	Middle of Topic	End of Topic
explaining that the parent function for an exponential relationship is $f(x) = ab^x$, where $a = 1$ and $b > 0$.	◯	◯	◯
demonstrating that an exponential function has a constant ratio over equal intervals.	◯	◯	◯
identifying an exponential function from a table of values, an equation, a graph, or a scenario.	◯	◯	◯
simplifying square roots by extracting perfect squares.	◯	◯	◯
using the rules of exponents to rewrite numeric and algebraic expressions with rational exponents.	◯	◯	◯
applying the rule $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ to rewrite expressions.	◯	◯	◯

TOPIC 1 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Introduction to Exponential Functions* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 1 SUMMARY

Introduction to Exponential Functions Summary

LESSON 1 Properties of Powers with Integer Exponents

An expression used to represent the product of a repeated multiplication is a *power.* A **power** has a base and an *exponent*. The **base** of a power is the expression that is used as a factor in the repeated multiplication. The **exponent** of a power is the number of times that the base is used as a factor in the repeated multiplication.

You can write a power as a product by writing out the repeated multiplication.

$$2^7 = (2)(2)(2)(2)(2)(2)(2)$$

The power 2^7 can be read as:

- "two to the seventh power."
- "the seventh power of two."
- "two raised to the seventh power."

Parentheses can change the value of expressions containing exponents. When the negative sign is not in parentheses, it's not part of the base. For example, $-1^2 = -1$, but $(-1)^2 = 1$.

NEW KEY TERMS

- power [potencia]
- base [base]
- exponent [exponente]
- horizontal asymptote [asíntota horizontal]
- perfect square
- square root
- radical [radical]
- radicand [radicando]
- product property of radicals [propiedad del producto de radicales]
- extracting perfect squares
- index [índice]
- quotient property of radicals [propiedad del cociente de radicales]

Properties of Powers	Words	Rule
Product of powers rule	To multiply powers with the same base, keep the base and add the exponents.	$a^m a^n = a^{m+n}$
Power of a power rule	To simplify a power of a power keep the base and multiply the exponents.	$(a^m)^n = a^{mn}$
Quotient of powers rule	To divide powers with the same base, keep the base and subtract the exponents.	$\frac{a^m}{a^n} = a^{m-n}$, if $a \neq 0$
Zero power	The zero power of any number, except for 0, is 1.	$a^0 = 1$, if $a \neq 0$
Negative exponents in the numerator	An expression with a negative exponent in the numerator and a 1 in the denominator equals 1 divided by the power with its opposite exponent placed in the denominator.	$a^{-m} = \frac{1}{a^m}$, if $a \neq 0$ and $m > 0$
Negative exponents in the denominator	An expression with a negative exponent in the denominator and a 1 in the numerator equals the power with its opposite exponent.	$\frac{1}{a^{-m}} = a^m$, if $a \neq 0$ and $m > 0$

LESSON 2 Analyzing Properties of Powers

The properties of powers can be used to simplify numeric and algebraic expressions.

For example, you can simplify the expression $\left(\frac{x^5}{x^4}\right)^3$.

$\left(\frac{x^5}{x^4}\right)^3 = (x^1)^3$ quotient of powers rule

$= x^3$ power of a power rule

LESSON 3 Geometric Sequences and Exponential Functions

An exponential function is a function of the form $f(x) = ab^x$, where a and b are real numbers and b is greater than 0 but not equal to 1.

Geometric sequences with positive common ratios belong in the exponential function family. The common ratio of a geometric sequence is the base of an exponential function.

If a geometric sequence represents an exponential function, you can use the product of powers rule and the definition of negative exponents to rewrite the explicit formula for the sequence as an exponential function.

For example, to represent $g_n = 45(2)^{n-1}$ using function notation, first rewrite it as $f(n) = 45(2)^{n-1}$.

Next, rewrite the expression $45(2)^{n-1}$.

$f(n) = 45(2)^n(2)^{-1}$	product of powers rule
$f(n) = 45(2)^{-1}(2)^n$	commutative property
$f(n) = (45)(\frac{1}{2})(2)^n$	definition of negative exponent
$f(n) = \frac{45}{2}(2)^n$	multiply

So, $g_n = 45(2)^{n-1}$ written in function notation is $f(n) = \frac{45}{2}(2)^n$.

The variable a in $f(x) = ab^x$ is the y-intercept, and b is the constant ratio.

LESSON 4 Rewriting Square Roots

A **square root** is one of two equal factors of a given number. Every positive number has two square roots: a positive square root and a negative square root. The positive square root is called the *principal square root*.

The symbol $\sqrt{10}$ is called a **radical**. The **radicand** is the quantity under a radical. For example, the expression shown is read as "the square root of 25" or as "radical 25."

radical radicand

$\sqrt{25}$

A **perfect square** is a number that is equal to the product of an integer multiplied by itself. In the example above, 25 is a perfect square because it is equal to the product of 5 multiplied by itself.

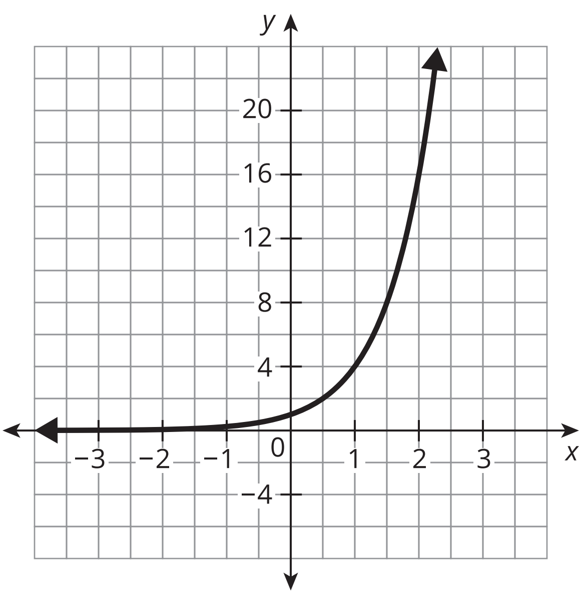
Rewrite radical expressions with an index of 2 by **extracting perfect squares.** This is the process of removing perfect squares from under the radical symbol.

For example, consider the irrational number represented by the radical expression $\sqrt{40}$.

$$\sqrt{40} = \sqrt{(4 \cdot 10)}$$
$$= \sqrt{4} \cdot \sqrt{10}$$
$$= 2\sqrt{10}$$

LESSON 5 Rational Exponents and Graphs of Exponential Functions

An exponential function is continuous, meaning that there is a value $f(x)$ for every real number value x. If the difference in the input values is the same, an exponential function shows a constant ratio between output values, no matter how large or how small the gap between input values. A constant ratio can be used to determine output values for integer and for non-integer inputs.



x	$f(x)$
−2	$\frac{1}{16}$
−1	$\frac{1}{4}$
0	1
1	4
2	16

An exponential function has a **horizontal asymptote**, which is a horizontal line that a function gets closer and closer to but never intersects.

Consider the table and graph represented by the function $f(x) = 4^x$.

There are no x-intercepts. The y-intercept is at (0, 1). The horizontal asymptote is $y = 0$. The domain is all real numbers, and the graph increases over the entire domain. The range is $y > 0$.

A rational exponent is an exponent that is a rational number. You can write each nth root using a rational exponent. If n is an integer greater than 1, then $\sqrt[n]{a} = a^{\frac{1}{n}}$.

For example, $\sqrt[4]{b} = b^{\frac{1}{4}}$ and $6^{\frac{1}{5}} = \sqrt[5]{6}$.

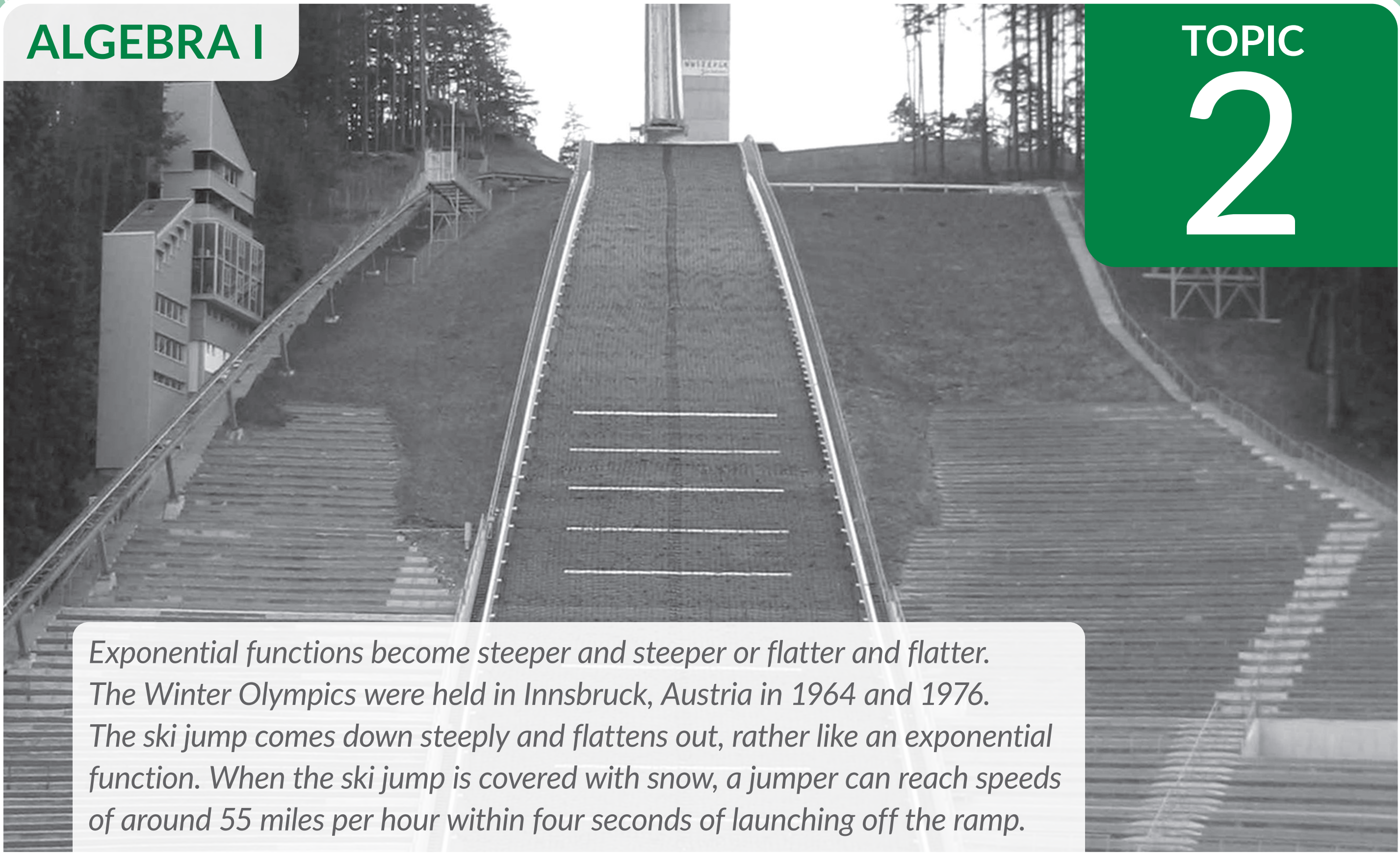
Write expressions with rational exponents in radical form using the known properties of integer exponents. Write the power as a product using a unit fraction. Use the power of a power rule and the definition of a rational exponent to write the power as a radical.

For example, consider the expressions $8^{\frac{2}{3}}$ and $(\sqrt[7]{c})^3$.

$$8^{\frac{2}{3}} = 8^{\left(\frac{1}{3}\right)^2} \qquad (\sqrt[7]{c})^3 = c^{\left(\frac{1}{7}\right)^3}$$
$$= (\sqrt[3]{8})^2 \qquad = c^{\frac{3}{7}}$$

ALGEBRA I

TOPIC 2



Exponential functions become steeper and steeper or flatter and flatter. The Winter Olympics were held in Innsbruck, Austria in 1964 and 1976. The ski jump comes down steeply and flattens out, rather like an exponential function. When the ski jump is covered with snow, a jumper can reach speeds of around 55 miles per hour within four seconds of launching off the ramp.

Using Exponential Equations

1 Exponential Equations for Growth and Decay

OBJECTIVES

- Classify exponential functions as increasing or decreasing.
- Compare formulas for simple interest and compound interest situations.
- Compare the average rate of change between common intervals of a linear and an exponential relationship.
- Write an exponential function that includes a percent increase or decrease with a *b*-value that is a decimal number.
- Solve exponential equations using graphs.

NEW KEY TERMS

- simple interest
- compound interest
- exponential growth function
- exponential decay function

You have analyzed linear and exponential functions and their graphs.

How can you compare linear and exponential functions as increasing and decreasing functions?

Getting Started

Up or Down?

Ask Yourself . . .
What does the structure of each function equation tell you?

Consider each function shown.

$f(x) = -2x + 5$	$g(x) = 2^x$	$h(x) = 0.95^x$
$p(x) = 6\left(\frac{5}{8}\right)^x$	$q(x) = 3(x - 4) - 1$	$r(x) = 2(1 - 0.5)^x$
$v(x) = 4 \cdot 1.10^{(x + 5)}$	$w(x) = -5 \cdot 3^x$	$z(x) = -x + 10$

1. Sort the functions into two groups. Justify your choices.

Increasing Functions   Decreasing Functions

ACTIVITY 1.1

Simple and Compound Interest

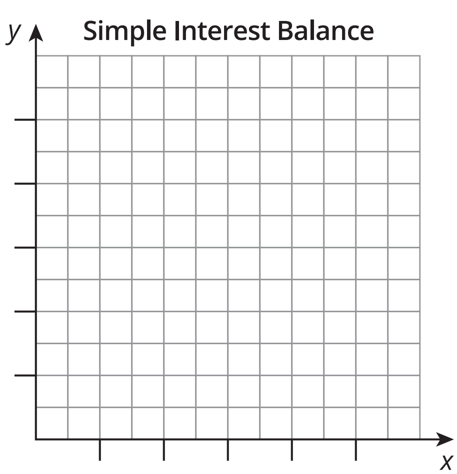
Suppose that your family deposited $10,000 in an interest-bearing account for your college fund that earns simple interest each year. A friend's family deposited $10,000 in an interest-bearing account for their child's college fund that earns compound interest each year.

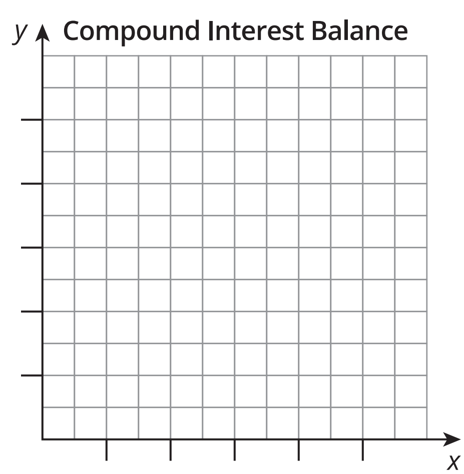
Time (years)	Simple Interest Balance (dollars)	Compound Interest Balance (dollars)
0	10,000	10,000
1	10,400	10,400
2	10,800	10,816
3	11,200	11,248.64
10	14,000	14,802.44

In a **simple interest** account, a percent of the starting balance is added to the account at each interval. The formula for simple interest is $I = Prt$, where P represents the starting amount, or principal, r represents the interest rate, t represents time, and I represents the interest earned. In a **compound interest** account, the balance is multiplied by the same amount at each interval.

1. Study the table of values.

   a. Sketch a graph of each account balance in dollars as a function of the time in years.





   b. Write a function, $s(x)$, to represent the simple interest account and a function, $c(x)$, to represent the compound interest account.

Think about . . .
How accurate does your answer need to be?

2. Use the functions $s(x)$ and $c(x)$ to determine each value.

a. $s(5)$

b. $c(5)$

c. $s(4)$

d. $c(4)$

3. Determine the average rate of change between each pair of values given for each relationship.

Time Intervals (years)	Simple Interest Function (dollars)	Compound Interest Function (dollars)
Between $t = 0$ and $t = 1$		
Between $t = 1$ and $t = 2$		
Between $t = 2$ and $t = 5$		
Between $t = 5$ and $t = 10$		

4. Compare the average rates of change for the simple and compound interest accounts.

a. What do you notice?

b. What does this tell you about the graphs of linear and exponential functions?

5. Use technology to determine when each account will reach the given dollar amount.

   a. When does the simple interest account reach $15,600?

   b. Approximately when does the compound interest account reach $100,000?

Ask Yourself . . .
What tools or strategies can you use to solve this problem?

6. Mei says that given any increasing linear function and any exponential growth function, the output of the exponential function will eventually be greater than the output of the linear function. Is Mei correct? Use examples to justify your thinking.

ACTIVITY 1.2

Identifying Exponential Growth and Decay

Remember . . .
Percent means out of 100. You can write 1.5% as $\frac{1.5}{100}$, which is equivalent to 0.015.

At this moment, the population of Downtown is 20,000, and the population of Uptown is 6000. But, over many years, people have been moving away from Downtown at a rate of 1.5% every year. At the same time, Uptown's population has been growing at a rate of 1.8% each year.

1. What are the independent and dependent quantities in each situation?

2. Which city's population can be represented as an increasing function, and which can be represented as a decreasing function?

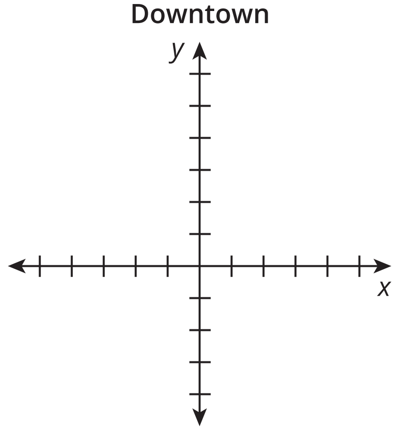
Remember . . .
You can use the y-intercept and the horizontal asymptote to help graph an exponential function.

Let's examine the properties of the graphs of the functions for Downtown and Uptown.

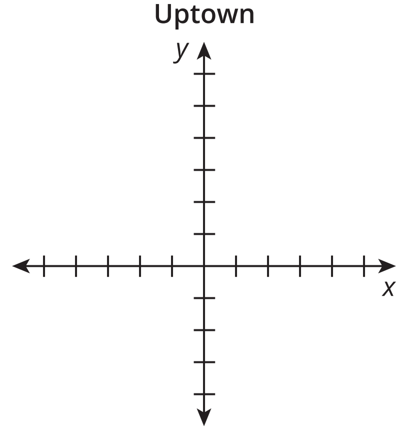
Downtown: $D(t) = 20{,}000(1 - 0.015)^t$ Uptown: $U(t) = 6000(1 + 0.018)^t$

3. Sketch a graph of each function. Label key points.

Downtown



Uptown



4. The functions $D(t)$ and $U(t)$ can each be written as an exponential function of the form $f(x) = ab^x$.

   a. What is the a-value for each function? What does each a-value mean in terms of this problem situation?

   b. What is the b-value for each function? What does each b-value mean in terms of this problem situation?

   c. Compare and explain the meanings of the expressions $(1 - 0.015)^t$ and $(1 + 0.018)^t$ in terms of this problem situation.

5. Analyze the y-intercepts of each function.

   a. Identify the y-intercepts.

   b. Interpret the meaning of each y-intercept in terms of the problem situation.

   c. Describe how you can determine the y-intercept of each function using only the formula for population increase or decrease.

Think About . . .

A decreasing exponential function is denoted by a decimal or fractional b-value between 0 and 1, not by a negative b-value.

An **exponential growth function** has a b-value greater than 1 and is of the form $y = a(1 + r)^x$, where r is the rate of growth. The b-value is $1 + r$. An **exponential decay function** has a b-value greater than 0 and less than 1 and is of the form $y = a(1 - r)^x$, where r is the rate of decay. The b-value is $1 - r$.

ACTIVITY
1.3 Comparing Exponential Functions

Functions

$f(x) = 7000 \cdot 0.969^x$

$f(x) = 7000(1 + 0.028)^x$

$f(x) = 7000 \cdot 1.012^x$

$f(x) = 7000(1 - 0.0175)^x$

$f(x) = 7000 \cdot 1.014^x$

$f(x) = 7000 \cdot 0.9875^x$

Consider the six different city population scenarios.

1. Match each situation with the appropriate function. Explain your reasoning.

   a. A city has a population of 7000. Its population is increasing at a rate of 1.4%.

   b. A city has a population of 7000. Its population is decreasing at a rate of 1.75%.

   c. A city has a population of 7000. Its population is increasing at a rate of 1.2%.

   d. A city has a population of 7000. Its population is decreasing at a rate of 3.1%.

   e. A city has a population of 7000. Its population is increasing at a rate of 2.8%.

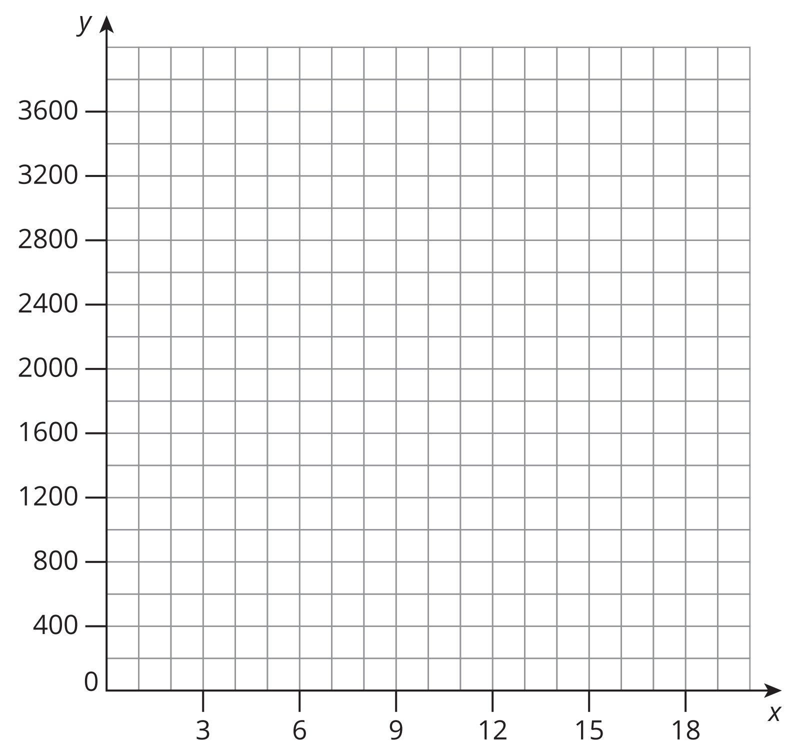
   f. A city has a population of 7000. Its population is decreasing at a rate of 1.25%.

Talk the Talk

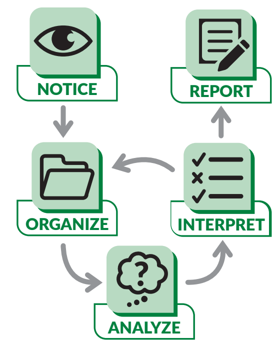
And More, Much More Than This . . .

A scientist is researching certain bacteria that have been found recently in the large animal cages at a local zoo. He starts with 200 bacteria that he intends to grow and study. He determines that every hour the number of bacteria increases by 25%.

1. Write a function and sketch a graph to represent this problem situation. Then, estimate the number of hours the scientist should let the bacteria grow to have no more than 2000 bacteria.



PROBLEM SOLVING



Review the steps and questions from the problem-solving model.

Ask Yourself . . .
Can others follow and understand your process and reasoning?

Lesson 1 Assignment

Write

Explain the difference between simple interest and compound interest.

Remember

An exponential growth function has a *b*-value greater than 1 and is of the form $y = a(1 + r)^x$, where r is the rate of growth. An exponential decay function has a *b*-value greater than 0 and less than 1 and is of the form $y = a(1 - r)^x$, where r is the rate of decay.

Practice

1. Valentina just received a $2500 bonus check from her employer. She is going to put it into an account that will earn interest. The Basic savings account at her bank earns 6% annual simple interest. The Gold savings account earns 4.5% interest compounded annually.

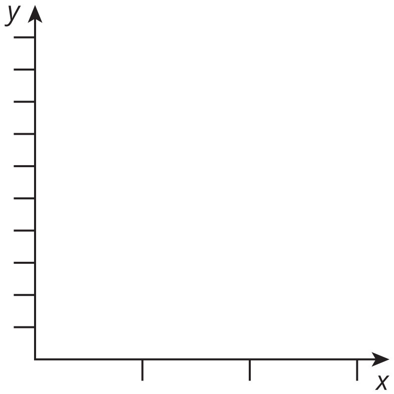
   a. Write a function for each account that can be used to determine the balance in the account based on the year, t. Describe each function.

Lesson 1 Assignment

b. Use your answers from part (a) to create a table of values for each function.

Time (years)	Basic Account Balance ($)	Gold Account Balance ($)

c. Use technology to graph the functions for the Basic and Gold savings accounts. Then, sketch the graphs.



d. Into which account would you recommend that Valentina deposit her money? Explain your reasoning.

e. After reading the pamphlet about the different accounts a little more closely, Valentina realizes that there is a one-time fee of $300 for depositing her money in the Gold account. Does this change the recommendation you made in part (d)? Why or why not?

Lesson 1 Assignment

f. Compare the rates of change for the Basic and Gold savings accounts. Explain what the rates of change tell you about the accounts.

g. What do the rates of change for linear and exponential functions tell you about the graphs of the functions?

2. Diego works for the owners of a bookstore. His starting salary was $24,500, and he gets a 3% raise each year.

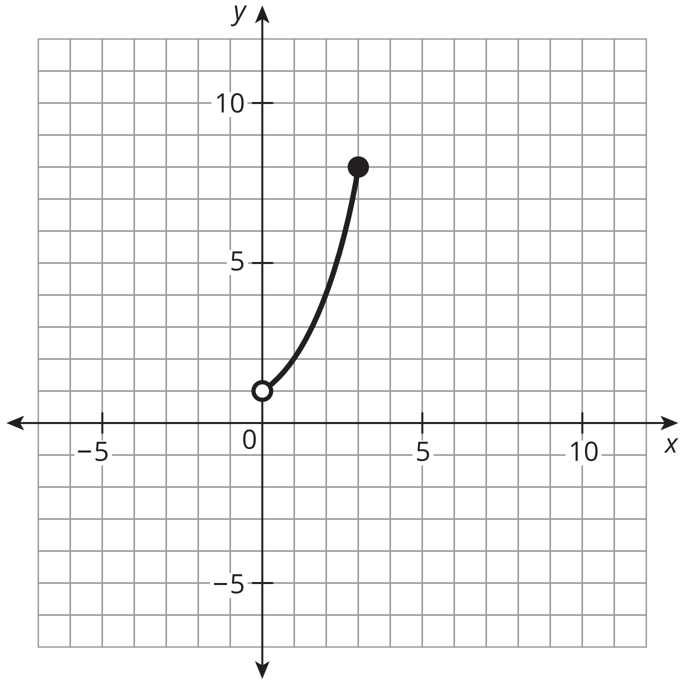
   a. Write an equation in function notation to represent Diego's salary as a function of the number of years he has been working at the bookstore.

   b. What will Diego's salary be when he begins his fourth year working at the bookstore? Show your work.

Lesson 1 Assignment

Prepare

A partial exponential function is graphed on the coordinate plane.



1. Identify the domain and range of the graph.

2

Interpreting Parameters in Context

OBJECTIVES

- Analyze equations and graphs of exponential functions.
- Match equations and graphs of exponential functions using the horizontal asymptote.
- Write and interpret exponential growth and decay functions.
- Use the properties of exponents to rewrite exponential functions.

You have written exponential functions for problem situations.

What strategies can you use to write and solve exponential equations?

Getting Started

Match Game

1. Match each graph with the correct function.

   Functions:

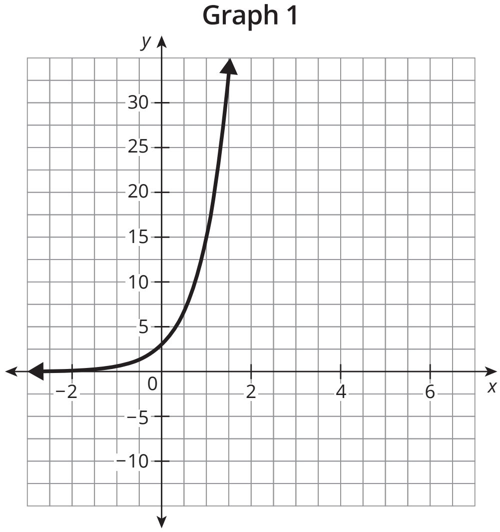
   $f(x) = 3(5)^x$

   $f(x) = 5(3)^x$

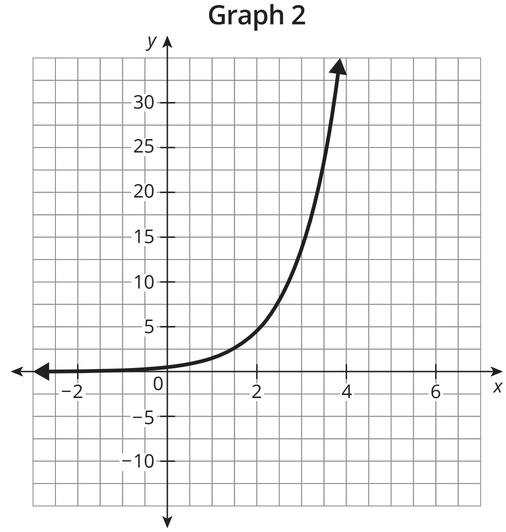
   $f(x) = 3\left(\frac{1}{2}\right)^x$

   $f(x) = \frac{1}{2} \cdot 3^x$

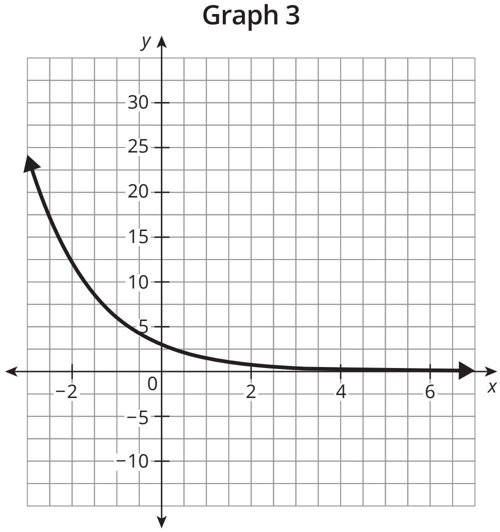
Graph 1



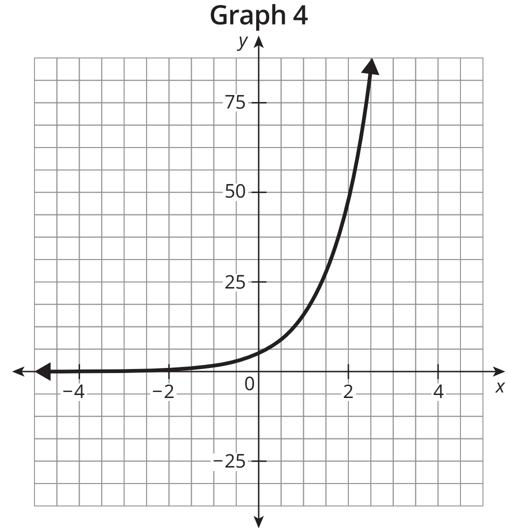
Graph 2



Graph 3



Graph 4



2. Sketch and label the y-intercept and asymptote on each graph.

3. Identify the domain and range of each graph.

Graph 1

Graph 2

Graph 3

Graph 4

4. Match each scenario with the correct function.

Scenario A: Nahimana spends one-half hour writing today. From day to day, she plans to triple the amount of time she spends writing. How much time, in hours, will Nahimana spend writing at the end of *x* days?

Scenario B: Luna has 3 hats in her collection. Each year, she plans to purchase 5 times the number of hats in her collection from the previous year. How many hats will Luna have at the end of *x* years?

Scenario C: Jasmine has 5 collectible figurines. Each year, she plans to triple the number of figurines she has in her collection from the previous year. How many figurines will Jasmine have at the end of *x* years?

Scenario D: Lucas spends 3 hours walking today. Each day, he wants to spend more time running so he plans to reduce the amount of time he spends walking each day by $\frac{1}{2}$ from the previous day. How much time will Lucas spend walking at the end of *x* days?

ACTIVITY 2.1

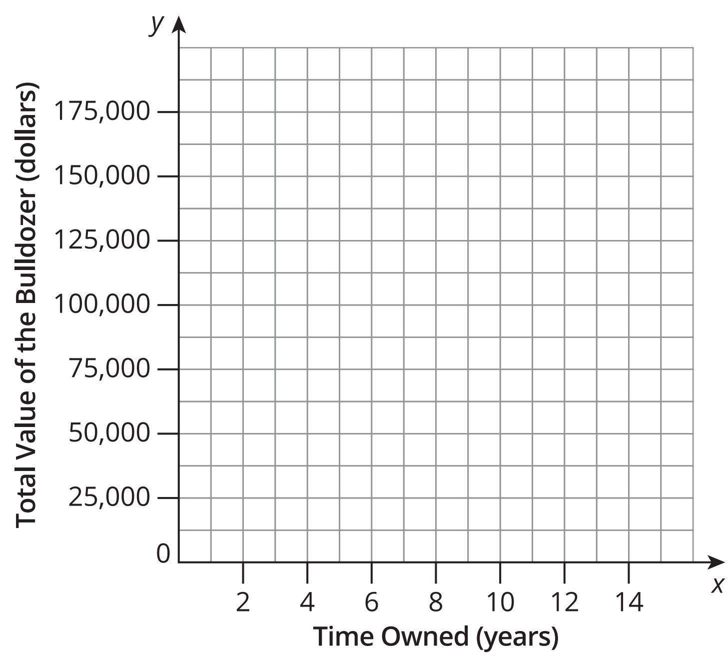
Solving Exponential Equations by Graphing

Depreciation is a decline in the value of something. Vehicles usually depreciate over time, meaning their value decreases over time. This decrease can often be represented by an exponential decay function.

A construction company bought a new bulldozer for $125,000. The bulldozer depreciates exponentially, and after 2 years, the value of the bulldozer is $80,000.

1. Write a function to represent the value of the bulldozer as a function of the number of years it is owned. Then, complete the table and graph.

Number of Years Owned	Value of Bulldozer
0	
2.5	
5	
7	
8.5	
10	
12.5	



Ask yourself . . .
What does each point on the graph represent?

2. The company wants to sell the bulldozer and get at least $25,000 from the sale. Use the graph to estimate the amount of time the company has to achieve this goal.

3. Estimate when the bulldozer will be worth:

   a. $50,000.

   b. $10,000.

4. When will the bulldozer be worth $0?

5. Identify and interpret each key feature in terms of the problem situation.

   a. *y*-intercept

   b. Asymptote

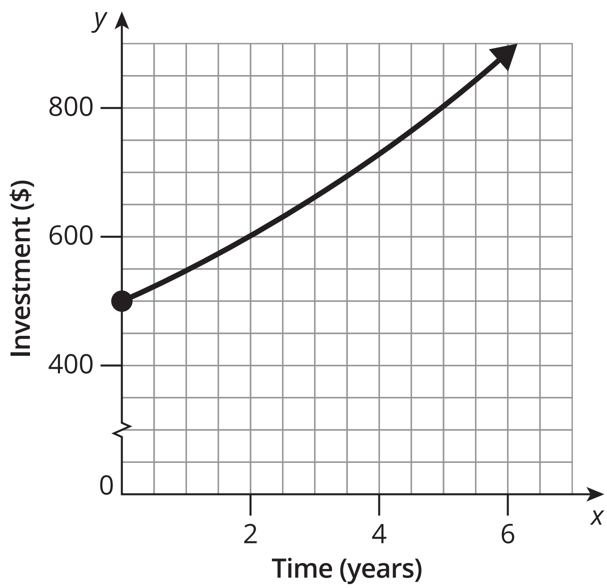
   c. *x*-intercept

ACTIVITY
2.2 Interpreting the *b*-value

Adriana has invested \$500 in a mutual fund which has shown an annual increase of about 10%.

1. Write a function, $f(t)$, that represents Adriana's investment in terms of t, time in years.

   The graph also represents Adriana's investment in terms of t, time in years.



2. Why is the graph a partial exponential function?

3. Use the graph to write the domain and range of the problem situation.

Suppose Adriana is interested in determining the monthly rate of increase. What is the approximate equivalent monthly rate of increase for her mutual fund?

4. Consider the responses from two of Adriana's friends. Describe the differences in their reasoning and why Logan is correct.

Emily

Because we are dividing up the annual rate of increase over twelve months, divide the constant ratio by 12.

$$\frac{1.10}{12}$$

Logan

Because the annual rate of increase is represented as a multiplier, take the 12th root of the constant ratio.

$$1.10^{\frac{1}{12}}$$

Avery wants to write a function that is equivalent to the annual rate of increase but reveals the monthly rate of increase.

5. Explain why Avery's reasoning is not correct.

Avery

Since Adriana's monthly rate of increase is the twelfth root of the annual rate of increase, I can use the function

$$f(t) = 500\left(1.10^{\frac{1}{12}}\right)^t.$$

To rewrite the function representing Adriana's annual increase as an equivalent function that reflects the monthly rate of increase, you must change the *b*-value. The *b*-value of an exponential function can be written as the coefficient of *x*.

$$f(x) = a(b)^{Bx}$$

WORKED EXAMPLE

You can use what you know about common bases to rewrite the expression in an equivalent form.

$\left(1.10^{\frac{1}{12}}\right)^{Bx} = (1.10)^x$

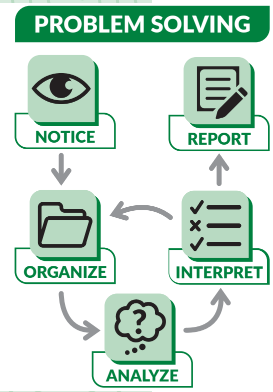
$(1.10)^{\frac{Bx}{12}} = (1.10)^x$	Apply the power to a power rule.
$\frac{Bx}{12} = x$	The bases are the same, so the exponents must be equivalent expressions.
$Bx = 12x$	Multiply both sides by 12.
$B = 12$	Divide both sides by *x*.

So, the function $f(x) = 500\left(1.10^{\frac{1}{12}}\right)^{12x}$ is equivalent to the function $f(x) = 500(1.10)^x$.

6. Suppose Adriana wants to determine how much her mutual fund increases each quarter. Rewrite the original function in an equivalent form that reveals the approximate equivalent quarterly rate of increase.

7. What is Adriana's monthly increase, as a percent?

Talk the Talk



Is This Uptown or Downtown?

In 2005, the population of a city was 42,500. By 2010, the population had grown to approximately 51,708 people.

1. Identify any equations that are appropriate exponential models for the population of the city. Explain why. Then, explain why the equations you did not choose are not appropriate models for the situation.

$f(t) = 51{,}708(1.04)^{t}$

$f(t) = 51{,}708(0.96)^{t}$

$f(t) = 42{,}500(1.04)^{5t}$

$f(t) = 51{,}708(1.04)^{\frac{1}{5}t}$

$f(t) = 42{,}500(1.04)^{t}$

$f(t) = 42{,}500(0.96)^{t}$

Lesson 2 Assignment

Write

Explain how an asymptote can be identified from an exponential equation and its graph.

Remember

You can estimate the solution to an exponential equation graphically. First, graph both the exponential function and the constant function for the given *y*-value. Next, determine the point of intersection of the graph of the exponential function and the horizontal line. Lastly, identify the *x*-value of the coordinate pair as the solution.

Practice

1. Sarah bought a brand new car for \$18,000. Its value depreciated at a rate of 1.2%.

   a. Write a function to represent the value of the car as a function of time. Use technology to estimate the number of years it will take for the value to reach each given amount.

   b. \$17,000

   c. \$15,000

   d. One-half of the starting value

   e. One-third the starting value

   f. \$0

   g. \$10,000

Lesson 2 Assignment

2. In 2012, the population of a city was 63,000. By 2017, the population was reduced to approximately 54,100. Identify any equations that are appropriate models for the population of the city, and explain why the others are not.

   a. $f(x) = 63{,}000(1.03)^t$

   b. $f(x) = 54{,}100(1.03)^t$

   c. $f(x) = 63{,}000(0.97)^t$

   d. $f(x) = 54{,}100(0.97)^t$

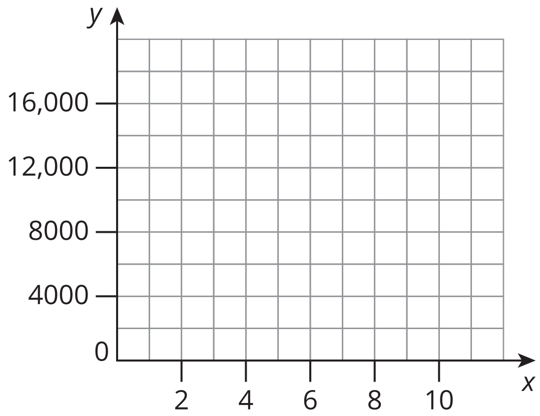
   e. $f(x) = 63{,}000(0.97)^{\frac{1}{5t}}$

   f. $f(x) = 54{,}100(0.97)^{5t}$

3. Camilla wants to own a bee colony so that she can extract honey from the hive. She starts a colony with 5000 bees. The number of bees grows exponentially with a growth factor of 12% each month.

   a. Write a function, $f(x)$, for the bee population that can be used to determine the number of bees in the colony, based on the month, x.

   b. Use technology to graph the function, $f(x)$.



Lesson 2 Assignment

c. What is the domain and range of the problem situation?

d. Identify and interpret the *y*-intercept.

e. Camilla feels that to get a decent amount of honey, there should be at least 15,000 bees in the colony. Estimate how many months it will take Camilla until she has 15,000 bees.

Prepare

While driving to their vacation spot, the Williams family kept a record of their gas purchases. Their information is recorded in the table.

Amount of Gas (gallons)	Total Cost (dollars)
9	21.33
16.6	40.00
11.8	26.08
10	24.70
13.2	28.91

1. Use technology to write the linear regression function.

2. What is the correlation coefficient, *r*? What does this value imply?

3 Modeling Using Exponential Functions

OBJECTIVES

- Write an exponential function to model a table of values and a graph.
- Add an exponential function and a constant function.
- Write an exponential function to model a data set.
- Use exponential models to solve problems.

You can interpret exponential scenarios, equations, tables, and graphs.

How can you use an exponential function to model real-world data?

Getting Started

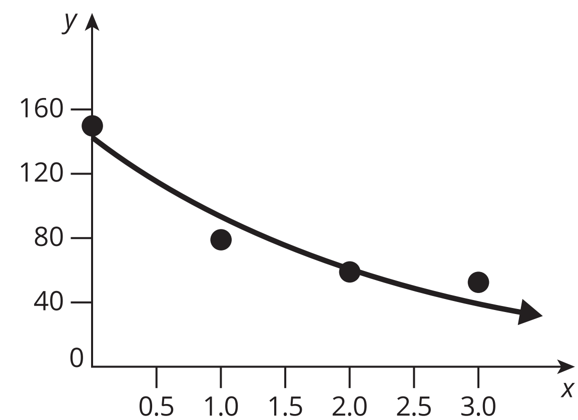
The Elephants in the Room

Two elephant populations over time are shown, each within a different small area in Africa.

Population A

Time (years)	Elephant Population
3	3218
5	3628
7	3721
9	3871

Population B



1. Estimate an exponential function for each population change over time. Explain how you determined your functions.

2. Use your functions and the features of each situation to describe the change in the populations over time.

ACTIVITY 3.1 Modeling a Decreasing Exponential Function

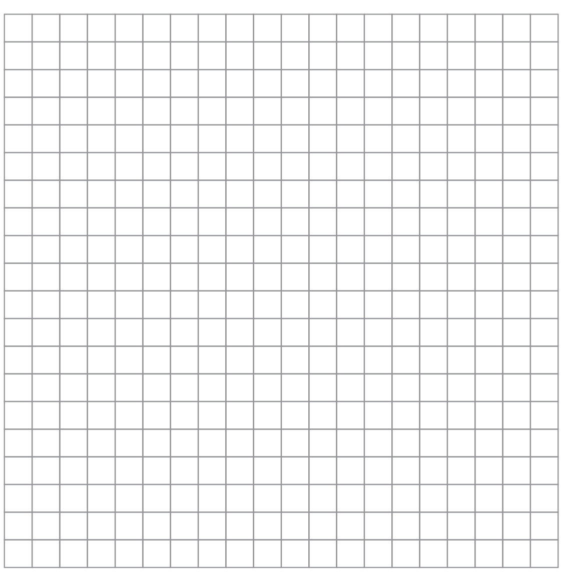
Avery loves drinking green tea. One morning, after making herself a cup of hot tea, she sat in her kitchen to enjoy it.

The table shows the temperature of a cup of Avery's tea over time.

Time (minutes)	Temperature (degrees Fahrenheit)
0	180
5	169
11	149
15	142
18	135
25	124
30	116
34	113
42	106
45	102
50	101

1. Model this situation.

   a. Create a scatterplot. Label your axes.



b. Use technology to write the exponential regression function. Define the variables. Identify the correlation coefficient.

c. Sketch the function on the same graph as your scatterplot.

2. State the domain and range of the function you sketched. How do they compare to the domain and range of this problem situation?

3. Use the function to predict the temperature of Avery's tea after an hour.

4. Use the function to predict the temperature of Avery's tea after 4 hours.

5. Does your prediction make sense in terms of this problem situation? Explain your reasoning.

ACTIVITY 3.2

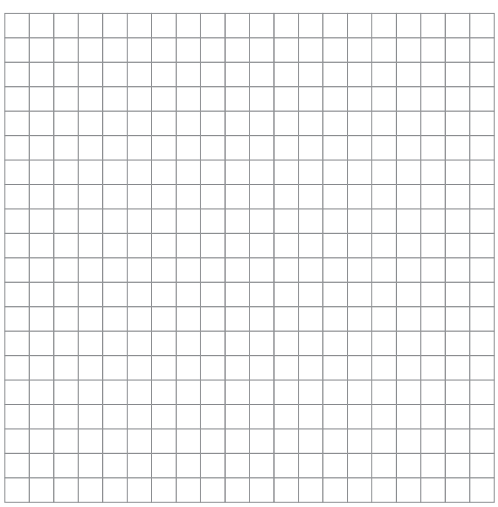
Determining the Best Fit to Model Data

The table shows the carbon dioxide concentration in Earth's atmosphere in parts per million from 1860 to 2017.

Year	Carbon Dioxide Concentration (parts per million)
1860	286
1880	291
1900	296
1920	303
1940	311
1960	317
1980	339
2000	370
2010	389
2017	406

1. Model this situation.

   a. Create a scatterplot using 1860 as Year 0. Label your axes.

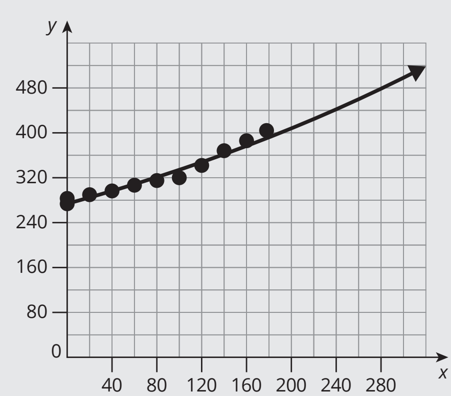


b. Use technology to choose the correct function to model this data. Write your regression function. Define the variables. Identify the correlation coefficient.

c. Sketch the function on the same graph as your scatterplot.

2. Parker used an exponential function to model the situation. Carlos used a linear function to model the situation. Who is correct? Explain your reasoning.

Parker



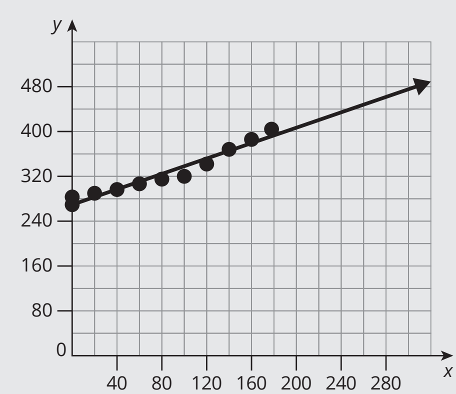
Exponential Regression Model

$f(x) = 272.67(1.00215)^x$

$r = 0.94473$

$r^2 = 0.8925$

Carlos



Linear Regression Model

$f(x) = 0.722x + 268.24$

$r = 0.93106$

$r^2 = 0.8669$

3. What other information would help you to make the decision as to whether a linear or exponential function is best to model this context and data?

Ask Yourself . . .
How can you use regression models in everyday life?

4. Why does the exponential function look very similar to the linear function?

5. Use each function to predict the concentration of carbon dioxide for each given year.

   a. During 2160

   b. During 2500

Talk the Talk

Making a List, Checking It Twice

Reflect on the exponential situations and graphs you have encountered in past lessons.

1. Consider the contexts.

   a. Describe the contexts that can be modeled by an exponential function.

   b. What do the contexts have in common that identify them as being exponential functions?

2. Consider the graphs.

   a. How can you tell from a scatterplot that it can be modeled by an exponential function?

   b. Sketch four different possible graphs of an exponential function of the form $y = ab^x$. Describe the a-value and common multiplier in each.

Lesson 3 Assignment

Write

Describe the information that can be used to determine whether a linear or exponential function is best to model a context and data.

Remember

You can use exponential functions to model scenarios that involve a percent increase or decrease, such as compound interest and population growth or decay.

Sometimes, it may be difficult to determine whether a scatterplot is best modeled by a linear or exponential function. In these cases, sometimes knowing the scenario can help, while in other cases, more data points or information may be needed.

Practice

1. The table shows the number of U.S. Post Offices at the beginning of each decade from 1900 to 2000.

Year	Number of U.S. Post Offices
1900	76,688
1910	59,580
1920	52,641
1930	49,063
1940	44,024
1950	41,464
1960	35,238
1970	32,002
1980	30,326
1990	28,959
2000	27,876

   a. Create a scatterplot of the data.

   b. Determine the exponential regression function and the value of the correlation coefficient, *r*. Then, graph the equation on the grid with the scatterplot.

Lesson 3 Assignment

c. Predict the number of U.S. Post Offices in the year 2050.

d. Predict when the number of U.S. Post Offices will reach 20,000.

e. What do you think is causing the decline in the number of U.S. Post Offices?

Prepare

Consider $f(x) = x^2 + 3x + 4$. Evaluate the function for each given value.

1. $f(1)$
2. $f(-1)$
3. $f(2)$
4. $f(-2)$

TOPIC 2 SELF-REFLECTION

Name: ______________________________

Using Exponential Equations

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Using Exponential Equations* topic by:

TOPIC 2: *Using Exponential Equations*	Beginning of Topic	Middle of Topic	End of Topic
using graphs, tables, and equations to compare the output values of linear and exponential functions.	○	○	○
interpreting exponential expressions by viewing one or more of their parts as a single entity.	○	○	○
writing an exponential equation to model a real-world problem.	○	○	○
graphing exponential equations that model growth and decay on a coordinate plane and using the graph to answer questions about the real-world situation that it models.	○	○	○
explaining that every point on the graph of an exponential equation is a solution to that equation.	○	○	○
using the intersection of a graph of an exponential function and a horizontal line to determine a solution to an exponential equation.	○	○	○
explaining the meaning of the values of *a* and *b* in the exponential function $f(x) = ab^x$.	○	○	○

continued on the next page

TOPIC 2 SELF-REFLECTION *continued*

TOPIC 2: *Systems of Linear Equations and Inequalities*	Beginning of Topic	Middle of Topic	End of Topic
using technology to determine a curve of best fit for a set of data that appears to be exponential.	○	○	○
using an exponential regression model to make predictions for the real-world problem it represents.	○	○	○

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Using Exponential Equations* topic.

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2. What mathematical understandings from the topic do you feel you are making the most progress with?

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TOPIC 2 SELF-REFLECTION *continued*

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

Using Exponential Equations Summary

NEW KEY TERMS

- simple interest [interés simple]
- compound interest [interés compuesto]
- exponential growth function
- exponential decay function [función de decaimiento exponencial]

LESSON 1 Exponential Equations for Growth and Decay

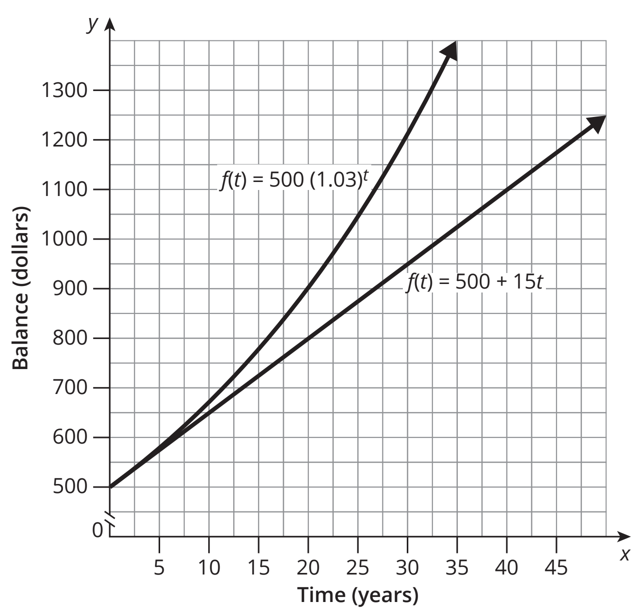
In a **simple interest** account, a percent of the starting balance is added to the account at each interval. The formula for simple interest is $I = Prt$, where P represents the starting amount, or principal, r represents the interest rate, t represents time, and I represents the interest earned.

In a **compound interest** account, the balance is multiplied by the same amount at each interval. Because the entire balance is multiplied by the same percent for each interval, the formula is represented by an exponential equation: $I = P(1 + r)^t$.

The rate of change for a simple interest account is constant. The rate of change between the values for the compound interest account is increasing as t becomes larger. A constant rate of change means that the graph of the linear equation is a straight line. An increasing rate of change means that the graph of the exponential function is a smooth curve.

For example, consider two accounts that each have an initial balance of \$500. One account earns 3% simple interest each year, while the other earns 3% compound interest each year.

Time (years)	Simple Interest Balance (dollars)	Compound Interest Balance (dollars)
0	500	500
1	515	515
2	530	530.45
10	650	671.96
100	2000	9609.32

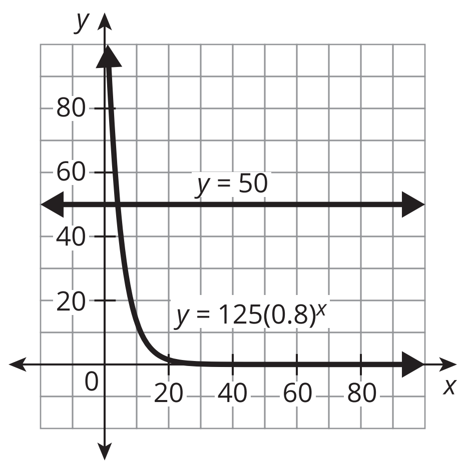


An **exponential growth function** has a b-value greater than 1 and is in the form $y = a(1 + r)^x$, where r is the rate of growth. An **exponential decay function** has a b-value greater than 0 and less than 1 and is in the form $y = a(1 - r)^x$, where r is the rate of decay.

For example, consider a population that starts at 50,000 and grows at a rate of 2% every year. The scenario can be modeled by the function $f(x) = 50{,}000(1.02)^x$. The a-value of the exponential function is the initial population and since the population grows, the b-value is $(1 + 0.02)$ or 1.02.

LESSON 2 Interpreting Parameters in Context

Graphs can be used to solve exponential equations by estimating the intersection point of the graph of an exponential function with a constant function.



For example, to determine the solution to $125(0.8)^x = 50$, graph each side of the equation as separate functions on the same coordinate plane.

The solution to the equation is $x \approx 4$.

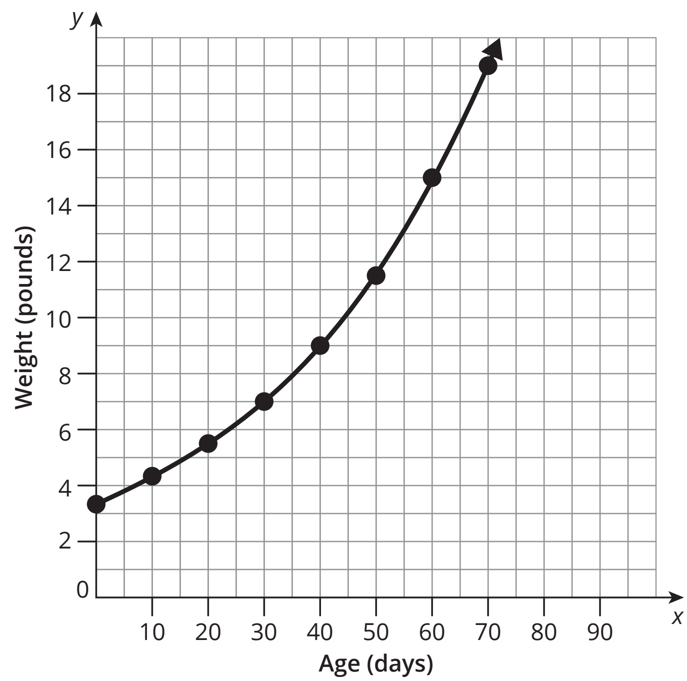
LESSON 3 Modeling Using Exponential Functions

Technology can be used to determine which exponential regression function best models real-world data. This regression function can be used to predict future values.

For example, the table shows the weight of a golden retriever puppy as recorded during its growth. A scatterplot and regression function of the data are shown.

The exponential regression function for this scenario is $f(x) = 3.31(1.025)^x$. The correlation coefficient, r, is 0.9999.

Age (days)	Weight (pounds)
0	3.25
10	4.25
20	5.5
30	7
40	9
50	11.5
60	15
70	19



The equation can be used to predict the puppy's weight at 80 days.

$f(80) = 3.31(1.025)^{80}$
$f(80) = 23.9$

The puppy's weight will be approximately 23.9 pounds on day 80.

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