MODULE 5

Maximizing and Minimizing

TOPIC 1 Introduction to Quadratic Functions 657

TOPIC 2 Polynomial Operations 757

TOPIC 3 Solving Quadratic Equations 825

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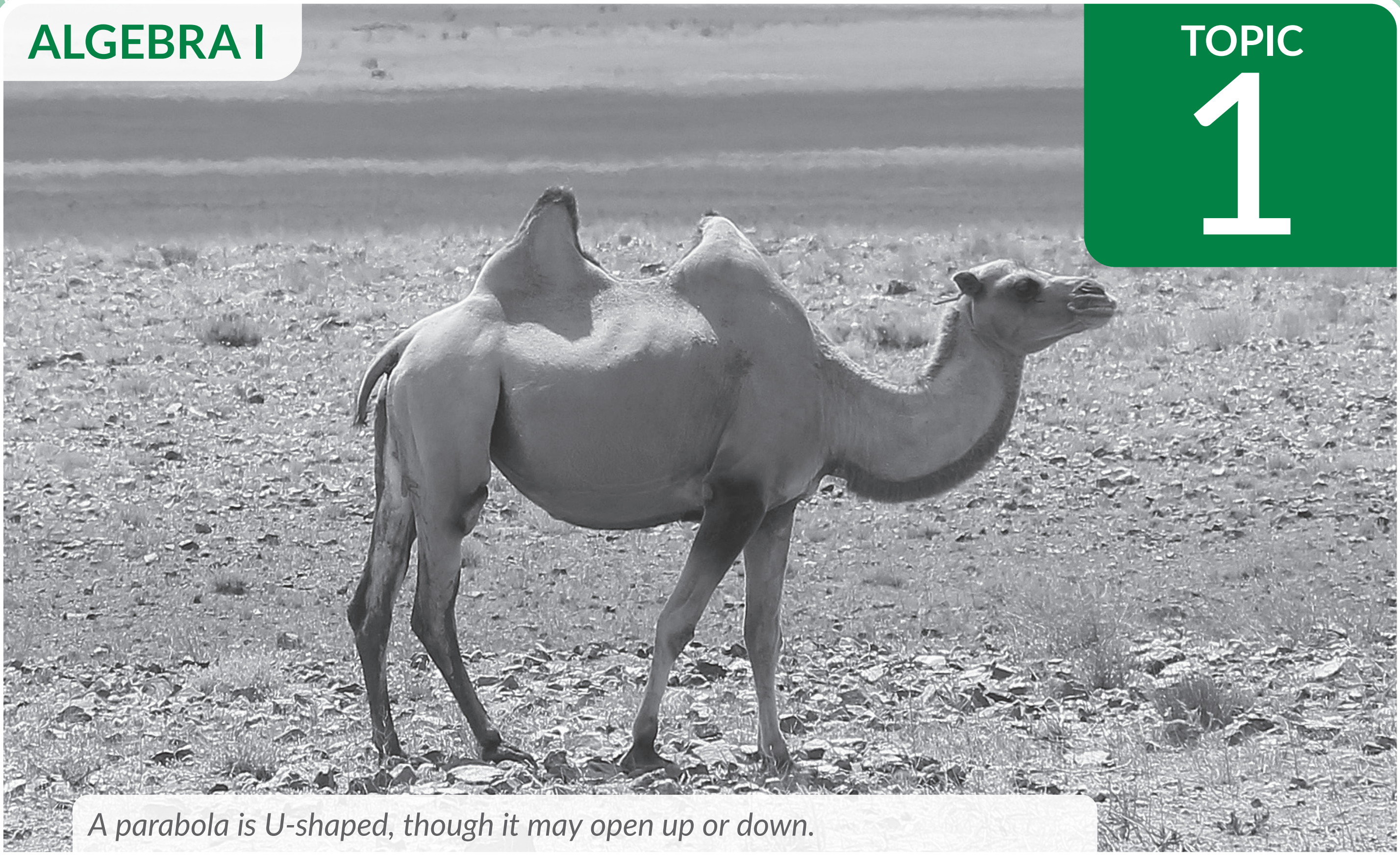
TOPIC 1



A parabola is U-shaped, though it may open up or down.

Introduction to Quadratic Functions

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LESSON 4 Horizontal Transformations and Vertex Form 723

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1 Exploring Quadratic Functions

OBJECTIVES

- Write quadratic functions to model contexts.
- Graph quadratic functions using technology.
- Interpret the key features of quadratic functions in terms of a context.
- Identify the domain and range of quadratic functions and their contexts.

NEW KEY TERMS

- parabola
- vertical motion model
- roots

You have used linear functions to model situations with constant change, and you have used exponential functions to model growth and decay situations.

What type of real-world situations can you model using quadratic functions?

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Getting Started

Squaring It Up

Destiny is using pennies to create a pattern.

Figure 1	Figure 2	Figure 3	Figure 4
			

1. Analyze the pattern and explain how to create Figure 5.

2. How many pennies would Destiny need to create Figure 5? Figure 6? Figure 7?

3. Which figure would Destiny create with exactly $4.00 in pennies?

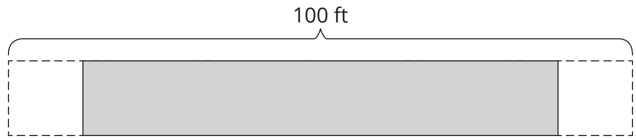
4. Write an equation to determine the number of pennies for any figure number. Define your variables.

5. Describe the function family to which this equation belongs.

ACTIVITY 1.1

Using Area to Introduce Quadratic Functions

A dog trainer is fencing in an enclosure, represented by the shaded region in the diagram. The trainer will also have two square-shaped storage units on either side of the enclosure to store equipment and other materials. She can make the enclosure and storage units as wide as she wants, but she can't exceed 100 feet in total length.



1. Let s represent a side length, in feet, of one of the storage units.

   a. Label the length and width of the enclosure in terms of *s*.

   b. Write the function $L(s)$ to represent the length of the enclosure as a function of side length, *s*.

   c. Sketch and label a graph of the function on the given coordinate plane. Identify any key points.

2. Describe the domain and range of the context and of the function.

Ask Yourself . . .
To identify key points on the graph, think about the function you are representing. Are there any intercepts? Are there any other points of interest?

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The progression of diagrams below shows how the area of the enclosure, $A(s)$, changes as the side length, s, of each square storage unit increases.

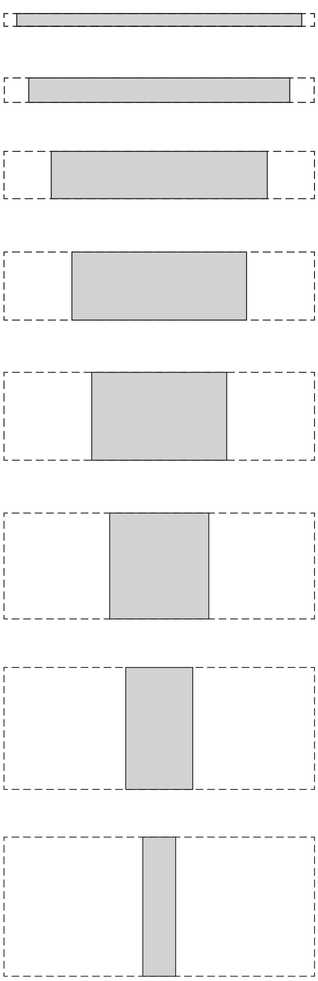
3. Identify each key characteristic of the graph. Then, interpret the meaning of each in terms of the context.

   a. Slope

   b. y-intercept

   c. Increasing or decreasing

   d. x-intercept



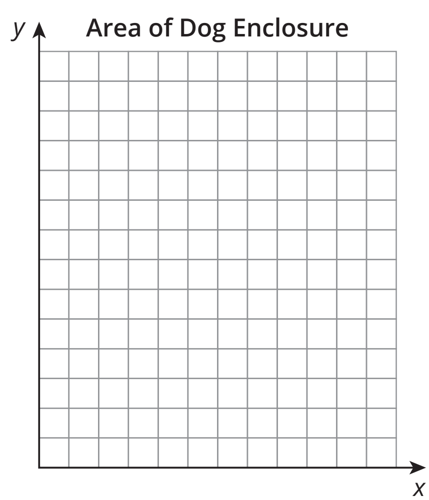
4. Write the function $A(s)$ to represent the area of the enclosure as a function of side length, s.

5. Describe how the area of the enclosure changes as the side length increases.

6. Consider the graph of the function, $A(s)$.

   a. Predict what the graph of the function will look like.

   b. Use technology to graph the function $A(s)$. Then, sketch the graph and label the axes.



7. Describe what all the points on the graph represent.

The function $A(s)$ that you wrote to model area is a quadratic function. The shape that a quadratic function forms when graphed is called a **parabola**.

Think about . . .
Quadratic functions model area because area is measured in square units.

8. Think about the possible areas of the enclosure.
   a. Is there a maximum area that the enclosure can contain? Explain your reasoning in terms of the graph and in terms of the context.
   b. Use technology to determine the maximum of $A(s)$. Describe what the x- and y-coordinates of the maximum represent in this context.
   c. Determine the dimensions of the enclosure that will provide the maximum area. Show your work and explain your reasoning.

9. Identify the domain and range of the context and of the function.

10. Identify each key characteristic of the graph. Then, interpret the meaning of each in terms of the context.
   a. y-intercept
   b. Increasing and decreasing intervals
   c. Symmetry
   d. x-intercepts

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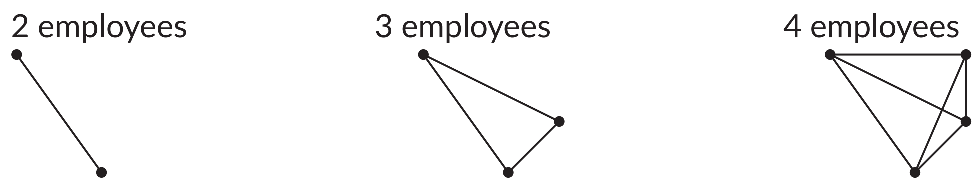
ACTIVITY
1.2 Writing and Interpreting a Quadratic Function

Suppose that there is a monthly meeting at CIA headquarters for all employees. How many handshakes will it take for every employee at the meeting to shake the hand of every other employee at the meeting once?

1. Use the figures shown to determine the number of handshakes that will occur between 2 employees, 3 employees, and 4 employees.

Ask Yourself . . .
Can you tell what shape the graph will be?



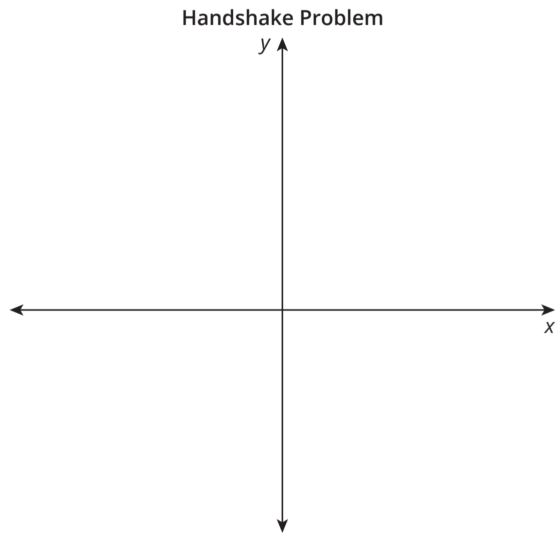
2. Draw figures to represent the number of handshakes that occur between 5 employees, 6 employees, and 7 employees, and determine the number of handshakes that will occur in each situation.

3. Enter your results in the table.

Number of Employees	2	3	4	5	6	7	n
Number of Handshakes							

4. Write a function to represent the number of handshakes given any number of employees. Enter your function in the table.

5. Use technology to graph the function you wrote in Question 4. Sketch the graph and label the axes.



6. How is the orientation of this parabola different from the parabola for the area of the dog enclosure? How is this difference reflected in their corresponding equations?

7. Determine the minimum of your function. Then, describe what the *x*- and *y*-coordinates of this minimum represent in this problem situation.

8. Identify the domain and range of the problem situation and of the function.

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ACTIVITY 1.3

Using a Quadratic Function to Model Vertical Motion

You can model the motion of a pumpkin released from a catapult using a vertical motion model. A **vertical motion model** is a quadratic equation that models the height of an object at a given time. The equation is of the form shown.

$$y = -16t^2 + v_0t + h_0$$

In this equation, y represents the height of the object in feet, t represents the time in seconds that the object has been moving, v_0 represents the initial vertical velocity (speed) of the object in feet per second, and h_0 represents the initial height of the object in feet.

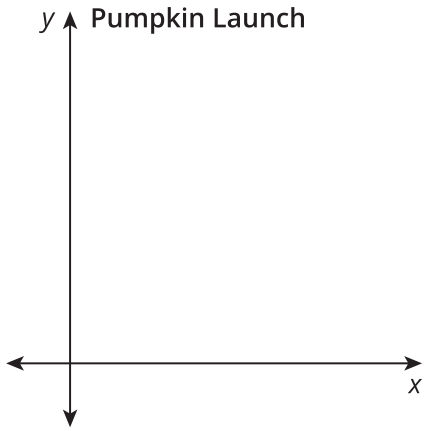
1. What characteristics of this situation indicate that you can model it using a quadratic function?

Suppose that a catapult hurls a pumpkin from a height of 68 feet at an initial vertical velocity of 128 feet per second.

2. Write a function for the height of the pumpkin, $h(t)$, in terms of time, t.

3. Does the function you wrote have a minimum or maximum? How can you tell from the form of the function?

4. Use technology to graph the function. Sketch your graph and label the axes.



Ask Yourself . . .
What do all the points on this graph represent?

5. Use technology to determine the maximum or minimum and label it on the graph. Explain what it means in terms of the problem situation.

Ask Yourself . . .
How can you use a maximum value of a function in everyday life?

6. Determine the y-intercept and label it on the graph. Explain what it means in terms of the problem situation.

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7. Use a horizontal line to determine when the pumpkin reaches each height after being catapulted. Label the points on the graph.

   a. 128 feet

   b. 260 feet

   c. 55 feet

Remember . . .
The zeros of a function are the *x*-values when the function equals 0.

8. Explain why the *x*- and *y*-coordinates of the points where the graph and each horizontal line intersects are solutions.

9. When does the catapulted pumpkin hit the ground? Label this point on the graph. Explain how you determined your answer.

The time when the pumpkin hits the ground is one of the *x*-intercepts, (x, 0). When an equation is used to model a situation, the *x*-coordinate of the *x*-intercept is referred to as a *root*. The **root** of an equation indicates where the graph of the equation crosses or touches the *x*-axis.

ACTIVITY 1.4

Expressing a Quadratic Function as the Product of Two Linear Functions

The Alamo is a historic landmark in San Antonio, Texas that is now a museum. The museum charges $50 per tour, using the proceeds for maintenance and repairs of the Alamo property. Each summer, he books 100 tours at that price. Michael is considering a decrease in the price per tour because he thinks it will help him book more tours. He estimates that he will gain 10 tours for every $1 decrease in the price per tour.

1. According to the scenario, how much money does the museum currently generate each summer with their tours?

Revenue is the amount of money regularly coming into a business. At the Alamo, the revenue is the number of tours multiplied by the price per tour. Your response to Question 1 can be referred to as revenue. Because Michael is considering different numbers of tours and prices per tour, the revenue can be modeled by a function.

2. Write a function, $r(x)$, to represent the revenue for the Alamo.

   a. Let x represent the decrease in the price per tour. Write an expression to represent the number of tours booked when the decrease in price is x dollars per tour.

   b. Write an expression to represent the price per tour when Michael decreases the price x dollars per tour.

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Think About . . .
You can always check your function by testing it with values of x. What is the value of $r(x)$ when $x = 0$? Does it make sense?

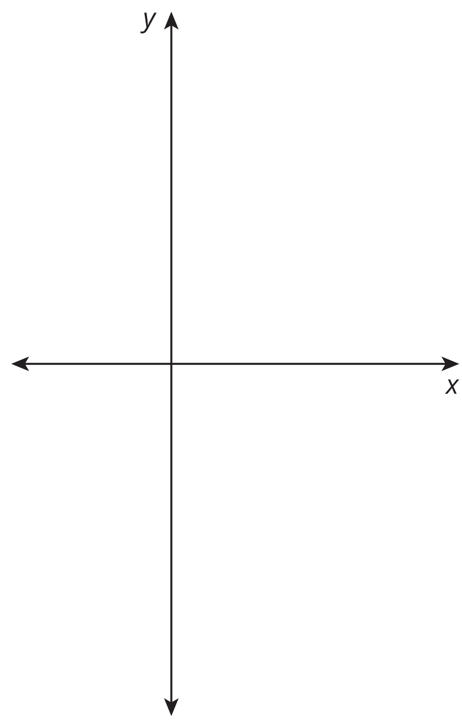
c. Use your expressions from parts (a) and (b) to represent the revenue, $r(x)$, as the number of tours times the price per tour.

Revenue	=	**Number of Tours**	·	**Price per Tour**
________	=	________________	·	________________

3. Use technology to graph the function $r(x)$. Sketch your graph and label the axes.

Remember . . .
Don't forget to label key points!

Alamo Tours



Ask Yourself . . .
What tools or strategies can you use to solve this problem?

4. Assume that Michael's estimate, that for every $1 decrease in the price per tour he will gain 10 tours, is accurate.

   a. What is the maximum revenue that the Alamo could earn for the summer?

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b. Ana and Javier are calculating the number of tours that would yield the maximum revenue.

Ana said that according to the graph, a tour should cost \$20. Since $\$9000 \div \$20 = 450$, the number of tours would be 450.

Javier said that the cost of a tour should be \$30, and \$9000 divided by \$30 per tour is 300 tours.

Who is correct? Explain your reasoning.

c. Would you advise Michael to adjust the cost per tour to make the maximum revenue? Why or why not?

5. Identify each key characteristic of the graph. Then, interpret its meaning in terms of the context.

a. x-intercepts/zeros

b. y-intercept

c. Increasing and decreasing intervals

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Talk the Talk

Making Connections

Analyze the graphs of the four quadratic functions in this lesson.

1. Summarize what you know about the graphs of quadratic functions. Include a sketch or sketches and list any characteristics.

2. Compare your sketch or sketches and list with your classmates. Did you all sketch the same parabola? Why or why not?

Lesson 1 Assignment

Write

Fill in the blank.

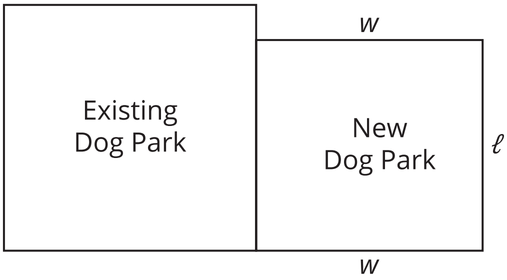
1. The x-intercepts of a graph of a quadratic function are the ______________ of the quadratic function.
2. A quadratic equation that models the height of an object at a given time is a ________________.
3. The shape that a quadratic function forms when graphed is called a ______________.
4. The ______________ of an equation indicate where the graph of the equation crosses or touches the x-axis.

Remember

The graph of a quadratic function is called a *parabola*. Parabolas are smooth curves that have an absolute maximum or minimum, both increasing and decreasing intervals, up to two x-intercepts, and symmetry.

Practice

1. The citizens of a local community have an existing dog park for dogs to play, but they have decided to build another one so that one park will be for small dogs and the other will be for large dogs. The plan is to build a rectangular fenced-in area adjacent to the existing dog park, as shown in the sketch. The county has enough money in the budget to buy 1000 feet of fencing.



   a. Determine the length of the new dog park, ℓ, in terms of the width, w.

   b. Write the function $A(w)$ to represent the area of the new dog park as a function of the width, w. Does this function have a minimum or a maximum? Explain your answer.

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Lesson 1 Assignment

c. Sketch the graph of the function. Label the axes, the maximum or minimum, the x-intercepts, and the y-intercept.

d. Determine the x-intercepts of the function. Explain what each x-intercept means in terms of the problem situation.

e. What should the dimensions of the dog park be to maximize the area? What is the maximum area of the park?

f. Use the graph to determine the dimensions of the park when you restrict the area to 105,000 square feet.

Prepare

Determine the slope and y-intercept of each linear function.

1. $h(x) = 3x$

2. $g(x) = \frac{1}{2}(x - 5)$

3. $k(x) = x - 2$

4. $m(x) = \frac{8x}{4} + 1$

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2 Key Characteristics of Quadratic Functions

OBJECTIVES

- Identify the factored form and standard form of an equation for a quadratic function.
- Determine the equation for the axis of symmetry of a quadratic function given the equation in standard form or factored form.
- Determine the absolute minimum or absolute maximum point on the graph of a quadratic function and identify this point as the vertex.
- Describe intervals of increase and decrease in relation to the axis of symmetry on the graph of a quadratic function.
- Use key characteristics of the graph of a quadratic function to write an equation in factored form.

NEW KEY TERMS

- second differences
- concave up
- concave down
- standard form of a quadratic function
- factored form
- vertex of a parabola
- axis of symmetry

You have identified key characteristics of linear and exponential functions.

What are the key characteristics of quadratic functions?

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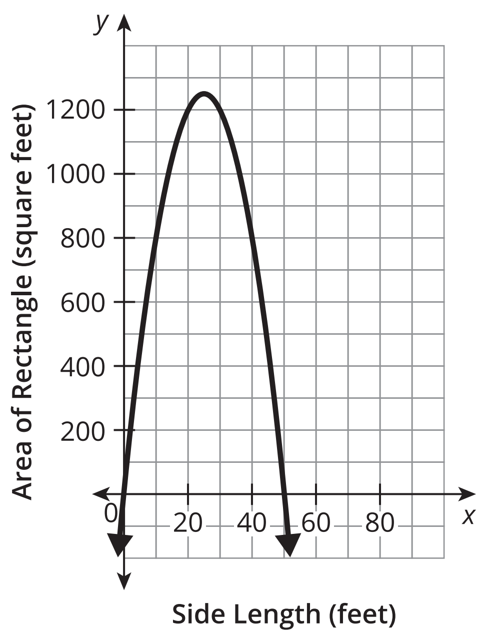
Getting Started

Dogs, Handshakes, Pumpkins, and the Alamo

Consider the four quadratic models you investigated in the previous lesson. There are multiple equivalent ways to write the equation to represent each situation and a unique parabola to represent the equivalent equations. You can also represent the function using a table of values.

Area of Dog Enclosure

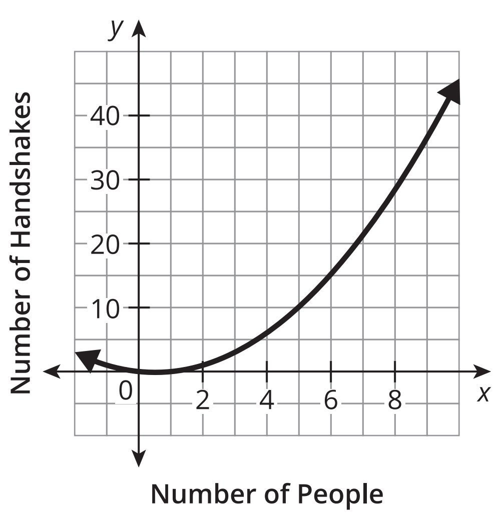
$A(s) = -2s^2 + 100s$
$\quad = -2(s)(s - 50)$



s	$A(s)$
0	0
1	98
2	192
3	282
4	368

Handshake Problem

$f(n) = \frac{1}{2}n^2 - \frac{1}{2}n$
$\quad = \frac{1}{2}(n)(n - 1)$

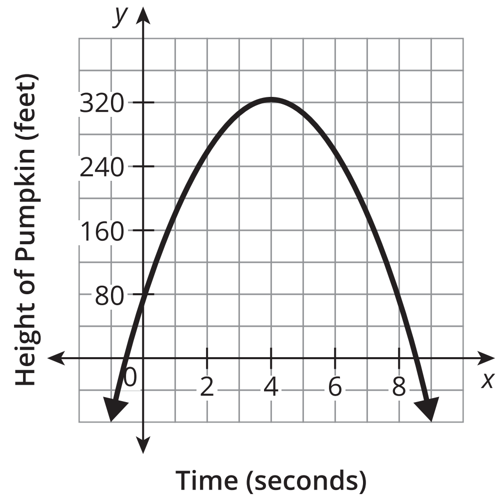


n	$f(n)$
0	0
1	0
2	1
3	3
4	6

Pumpkin Launch

$h(t) = -16t^2 + 128t + 68$

$= -16\left(t - \frac{17}{2}\right)\left(t + \frac{1}{2}\right)$

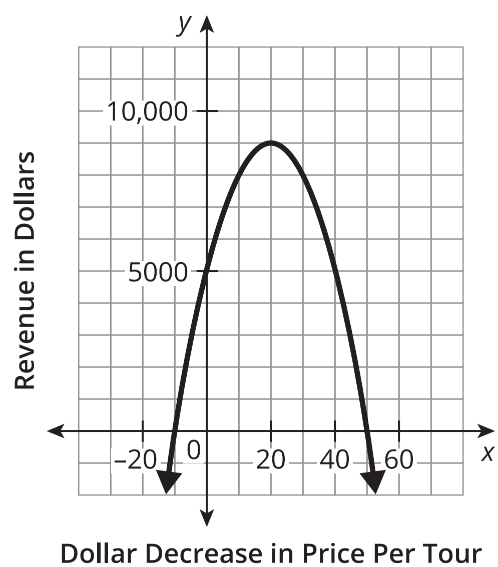


t	$h(t)$
0	68
1	180
2	260
3	308
4	324

Alamo Tours

$r(x) = -10(x + 10)(x - 50)$

$= -10x^2 + 400x + 5000$



x	$r(x)$
0	5000
1	5390
2	5760
3	6110
4	6440

1. Consider each representation.

   a. How can you tell from the structure of the equation that it is quadratic?

   b. What does the structure of the equation tell you about the shape and characteristics of the graph?

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c. How can you tell from the shape of the graph that it is quadratic?

d. How can you tell from the table that the relationship is quadratic?

ACTIVITY **2.1**

Second Differences

Let's explore how a table of values can show that a function is quadratic. Consider the table of values represented by the basic quadratic function. This table represents the first differences between seven consecutive points.

x	$f(x)$
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9

First Differences

$4 - 9 = -5$

$1 - 4 = -3$

$0 - 1 = -1$

$1 - 0 = 1$

$4 - 1 = 3$

$9 - 4 = 5$

Remember . . .

You can tell whether a table represents a linear function by analyzing first differences. First differences imply the calculation of $y_2 - y_1$.

1. What do the first differences tell you about the relationship of the table of values?

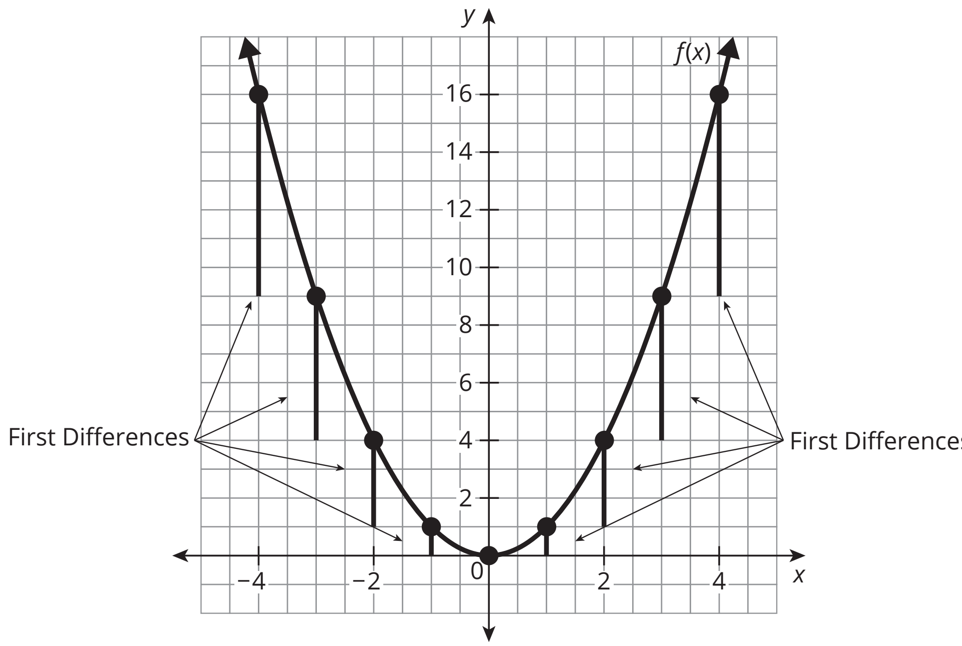
Let's consider the *second differences*. The **second differences** are the differences between consecutive values of the first differences.

2. Calculate the second differences for $f(x)$. What do you notice?

You know that with linear functions, the first differences are constant. For quadratic functions, the second differences are constant.

Let's consider the graph of the basic quadratic function, $f(x) = x^2$, and the distances represented by the first and second differences. Graph 1 shows the distances between consecutive values of $f(x)$. The vertical line segments are different lengths because the first differences are not the same.

Graph 1

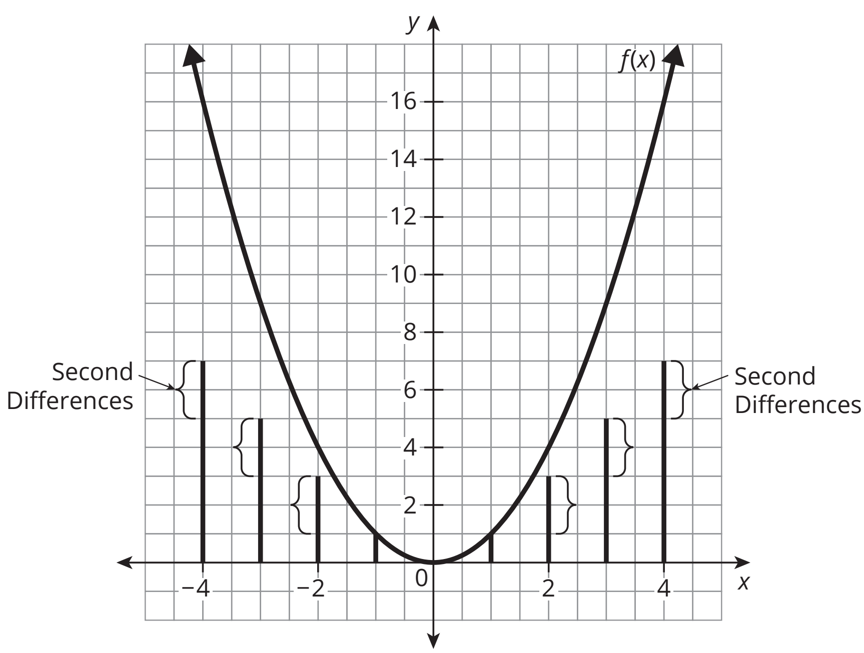


Think About . . .

Quadratic equations are polynomials with a degree of 2. Their second differences are constant. Linear functions are polynomials with a degree of 1, and their first differences are constant.

Graph 2 shows the lengths of the first differences positioned along the x-axis. By comparing these lengths, you can see the second differences.

Graph 2



3. How does the representation in Graph 1 support the first differences calculated from the table of values?

4. How does the representation in Graph 2 support the second differences you calculated in the table?

5. Identify each equation as linear or quadratic. Complete the table to calculate the first and second differences. Then, sketch the graph.

a. $y = 2x$ ____________

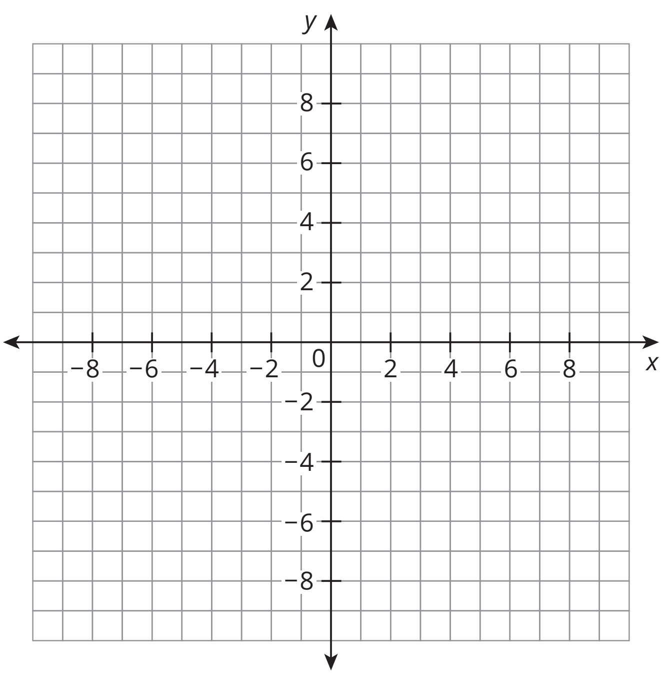
x	y
−3	−6
−2	−4
−1	−2
0	0
1	2
2	4
3	6

First Differences

_____
_____
_____
_____
_____
_____

Second Differences

_____
_____
_____
_____
_____



b. $y = 2x^2$ ____________

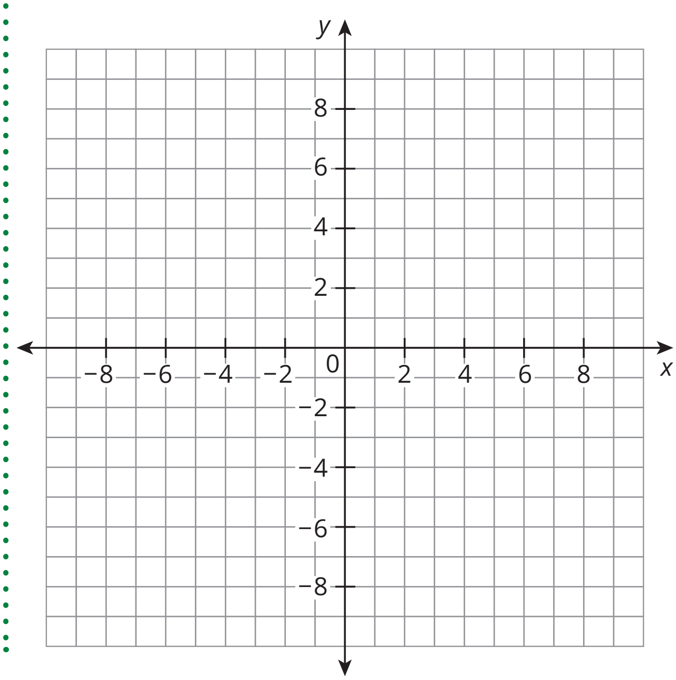
x	y
−3	18
−2	8
−1	2
0	0
1	2
2	8
3	18

First Differences

_____
_____
_____
_____
_____
_____

Second Differences

_____
_____
_____
_____
_____

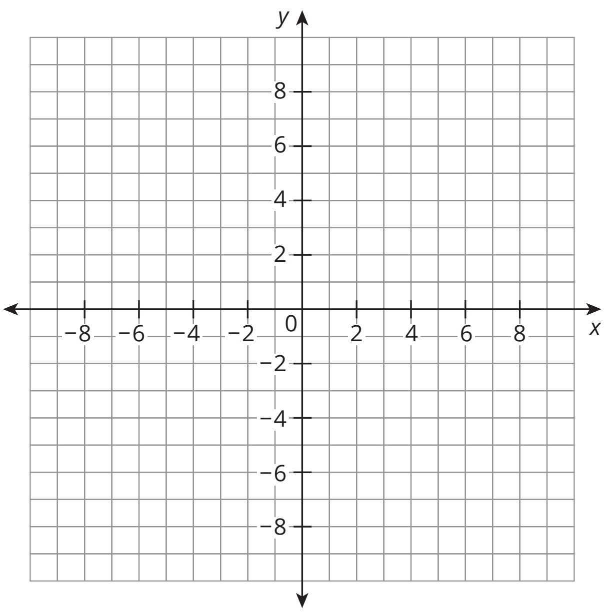


c. $y = -x + 4$ ____________

x	y
−3	7
−2	6
−1	5
0	4
1	3
2	2
3	1

First Differences

Second Differences

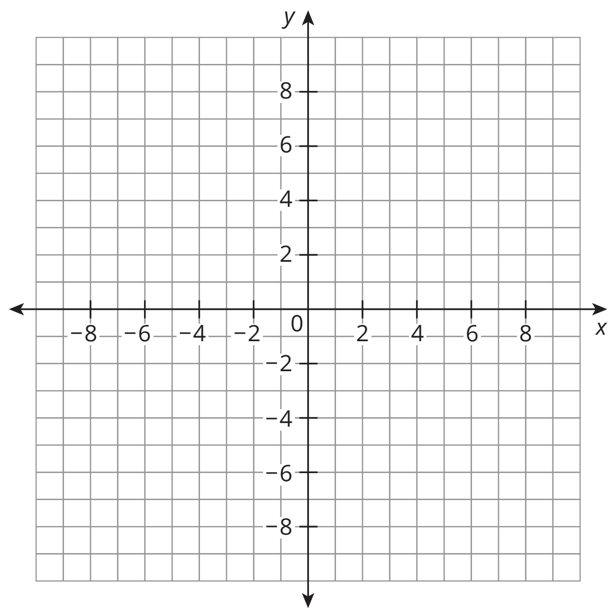


d. $y = -x^2 + 4$ ____________

x	y
−3	−5
−2	0
−1	3
0	4
1	3
2	0
3	−5

First Differences

Second Differences



A graph that opens upward is identified as being **concave up**. A graph that opens downward is identified as being **concave down**.

6. Compare the signs of the first and second differences for each function and its graph.

   a. How do the signs of the first differences for a linear function relate to the graph either increasing or decreasing?

   b. How do the signs of the second differences for quadratic functions relate to whether the parabola is opening upward or downward?

ACTIVITY 2.2 Analyzing the Leading Coefficient

You know that different forms of an equation can reveal different characteristics about functions. Quadratic functions can be written in different forms.

A quadratic function written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$, is in **standard form**. In this form, a and b are numerical coefficients and c is a constant.

A quadratic function written in **factored form** is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$.

Remember . . .
The leading coefficient of an equation is the numeric coefficient of the term with the greatest power.

1. Identify the standard form and factored form of each equation in the Getting Started.

2. Consider the leading coefficient of each function equation in both standard form and factored form.

   a. What does the leading coefficient tell you about the graph of each function?

   b. How is the leading coefficient related to the absolute minimum or absolute maximum of each function?

   c. How can you determine the y-intercept of the graph using standard form?

The graph of a quadratic function has either an absolute maximum or absolute minimum.

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3. Determine from the equation whether each quadratic function has an absolute maximum or absolute minimum. Explain how you know.

a. $f(n) = 2n^2 + 3n - 1$

b. $g(x) = -2x^2 - 3x + 1$

c. $r(x) = -\frac{1}{2}x^2 - 3x + 1$

d. $b(x) = -0.009(x + 50)(x - 250)$

e. $f(t) = \frac{1}{3}(x - 1)(x - 1)$

f. $j(x) = -2x(1 - x)$

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ACTIVITY
2.3 Axis of Symmetry

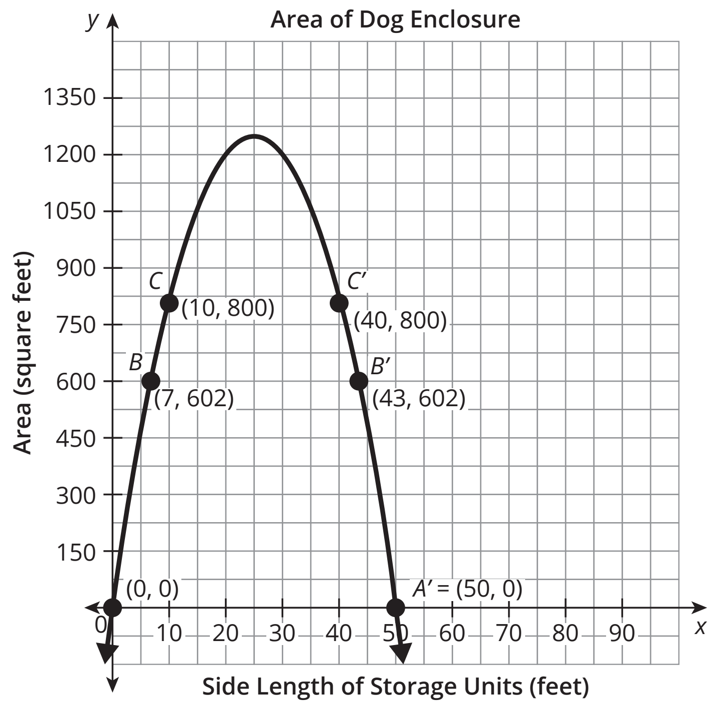
The **vertex of a parabola** is the lowest or highest point on the graph of the quadratic function. The **axis of symmetry** or the line of symmetry of a parabola is the vertical line that passes through the vertex and divides the parabola into two mirror images. Because the axis of symmetry always divides the parabola into two mirror images, you can say that a parabola has reflectional symmetry.

The vertex is identified as either the absolute minimum or absolute maximum.

1. Use patty paper to trace the graph representing the area of the dog enclosure. Then, fold the graph to show the symmetry of the parabola and trace the axis of symmetry.

   a. Place the patty paper over the original graph. What is the equation of the axis of symmetry?

   b. Draw and label the axis of symmetry on the graph from your patty paper.



2. Analyze the symmetric points labeled on the graph.

   a. What do you notice about the y-coordinates of the points?

   b. What do you notice about each point's horizontal distance from the axis of symmetry?

   c. How does the x-coordinate of each symmetric point compare to the x-coordinate of the vertex?

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For a function in factored form, $f(x) = a(x - r_1)(x - r_2)$, the equation for the axis of symmetry is given by $x = \frac{r_1 + r_2}{2}$. For a quadratic function in standard form, $f(x) = ax^2 + bx + c$, the equation for the axis of symmetry is $x = \frac{-b}{2a}$.

3. Identify the axis of symmetry of the graph of each situation from the Getting Started using the factored form of each equation.

4. Describe the meaning of the axis of symmetry in each situation, when possible.

5. Describe how you can use the axis of symmetry to determine the ordered pair location of the absolute maximum or absolute minimum of a quadratic function given the equation for the function in factored form.

As you analyze a parabola from left to right, it will have either an interval of increase followed by an interval of decrease, or an interval of decrease followed by an interval of increase.

6. How does the absolute maximum or absolute minimum help you determine each interval?

Consider the graph of the quadratic function representing the Pumpkin Launch problem situation.

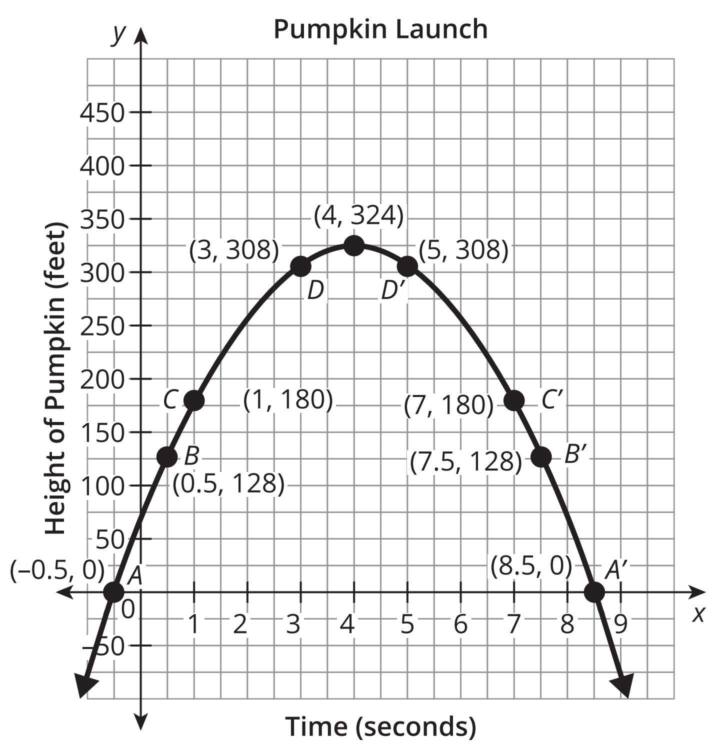
7. Determine the average rate of change between each pair. Then, summarize what you notice.

   a. Points A and B

   b. Points A' and B'

   c. Points B and C

   d. Points B' and C'



   e. What do you notice about the average rates of change between pairs of symmetric points?

The formula for the average rate of change is $\frac{f(b) - f(a)}{b - a}$.

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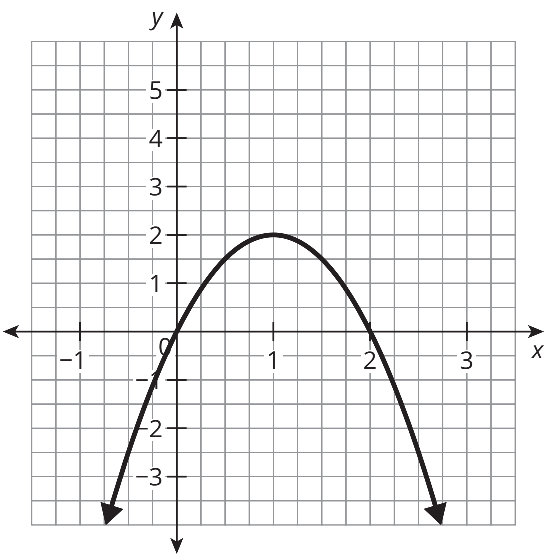
8. For each function shown, identify the domain, range, x-intercepts, y-intercept, axis of symmetry, vertex, and interval of increase and decrease.

a. The graph shown represents the function $f(x) = -2x^2 + 4x$.



Domain:

Range:

***x*-intercepts:**

***y*-intercept:**

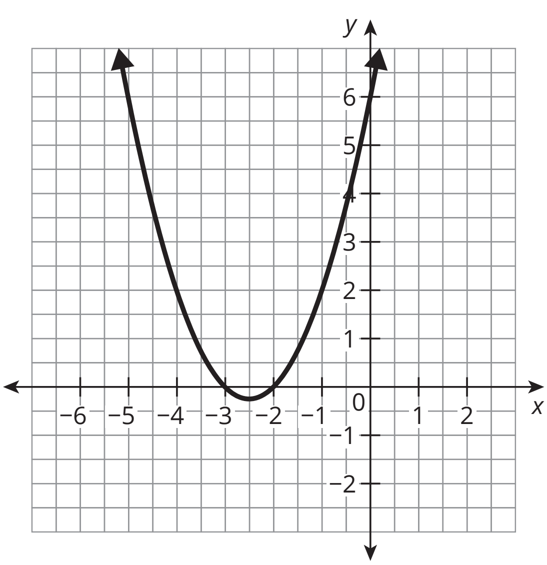
Axis of symmetry:

Vertex:

Interval of increase:

Interval of decrease:

b. The graph shown represents the function $f(x) = x^2 + 5x + 6$.



Domain:

Range:

***x*-intercepts:**

***y*-intercept:**

Axis of symmetry:

Vertex:

Interval of increase:

Interval of decrease:

c. The graph shown represents the function $f(x) = x^2 - x - 2$.

Domain:

Range:

***x*-intercepts:**

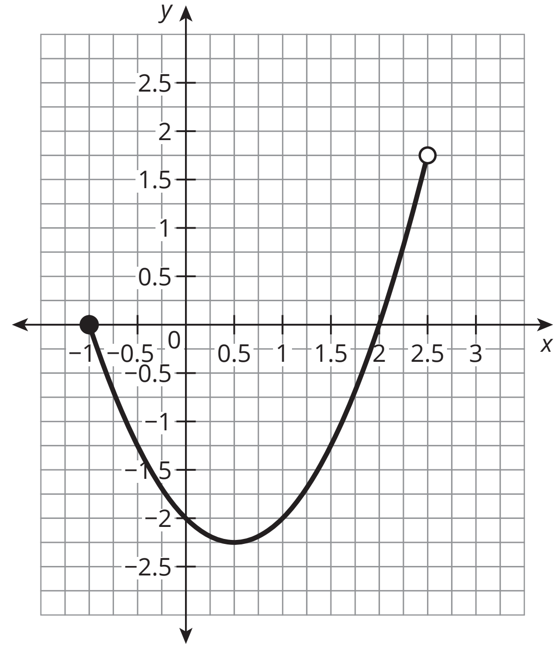
***y*-intercept:**

Axis of symmetry:

Vertex:

Interval of increase:

Interval of decrease:



Ask Yourself . . .
How does an open or closed circle on a graph reflect the domain? The range?

d. The graph shown represents part of the function $f(x) = x^2 - 3x + 2$.

Domain:

Range:

x-intercepts:

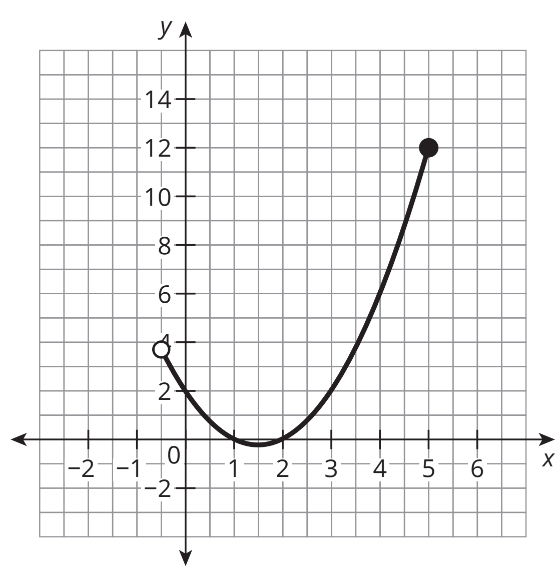
y-intercept:

Axis of symmetry:

Vertex:

Interval of increase:

Interval of decrease:



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ACTIVITY 2.4 Exploring Factored Form

When given a function $g(x)$ has a zero at $x = 4$, then $g(4) = 0$. This can also be interpreted as an x-intercept at $(4, 0)$.

You have analyzed quadratic functions and their equations. Let's look at the factored form of a quadratic function in more detail.

1. A group of students each write a quadratic function in factored form to represent a parabola that opens downward and has zeros at $x = 4$ and $x = -1$.

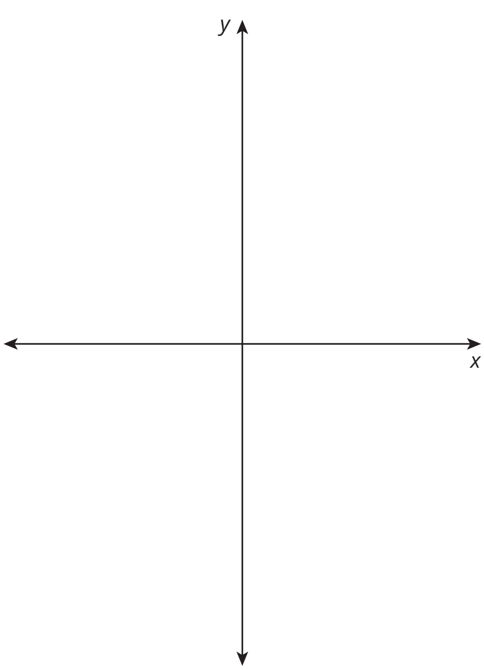
Gabriela
My function is
$k(x) = -(x - 4)(x + 1)$.

Alexander
My function is
$g(x) = -2(x - 4)(x + 1)$.

Fernando
My function is
$m(x) = 2(x - 4)(x + 1)$.

Michael
My function is
$f(x) = -(x + 4)(x - 1)$.

a. Sketch a graph of each student's function and label key points. What are the similarities among all the graphs? What are the differences among the graphs?



b. What would you tell Fernando and Michael to correct their functions?

c. How is it possible to have more than one correct function?

d. How many possible functions can represent the given characteristics? Explain your reasoning.

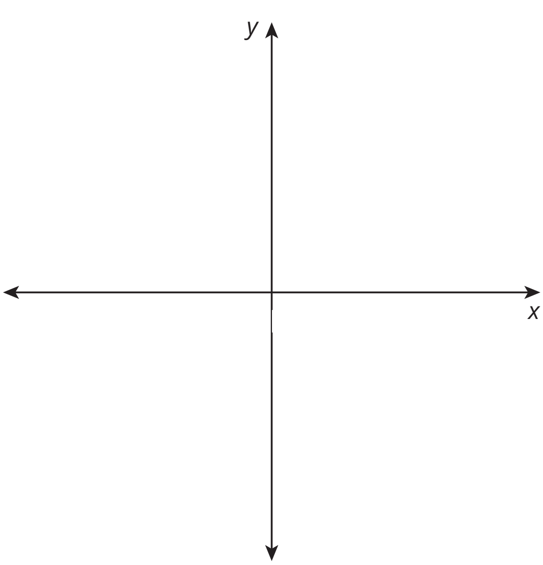
2. Consider a quadratic function written in factored form, $f(x) = a(x - r_1)(x - r_2)$.

   a. What does the sign of the a-value tell you about the graph?

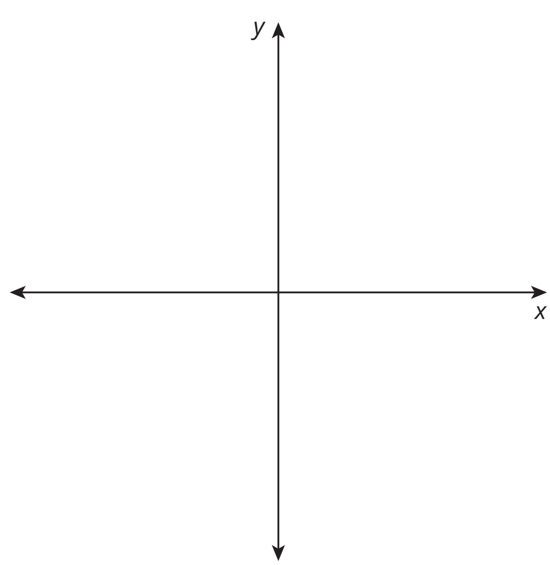
   b. What do r_1 and r_2 tell you about the graph?

3. Use the given information to write a function in factored form. Sketch a graph of each function and label key points, which include the vertex and the x- and y-intercepts.

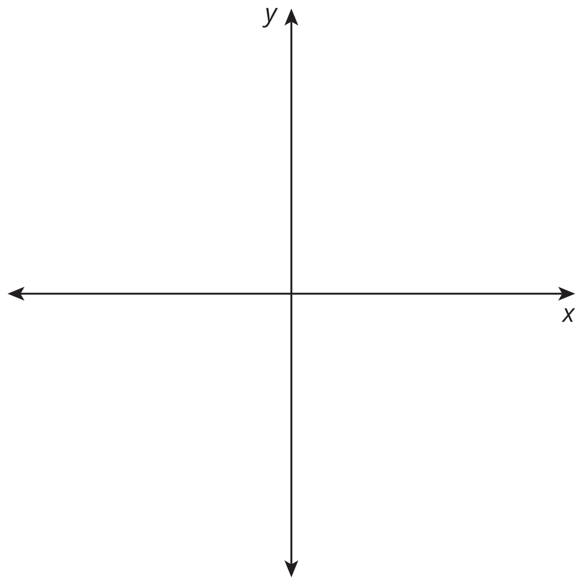
   a. The parabola opens upward, and the zeros are at $x = 2$ and $x = 4$.



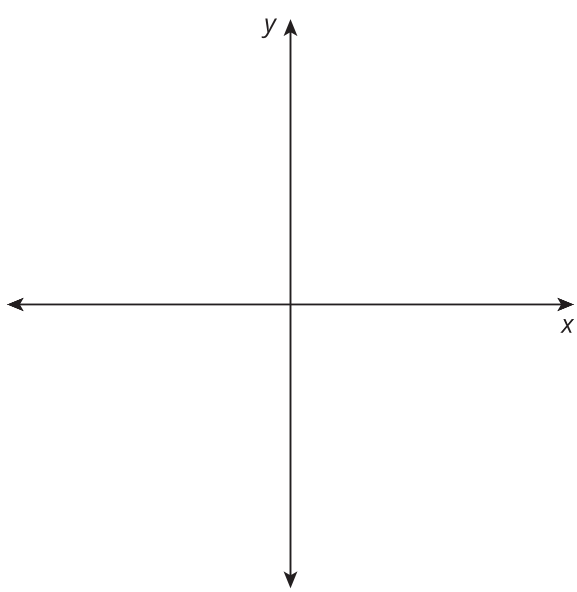
   b. The parabola opens downward, and the zeros are at $x = -3$ and $x = 1$.



c. The parabola opens downward, and the zeros are at $x = 0$ and $x = 5$.



d. The parabola opens upward, and the zeros are at $x = -2.5$ and $x = 4.3$.



4. Compare your quadratic functions with your classmates' functions. How does the *a*-value affect the shape of the graph?

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5. For each quadratic function,

   - Use the standard form to determine the axis of symmetry, the absolute maximum or absolute minimum, and the y-intercept. Graph and label each characteristic.
   - Use technology to identify the zeros. Label the zeros on the graph.
   - Draw the parabola. Use the curve to write the function in factored form.
   - Verify the function you wrote in factored form is equivalent to the given function in standard form.

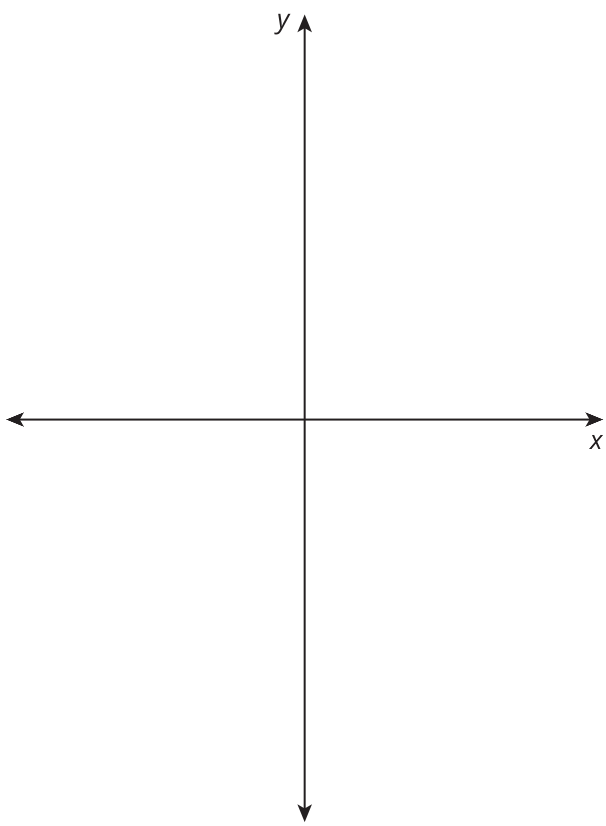
Remember . . .

A function written in standard form $f(x) = ax^2 + bx + c$ has an axis of symmetry at $x = \frac{-b}{2a}$.

a. $h(x) = x^2 - 8x + 12$

Zeros: ____________________

Factored form: ____________________



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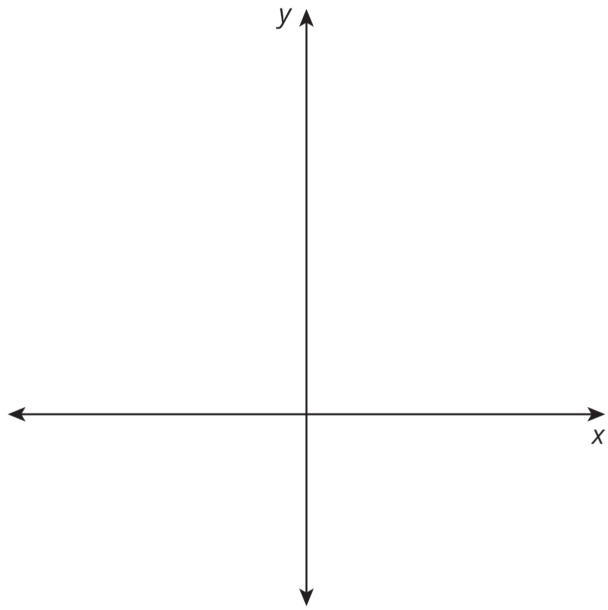
b. $r(x) = -2x^2 + 6x + 20$

Zeros: ______________________

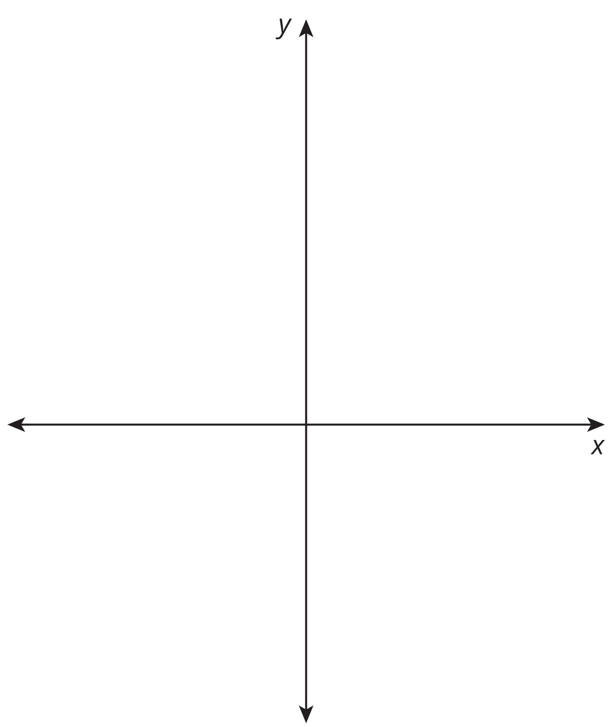
Factored form: ______________________



c. $w(x) = -x^2 - 4x$

Zeros: ______________________

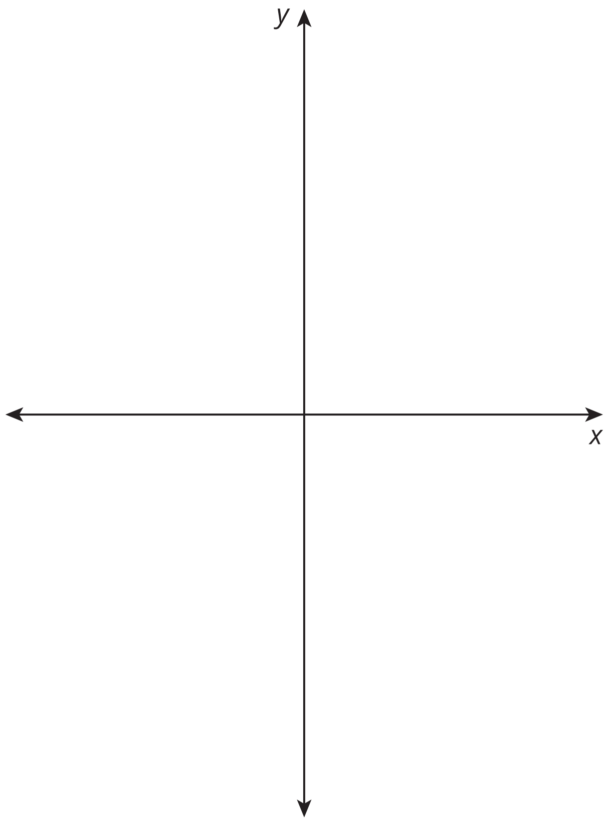
Factored form: ______________________



d. $c(x) = 3x^2 - 3$

Zeros: ____________________

Factored form: ____________________



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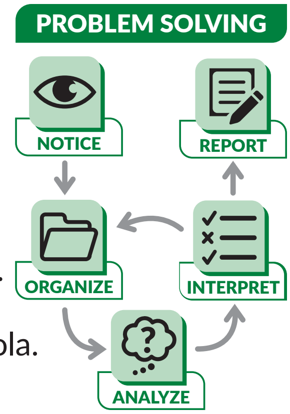
Talk the Talk



Quadratic Sleuthing

Think About . . .
Sketch a graph by hand when you need a model.

Use the given information to answer each question. Do not use technology. Show your work.

1. Determine the axis of symmetry of each parabola.

   a. The x-intercepts of the parabola are (1, 0) and (5, 0).

   b. The x-intercepts of the parabola are (−3.5, 0) and (4.1, 0).

   c. Two symmetric points on the parabola are (−7, 2) and (0, 2).

2. Describe how to determine the axis of symmetry given the x-intercepts of a parabola.

3. Determine the location of the vertex of each parabola.

   a. The function $f(x) = x^2 + 4x + 3$ has the axis of symmetry of $x = -2$.

   b. The equation of the parabola is $y = x^2 - 4$, and the x-intercepts are (−2, 0) and (2, 0).

   c. The function $f(x) = x^2 + 6x - 5$ has two symmetric points, (−1, −10) and (−5, −10).

4. Describe how to determine the vertex of a parabola given the equation and the axis of symmetry.

5. Determine another point on each parabola.

   a. The axis of symmetry is $x = 2$, and a point on the parabola is (0, 5).

   b. The vertex is (0.5, 9), and an x-intercept is (−2.5, 0).

   c. The vertex is (−2, −8), and a point on the parabola is (−1, −7).

6. Describe how to determine another point on a parabola when you are given one point and the axis of symmetry.

Ask Yourself . . .
What precise mathematical language do you need to communicate your mathematical reasoning?

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Lesson 2 Assignment

Write

1. Describe the characteristics of a quadratic function that you can determine from its equation in standard form.
2. Describe the characteristics of a quadratic function that you can determine from its equation in factored form.

Remember

The sign of the leading coefficient of a quadratic function in standard form or factored form describes whether the function has an absolute maximum or absolute minimum.

A parabola is a smooth curve with reflectional symmetry. The axis of symmetry contains the vertex of the graph of the function, which is located at the absolute minimum or absolute maximum of the function.

Practice

1. Analyze each quadratic function.

$g(x) = 12x - 4x^2 + 16$ $\qquad$ $h(x) = \frac{1}{4}(x - 3)(x + 2)$

a. Identify the quadratic function as standard form or factored form.

b. Does the quadratic function have an absolute maximum or absolute minimum?

c. Does the graph open upward or downward?

d. Determine any intercepts from the given form of the function.

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Lesson 2 Assignment

2. Analyze each quadratic function.

   $f(x) = -\frac{2}{3}x^2 - 3x + 15$ $\qquad$ $g(x) = \frac{3}{4}x^2 + 12x - 27$

   a. Identify the axis of symmetry.

   b. Use the axis of symmetry to determine the ordered pair of the absolute maximum or absolute minimum value.

   c. Describe the intervals of increase and decrease.

   d. Sketch the graph based on the information you just calculated.

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Lesson 2 Assignment

e. Use technology to identify the zeros.

f. Place two pairs of symmetric points on your graph. What is the average rate of change between these pairs of symmetric points?

g. Write the function in factored form.

3. A parabola opens downward and has zeros at $x = -2$ and $x = 3$.

a. Represent it as a quadratic equation in factored form.

b. Sketch a graph of the quadratic function.

c. What is the axis of symmetry and y-intercept of the quadratic function?

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Lesson 2 Assignment

Prepare

1. Describe the effect of changing the *a*-value of the function $a \cdot f(x)$ given the parent function $f(x) = x$.

2. Describe the effect of changing the *d*-value of the function $f(x) + d$ given the parent function $f(x) = x$.

3 Quadratic Function Transformations

OBJECTIVES

- Experiment with transformations of quadratic functions using technology.
- Graph quadratic functions and transformations of quadratic functions.
- Determine the effect of replacing the parent quadratic function $f(x) = x^2$ with $f(x) + d$, $af(x)$ and $f(x - c)$ for different values of a, c, and d.
- Distinguish between function transformations that occur outside the function and inside the argument of the function.

NEW KEY TERMS

- argument of a function
- reflection
- line of reflection

You know how to transform linear functions.

How can you define quadratic functions and show transformations of this function type?

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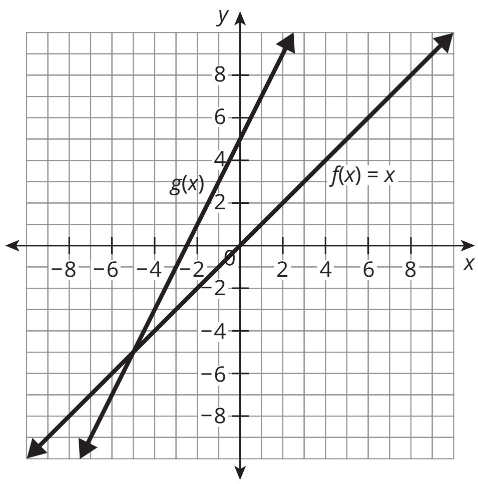
Getting Started

A Blast From the Past

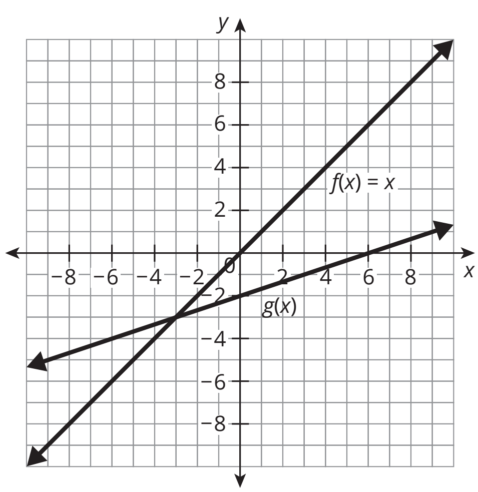
Let $f(x) = x$ and $g(x) = a \cdot f(x) + d$.

For each function and graph on the coordinate plane, describe the transformations.

1. $g(x) = 2(x) + 5$



2. $g(x) = \frac{1}{3}(x) - 2$



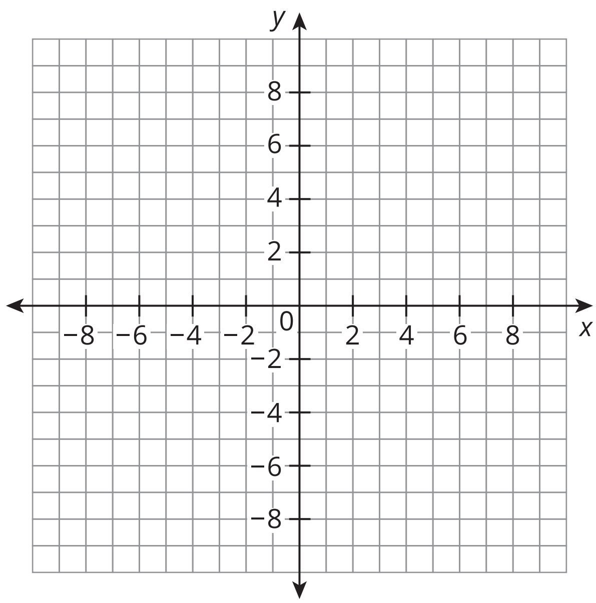
ACTIVITY 3.1

Transformations Outside the Function

Consider the three quadratic functions shown.

$g(x) = x^2$ $c(x) = x^2 + 3$ $d(x) = x^2 - 3$

1. Use technology to graph each function. Then, sketch and label the graph of each function.



2. Write the functions $c(x)$ and $d(x)$ in terms of the parent function $g(x)$. Then, describe the transformations of each function.

3. Describe the similarities and differences between the three graphs. How do these similarities and differences relate to the equations of the functions $g(x)$, $c(x)$ and $d(x)$?

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Recall that the function $t(x)$ of the form $t(x) = f(x) + d$ is a vertical translation of the function $f(x)$. The value $|d|$ describes how many units up or down the graph of the original function is translated.

4. Describe each graph in relation to the parent function, $g(x) = x^2$. Then, use coordinate notation to represent the vertical translation.

   a. $f(x) = g(x) + d$, when $d > 0$

   b. $f(x) = g(x) + d$, when $d < 0$

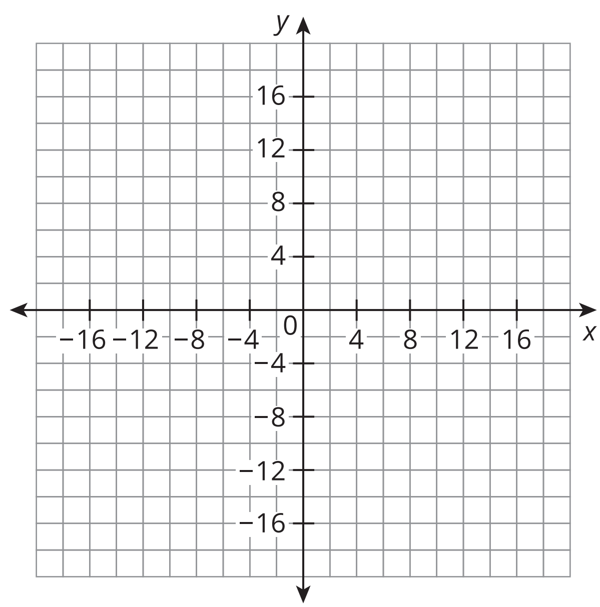
   c. Each point (x, y) on the graph of $g(x)$ becomes the point ______________ on $f(x)$.

Consider these quadratic functions.

$g(x) = x^2$ $\quad k(x) = \frac{1}{2}x^2$

$j(x) = 2x^2$ $\quad p(x) = -x^2$

5. Use technology to graph each function. Then, sketch and label the graph of each function.



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6. Write the functions $j(x)$, $k(x)$, and $p(x)$ in terms of the parent function $g(x)$. Then, describe the transformations of each function.

Recall that a function $t(x)$ of the form $t(x) = a \cdot f(x)$ is a vertical dilation of the function $f(x)$. The a-value describes the vertical dilation of the graph of the original function.

7. Describe each graph in relation to the parent function $g(x) = x^2$. Then, use coordinate notation to represent the vertical dilation.

   a. $f(x) = a \cdot g(x)$, when $a > 1$

   b. $f(x) = a \cdot g(x)$, when $a < 0$

   c. $f(x) = a \cdot g(x)$, when $0 < a < 1$

   d. Each point (x, y) on the graph of $g(x)$ becomes the point

   ________________ on $f(x)$.

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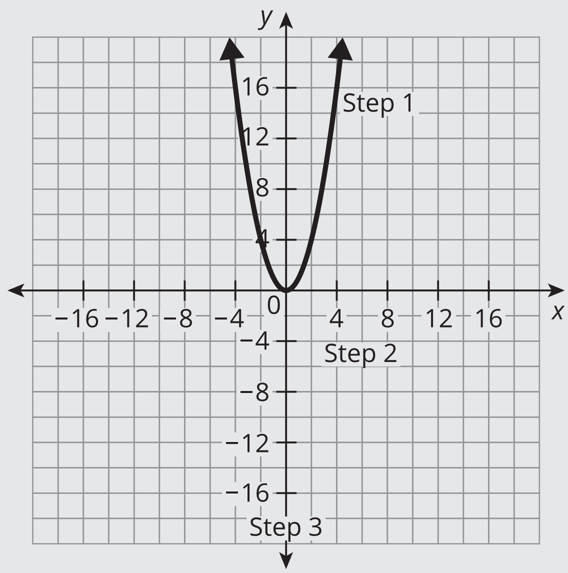
You know that changing the *a*-value of a function to its opposite reflects the function across a horizontal line. But, the line of reflection for the function might be different, depending on how you write the transformation and the order in which the transformations are applied.

8. Jacob and Hannah each sketched a graph of the function $b(x) = -(x^2) - 3$ using different strategies. Write the step-by-step reasoning that each student used.

A *reflection of a graph* is the mirror image of the graph about a line of reflection.

Jacob



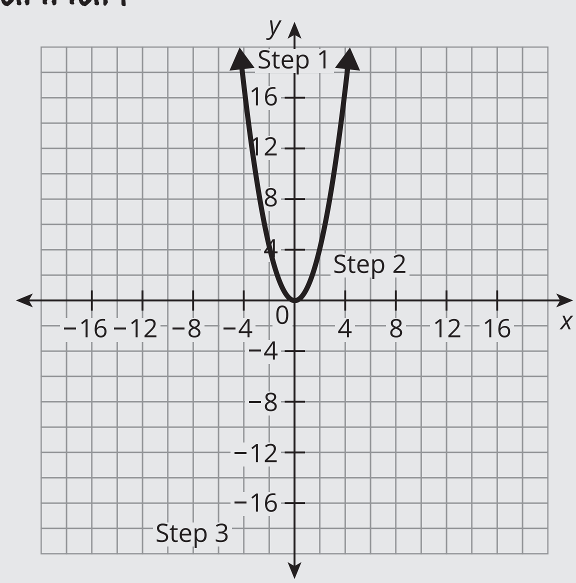
Step 1:

Step 2:

Step 3:

A *line of reflection* is the line that the graph is reflected across. A horizontal line of reflection affects the *y*-coordinates.

Hannah



Step 1:

Step 2:

Step 3:

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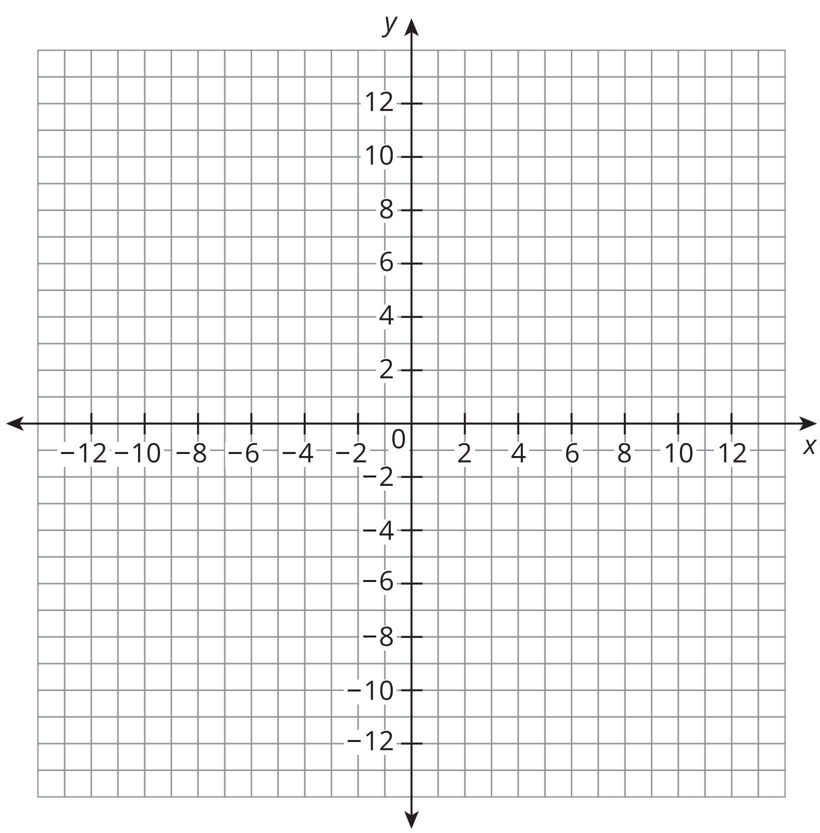
9. Explain how changing the order of the transformations affects the line of reflection.

Given the function $f(x) = x^2$, use the coordinate plane below to answer Questions 10 through 14.

10. Consider the function $a(x) = 2f(x) + 1$.

    a. Use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of $a(x)$.

    b. Graph and label $a(x)$ on the coordinate plane below.



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11. Consider the function $b(x) = -2f(x) + 1$.

a. Use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of $b(x)$.

b. Graph and label $b(x)$ on the coordinate plane above.

12. Describe the graph of $b(x)$ in terms of $a(x)$.

13. Consider the function $-a(x)$.

a. Use coordinate notation to describe how each point (x, y) on the graph of $a(x)$ becomes a point on the graph of $-a(x)$.

b. Graph and label $-a(x)$ on the coordinate plane on the previous page.

14. Describe the graph of $-a(x)$ in terms of $a(x)$.

ACTIVITY
3.2 Horizontal Translations of Quadratic Functions

Consider the three quadratic functions shown, where $h(x) = x^2$ is the parent function. The operations are performed on x, which is the argument of the function.

- $h(x) = x^2$
- $v(x) = (x + 3)^2$
- $w(x) = (x - 3)^2$

The **argument of a function** is the expression inside the parentheses. For $y = f(x - c)$ the expression $x - c$ is the argument of the function.

You can write the given functions $v(x)$ and $w(x)$ in terms of the parent function $h(x)$.

WORKED EXAMPLE

To write $v(x)$ in terms of $h(x)$, you just substitute $x + 3$ into the argument for $h(x)$, as shown.

$$h(x) = x^2$$

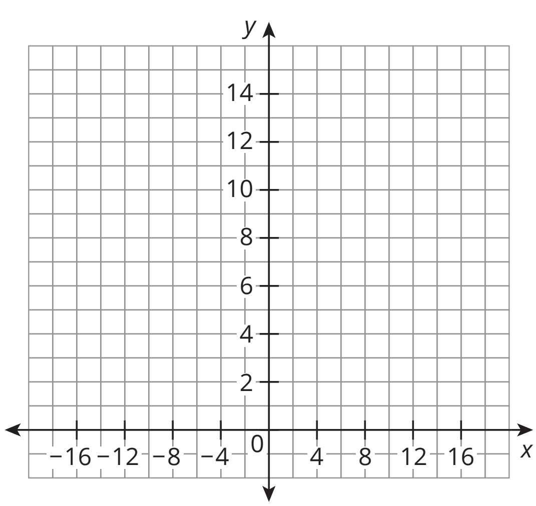
$$v(x) = h(x + 3) = (x + 3)^2$$

So, $x + 3$ replaces the variable x in the function $h(x) = x^2$.

1. Write the function $w(x)$ in terms of the parent function $h(x)$.

Sketch the graphs one at a time to help you see which is which.

2. Sketch and label the graph of each function. Identify key points.



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3. Compare the graphs of $v(x)$ and $w(x)$ to the graph of the parent function. What do you notice?

Think About . . .

Notice there are no negative y-values in this table. Are negative values included in the range of h, v, or w?

4. Write the x-value of each of the corresponding reference points on $v(x)$ and $w(x)$.

$h(x) = x^2$	$v(x) = (x + 3)^2$	$w(x) = (x - 3)^2$
$(-2, 4)$	$(\_\_\_\_, 4)$	$(\_\_\_\_, 4)$
$(-1, 1)$	$(\_\_\_\_, 1)$	$(\_\_\_\_, 1)$
$(0, 0)$	$(\_\_\_\_, 0)$	$(\_\_\_\_, 0)$
$(1, 1)$	$(\_\_\_\_, 1)$	$(\_\_\_\_, 1)$
$(2, 4)$	$(\_\_\_\_, 4)$	$(\_\_\_\_, 4)$

5. Use the table to compare the ordered pairs of the graphs of $v(x)$ and $w(x)$ to the ordered pairs of the graph of the parent function $h(x)$. What do you notice?

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6. Complete each sentence with the coordinate notation to represent the horizontal translation of each function.

   a. $v(x) = h(x + 3)$
      Each point (x, y) on the graph of $h(x)$ becomes the point
      ________________ on $v(x)$.

   b. $w(x) = h(x - 3)$
      Each point (x, y) on the graph of $h(x)$ becomes the point
      ________________ on $w(x)$.

When a constant is added or subtracted outside a function, like $g(x) + 3$ or $g(x) - 3$, then only the y-values change, resulting in a vertical translation. When a constant is added or subtracted inside a function, like $g(x + 3)$ or $g(x - 3)$, then only the x-values change, resulting in a horizontal translation.

7. Describe each graph in relation to the parent function $h(x) = x^2$.

   a. Compare $f(x) = h(x - c)$ to the parent function for $c > 0$.

   b. Compare $f(x) = h(x - c)$ to the parent function for $c < 0$.

Recall that for the parent function, the c-value of the transformed function $y = f(x - c)$ affects the input values of the function. The value $|c|$ describes the number of units the graph of $f(x)$ is translated right or left. When $c > 0$, the graph is translated to the right; when $c < 0$, the graph is translated to the left.

8. What generalization can you make about the effects of horizontal translations on the domain, range, and vertex of quadratic functions?

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ACTIVITY

3.3

Reflections of Quadratic Functions

Consider the three quadratic functions below, where $h(x) = x^2$ is the parent function.

- $h(x) = x^2$
- $m(x) = -(x^2)$
- $n(x) = (-x)^2$

1. Write the functions $m(x)$ and $n(x)$ in terms of the parent function $h(x)$.

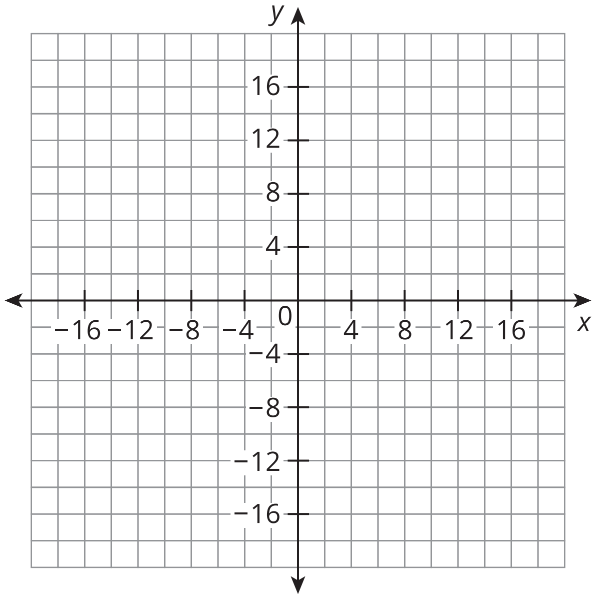
    a. $m(x) =$

    b. $n(x) =$

2. Compare $m(x)$ to $h(x)$. Does an operation performed on $h(x)$ or on the argument of $h(x)$ result in the equation for $m(x)$? What is the operation?

3. Compare $n(x)$ to $h(x)$. Does an operation performed on $h(x)$ or on the argument of $h(x)$ result in the equation for $n(x)$? What is the operation?

4. Use technology to sketch and label each function.



5. Compare the graphs of $m(x)$ and $n(x)$ to the graph of the parent function $h(x)$. What do you notice?

6. Write the x- or y-value of each of the corresponding reference points on $m(x)$ and $n(x)$.

$h(x) = x^2$	$m(x) = -(x^2)$	$n(x) = (-x)^2$
(−2, 4)	(−2, ____)	(____, 4)
(−1, 1)	(−1, ____)	(1, 1)
(0, 0)	(0, ____)	(____, 0)
(1, 1)	(1, ____)	(____, 1)
(2, 4)	(2, ____)	(____, 4)

7. Use the table to compare the ordered pairs of the graphs of $m(x)$ and $n(x)$ to the ordered pairs of the graph of the parent function $h(x)$. What do you notice?

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When the negative is on the outside of the function, like $-g(x)$, all the y-values become the opposite of the y-values of $g(x)$. The x-values remain unchanged.

Remember, a **reflection** of a graph is a mirror image of the graph about a line of reflection. A **line of reflection** is the line that the graph is reflected across. A horizontal line of reflection affects the y-coordinates, and a vertical line of reflection affects the x-coordinates.

8. Consider the graphs of $m(x)$ and $n(x)$.

   a. Which function represents a reflection of $h(x)$ across a horizontal line? Name the line of reflection.

When the negative is on the inside of the function, like $g(-x)$, all the x-values become the opposite of the x-values of $g(x)$. The y-values remain unchanged.

   b. Which function represents a reflection of $h(x)$ across a vertical line? Name the line of reflection.

9. Complete each sentence with the coordinate notation to represent the reflection of each function.

   a. $m(x) = -h(x)$
   Each point (x, y) on the graph of $h(x)$ becomes the point
   ______________ on $m(x)$.

   b. $n(x) = h(-x)$
   Each point (x, y) on the graph of $h(x)$ becomes the point
   ______________ on $n(x)$.

Talk the Talk

Write Them Up!

1. Consider the function, $f(x) = x^2$. Write the function in transformation function form in terms of the transformations described, then write an equivalent equation.

Transformation	Transformation Function Form	Equation
a. Reflection across the x-axis		
b. Horizontal translation of 2 units to the left and a vertical translation of 3 units up		
c. Vertical stretch of 2 units and a reflection across the line $y = 0$		
d. Vertical dilation of 2 units and a reflection across the line $y = 3$		
e. Horizontal translation of 3 units to the right, a vertical translation down 2 units, and a vertical dilation of $\frac{1}{2}$		
f. Vertical compression by a factor of 4		
g. Vertical stretch by a factor of 4		

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Lesson 3 Assignment

Write

Given a parent function $y = f(x)$ and a function written in transformation form $g(x) = a \cdot f(x - c) + d$, describe how the transformations that are inside a function affect a graph differently than those on the outside of the function.

Remember

The parent quadratic function is $f(x) = x^2$.

The transformed function $y = f(x) + d$ shows a vertical translation of the function.

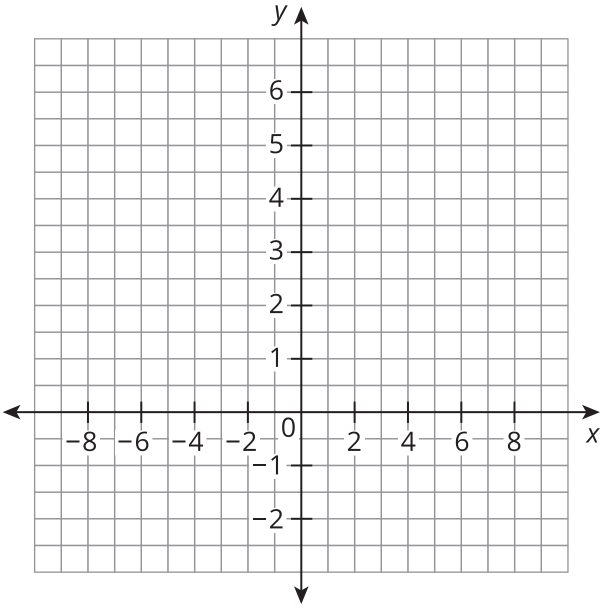
The transformed function $y = af(x)$ shows a vertical dilation of the function when $a > 0$, and when $a < 0$, it shows a vertical dilation and reflection across the x-axis.

The transformed function $y = f(x - c)$ shows a horizontal translation of the function.

Practice

Given the parent function $f(x) = x^2$, consider each transformation. Describe how the transformations affected $f(x)$. Then, use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of the transformed function. Finally, sketch a graph of each new function.

1. $g(x) = \frac{1}{3}f(x) - 2$



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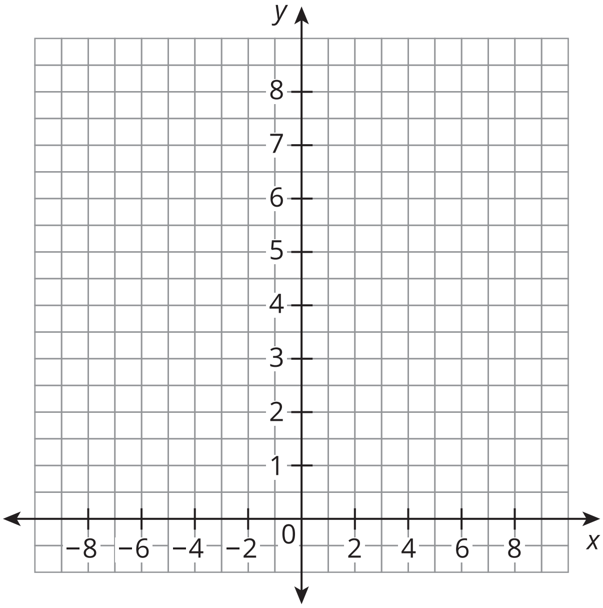
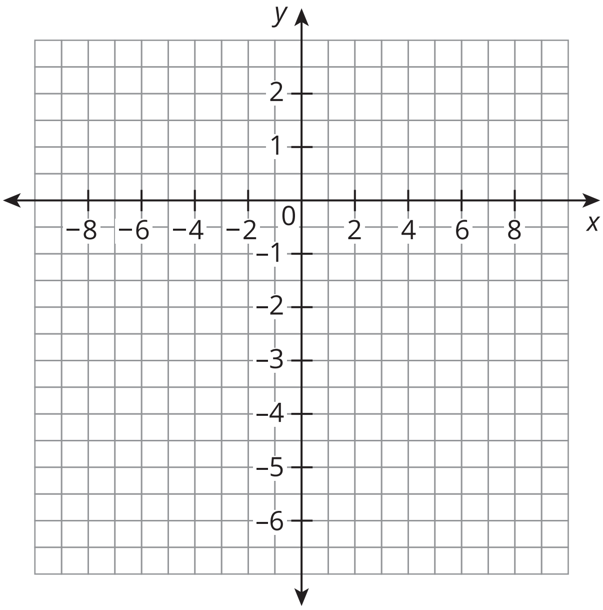
Lesson 3 Assignment

2. $j(x) = 2f(x + 1) + 4$

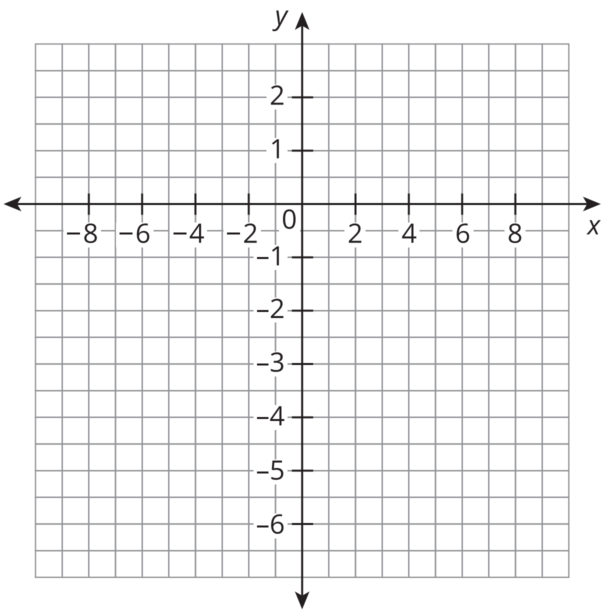


3. $m(x) = -\frac{1}{2}f(x - 3) - 1$



Lesson 3 Assignment

4. $p(x) = -f(x + 4) + 3$



Prepare

Write the equation for the axis of symmetry given each quadratic function.

1. $f(x) = -3x^2 - 4x + 5$

2. $f(x) = \frac{1}{4}(x - 1)(x + 2)$

3. $f(x) = -x^2 + 3$

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4 Horizontal Transformations and Vertex Form

OBJECTIVES

- Dilate quadratic functions horizontally.
- Write equations of quadratic functions given multiple transformations.
- Graph quadratic functions given multiple transformations.
- Identify multiple transformations of quadratic functions given equations.
- Understand the form in which a quadratic function is written can reveal different key characteristics.
- Write quadratic equations in vertex and factored form.

NEW KEY TERM

- vertex form

You know how to transform linear functions.

How can you apply what you know about the transformation form of a function, $g(x) = a \cdot f(b(x - c)) + d$, to quadratic functions?

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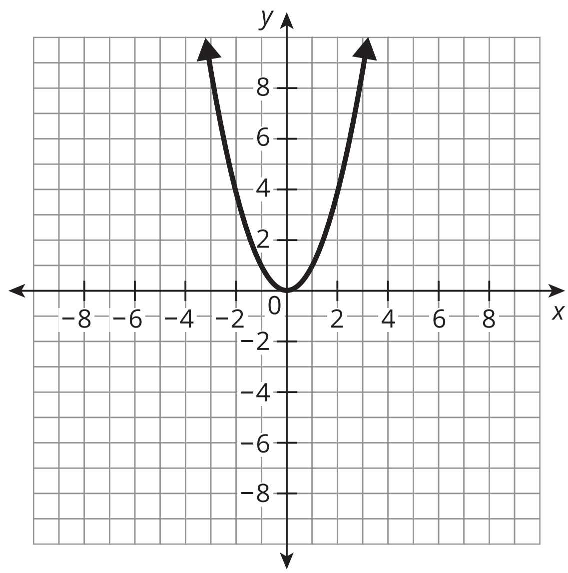
Getting Started

H, I, J, . . .

Consider the function graphed.



1. Identify each part of the graphed function.

   a. Domain

   b. Range

   c. Vertex

Recall that the transformation form of a function $y = f(x)$ can be written as shown.

Outside the function

$$g(x) = a \cdot f(x - c) + d$$

Inside the function

2. How do the a-, c-, and d-values affect the graph of $f(x)$?

ACTIVITY 4.1

Horizontal Dilations of Quadratic Functions

In the previous lesson, you considered a transformation that affects the input values of a function. For the parent function, the b-value of the transformed function $y = f(bx)$ affects the input values of the function. For $b = -1$, the graph is reflected across the line $x = 0$, or the y-axis.

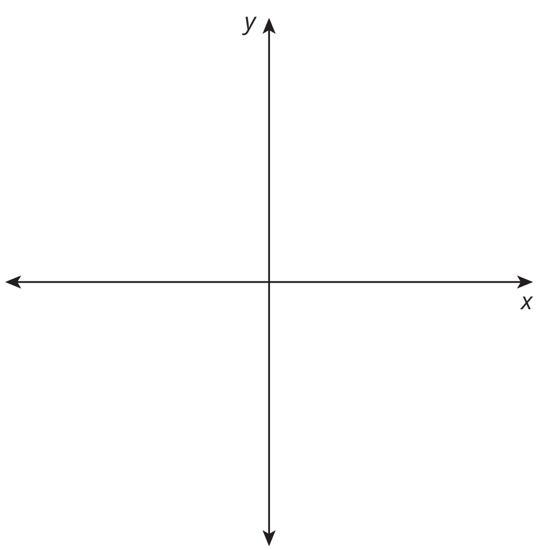
Let's consider different b-values and their effect on a parent function.

1. Consider the three quadratic functions, where $f(x) = x^2$ is the parent function.

   - $f(x) = x^2$
   - $t(x) = (3x)^2$
   - $q(x) = \left(\frac{1}{3}x\right)^2$

   a. Write the functions $t(x)$ and $q(x)$ in terms of the parent function $f(x)$. For each, determine whether you are performing an operation on the function $f(x)$ or on the argument of the function $f(x)$. Describe the operation.

   b. Sketch the graph of each function. Label each graph and include key points.



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c. Use coordinate notation to represent the dilation of each function. Each point (x, y) on the graph of $f(x)$:

- becomes the point ________________________________ on the graph of $t(x)$.
- becomes the point ________________________________ on the graph of $q(x)$.

A horizontal dilation is a type of transformation that stretches or compresses the entire graph. Horizontal stretching is the stretching of a graph away from the y-axis. Horizontal compression is the squeezing of a graph toward the y-axis.

Consider the quadratic functions, where $h(x) = x^2$ is the parent function.

- $w(x) = \left(\frac{1}{4}x\right)^2$
- $z(x) = (4x)^2$

2. Write the functions $w(x)$ and $z(x)$ in terms of $h(x)$.

3. Write the x-value of each of the corresponding reference points on $w(x)$ and $z(x)$.

$h(x) = x^2$	$w(x) = \left(\frac{1}{4}x\right)^2$	$z(x) = (4x)^2$
(−2, 4)	(___, 4)	(___, 4)
(−1, 1)	(___, 1)	(___, 1)
(0, 0)	(___, 0)	(___, 0)
(1, 1)	(___, 1)	(___, 1)
(2, 4)	(___, 4)	(___, 4)

4. Use the table to compare the ordered pairs of the graphs of $w(x)$ and $z(x)$ to the ordered pairs of the graph of the parent function $h(x)$. What do you notice?

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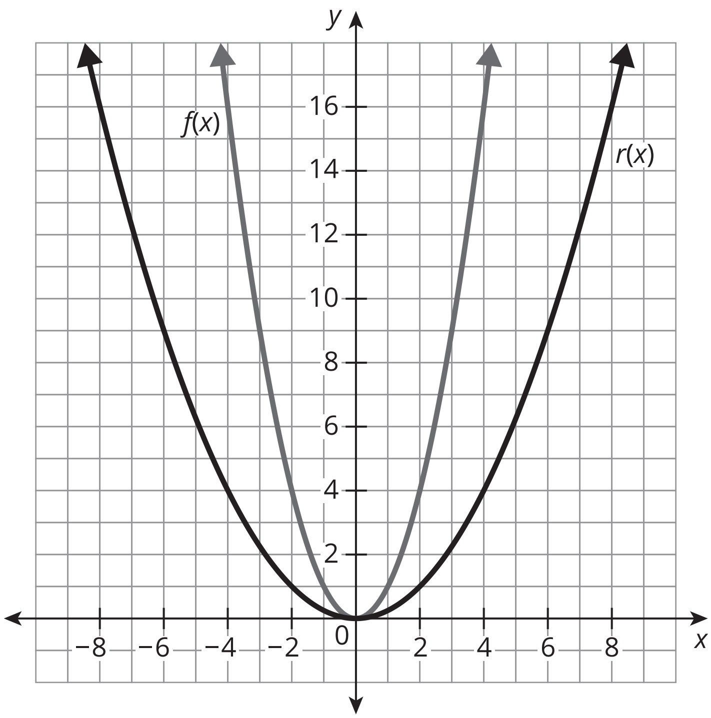
5. Given that the point (x, y) is on the graph of the function $y = f(x)$, what ordered pair describes a point on the graph of $g(x) = f(bx)$?

6. Now, let's compare the graph of $f(x) = x^2$ with $r(x) = f\left(\frac{1}{2}x\right)$.



a. Analyze the table of values that correspond to the graph.

Circle instances where the y-values for each function are the same. Then, list all the points where $f(x)$ and $r(x)$ have the same y-value. The first instance has been circled for you.

x	$f(x) = x^2$	$r(x) = f\left(\frac{1}{2}x\right)$
0	0	0
1	1	0.25
2	4	1
3	9	2.25
4	16	4
5	25	6.25
6	36	9

b. How do the x-values compare when the y-values are the same?

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c. Complete the statement.
The function $r(x)$ is a ____________________ of $f(x)$ by a factor of ________.

d. How does the factor of stretching or compression compare to the b-value in $r(x)$?

Compared with the graph of $f(x)$, the graph of $f(bx)$ is:

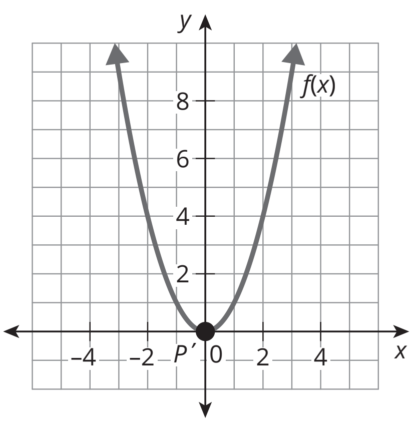
- horizontally compressed by a factor of $\frac{1}{|b|}$ when $|b| > 1$.
- horizontally stretched by a factor of $\frac{1}{|b|}$ when $0 < |b| < 1$.

WORKED EXAMPLE

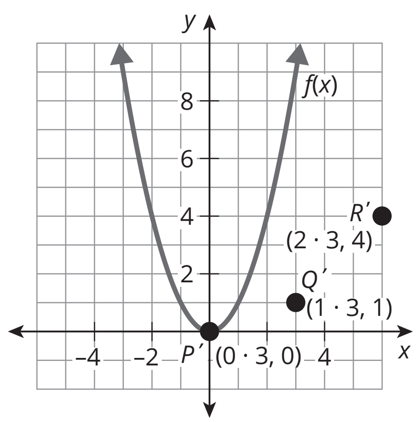
You can use reference points to graph the function $q(x) = f\left(\frac{1}{3}x\right)$ when $f(x) = x^2$.

From $q(x)$, you know that $c = 0$, $d = 0$, and $b = \frac{1}{3}$. The vertex for $q(x)$ is (0, 0).

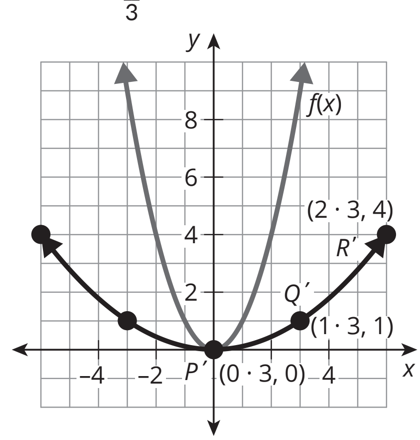
Notice $0 < |b| < 1$, so the graph will horizontally stretch by a factor of $\frac{1}{\frac{1}{3}}$, or 3.



The function $f(x)$ is shown. First, plot the new vertex (c, d). This point establishes the new set of axes.



Next, think about b. To plot Q', move right $1 \cdot 3$ units and up 1 unit from the vertex because all x-coordinates are being stretched by a factor of 3. To plot R', start at the vertex, move to the right $2 \cdot 3$ units, and go up 4 units.



Finally, use symmetry to complete the graph.

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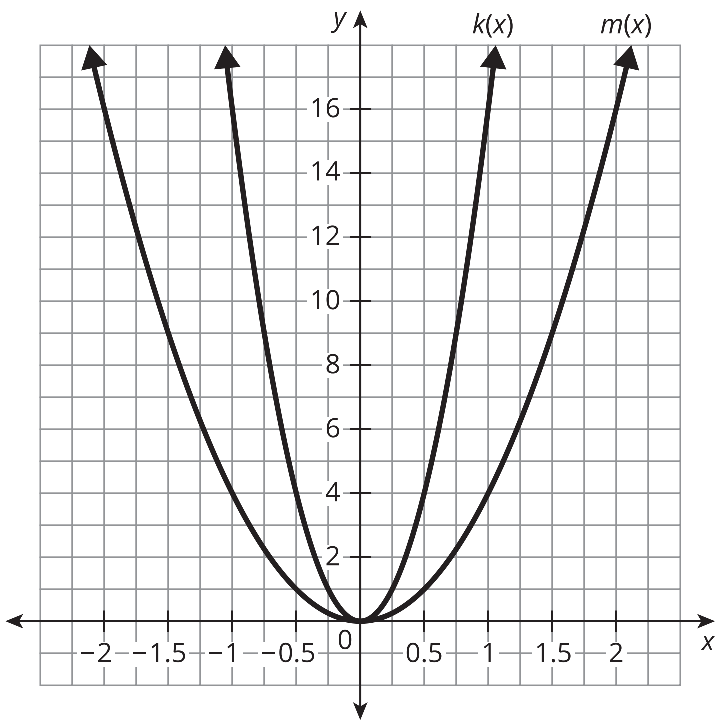
7. Suppose you were going to graph $p(x) = f(3x)$. Describe how the graph would change. When (x, y) is any point on $f(x)$, describe any point on $p(x)$.

8. Consider the graph showing the quadratic functions $k(x)$ and $m(x)$. Amir and Madison are writing the function $m(x)$ in terms of $k(x)$.



Amir says that $m(x)$ is a transformation of the a-value.

$$m(x) = \frac{1}{4}k(x)$$

Madison says that $m(x)$ is a transformation of the b-value.

$$m(x) = k\left(\frac{1}{2}x\right)$$

Who's correct? Justify your reasoning.

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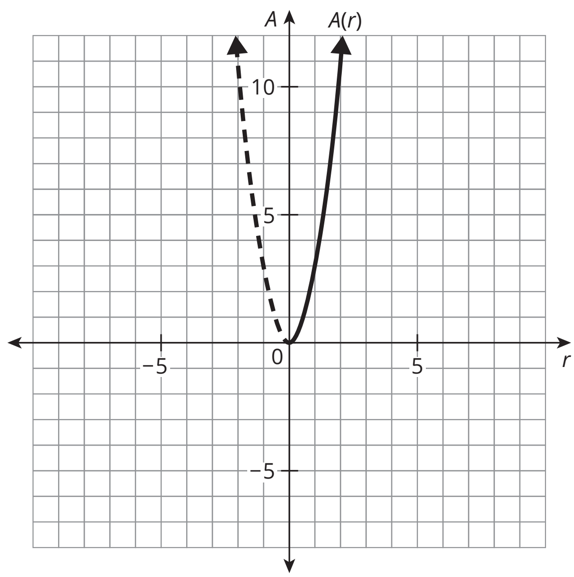
9. Describe how you can rewrite a quadratic function with a b-value transformation as a quadratic function with an a-value transformation.

10. Rewrite the function from the Worked Example, $q(x) = f\left(\frac{1}{3}x\right)$, without a b-value.

Consider the formula to calculate the area of a circle, $a = \pi r^2$. You can represent the area formula as the function $A(r) = \pi r^2$ and represent it on a coordinate plane.

Ask Yourself . . .
Why is part of the graph represented with a dashed smooth curve?



11. How would doubling the radius affect the area? Explain the change in area in terms of a transformation of the graph.

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ACTIVITY
4.2

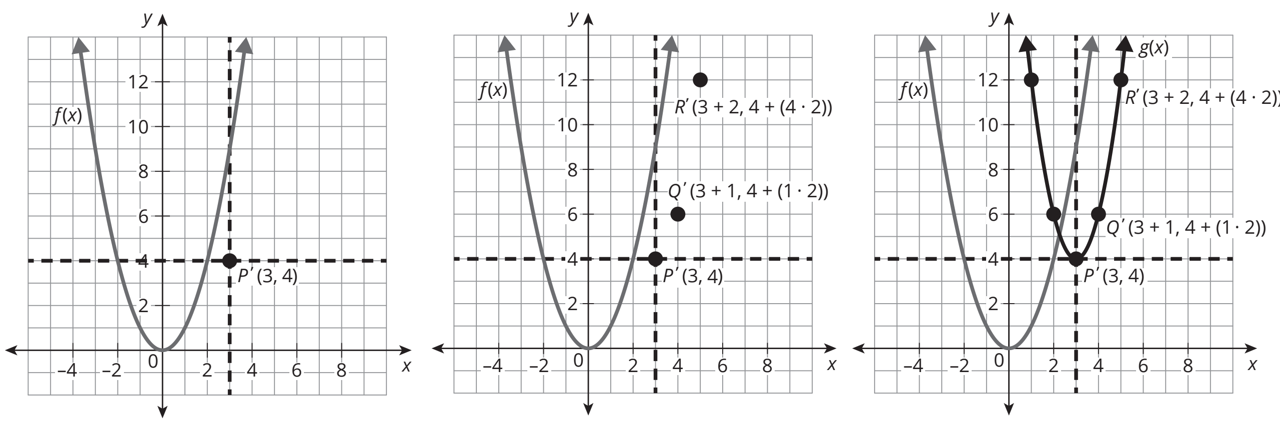
Using Reference Points to Graph Quadratic Functions

Given $y = f(x)$ is the parent quadratic function, you can use reference points to graph $y = af(b(x - c)) + d$. Any point (x, y) on $f(x)$ maps to the point $\left(\frac{1}{b}x + c, ay + d\right)$.

Think About . . .
What is the pattern of the a-value when transforming the parent quadratic function?

WORKED EXAMPLE

Given $f(x) = x^2$, graph the function $g(x) = 2f(x - 3) + 4$.
You can use reference points for $f(x)$ and your knowledge about transformations to graph the function $g(x)$.
From $g(x)$, you know that $a = 2$, $c = 3$, and $d = 4$.
The vertex for $g(x)$ will be at $(3, 4)$. Notice $a > 0$, so the graph of the function will vertically stretch by a factor of 2.



First, plot the new vertex, (c, d). This point establishes the new set of axes.

Next, think about the reference points for the parent quadratic function and that $a = 2$. To plot point Q', move right 1 unit and up, not 1, but $1 \cdot 2$ units from the vertex P' because all y-coordinates are being multiplied by a factor of 2. To plot point R', move right 2 units from P' and up, not 4, but $4 \cdot 2$ units.

Finally, use symmetry to complete the graph.

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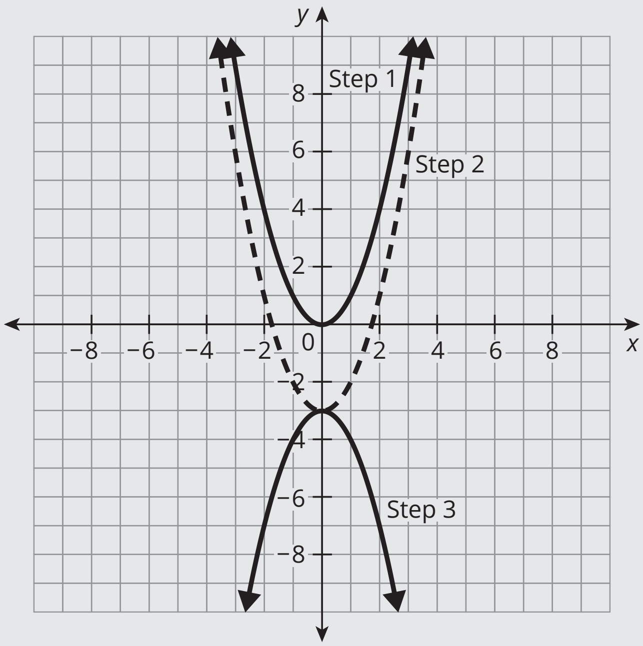
1. William, Maria, and Abby each sketched a graph of the equation $y = -x^2 - 3$ using different strategies. Provide the step-by-step reasoning that each student used.

William

$a = -1$ and $d = -3$



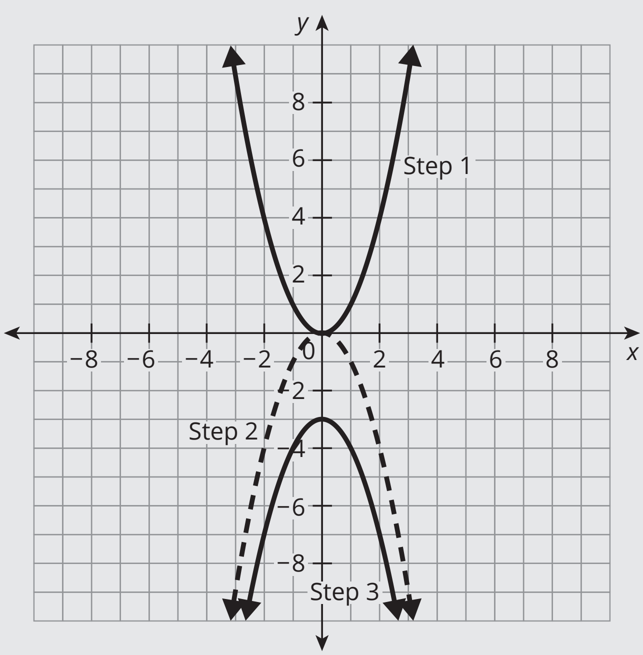
Step 1:

Step 2:

Step 3:

Maria

$d = -3$ and $a = -1$



Step 1:

Step 2:

Step 3:

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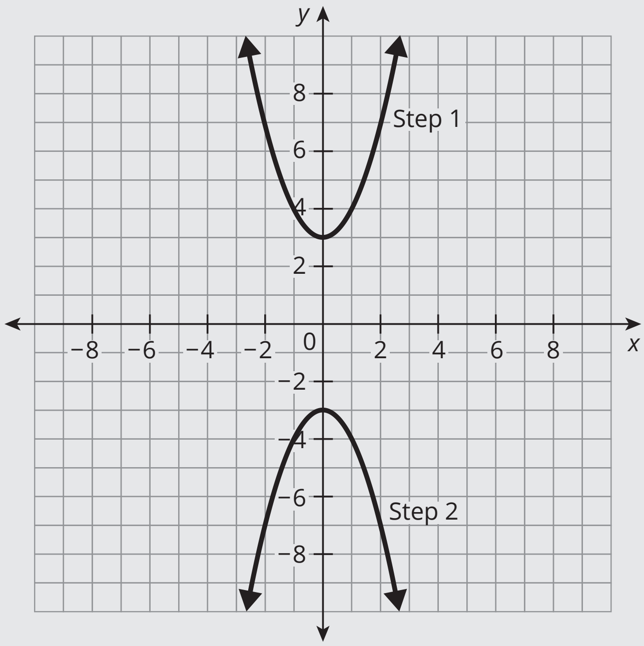
Abby

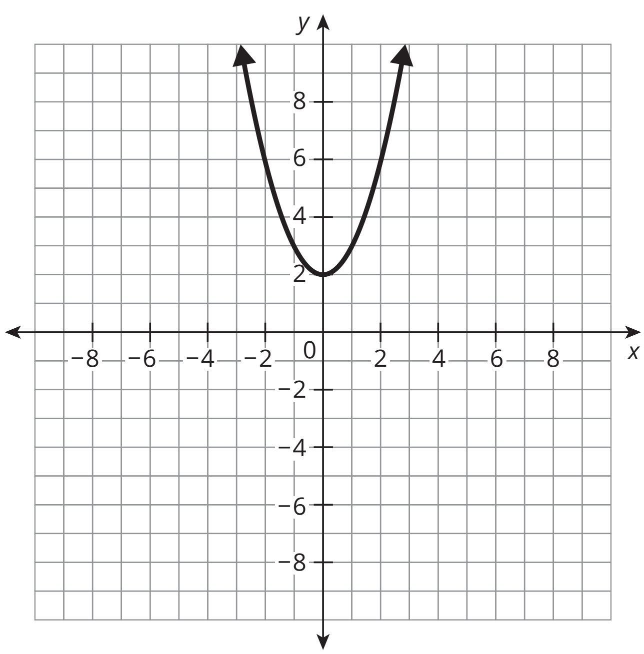
I rewrote the equation as $y = -(x^2 + 3)$.



Step 1:

Step 2:

2. Given $y = p(x)$, sketch $m(x) = -p(x + 3)$. Describe the transformations you performed.



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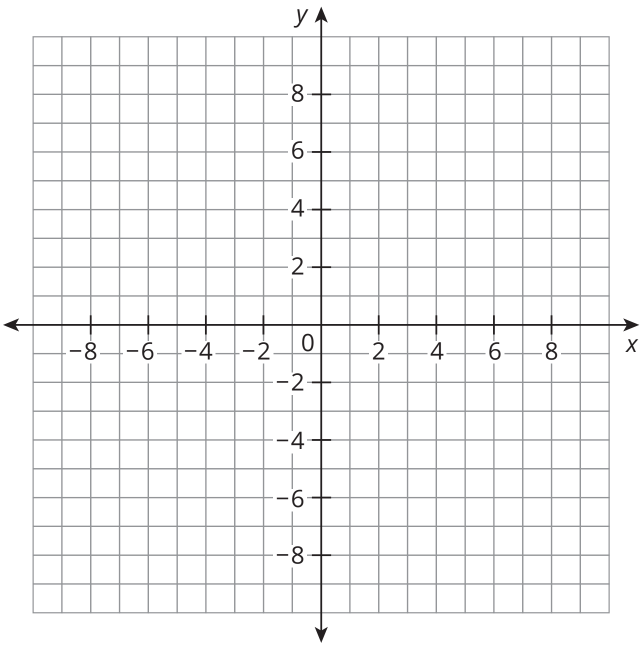
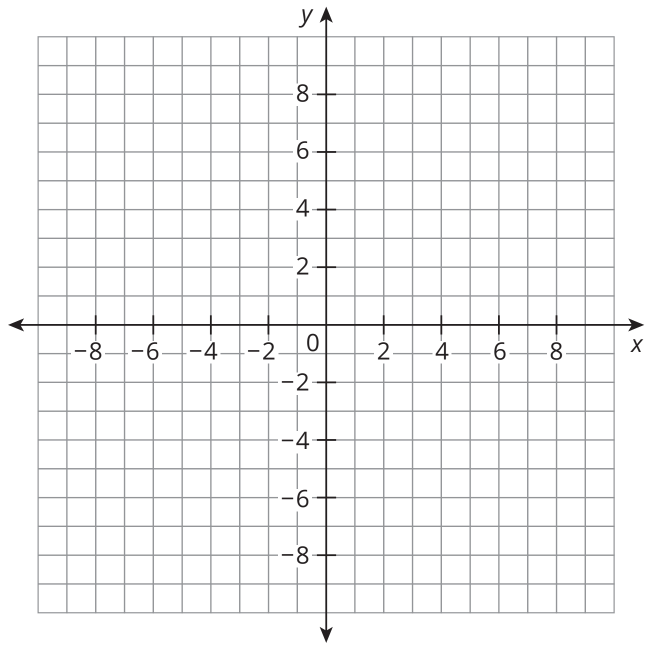
3. Given $f(x) = x^2$, graph each function. Then, write each corresponding quadratic equation.

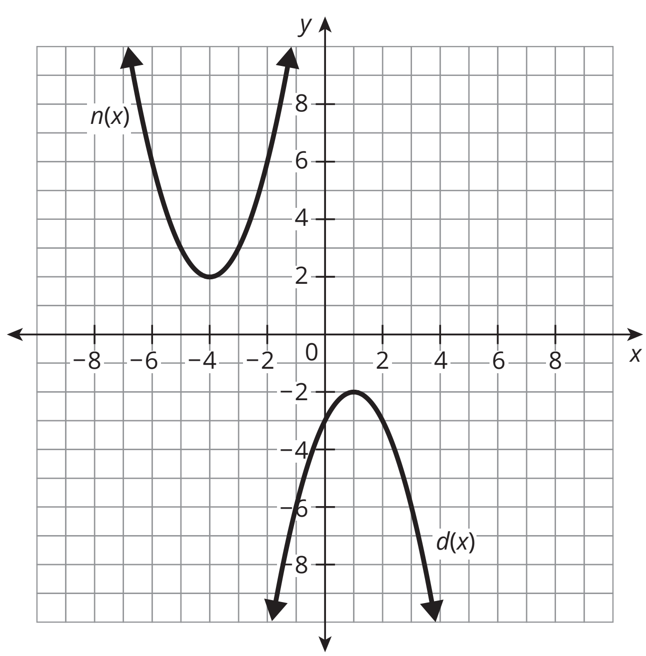
a. $f'(x) = \frac{1}{2}f(x - 2) + 3$



b. $f'(x) = -3f(x + 1) + 1$



4. Write $n(x)$ in terms of $d(x)$. Then, write the quadratic equation for $n(x)$.



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ACTIVITY
4.3
Vertex Form of a Quadratic Function

Given a parent function $y = f(x)$, you have learned how to identify the effects and graph a function written in the transformation form $g(x) = af(x - c) + d$. For quadratic functions written in transformation form, $a \neq 0$.

For quadratic functions specifically, you will also see them written in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$. This is referred to as **vertex form**.

In vertex form, the coefficient of x is always 1. Therefore, the B-value in the transformation form in this case is also 1 and is left out of the expression.

1. What does the variable h represent in the vertex form of a quadratic function?

Think About . . .
Do you see how this form of the function tells you about the vertex?

2. What does the variable k represent in the vertex form of a quadratic function?

3. What key characteristics can you determine directly from the quadratic function when it is written in vertex form?

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4. Koda, Huyen, Angel, Destiny, and Luis are working together to write a quadratic function to represent a parabola that opens upward and has a vertex at (−6, −4).

Koda

My function is

$s(x) = 3(x + 6)^2 - 4.$

Huyen

My function is

$t(x) = \frac{1}{4}(x + 6)^2 - 4.$

Angel

My function is

$j(x) = -3(x + 6)^2 - 4.$

Destiny

My function is

$D(x) = (x + 6)^2 - 4.$

Luis

My function is

$z(x) = 2(x - 6)^2 - 4.$

a. What are the similarities among all the graphs of the functions? What are the differences among the graphs?

b. How is it possible to have more than one correct function?

c. What would you tell Angel and Luis to correct their functions?

d. How many possible functions can you write for the parabola described in this problem? Explain your reasoning.

5. Use technology to graph each function. Then, use the graph to rewrite the function in vertex form and in factored form.

a. $h(x) = x^2 - 8x + 12$

- Vertex: ______
- Vertex form: ______
- Zero(s): ______
- Factored form: ______

b. $r(x) = -2x^2 + 6x + 20$

- Vertex: ______
- Vertex form: ______
- Zero(s): ______
- Factored form: ______

c. $w(x) = -x^2 - 4x$

- Vertex: ______
- Vertex form: ______
- Zero(s): ______
- Factored form: ______

d. $c(x) = 3x^2 - 3$

- Vertex: ______
- Vertex form: ______
- Zero(s): ______
- Factored form: ______

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6. Identify the form(s) of each quadratic function as either standard form, factored form, or vertex form. Then, state everything you know about each quadratic function's key characteristics, based only on the given equation of the function.

Ask Yourself . . .
What tools or strategies can you use to solve this problem?

a. $g(x) = -(x - 1)^2 + 9$

b. $g(x) = x^2 + 4x$

c. $g(x) = -\frac{1}{2}(x - 3)(x + 2)$

d. $g(x) = x^2 - 5$

ACTIVITY
4.4

Writing Equations in Vertex and Factored Forms

You can write a quadratic function in vertex form when you know the coordinates of the vertex and another point on the graph.

WORKED EXAMPLE

Write an equation for a quadratic function with vertex (1, −2) that passes through the point (0, 1).

Step 1: Substitute the coordinates of the vertex into vertex form of a quadratic function.

$$y = a(x - h)^2 + k$$
$$y = a(x - 1)^2 - 2$$

Step 2: Substitute the coordinates of the other point on the graph for *x* and *y*.

$$1 = a(0 - 1)^2 - 2$$

Step 3: Solve for the value of *a*.

$$1 = a(-1)^2 - 2$$
$$1 = a(1) - 2$$
$$1 = a - 2$$
$$3 = a$$

Step 4: Rewrite the equation in vertex form, substituting the vertex and the value of *a*.

$$f(x) = 3(x - 1)^2 - 2$$

1. How would you determine an equation of a quadratic function in factored form given the zeros and another point on the graph?

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2. Malik and Michael each wrote an equation for the function represented by the graph shown.

Malik

$y = a(x + 1)^2 - 3$

$0 = a(1 + 1)^2 - 3$

$0 = 4a - 3$

$3 = 4a$

$a = \frac{3}{4}$

$y = \frac{3}{4}(x + 1)^2 - 3$

Michael

$y = a(x + 3)(x - 1)$

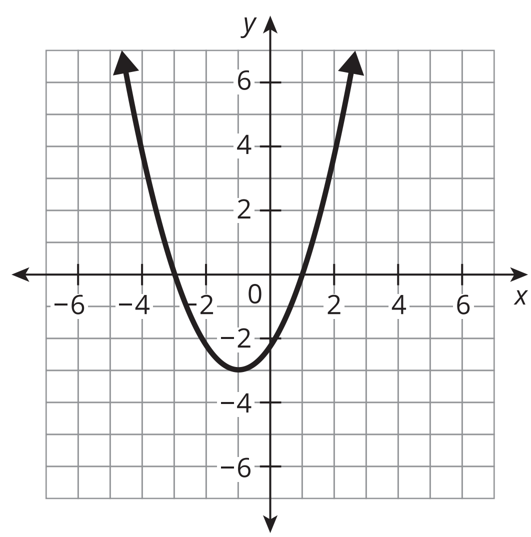
$-3 = a(-1 + 3)(-1 - 1)$

$-3 = a(2)(-2)$

$-3 = -4a$

$a = \frac{3}{4}$

$y = \frac{3}{4}(x + 3)(x - 1)$



a. Explain Malik's reasoning.

b. Explain Michael's reasoning.

c. Use technology to show that Mailk's equation and Michael's equation are equivalent.

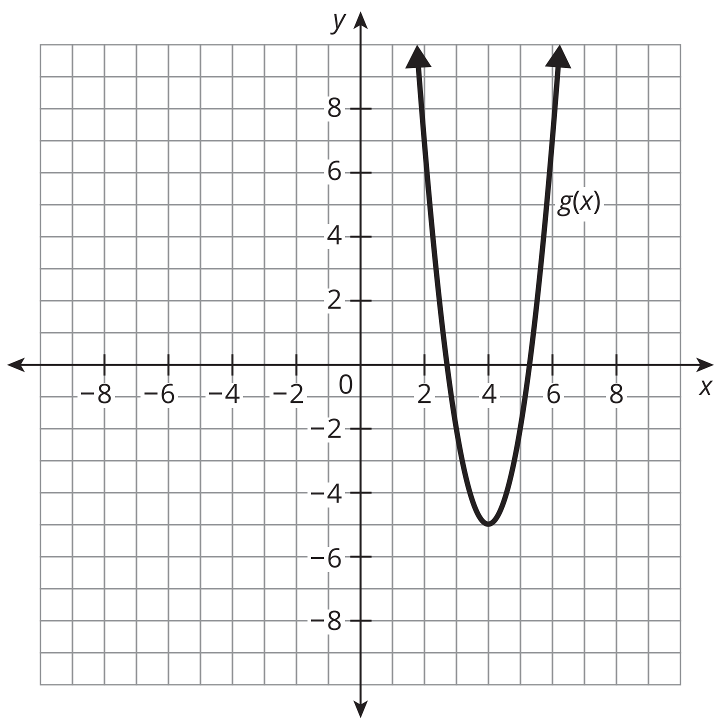
Ask Yourself . . .
How does representing mathematics in multiple ways help to communicate reasoning?

3. Write an equation for a quadratic function in vertex form with vertex (3, 1) that passes through the point (1, 9).

4. Write an equation for a quadratic function in factored form with zeros at $x = -4$ and $x = 0$ that passes through the point $(-3, 6)$.

5. Write an equation for a quadratic function in vertex form with vertex $(-1, 6)$ that passes through the point $(-3, 4)$.

6. Write an equation for a quadratic function $g(x)$ in vertex form given the graph of $g(x)$.



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Talk the Talk

Show What You Know

1. Based on the equation of each function, describe how the graph of each function compares to the graph of $f(x) = x^2$.

   a. $z(x) = -(x - 1)^2 - 10$

   b. $r(x) = \frac{1}{2}(x + 6)^2 + 7$

   c. $m(x) = (4x)^2 + 5$

2. Describe each transformation in relation to the parent function $f(x) = x^2$.

   a. $h(x) = f(x) + d$, when $d > 0$

   b. $h(x) = f(x) + d$, when $d < 0$

   c. $h(x) = f(x - c)$, when $c > 0$

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d. $h(x) = f(x - c)$, when $c < 0$

e. $h(x) = af(x)$, when $|a| > 1$

f. $h(x) = af(x)$, when $0 < |a| < 1$

g. $h(x) = af(x)$, when $a = -1$

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Lesson 4 Assignment

Write

Describe the connections between the vertex form, $f(x) = a(x - h)^2 + k$, and the transformation form, $g(x) = a \cdot f(x - c) + d$, of the parent quadratic function, $y = f(x)$.

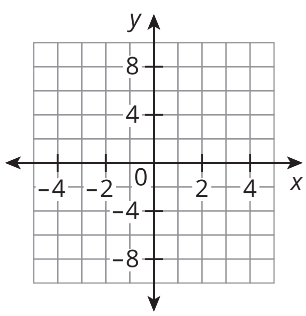
Remember

Transformations performed on any function $f(x)$ can be described by the transformation function $g(x) = af(b(x - c)) + d$ where the c-value translates the function $f(x)$ horizontally, the d-value translates $f(x)$ vertically, the a-value vertically stretches or compresses $f(x)$, and the b-value horizontally stretches or compresses $f(x)$. When the a-value is negative, the function $f(x)$ is reflected across a horizontal line of reflection, and when the b-value is negative, the function $f(x)$ is reflected across a vertical line of reflection.

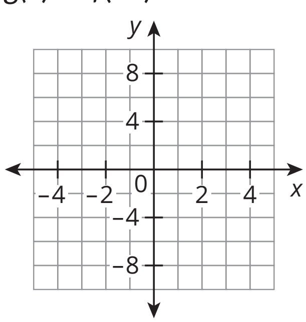
Practice

1. Given $f(x) = x^2$, graph each function and write the corresponding quadratic equation.

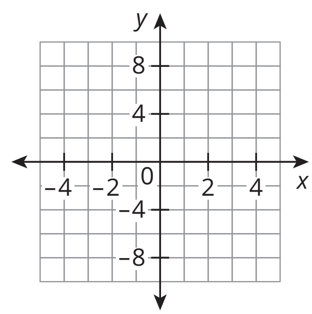
   a. $g(x) = 3f(x - 1)$



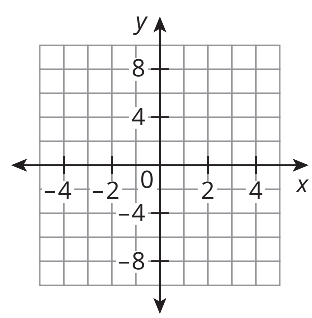
   b. $g(x) = f(3x) - 1$



   c. $g(x) = \frac{1}{2} f(x) + 5$



   d. $g(x) = 2f(x - 3) + 1$



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Lesson 4 Assignment

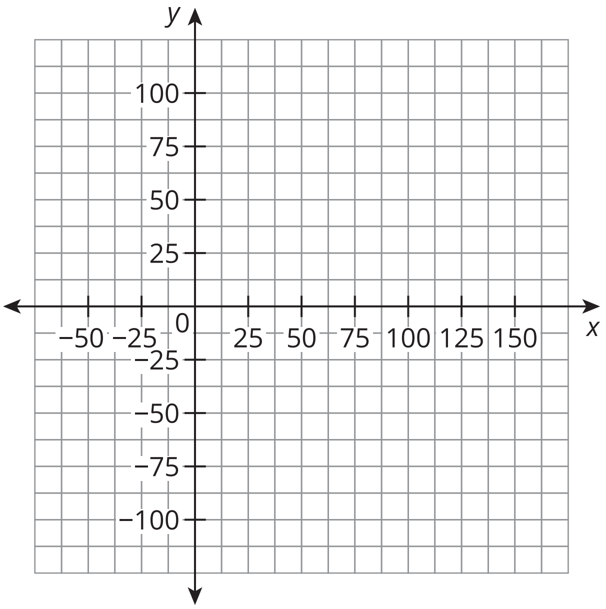
2. The graph shows the parent function $f(x) = x^2$ as well as the function $h(x)$.

   c. Describe the types of transformations performed on $f(x)$ to result in $h(x)$.

   b. When the dilation factor is 16, write the function $h(x)$.

3. Use the given characteristics to write a function $R(x)$ in vertex form. Then, sketch the graph of $R(x)$ and the parent function $f(x) = x^2$.

   - The function has an absolute maximum.
   - The function is translated 70 units up and 100 units to the right.
   - The function is vertically dilated by a factor of $\frac{1}{5}$.



Prepare

Rewrite each expression by combining like terms.

1. $-3x + 4y - 9x - 5y$

2. $2xy^2 + 5x^2y - 7xy + xy^2$

3. $6 - m^2 + 5m^2$

4. $-8 - (-4k) + 7 + 1 - 4k$

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TOPIC 1 SELF-REFLECTION

Name: ______________________________

Introduction to Quadratic Functions

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Introduction to Quadratic Functions* topic by:

TOPIC 1: *Introduction to Quadratic Functions*	Beginning of Topic	Middle of Topic	End of Topic
defining the terms and coefficients of a quadratic equation.	○	○	○
using context to define the variables and quantities for a quadratic function.	○	○	○
recognizing real-world problem situations that can be best represented with quadratic functions.	○	○	○
identifying the defining characteristics of the graph of a quadratic function and the equation it represents.	○	○	○
interpreting key characteristics of a graph of a quadratic function in terms of a given problem situation.	○	○	○
graphing quadratic functions on the coordinate plane and using the graph to identify key attributes, when possible, including the x-intercept, y-intercept, zeros, maximum or minimum value, vertex, and the equation of the axis of symmetry.	○	○	○
determining the domain and range of quadratic functions.	○	○	○

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TOPIC 1 SELF-REFLECTION *continued*

TOPIC 1: *Introduction to Quadratic Functions*	Beginning of Topic	Middle of Topic	End of Topic
recognizing the different forms of a quadratic function, i.e., standard form, factored form, and vertex form.	◯	◯	◯
rewriting a quadratic function from one form to another using graphs and technology.	◯	◯	◯
writing the equation of a quadratic function given the vertex and another point on the graph.	◯	◯	◯
evaluating a quadratic function for a given value.	◯	◯	◯
using second differences to determine whether a table of values represents a quadratic function.	◯	◯	◯
understanding that vertex form is a special name given to transformations formed on the parent function $f(x) = x^2$.	◯	◯	◯
transforming a quadratic function of the form $y = a \cdot f(b(x - c)) + d$ and understanding how the a-, b-, c-, and d-values affect the original function.	◯	◯	◯
graphing the transformation of a given quadratic function using reference points.	◯	◯	◯

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TOPIC 1 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Introduction to Quadratic Functions* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

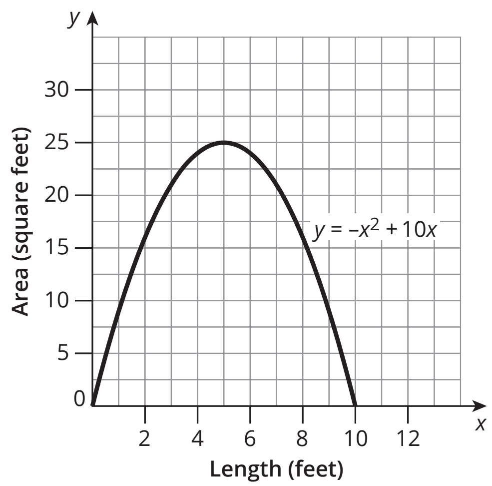
TOPIC 1 SUMMARY

Introduction to Quadratic Functions Summary

LESSON 1 Exploring Quadratic Functions

The shape that a quadratic function forms when graphed is called a *parabola*. A **parabola** is a smooth U-shaped curve that has symmetry. The parabola can open upward, decreasing to a minimum point before increasing, or can open downward, increasing to a maximum point before decreasing. The domain of a quadratic function is all real numbers. The range of a quadratic function is all real numbers greater than or equal to the minimum y-value or less than or equal to the maximum y-value.

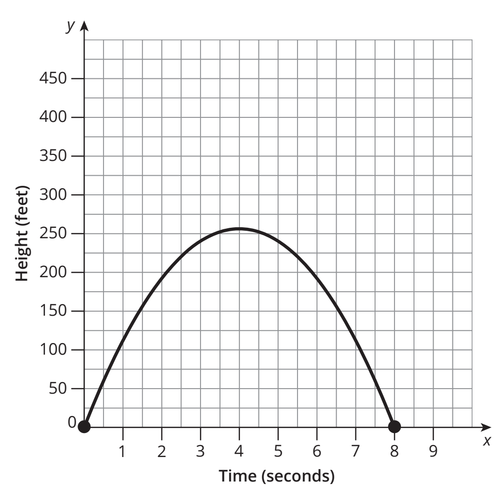
The graph of a quadratic function has one y-intercept and, at most, two x-intercepts.



Quadratic functions model area because area is measured in square units.

For example, suppose you have 20 feet of fencing with which to enclose a rectangular area.

The graph represents a quadratic function for the area of the rectangle given possible lengths of the rectangle. The maximum of the parabola is at the point (5, 25). It has x-intercepts at (0, 0) and (10, 0) and a y-intercept at (0, 0).



A **vertical motion model** is a quadratic equation that models the height of an object at a given time. The equation is of the form $y = -16t^2 + v_0t + h_0$, where y represents the height of the object in feet, t represents the time in seconds that the object has been

NEW KEY TERMS

- parabola [parábola]
- vertical motion model [modelo de movimiento vertical]
- roots [raíces]
- second differences [segundas diferencias]
- standard form of a quadratic function [forma estándar de una función cuadrática]
- factored form [forma factorizada]
- concave down
- concave up
- vertex [vértice]
- axis of symmetry [eje de simetría]
- argument of a function [argumento de una función]
- reflection [reflexión]
- line of reflection [línea de reflexión]
- vertex form [forma de vértice]

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moving, v_0 represents the initial vertical velocity of the object in feet per second, and h_0 represents the initial height of the object in feet.

For example, suppose a firework is launched into the air from the ground with a vertical velocity of 128 feet per second. The function that describes the height of the firework in terms of time is $g(t) = -16t^2 + 128t$.

The x-intercepts of a graph of a quadratic function are also called the *zeros* of the quadratic function. The x-intercepts, or zeros, of $g(t) = -16t^2 + 128t$ are (0, 0) and (8, 0).

When an equation is used to model a situation, the x-intercepts are referred to as roots. The **roots** of an equation indicate where the graph of the equation crosses or touches the x-axis.

LESSON 2 Key Characteristics of Quadratic Functions

First differences are the differences between successive output values when successive input values have a difference of 1. **Second differences** are the differences between consecutive values of first differences. Linear functions have constant first differences and second differences of 0. Quadratic functions have changing first differences and constant second differences.

Linear function: $f(x) = -x + 2$

x	$f(x)$
0	2
1	1
2	0
3	−1
4	−2

First Differences
−1
−1
−1
−1

Second Differences
0
0
0

Quadratic function: $f(x) = 2x^2 - 3x$

x	$f(x)$
0	0
1	−1
2	2
3	9
4	20

First Differences
−1
3
7
11

Second Differences
4
4
4

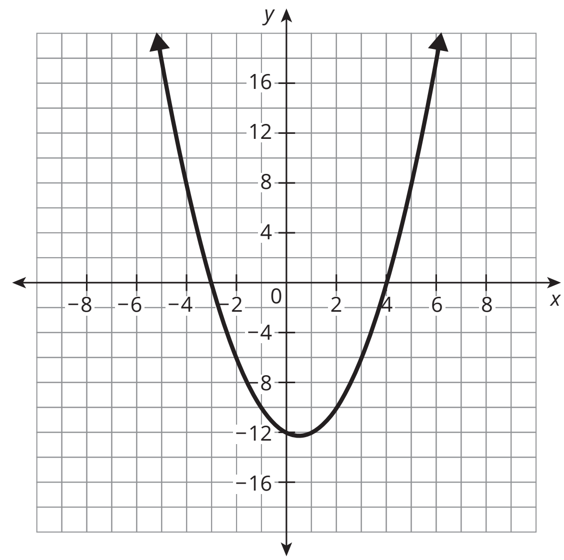
The **standard form of a quadratic function** is $f(x) = ax^2 + bx + c$, where $a \neq 0$. In this form, a and b represent numerical coefficients and c represents a constant. A quadratic function written in **factored form** is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$ and r_1 and r_2 represent the roots.

When the leading coefficient a is negative, the graph of the quadratic function is identified as **concave down**, which means it opens downward and has a maximum. When a is positive, the graph of the quadratic function is identified as **concave up**, which means it opens upward and has a minimum. When a quadratic function is written in standard form, the constant c is the y-intercept.

The **vertex** of a parabola is the lowest or highest point on the graph of the quadratic function. The **axis of symmetry** of a parabola is the vertical line that passes through the vertex and divides the parabola into two mirror images.

For a quadratic function in factored form, the equation for the axis of symmetry is given by $x = \frac{r_1 + r_2}{2}$. For a quadratic function in standard form, the equation for the axis of symmetry is $x = \frac{-b}{2a}$.

The graph represents the function $f(x) = x^2 - x - 12$. The axis of symmetry is $x = -\frac{(-1)}{2(1)} = \frac{1}{2}$. The vertex is $\left(\frac{1}{2}, -12\frac{1}{4}\right)$. The x-intercepts, or zeros, of the function are $x = -3$ and $x = 4$, so the function can be written in factored form as $f(x) = (x + 3)(x - 4)$. The y-intercept is $(0, -12)$. The domain of the function is all real numbers, and the range is all real numbers greater than or equal to $-12\frac{1}{4}$. The graph has an interval of decrease from $-\infty$ to $\frac{1}{2}$ and an interval of increase from $\frac{1}{2}$ to ∞.



You can use the fact that the graph of a quadratic function is symmetric about the axis of symmetry to determine a second point on the parabola given a point on the parabola and to determine the axis of symmetry given two symmetric points on the parabola.

For example, the vertex of a parabola is (3, 5). A point on the parabola is (0, 3). Another point on the parabola is (6, 3).

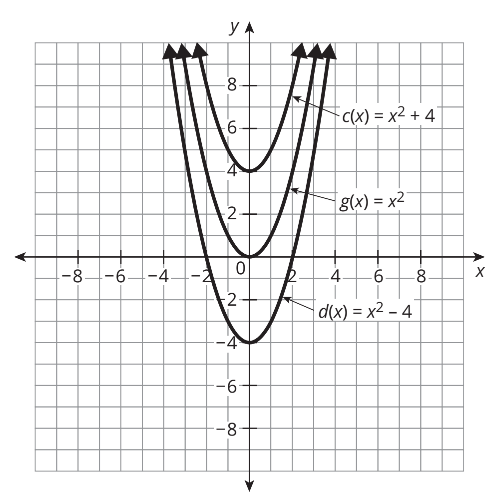
$$\frac{0 + a}{2} = 3$$
$$0 + a = 6$$
$$a = 6$$

Suppose two symmetric points on a parabola are (−7, 20) and (4, 20). The axis of symmetry is $x = -\frac{3}{2}$ because $\frac{-7 + 4}{2} = -\frac{3}{2}$.

LESSON 3 Quadratic Function Transformations

You can use function transformation form, $a \cdot f(b(x - c)) + d$, to transform quadratic functions.

The function $f(x)$ of the form $f(x) = g(x) + d$ is a vertical translation of the function $g(x)$. The value $|d|$ describes how many units up or down the graph of the original function is translated. Vertical translations are performed on a parent quadratic function $g(x) = x^2$ by adding a constant to, or subtracting a constant from, the function. Adding to the function translates it up, and subtracting translates it down.

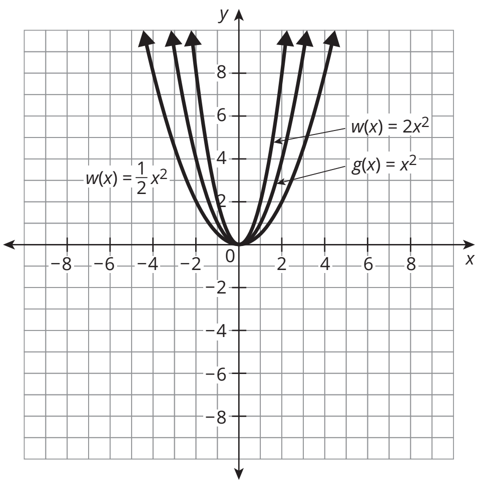


Vertical translations

$g(x) = x^2$	Parent function
$c(x) = g(x) + 4$	$g(x)$ translated 4 units up, so $(x, y) \longrightarrow (x, y + 4)$
$d(x) = g(x) - 4$	$g(x)$ translated 4 units down, so $(x, y) \longrightarrow (x, y - 4)$

The function $f(x)$ in the form $f(x) = a \cdot g(x)$ is a vertical dilation of the function $g(x)$. The a-value describes the vertical dilation of the graph of the original function. A **vertical dilation** of a function is a transformation in which the y-coordinate of every point on the graph of the function is multiplied by a common factor called the **dilation factor**. A vertical dilation stretches or shrinks the graph of a function vertically. For the transformed parent quadratic function, $f(x) = a \cdot g(x)$, when $|a| > 1$, the graph of $g(x)$ is stretched vertically. When $0 < |a| < 1$, $g(x)$ shrinks vertically. You can use the coordinate notation shown to indicate a vertical dilation.

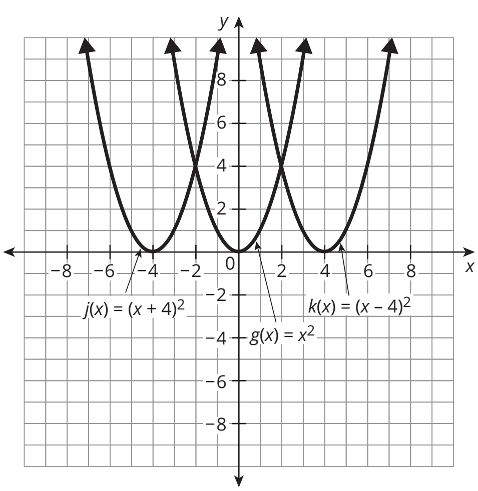
$(x, y) \longrightarrow (x, ay)$, where a is the dilation factor.



Vertical dilations

$g(x) = x^2$	Parent function
$v(x) = 2g(x)$	$g(x)$ stretched by a dilation factor of 2, so $(x, y) \longrightarrow (x, 2y)$
$w(x) = \frac{1}{2}g(x)$	$g(x)$ shrunk by a dilation factor of $\frac{1}{2}$, so $(x, y) \longrightarrow (x, \frac{1}{2}y)$

Horizontal translations are performed on the parent quadratic function $g(x) = x^2$ by adding a constant to or subtracting a constant from the argument, x, of the function. The **argument of a function** is the expression inside the parentheses. For $f(x) = g(x - c)$, the expression $x - c$ is the argument of the function. The c-value of the transformed function affects the input values of the function. The value $|c|$ describes the number of units the graph of $g(x)$ is translated right or left. When $c > 0$, the graph is translated to the right. When $c < 0$, the graph is translated to the left.

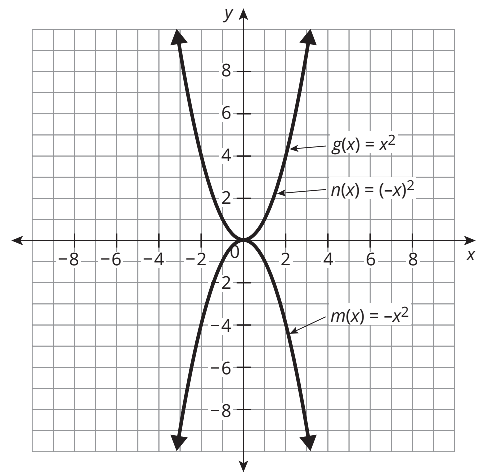


Horizontal translations

$g(x) = x^2$	Parent function
$j(x) = g(x + 4)$	$g(x)$ translated 4 units left, so $(x, y) \longrightarrow (x - 4, y)$
$k(x) = g(x - 4)$	$g(x)$ translated 4 units right, so $(x, y) \longrightarrow (x + 4, y)$

Changing the a-value of a function to its opposite reflects the function across a horizontal line. A **reflection** of a graph is the mirror image of the graph about a line of reflection. The line that the graph is reflected across is called the **line of reflection**. A horizontal line of reflection affects the y-coordinates, and a vertical line of reflection affects the x-coordinates. The line of reflection for the function might be different, depending on how you write the transformation and the order in which the transformations are applied.

Multiplying the parent quadratic function, $g(x) = x^2$, by -1 results in a reflection across the line $y = 0$. Multiplying the argument of the parent quadratic function by -1 results in a reflection across the line $x = 0$, which ends up being the same as the original function because quadratic functions have a vertical axis of symmetry.



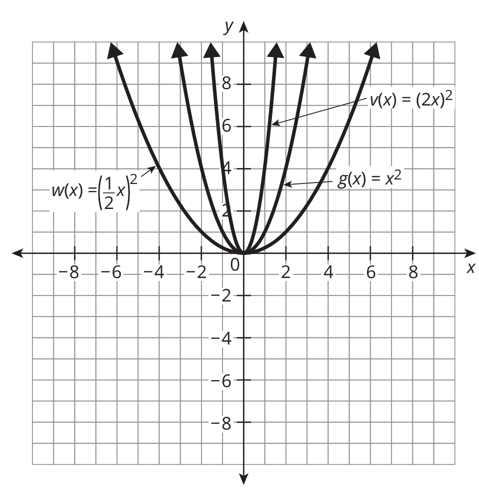
Reflections

$g(x) = x^2$	Parent function
$m(x) = -g(x)$	$g(x)$ is reflected across $y = 0$, so $(x, y) \longrightarrow (x, -y)$
$n(x) = g(-x)$	$g(x)$ is reflected across $x = 0$, so $(x, y) \longrightarrow (-x, y)$

LESSON 4 Horizontal Transformations and Vertex Form

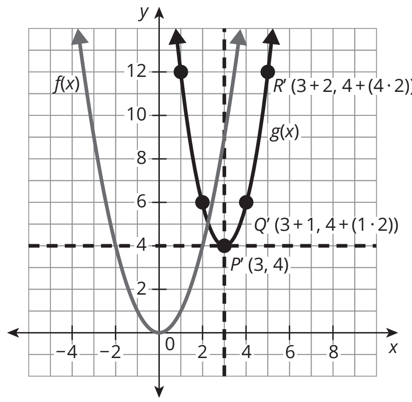
A **horizontal dilation** is a type of transformation that stretches or compresses the entire graph. **Horizontal stretching** is the stretching of a graph away from the y-axis. **Horizontal compression** is the squeezing of a graph towards the y-axis. A horizontal dilation of a function is a transformation in which the x-coordinate of every point on the graph of the function is multiplied by a dilation factor. For the transformed parent quadratic function, $f(x) = g(bx)$, when $|b| > 1$, the graph of the function is compressed horizontally. When $0 < |b| < 1$, the function is stretched horizontally. You can use the coordinate notation shown below to indicate a horizontal dilation.

$(x, y) \longrightarrow \left(\frac{1}{|b|}x, y\right)$, where $\frac{1}{|b|}$ is the dilation factor.



Horizontal dilations

$g(x) = x^2$	Parent function
$v(x) = g(2x)$	$g(x)$ compressed by a dilation factor of $\frac{1}{2}$, so $(x, y) \longrightarrow \left(\frac{1}{2}x, y\right)$
$w(x) = g\left(\frac{1}{2}x\right)$	$g(x)$ stretched by a dilation factor of 2, so $(x, y) \longrightarrow (2x, y)$



Given $y = f(x)$ is the parent function, you can use reference points to graph $y = a \cdot f(b(x - c)) + d$ without using technology. Any point (x, y) on $f(x)$ maps to the point $\left(\frac{1}{b}x + c, ay + d\right)$.

Given $f(x) = x^2$, the function $g(x) = 2f(x - 3) + 4$ has been graphed.

A quadratic function written in **vertex form** is in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The variable h represents the x-coordinate of the vertex. The variable k represents the y-coordinate of the vertex.

You can write a quadratic function in vertex form when you know the coordinates of the vertex and another point on the graph.

For example, write an equation for a quadratic function with the vertex $(1, -2)$ that passes through the point $(0, 1)$.

Step 1: Substitute the coordinates of the vertex into vertex form of a quadratic function.

$$y = a(x - h)^2 + k$$
$$y = a(x - 1)^2 - 2$$

Step 2: Substitute the coordinates of the other point on the graph for x and y.

$$1 = a(0 - 1)^2 - 2$$

Step 3: Solve for the value of a.

$$1 = a(-1)^2 - 2$$
$$1 = a(1) - 2$$
$$1 = a - 2$$
$$3 = a$$

Step 4: Rewrite the equation in vertex form, substituting the vertex and the value of a.

$$f(x) = 3(x - 1)^2 - 2$$

ALGEBRA I

TOPIC 2

A parabola is the shape a quadratic function forms when graphed. It is a smooth curve with reflectional symmetry. The Margaret Hunt Hill Bridge in Dallas, Texas, features an arch shaped like a parabola.

Polynomial Operations

LESSON 1 Adding and Subtracting Polynomials **759**

LESSON 2 Multiplying Polynomials **781**

LESSON 3 Polynomial Division **801**

1 Adding and Subtracting Polynomials

OBJECTIVES

- Name polynomials by number of terms or degree.
- Understand that you can perform operations on functions as well as numbers.
- Add and subtract polynomials.

NEW KEY TERMS

- polynomial
- monomial
- binomial
- trinomial
- degree of a polynomial

You know that a linear expression is one type of polynomial expression.

What are other polynomial expressions, and how do you add and subtract them?

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Getting Started

Sorting It Out

You are familiar with many types of mathematical expressions. Cut out the 12 expressions located at the end of this lesson. Analyze and sort them into groups based on common characteristics.

1. Summarize the groups you formed by listing the expressions that you grouped together and your description for each group. Use mathematical terms in your descriptions.

2. Compare your groups of expressions to your classmates' groups. Describe any similarities and differences.

3. Michael and Javier agree that $4x - 6x^2$ and $25 - 18m^2$ belong in the same group. They each are adding the expressions shown to the group. Who is correct? Explain your reasoning.

Michael	Javier
$5 - 7h$	$y^2 - 4y + 10$
$78j^3 - 3j$	$-3 + 7n + n^2$
$-13s + 6$	

4. What characteristics do all twelve expressions share?

ACTIVITY 1.1

Categorizing Polynomials

Previously, you worked with linear expressions in the form $ax + b$ and quadratic expressions in the form $ax^2 + bx + c$. Each is also part of a larger group of expressions known as *polynomials*.

A **polynomial** is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable is the sum of terms of the form ax^k, where a is any real number and k is a non-negative integer. In general, a polynomial is of the form $a_1x^k + a_2x^{k-1} + \ldots + a_nx^0$. Within a polynomial, each product is a term and the number being multiplied by a power is a coefficient.

WORKED EXAMPLE

The polynomial $m^3 + 8m^2 - 10m + 5$ has four terms. Each term is written in the form ax^k.

- The first term is m^3.
- The power is m^3, and its coefficient is 1.
- In this term, the variable is m and the exponent is 3.

1. Write each term from the Worked Example and identify the coefficient, the power, and the exponent. The first term has already been completed for you.

	1st	2nd	3rd	4th
Term	m^3			
Coefficient	1			
Variable	m			
Power	m^3			
Exponent	3			

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2. Identify the terms and coefficients in each polynomial.

   a. $-2x^2 + 100x$

   b. $4m^3 - 2m^2 - 5$

   c. $y^5 - y + 3$

Polynomials are named according to the number of terms they have. Polynomials with only one term are **monomials**. Polynomials with exactly two terms are **binomials**. Polynomials with exactly three terms are **trinomials**.

The degree of a term in a polynomial is the exponent of the term. The greatest exponent in a polynomial determines the **degree of the polynomial**. In the polynomial $4x + 3$, the greatest exponent is 1, so the degree of the polynomial is 1.

3. Fernando says that $3x^{-2} + 4x - 1$ is a trinomial with a degree of 1 because 1 is the greatest exponent. Olivia disagrees and says that this is not a polynomial because the power on the first term is not a whole number. Who is correct? Explain your reasoning.

4. Determine whether each expression is a polynomial. Explain your reasoning.

   $5^x + 4^{x-1} + 3^{x-2}$ $\qquad$ $x^2 + \sqrt{x}$ $\qquad$ $x^4 y + x^3y^2 + x^2y$

A polynomial is written in standard form when the terms are in descending order, starting with the term with the largest degree and ending with the term with the smallest degree.

5. Revisit the cards you sorted in the Getting Started.

   a. Identify any polynomial not written in standard form and rewrite it in standard form on the card.

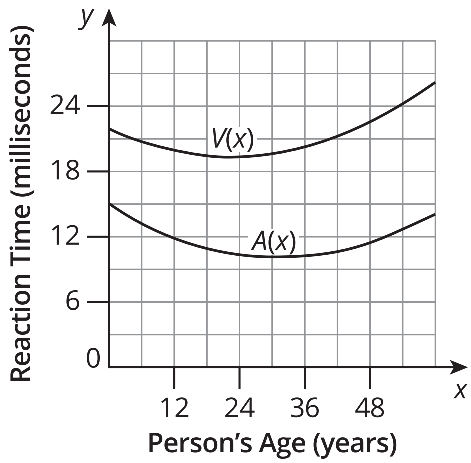
   b. Identify the degree of each polynomial and write the degree on the card.

   c. Glue each card in the appropriate column based on the number of terms in each polynomial. Write your own polynomial to complete any empty boxes.

Monomial	Binomial	Trinomial

ACTIVITY 1.2 Interpreting the Graphs of Polynomial Functions

The graphs of functions $V(x)$ and $A(x)$ are shown. The function $V(x)$ models people's reaction times to visual stimuli in milliseconds based on the age of a person in years. The function $A(x)$ models people's reaction times to audio stimuli in milliseconds based on the age of a person in years.



1. Interpret the graphs of the functions.

   a. Describe the functions $V(x)$ and $A(x)$.

   b. Write a summary to describe people's reaction times to visual stimuli and audio stimuli.

   c. Do you think a person would react faster to a car horn or a flashing light? Explain your reasoning.

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Ask Yourself . . .
How can you incorporate information about auto insurance rates and a driver's age in your report?

2. Estimate the age that a person has the quickest reaction time to each stimuli. Explain how you determined each answer.

 a. Visual stimuli

 b. Audio stimuli

Many times, auto insurance companies use test results similar to the ones shown to create insurance policies for different drivers.

3. How do you think the information provided in the graphic representation may be used by an auto insurance company?

4. Consider a new function $h(x)$, where $h(x) = V(x) - A(x)$. What does $h(x)$ mean in terms of the problem situation?

5. Write a report about drivers' reaction times to visual and audio stimuli. Discuss actions that may improve drivers' reaction times and distractions that may worsen drivers' reaction times. Discuss the importance of flashing lights and sirens on emergency vehicles.

ACTIVITY 1.3

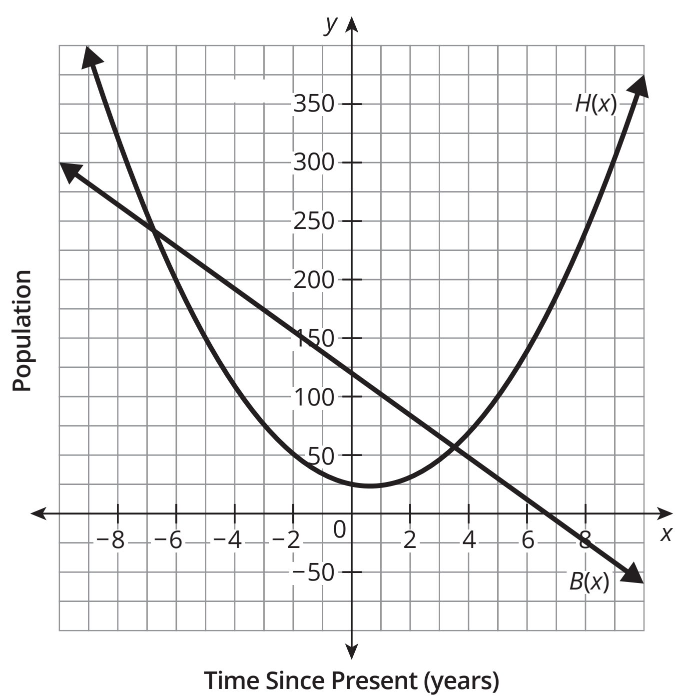
Adding and Subtracting Polynomial Functions

You are playing a new virtual reality game called *Species*. You are an environmental scientist who is responsible for tracking two species of endangered parrots, the orange-bellied parrot and the yellow-headed parrot. Suppose the orange-bellied parrots' population can be modeled by the function $B(x)$, where x represents the number of years since the current year. Suppose that the population of the yellow-headed parrot can be modeled by the function $H(x)$.

$$B(x) = -18x + 120$$

$$H(x) = 4x^2 - 5x + 25$$

The two polynomial functions are shown on the coordinate plane.



Your new task in this game is to determine the total number of these endangered parrots each year over a six-year span. You can calculate the total population of parrots using the two graphed functions.

1. Use the graphs of $B(x)$ and $H(x)$ to determine the function, $T(x)$, to represent the total population of parrots.

   a. Write $T(x)$ in terms of $B(x)$ and $H(x)$.

Ask Yourself . . .
One place to start the sketch of $T(x)$ would be to consider the y-intercept for each function. What would the new y-intercept be for $T(x)$?

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b. Predict the shape of the graph of $T(x)$.

c. Sketch a graph of $T(x)$ on the coordinate plane shown. First, choose any 5 x-values and add their corresponding y-values to create a new point on the graph of $T(x)$. Then, connect the points with a smooth curve. Record the values in the table.

x	$B(x)$	$H(x)$	$T(x)$

d. Did the graph of $T(x)$ match your prediction in part (b)? Identify the function family to which $T(x)$ belongs.

You can write a function, $T(x)$, in terms of x to calculate the total number of parrots at any time.

WORKED EXAMPLE

$T(x) = B(x) + H(x)$	Write $T(x)$ in terms of two known functions.
$T(x) = (-18x + 120) + (4x^2 - 5x + 25)$	Substitute the functions in terms of x.
$T(x) = 4x^2 + (-18x + (-5x)) + (120 + 25)$	Use the commutative property to reorder and the associative property to group like terms.
$T(x) = 4x^2 - 23x + 145$	Combine like terms.

Remember . . .
The table feature on a graphing calculator is an efficient tool to determine y-values.

2. Choose any two x-values in your table. Use the new polynomial function, $T(x)$, to confirm that your solution in the table for those times is correct. Show your work.

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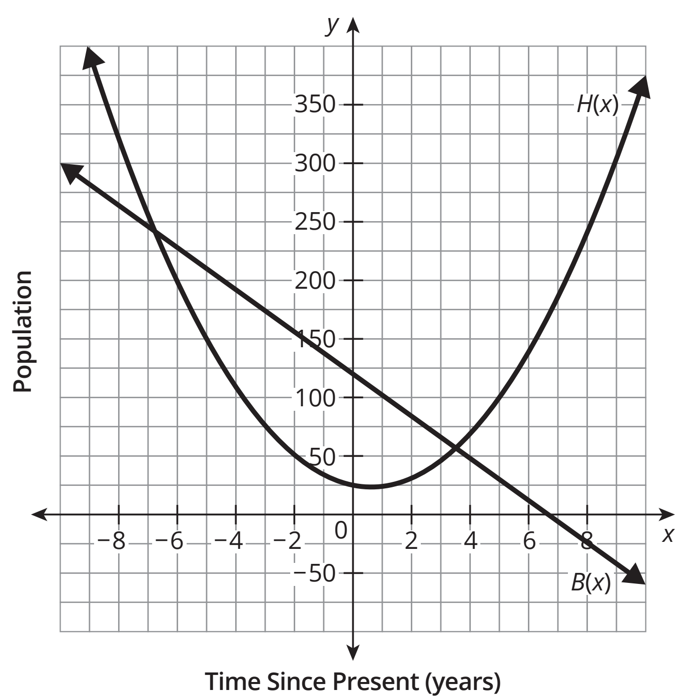
3. Use technology to confirm that your graph and the remaining solutions in the table are correct. Explain any discrepancies and how you corrected them.

4. Daniela says that using $T(x)$ will not work for any time after 6 years from now because by that point the orange-bellied parrot will be extinct. Is Daniela's statement correct? Why or why not?

Throughout the game *Species*, you must always keep track of the difference between the population of each type of species. If the difference gets to be too great, you lose the game. The graphs of $B(x) = -18x + 120$ and $H(x) = 4x^2 - 5x + 25$ are shown.



5. Use the graphs of $B(x)$ and $H(x)$ to determine the function $D(x)$ to represent the difference between the populations of each type of species.

   a. Write $D(x)$ in terms of $B(x)$ and $H(x)$.

   b. Predict the shape of the graph of $D(x)$.

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c. Sketch a graph of $D(x)$ on the coordinate plane shown. First, choose any 5 x-values and subtract their corresponding y-values to create a new point on the graph of $D(x)$. Then, connect the points with a smooth curve. Record the values in the table.

x	$B(x)$	$H(x)$	$D(x)$

d. Did the graph of $D(x)$ match your prediction in part (b)? Identify the function family to which $D(x)$ belongs.

Think About . . .

Refer to the Worked Example for adding polynomials as a guide.

6. Write a function, $D(x)$, in terms of x to calculate the difference between the population of the orange-bellied parrots and the yellow-headed parrots. Write $D(x)$ as a polynomial in standard form.

7. Choose any two x-values in your table. Use your new polynomial function to confirm that your solution in the table for those times is correct. Show your work.

8. Use technology to confirm that your graph and the remaining solutions in the table are correct. Explain any discrepancies and how you corrected them.

9. Alexander uses his function $D(x) = -4x^2 - 13x + 95$ to determine that the difference between the number of orange-bellied parrots and the number of yellow-headed parrots 7 years from now will be -192. Is Alexander correct or incorrect? If he is correct, explain to him what his answer means in terms of the problem situation. If he is incorrect, explain where he made his error and how to correct it.

10. The next round of the *Species* game included the red-winged parrot, whose population can be modeled by the function $W(x) = -9x + 80$, and the rainbow lorikeet parrot, whose population can be modeled by the function $L(x) = 2x^2 - 4x + 10$. In both cases, x represents the number of years since the current year.

    a. Write a function, $S(x)$, in terms of x to calculate the total number of red-winged parrots and rainbow lorikeet parrots at any time.

    b. Write a function, $M(x)$, in terms of x to calculate the difference in the number of red-winged parrots and rainbow lorikeet parrots at any time.

    c. Calculate $S(4)$ and $M(4)$. Interpret the meaning of your results.

    d. In four years, how many red-winged parrots will there be? How many rainbow lorikeet parrots will there be?

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ACTIVITY
1.4 Just the Math

In this activity, you will practice adding and subtracting polynomials.

WORKED EXAMPLE

You can add and subtract polynomials by grouping terms that have the same variable and exponent.

To determine the sum $(3x^2 - 2x + 1) + (2x^2 + 5x - 5)$:

Step 1: Use the distributive property to rewrite the expressions without grouping symbols. $3x^2 - 2x + 1 + 2x^2 + 5x - 5$

Step 2: Use the commutative property of Addition to rearrange the terms. $3x^2 + 2x^2 - 2x + 5x + 1 - 5$

Step 3: Use the associative property of addition to group like terms. $(3x^2 + 2x^2) + (-2x + 5x) + (1 - 5)$

Step 4: Simplify the like terms. $5x^2 + 3x - 4$

To determine the difference $(2x^2 - 4x + 3) - (5x^2 - 3x + 6)$:

Step 1: Use the distributive property to rewrite the expressions without grouping symbols. $2x^2 - 4x + 3 - 5x^2 + 3x - 6$

Step 2: Rewrite the terms using addition. $2x^2 + (-4x) + 3 + (-5x^2) + 3x + (-6)$

Step 3: Use the commutative property of addition to rearrange the terms. $2x^2 + (-5x^2) + (-4x) + 3x + 3 + (-6)$

Step 4: Use the associative property of addition to group like terms. $(2x^2 + (-5x^2)) + ((-4x) + 3x) + (3 + (-6))$

Step 5: Simplify the grouped terms. $-3x^2 - x - 3$

1. Analyze each student's work. Determine the error and make the necessary corrections.

Jacob

$3x^2 + 5x^2 = 8x^4$

Amir

$2x - (4x + 5)$
$2x - 4x + 5$
$-2x + 5$

Destiny

$(4x^2 + 2x - 5) + (3x^2 + 7)$
$(4x^2 + 3x^2) - (2x) - (5 + 7)$
$7x^2 - 2x - 12$

2. Determine the sum or difference of each. Show your work.

a. $(3x - 7) + (5x + 4) - (6x - 2)$

b. $(7xy - 9yz + 2xz) - (10xy + 4yz - 3xy)$

c. $(5x^2 - 2x + 7) - (4x + 3)$

d. $(-2x^2 + 3x - 1) - (3x - 1)$

e. $(x^2 - 2x - 3) + (2x^2 + 1)$

f. $(2x^2 + 3x - 4) - (2x^2 + 5x - 6)$

g. $(4x^3 + 5x - 2) + (7x^2 - 8x + 9)$

h. $(9x^4 - 5) - (8x^4 - 2x^3 + x)$

Since the sum of the exponents for each item in part (b) is 2, each term has a degree of 2.

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Talk the Talk

Putting It Into Practice

Match each expression with the equivalent polynomial.

Expressions	Polynomials
1. $(x^2 - 3) + (x^2 + 2)$	A. -1
2. $(x^2 - 3) - (x^2 + 2)$	B. $-2x^2 - 1$
3. $(x^2 - 3) - (x^2 - 2)$	C. $-2x^2 - 5$
4. $(x^2 - 3) + (x^2 - 2)$	D. $2x^2 - 1$
5. $-(x^2 + 3) - (x^2 - 2)$	E. $2x^2 - 5$
6. $-(x^2 + 3) - (x^2 + 2)$	F. -5

7. Determine the sum or difference of each. Show your work.

a. $(6x + 9) + (-2x + 17)$

b. $(-4x^2 + 5x - 1) - (2x^2 + 7x)$

c. $(10x^3 - 6x + 4) + (-2x^2 - 4)$

d. $(2x^4 + 3x) - (-9x^2 + 11x - 1)$

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Expression Cards

$4x - 6x^2$	$125p$	$\frac{4}{5}r^3 + \frac{2}{5}r - 1$
$-\frac{2}{3}$	$y^2 - 4y + 10$	$5 - 7h$
$-3 + 7n + n^2$	-6	$-13s + 6$
$12.5t^3$	$78j^3 - 3j$	$25 - 18m^2$

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So you can cut out the Expression Cards on the other side

Lesson 1 Assignment

Write

Match each definition with its corresponding term.

1. polynomial	a. a polynomial with only 1 term
2. term	b. the degree of the term with the greatest exponent
3. coefficient	c. a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients
4. monomial	d. a polynomial with exactly 3 terms
5. binomial	e. any number being multiplied by a variable to a power within a polynomial expression
6. trinomial	f. each product in a polynomial expression
7. degree of a term	g. a polynomial with exactly 2 terms
8. degree of a polynomial	h. the exponent of a term in a polynomial

Remember

A *polynomial* is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. You can operate with polynomial expressions just as you would with numeric expressions.

Practice

1. Koda and William each build a rocket launcher. They launch a model rocket using Koda's launcher, and on its way back down, it lands on the roof of a building that is 320 feet tall. The height of the rocket can be represented by the equation $H_1(x) = -16x^2 + 200x$, where x represents the time in seconds and $H_1(x)$ represents the height. Koda and William take the stairs to the roof of the building and relaunch the rocket using William's rocket launcher. The rocket lands back on the ground. The height of the rocket after this launch can be represented by the equation $H_2(x) = -16x^2 + 192x + 320$.

   a. Compare and contrast the polynomial functions.

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Lesson 1 Assignment

b. Use technology to sketch a graph of the functions.

c. Does it make sense in terms of the problem situation to graph the functions outside of Quadrant I? Explain your reasoning.

d. Explain why the graphs of these functions do not intersect.

e. Koda believes that she can add the two functions to determine the total height of the rocket at any given time. Write a function $S(x)$ that represents the sum of $H_1(x)$ and $H_2(x)$. Show your work.

f. Is Koda correct? Explain your reasoning.

Lesson 1 Assignment

g. Subtract $H_1(x)$ from $H_2(x)$ and write a new function, $D(x)$, that represents the difference. Then, explain what this function means in terms of the problem situation.

2. Determine whether each expression is a polynomial. If so, identify the terms, coefficients, and degree of the polynomial. If not, explain your reasoning.

a. $-2b^4 + 4b - 1$

b. $6 - g^{-2}$

c. $8h^4$

d. $9w - w^3 + 5w^2$

e. $x^{\frac{1}{2}} + 2$

f. $\frac{4}{5}y + \frac{2}{3}y^2$

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Lesson 1 Assignment

3. Given $A(x) = x^2 - 5x + 4$, $B(x) = 2x^2 + 5x - 6$, and $C(x) = -x^2 + 3$, determine each function. Write your answer in standard form.

   a. $D(x) = B(x) + C(x)$

   b. $E(x) = A(x) + B(x)$

   c. $F(x) = A(x) - C(x)$

   d. $G(x) = C(x) - B(x)$

   e. $H(x) = A(x) + B(x) - C(x)$

   f. $J(x) = B(x) - A(x) + C(x)$

4. Determine each sum or difference.

   a. $(8x - 4) + (9x + 12)$

   b. $(10p^2 - 5p + 2) - (6p^2 - 4p)$

   c. $(7rs + 9r) - (-rs + 3s)$

   d. $(-2z^2 + 6z - 3) + (2z^2 - 8z + 6)$

   e. $(-x^2 + 5x - 4) - (2x + 5)$

   f. $(6x^2 + 8x) - (2x + 3)$

Prepare

Determine the product.

1. $4x \cdot x$

2. $-3x \cdot 10x$

3. $-x(5 + 4x)$

4. $(2x - 8) \cdot (12x)$

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2 Multiplying Polynomials

OBJECTIVES

- Model the multiplication of a binomial by a binomial using algebra tiles.
- Use multiplication tables to multiply binomials.
- Use the distributive property to multiply binomials.
- Recognize and use special products when multiplying binomials.

NEW KEY TERMS

- difference of two squares
- perfect square trinomial

You have added and subtracted polynomials.

How do you multiply polynomials?

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Getting Started

Modeling Binomials

So far, you have learned how to add and subtract polynomials. But what about multiplying polynomials?

Let's consider the binomials $x + 1$ and $x + 2$. You can use algebra tiles to model the two binomials and determine their product.

WORKED EXAMPLE

Represent each binomial with algebra tiles.

x	1

$x + 1$

x	1	1

$x + 2$

Create an area model using each binomial.

$\cdot$	x	1	1
x	x^2	x	x
1	x	1	1

1. What is the product of $(x + 1)(x + 2)$?

2. How would the model change if the binomial $x + 2$ was changed to $x + 4$? What is the new product of $x + 1$ and $x + 4$?

ACTIVITY
2.1 Multiplying with Algebra Tiles

In the Getting Started, you determined the product of $x + 1$ and $x + 2$.

1. Angel represented the product of $(x + 1)$ and $(x + 2)$ as shown.

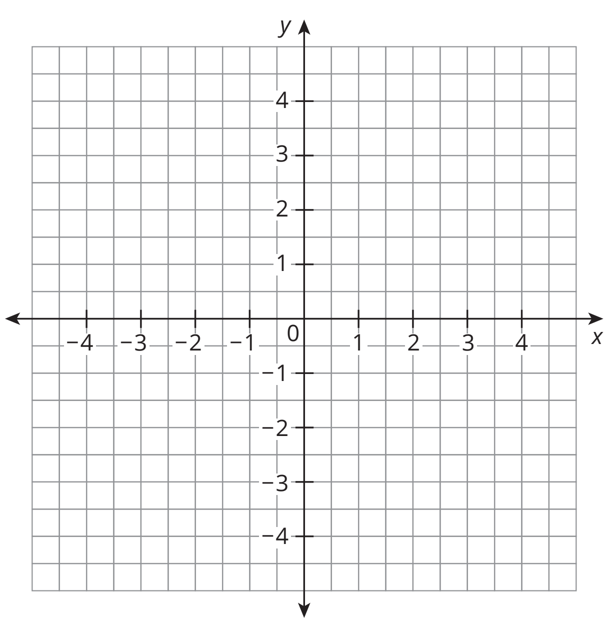
•	x	1
x	x^2	x
1	x	1
1	x	1

Gabriela told Angel that he incorrectly represented the area model because it does not look like the model in the Getting Started. Angel replied that it doesn't matter how the binomials are arranged in the model.

Who's correct?

2. Use graphing technology to verify.
$(x + 1)(x + 2) = x^2 + 3x + 2$

a. Sketch both graphs on the coordinate plane.



b. How do the graphs verify that $(x + 1)(x + 2)$ and $x^2 + 3x + 2$ are equivalent?

c. Plot and label the x-intercepts and the y-intercept on your graph. How do the forms of each expression help you identify these points?

3. Use algebra tiles to determine the product of the binomials in each.

a. $x + 2$ and $x + 3$

b. $x + 2$ and $x + 4$

c. $2x + 3$ and $3x + 1$

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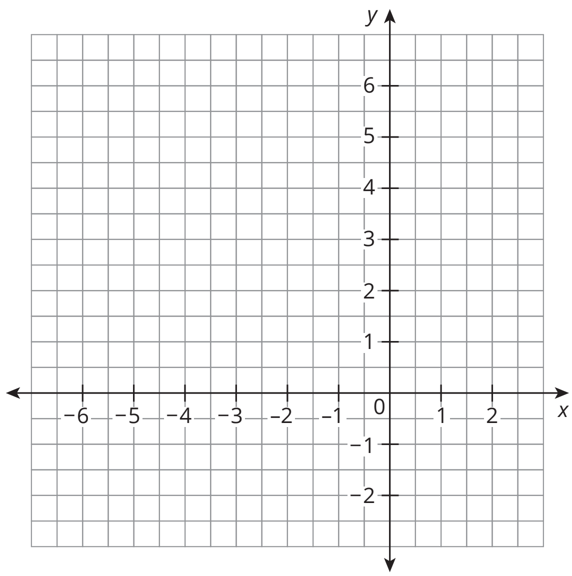
4. Verify that the products you determined in Question 3 are correct using graphing technology. Write each pair of factors and the product. Then, sketch each graph on the coordinate plane.

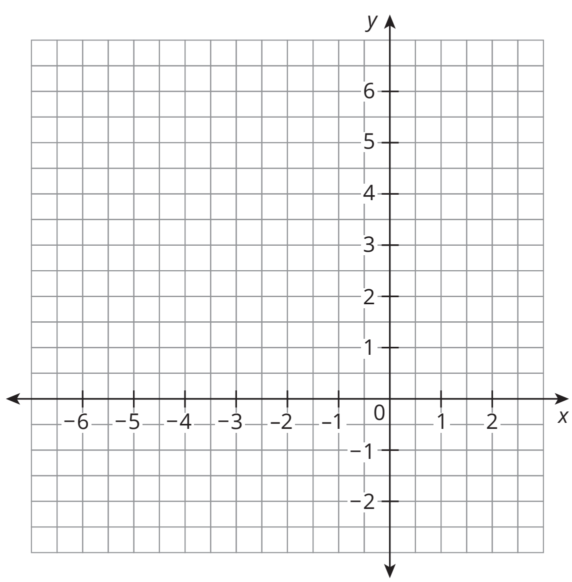
a.



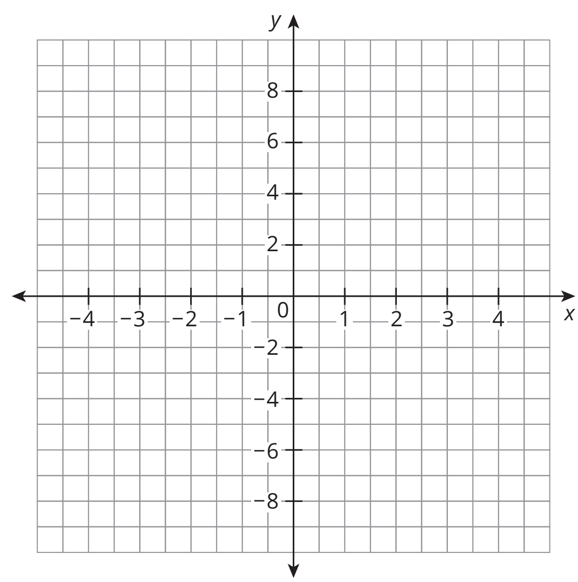
Think About . . .

How are the *x*-intercepts represented in the linear binomial expressions?

b.



c.



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ACTIVITY 2.2

Using Area Models and the Distributive Property

While using algebra tiles is one way to determine the product of polynomials, they can also become difficult to use when the terms of the polynomials become more complex.

Luis was calculating the product of the binomials $4x + 7$ and $5x - 3$. He thought he didn't have enough algebra tiles to determine the product. Instead, he performed the calculation using the model shown.

1. Describe how Luis calculated the product of $4x + 7$ and $5x - 3$.

Luis

$\cdot$	$5x$	-3
$4x$	$20x^2$	$-12x$
7	$35x$	-21

$20x^2 + 23x - 21$

2. How is Luis's method similar to and different from using the algebra tiles method?

Luis used a multiplication table to calculate the product of the two binomials. By using a multiplication table, you can organize the terms of the binomials as factors of multiplication expressions. You can then use the distributive property of multiplication to multiply each term of the first polynomial with each term of the second polynomial.

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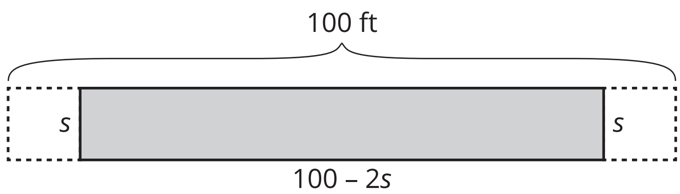
Consider the dog enclosure scenario from the previous topic.



The area of the enclosure is expressed as $A(s) = s(100 - 2s)$, or the product of a monomial and a binomial.

3. Consider how Malik and Hannah wrote an equivalent polynomial function in standard form by calculating the product.

Malik

$\cdot$	100	$-2s$
s	$100s$	$-2s^2$

$A(s) = -2s^2 + 100s$

Hannah

$A(s) = s(100 - 2s)$

$A(s) = 100s - 2s^2$

$A(s) = -2s^2 + 100s$

Remember . . .
Develop a habit of writing answers in standard form. It makes them easier to compare with others' answers.

a. Describe the strategy Malik used to calculate the product.

b. How is Malik's strategy similar to Hannah's strategy?

Consider the Alamo Tours scenario from the previous topic. The revenue for the business is expressed as the product of a binomial times a binomial.

Revenue	=	**Number of Tours**	·	**Price per Tour**
↓		↓		↓
$r(x)$	=	$(10x + 100)$	·	$(50 - x)$

4. Finish Malik's process to write an equivalent polynomial function for revenue in standard form.

The process of using a multiplication table to multiply polynomials is referred to as an *area model*.

·	50	$-x$
$10x$	$500x$	$-10x^2$
100		

5. Use an area model to calculate the product of each polynomial. Write each product in standard form.

Ask Yourself . . .
Does it matter where you place the polynomials in the multiplication table?

a. $(3x + 2)(x - 4)$

b. $(x - 5)(x + 5)$

c. $(2x + 3)^2$

d. $(4x^2 + x - 1)(3x - 7)$

In Question 3, Hannah uses the distributive property to multiply a monomial and a binomial. She wants to use the distributive property to multiply any polynomials.

Think About . . .
How can you use technology to check your answers?

WORKED EXAMPLE

Consider the polynomials $x + 5$ and $x - 2$. You can use the distributive property to multiply these polynomials.

Distribute x to each term of $(x - 2)$ and then distribute 5 to each term of $(x - 2)$.

$$\begin{aligned}(x + 5)(x - 2) &= (x)(x - 2) + (5)(x - 2)\\ &= x^2 - 2x + 5x - 10\\ &= x^2 + 3x - 10\end{aligned}$$

6. Use the distributive property to determine each product. Write the polynomial in standard form.

a. $6(7x - 3)$

b. $5(3x^2 - 2x + 7)$

c. $(5x - 1)(2x + 1)$

d. $(x - 7)(x + 7)$

e. $(x^2 + 5x + 2)(4x + 2)$

f. $(2x^2 + 1)(3x^2 + x - 1)$

7. Explain the mistake in Ana's thinking. Then, determine the correct product.

Ana

$(x + 4)^2 = x^2 + 16.$

I can just square each term to determine the product.

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ACTIVITY 2.3

Rewriting Quadratic Functions

You can use polynomial multiplication to rewrite quadratic functions written in factored form or vertex form.

WORKED EXAMPLE

Write the quadratic function $f(x) = (2x + 1)(x - 5)$ in standard form:

Step 1: Apply the distributive property. $f(x) = 2x(x - 5) + 1(x - 5)$

Step 2: Apply the distributive property. $f(x) = 2x^2 - 10x + x - 5$

Step 3: Combine like terms. $f(x) = 2x^2 - 9x - 5$

Write the quadratic function $g(x) = (x - 1)^2 + 6$ in standard form:

Step 1: Write the power as a multiplication. $g(x) = (x - 1)(x - 1) + 6$

Step 2: Apply the distributive property. $g(x) = x(x - 1) - 1\,(x - 1) + 6$

Step 3: Apply the distributive property. $g(x) = x^2 - x - x + 1 + 6$

Step 4: Combine like terms. $g(x) = x^2 - 2x + 7$

1. Write each quadratic equation in standard form.

a. $d(x) = (x - 4)(2x + 1)$

b. $p(x) = (4x + 2)(4x - 2)$

c. $k(x) = (2x - 3)^2 - 5$

d. $c(x) = (3x + 1)^2 + 4$

ACTIVITY 2.4

Special Products When Multiplying Binomials

In this activity, you will investigate the product of two linear factors when one is the sum of two terms and the other is the difference of the same two terms and when the two linear factors are the same.

1. Determine each product.

   a. $(x - 4)(x + 4) =$
   $(x + 4)(x + 4) =$
   $(x - 4)(x - 4) =$

   b. $(x - 3)(x + 3) =$
   $(x + 3)(x + 3) =$
   $(x - 3)(x - 3) =$

   c. $(3x - 1)(3x + 1) =$
   $(3x + 1)(3x + 1) =$
   $(3x - 1)(3x - 1) =$

   d. $(2x - 1)(2x + 1) =$
   $(2x + 1)(2x + 1) =$
   $(2x - 1)(2x - 1) =$

2. What patterns do you notice between the factors and the products?

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3. Multiply each pair of binomials.

   a. $(ax - b)(ax + b) =$

   b. $(ax + b)(ax + b) =$

   c. $(ax - \mathrm{b})(ax - b) =$

In Questions 1 and 3, you should have observed a few special products. The first type of special product is called the *difference of two squares.* The **difference of two squares** is an expression in the form $a^2 - b^2$ that has factors $(a - b)(a + b)$.

4. List the expressions in Questions 1 and 3 that are examples of the difference of two squares.

The second type of special product is called a *perfect square trinomial.* A **perfect square trinomial** is an expression in the form $a^2 + 2ab + b^2$ or the form $a^2 - 2ab + b^2$. A perfect square trinomial can be written as the square of a binomial.

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

5. List the expressions in Questions 1 and 3 that are examples of perfect square trinomials.

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Talk the Talk

Special Products

1. Use special products to determine each product.

a. $(x - 8)(x - 8)$

b. $(x + 8)(x - 8)$

c. $(x + 8)^2$

d. $(3x + 2)^2$

e. $(3x - 2)(3x - 2)$

f. $(3x - 2)(3x + 2)$

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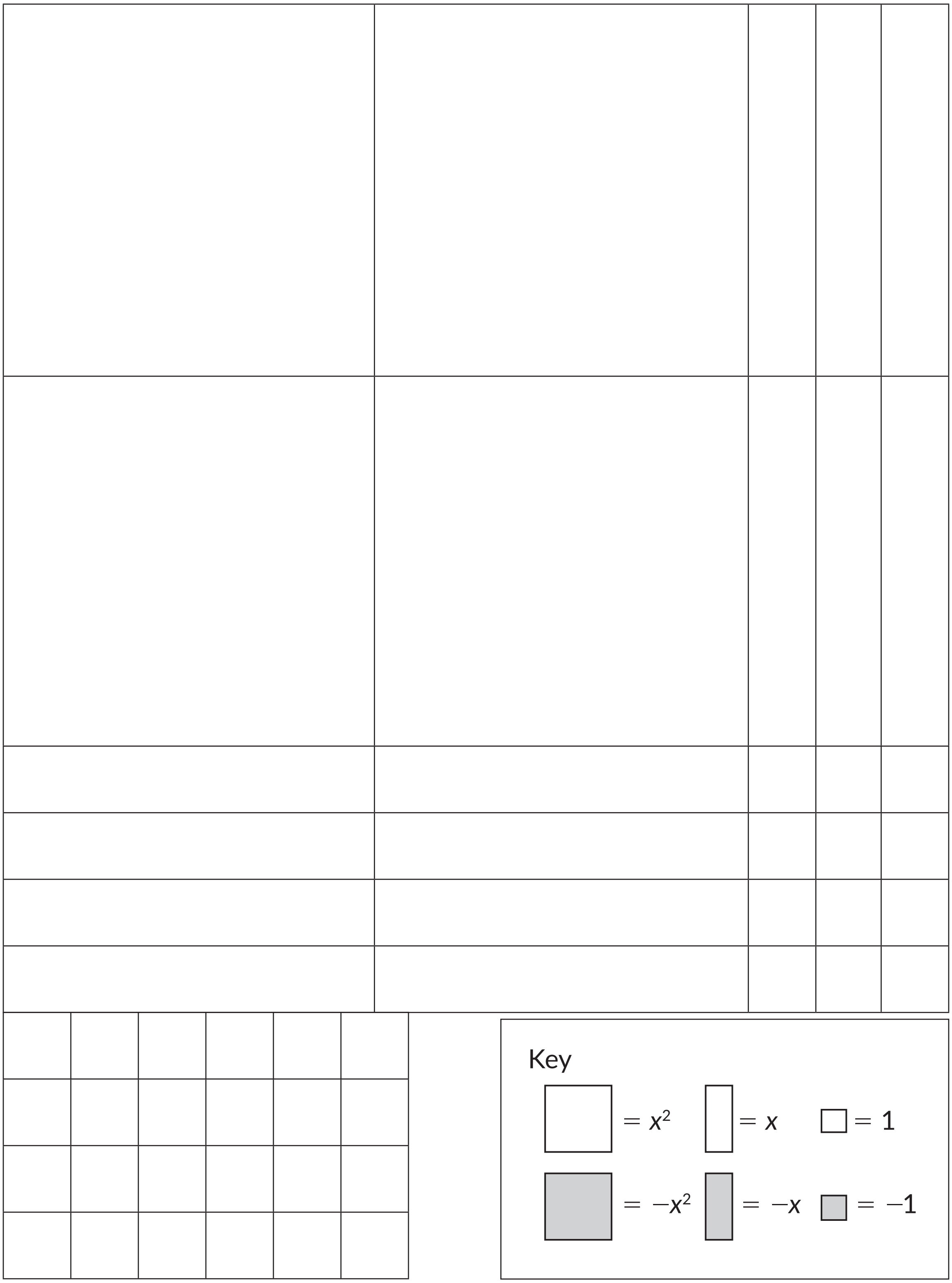
Algebra Tiles



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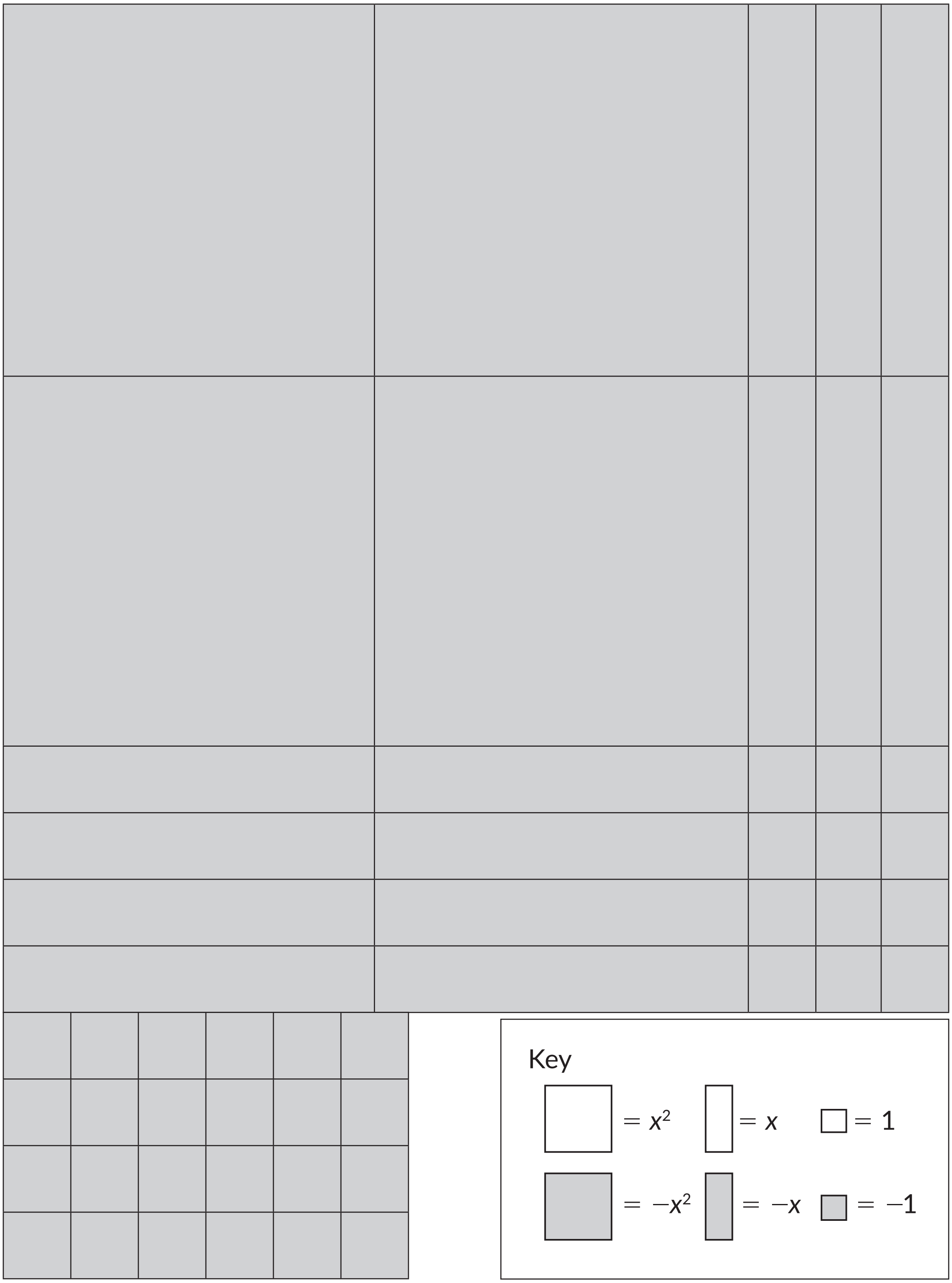
Algebra Tiles



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Lesson 2 Assignment

Write

In your own words, explain how to use the distributive property to multiply two binomials. Provide an example.

Remember

The difference of two squares is an expression in the form $a^2 - b^2$ that has factors $(a + b)(a - b)$.

A perfect square trinomial is an expression in the form $a^2 + 2ab + b^2$ or in the form $a^2 - 2ab + b^2$ that has the factors $(a + b)^2$ and $(a - b)^2$, respectively.

Practice

Use an area model to calculate each product.

1. $(4x - 1)(6 + 3x)$

$\cdot$		

2. $(-5x + 5)^2$

$\cdot$	$-5x$	5
$-5x$		
5		

3. $(x^2 + 3x - 2)(2x + 5)$

$\cdot$		

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Lesson 2 Assignment

4. $(x + 2)(x - 2)$

$\cdot$		

Use the distributive property to determine each product.
Write the polynomial in standard form.

5. $-3(4x - 5)$

6. $8(2x^2 - 4x + 3)$

7. $(x + 9)(x + 9)$

8. $(2x - 1)(x + 3)$

9. $(x + 4)(x^2 + 3x - 2)$

10. $(x^2 + 3)(3x^2 - 2x + 5)$

11. Rewrite the equation $y = (x - 3)^2 + 6$ in standard form.

Prepare

Determine each quotient.

1. $9x \div 3x$

2. $4x^2 \div 2x$

3. $x \div x$

3 Polynomial Division

OBJECTIVES

- Describe similarities between polynomials and integers.
- Determine factors of a polynomial using one or more roots of the polynomial.
- Compare polynomial long division to integer long division.
- Determine factors through polynomial long division.
- Use the remainder theorem to evaluate polynomial equations and functions.

NEW KEY TERMS

- factor theorem
- polynomial long division
- remainder theorem

You know how to divide integers using the long division algorithm.
How can you use a similar algorithm to divide polynomials?

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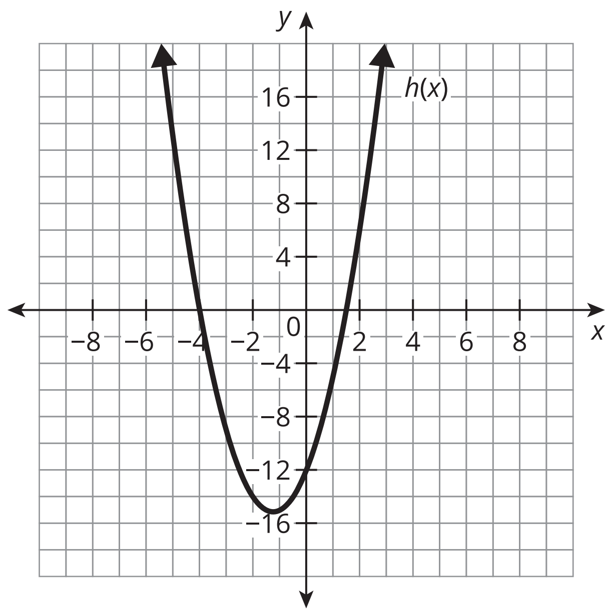
Getting Started

The $x - r$ Factor

You have analyzed the graphs of polynomials to determine the type and location of the zeros. How can you determine the factors of a polynomial given its algebraic representation and one of its factors?

1. Analyze the graph of the function $h(x) = 2x^2 + 5x - 12$.



a. Describe the key characteristics of $h(x)$.

b. Describe the number and types of zeros of $h(x)$.

The **factor theorem** states that a polynomial function $p(x)$ has $x - r$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$.

You can use the factor theorem to show that a linear expression is a factor of a polynomial.

WORKED EXAMPLE

Consider the graph of the polynomial function $h(x) = 2x^2 + 5x - 12$ in Question 1.

The graph appears to have a zero at $(-4, 0)$, so a possible linear factor of the polynomial is $(x + 4)$.

Determine the value of the polynomial at $x = -4$, or $h(-4)$.

$$\begin{aligned} h(-4) &= 2(-4)^2 + 5(-4) - 12 \\ &= 32 - 20 - 12 \\ &= 0 \end{aligned}$$

So, $(x + 4)$ is a linear factor of the polynomial function.

2. Consider that $d(x) = (x + 4)$ and $d(x) \cdot q(x) = h(x)$.

   a. What do you know about the function $q(x)$?

   b. Can you write the algebraic representation for $q(x)$? Explain your reasoning.

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ACTIVITY
3.1 Polynomial Long Division

To solve $0 = 2x^2 + 5x - 12$, you need to factor the polynomial and use the zero product property to determine its zeros.

Remember . . .
Recall that $a \div b$ is $\frac{a}{b}$, where $b \neq 0$.

The fundamental theorem of algebra states that every polynomial equation of degree n must have n roots. This means that every polynomial can be written as the product of n factors of the form $(ax + b)$. For example, $2x^2 - 3x - 9 = (2x + 3)(x - 3)$.

If 2 is a factor of 24, then 24 can be divided by 2 without a remainder. In the same way, the factors of a polynomial divide into that polynomial without a remainder.

Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree. The process is similar to integer long division.

WORKED EXAMPLE

<table>
<tr><th>Integer Long Division</th><th>Polynomial Long Division</th></tr>
<tr><td>

$3660 \div 12$

or

$\frac{3660}{12}$

$$\begin{array}{r} 305 \\ 12\overline{)3660} \\ -36 \\ 6 \\ \underline{-0} \\ 60 \\ \underline{-60} \\ 0 \end{array}$$

</td><td>

$(2x^2 + 5x - 12) \div (x + 4)$

or

$\frac{2x^2 + 5x - 12}{x + 4}$

$$\begin{array}{r} \text{Ⓐ}2x \ \text{Ⓓ}{-3} \\ x + 4\overline{)2x^2 + 5x - 12} \\ \text{Ⓑ}\underline{-(2x^2 + 8x)} \downarrow \ \text{Ⓒ} \\ -3x - 12 \\ \underline{-(-3x - 12)} \\ \text{Remainder } 0 \end{array}$$

A. Divide $\frac{2x^2}{x} = 2x$.
B. Multiply $2x(x + 4)$ and then subtract.
C. Bring down -12.
D. Divide $\frac{-3x}{x} = -3$.
E. Multiply $-3(x + 4)$ and then subtract.

</td></tr>
</table>

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1. Analyze the Worked Example that shows integer long division and polynomial long division.

   a. In what ways are the integer and polynomial long division algorithms similar?

   b. Rewrite each expression as a product of its factors.

   $3660 = \_\_\_\_\_\_ \cdot \_\_\_\_\_\_$ $2x^2 + 5x - 12 = \_\_\_\_\_\_\_\_ \cdot \_\_\_\_\_\_\_\_$

   c. Write the product of the linear factors of $2x^2 + 5x - 12$ as a function, $f(x)$.

   d. Determine the zeros of the function.

2. Determine whether $(x - 7)$ is a factor of $x^2 + 5x - 84$ using:

   a. Polynomial long division

   b. Factor theorem

   c. Which method do you prefer and why?

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3. Use polynomial long division or the factor theorem to answer each question. Show your work and explain your reasoning.

   a. Is the expression $(x - 2)$ a factor of $12x^2 - 19x + 10$?

   b. Is the expression $(x + 18)$ a factor of $x^2 - 324$?

ACTIVITY
3.2

The Remainder Theorem

You learned that the process of dividing polynomials is similar to the process of dividing integers. Sometimes, when you divide two integers there is a remainder, and sometimes, there is not a remainder. What does each case mean? In this activity, you will investigate what the remainder means in terms of polynomial division.

1. Determine the quotient for each. Show all your work.

   a. $(5x - 12) \div (x - 4)$

   b. $(4x^2 + 31x + 21) \div (x + 7)$

   c. $(5x^2 - 23x + 17) \div (x - 2)$

   d. $(8x^2 + 3x + 6) \div (x^2 - x + 4)$

2. Consider Question 1 parts (a) through (d) to answer each question.

   a. When there is a remainder, is the divisor a factor of the dividend? Explain your reasoning.

   b. Describe the remainder when you divide a polynomial by one of its factors.

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Remember from your experiences with division that:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

For example, $\frac{283}{12} = 23 + \frac{7}{12} = 23\frac{7}{12}$.

You can verify your solution remembering that:

dividend $=$ (divisor)(quotient) $+$ remainder.

$$283 = 12(23) + 7$$
$$283 = 276 + 7$$
$$283 = 283$$

3. Rewrite your solutions from Question 1 in the form $\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$ and then verify each solution.

It follows that any polynomial, $p(x)$, can be written in the form:

$$\frac{p(x)}{\text{linear expression}} = \text{quotient} + \frac{\text{remainder}}{\text{linear expression}}$$

or

$$p(x) = (\text{linear expression})\,(\text{quotient}) + \text{remainder}.$$

Generally, the linear expression is written in the form $(x - r)$, the quotient is represented by $q(x)$, and the remainder is represented by R.

$$p(x) = (x - r) \cdot q(x) + R$$

4. Consider each dividend in Question 1 as a function, $p(x)$.

   a. In part (a), evaluate $p(x)$ for $x = 4$.

   b. In part (b), evaluate $p(x)$ for $x = -7$.

   c. In part (c), evaluate $p(x)$ for $x = 2$.

   d. What is the relationship between the remainder and the divisor in each problem?

Remember . . .
The factor theorem states that a polynomial has a linear polynomial as a factor if and only if the remainder is zero. Therefore, if $R = 0$, then $f(r) = 0$, and $(x - r)$ is a factor of $f(x)$.

5. What conclusion can you make about any polynomial evaluated at r?

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The **remainder theorem** states that when any polynomial function, $f(x)$, is divided by a linear expression of the form $(x - r)$, the remainder $R = f(r)$, or the value of the function when $x = r$.

6. Use the remainder theorem to complete the table.

Dividend $p(x)$	Divisor $(x - r)$	r	Remainder, R $p(r)$
$4x^2 - 5x - 21$	$x - 3$		
$2x^2 + 3x - 49$	$x + 7$		

7. Determine the unknown in each.

a. $\frac{x}{7} = 18$ R 2. Determine x.

b. $\frac{p(x)}{x + 3} = 5x - 2$ R (-4). Determine the function $p(x)$.

c. Describe the similarities and differences in your solution strategies.

Talk the Talk

A Polynomial Divided

1. Given the information, determine whether each statement is true or false. Explain your reasoning.

$$p(x) = 3x^2 - 19x + 20, \text{ and}$$
$$p(x) \div (x + 6) = 3x - 1 + \frac{14}{x - 6}$$

   a. $p(6) = 14$

   b. $p(x) = (x + 6)(3x - 1) + 14$

   c. $(x - 2)$ is a factor of $p(x)$

   d. $(x - 5)$ is a factor of $p(x)$

2. Explain the difference between the remainder theorem and the factor theorem.

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Lesson 3 Assignment

Write

Write an example for each term using the dividend $x^2 - 2x + 4$ and the divisor $(x + 1)$.

1. Factor theorem
2. Polynomial long division
3. Remainder theorem

Remember

A polynomial function $p(x)$ has $(x - r)$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$. When any polynomial equation or function $f(x)$ is divided by a linear expression of the form $(x - r)$, the remainder is $R = f(r)$, or the value of the function when $x = r$.

Practice

1. Use the factor theorem to determine whether each linear expression is a factor of $2x^2 - 29x - 48$.

   a. $x + 3$

   b. $x - 16$

   c. $x + \frac{3}{2}$

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Lesson 3 Assignment

2. Determine the quotient for each problem. Show all your work.

a. $\dfrac{2x^2 + 6x - 7}{x + 1}$

b. $(3x^2 - \frac{5}{2}x - 2) \div (x + \frac{1}{2})$

c. $\dfrac{14x^2 + 10x - 1}{x - 2}$

d. $(3x^2 + 2x - 5) \div (x^2 - 4)$

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Lesson 3 Assignment

3. The table of values represents the function $f(x) = x^2 + 10x - 96$.

   a. Determine one of the factors of $f(x)$ without using a calculator. Explain your reasoning.

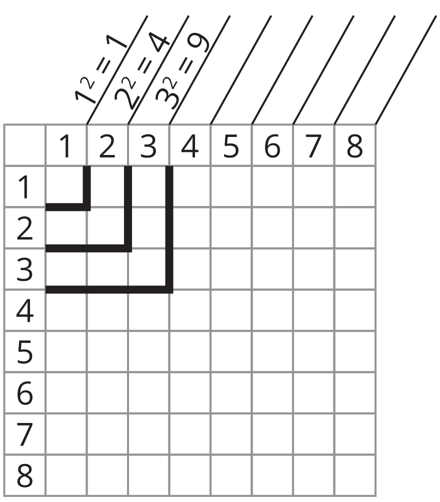
(x)	-18	-12	0	6	8
$f(x)$	48	-72	-96	0	48

   b. Completely factor $f(x)$ without using a calculator.

   c. Determine all of the zeros of $f(x)$ without using a calculator.

Prepare

1. Complete the grid by continuing to make squares with side lengths of 4 through 8. Connect the side lengths and then write an equation using exponents to represent each perfect square.



TOPIC 2 SELF-REFLECTION

Name: ____________________

Polynomial Operations

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Polynomial Operations* topic by:

TOPIC 2: *Polynomial Operations*	Beginning of Topic	Middle of Topic	End of Topic
classifying a polynomial as a monomial, binomial, or trinomial based on the number of terms and determine the degree of the polynomial.	◯	◯	◯
adding and subtracting polynomials.	◯	◯	◯
multiplying polynomials.	◯	◯	◯
dividing polynomials.	◯	◯	◯
recognizing special products, including the difference of two squares, and using them to multiply polynomials.	◯	◯	◯

continued on the next page

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TOPIC 2 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Polynomial Operations* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. What concepts are you working to understand from this topic? How will you work to build your understanding of these concepts?

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TOPIC 2 SUMMARY

Polynomial Operations

LESSON 1 Adding and Subtracting Polynomials

Linear and quadratic expressions are part of a larger group of expressions known as *polynomials*. A **polynomial** is an expression involving the sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable is the sum of terms of the form ax^k, where a, called the *coefficient*, is a real number and k is a non-negative integer. A polynomial is written in standard form when the terms are in descending order, starting with the term with the greatest degree and ending with the term with the least degree.

$$a_1x^k + a_2x^{k-1} + \dots a_nx^0$$

Each product in a polynomial is called a *term*. Polynomials are named according to the number of terms: a **monomial** has exactly 1 term, a **binomial** has exactly 2 terms, and a **trinomial** has exactly 3 terms.

The exponent of a term is the degree of the term, and the greatest exponent in a polynomial is the **degree of the polynomial**.

The characteristics of the polynomial $13x^3 + 5x + 9$ are shown in the chart.

	1st term	2nd term	3rd term
Term	$13x^3$	$5x$	9
Coefficient	13	5	9
Power	x^3	x^1	x^0
Exponent	3	1	0

This trinomial has a degree of 3 because 3 is the greatest degree of the terms in the trinomial.

Polynomials can be added or subtracted by identifying the like terms of the polynomial functions, using the associative property to group the like terms together and combining the like terms to simplify the expression.

For example, to add the polynomial expressions $(7x^2 - 2x + 12)$ and $(8x^3 + 2x^2 - 3x)$, use the associative property to combine like terms.

$$(7x^2 - 2x + 12) + (8x^3 + 2x^2 - 3x)$$
$$8x^3 + (7x^2 + 2x^2) + (-2x - 3x) + 12$$
$$8x^3 + 9x^2 - 5x + 12$$

NEW KEY TERMS

- polynomial [polinomio]
- monomial [monomio]
- binomial [binomio]
- trinomial [trinomio]
- degree of a polynomial [grado de un polinomio]
- difference of two squares
- perfect square trinomial
- factor theorem [teorema del factor]
- polynomial long division [división (larga) de polinomios]
- remainder theorem

LESSON 2

Multiplying Polynomials

The product of 2 binomials can be determined by using an area model with algebra tiles. Another way to model the product of 2 binomials is a multiplication table, which organizes the two terms of the binomials as factors of multiplication expressions.

Example 1

$(2x + 1)(x + 3)$

$\cdot$	x	1	1	1
x	x^2	x	x	x
x	x^2	x	x	x
1	x	1	1	1

$(2x + 1)(x + 3) = 2x^2 + 7x + 3$

Example 2

$(9x - 1)(5x + 7)$

$\cdot$	$9x$	-1
$5x$	$45x^2$	$-5x$
7	$63x$	-7

$$(9x - 1)(5x + 7) = 45x^2 - 5x + 63x - 7$$
$$= 45x^2 + 58x - 7$$

You can also use the distributive property to multiply polynomials.

To multiply $4(x - 3)$, use the distributive property once.

$4(x - 3) = 4(x) - 4(3) = 4x - 12$

Depending on the number of terms in the polynomials, you may use the distributive property multiple times.

For example, to multiply the polynomials $x + 5$ and $x - 2$, first, use the distributive property to multiply each term of $x + 5$ by the entire binomial $x - 2$.

$(x + 5)(x - 2) = (x)(x - 2) + (5)(x - 2)$

Next, distribute x to each term of $x - 2$ and distribute 5 to each term of $x - 2$.

$x^2 - 2x + 5x - 10$

Finally, collect the like terms and write the solution in standard form.

$x^2 + 3x - 10$

You can use the same process to multiply any polynomials

For example:

$(x^2 + 4x - 7)(x^2 - 3x + 1)$

$= x^2 (x^2 - 3x + 1) + 4x(x^2 - 3x + 1) - 7(x^2 - 3x + 1)$

$= (x^4 - 3x^3 + x^2) + (4x^3 - 12x^2 + 4x) + (-7x^2 + 21x - 7)$

$= x^4 + x^3 - 18x^2 + 25x - 7$

There are special products of degree 2 that have certain characteristics. The expression in the form $a^2 - b^2$ that has factors $(a + b)(a - b)$ is called the

difference of two squares. A **perfect square trinomial** is formed by multiplying a binomial by itself. It is an expression in the form $a^2 + 2ab + b^2$, or in the form $a^2 - 2ab + b^2$. A perfect square trinomial can be written as the square of a binomial. In these cases, the factors are $(a + b)^2$ and $(a - b)^2$, respectively.

You can use what you know about multiplying polynomials to rewrite quadratic functions in different forms.

Write the quadratic function $f(x) = (2x + 1)(x - 5)$ in standard form:

Step 1: Apply the distributive property. $f(x) = 2x(x - 5) + 1(x - 5)$

Step 2: Apply the distributive property. $f(x) = 2x^2 - 10x + x - 5$

Step 3: Combine like terms. $f(x) = 2x^2 - 9x - 5$

Write the quadratic function $g(x) = (x - 1)^2 + 6$ in standard form:

Step 1: Write the power as a multiplication. $g(x) = (x - 1)(x - 1) + 6$

Step 2: Apply the distributive property. $g(x) = x(x - 1) - 1\,(x - 1) + 6$

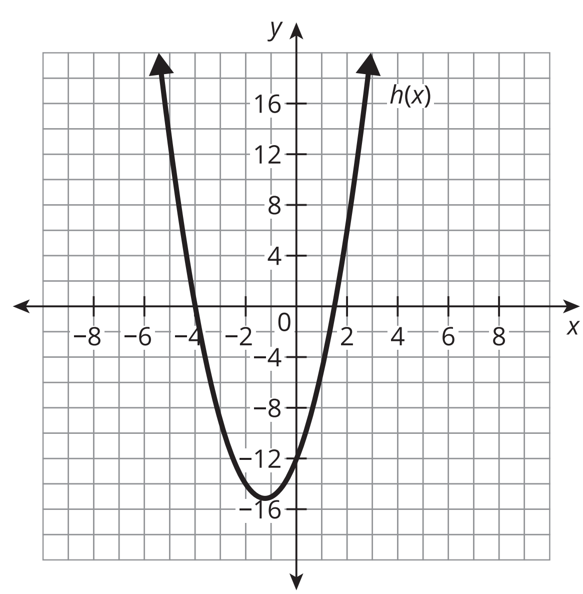
Step 3: Apply the distributive property. $g(x) = x^2 - x - x + 1 + 6$

Step 4: Combine like terms. $g(x) = x^2 - 2x + 7$

LESSON 3 Polynomial Division

The **factor theorem** states that a polynomial function $p(x)$ has $(x - r)$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$. You can use the factor theorem to show that a linear expression is a factor of a polynomial.

Consider the graph of the polynomial function $h(x) = 2x^2 + 5x - 12$.



The graph appears to have a zero at (−4, 0), so a possible linear factor of the polynomial is $(x - (-4))$, or $(x + 4)$.

Determine the value of the polynomial at $x = -4$, or $h(-4)$.

$$h(-4) = 2(-4)^2 + 5(-4) - 12$$

$$h(-4) = 32 - 20 - 12$$

$$h(-4) = 0$$

So, $(x + 4)$ is a linear factor of the polynomial function.

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Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree. The process is similar to integer long division. For example, when 2 is a factor of 24, then 24 can be divided by 2 without a remainder. In the same way, the factors of a polynomial divide into that polynomial without a remainder. Therefore, you can use polynomial long division to determine whether a linear expression is a factor of a polynomial.

Integer Long Division	Polynomial Long Division
$3660 \div 12$ or $\frac{3660}{12}$	$(2x^2 + 5x - 12) \div (x + 4)$ or $\frac{2x^2 + 5x - 12}{x + 4}$
$\begin{array}{r} 305 \\ 12\overline{)3660} \\ -36 \\ \hline 6 \\ -0 \\ \hline 60 \\ -60 \\ \hline 0 \end{array}$	$\begin{array}{rl} & \text{A}\ 2x - \text{D}\ 3 \\ x + 4 & \overline{)2x^2 + 5x - 12} \\ \text{B} & -(2x^2 + 8x) \downarrow \ \text{C} \\ \hline & -3x - 12 \\ & -(-3x - 12) \\ \hline & \text{Remainder } 0 \end{array}$ A. Divide $\frac{2x^2}{x} = 2x$. B. Multiply $2x(x + 4)$ and then subtract. C. Bring down -12. D. Divide $\frac{-3x}{x} = -3$. E. Multiply $-3(x + 4)$ and then subtract.

For the polynomial long division example above, the remainder is 0, which means that $(x + 4)$ is a factor of $2x^2 + 5x - 12$. The quotient is $(2x - 3)$, which is the other factor of the polynomial. Therefore, $2x^2 + 5x - 12 = (x + 4)(2x - 3)$.

Remember from your experiences with division that:

$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$.

It follows that any polynomial, $p(x)$, can be written in the form:

$$\frac{p(x)}{\text{linear expression}} = \text{quotient} + \frac{\text{remainder}}{\text{linear expression}}$$

or

$p(x) = (\text{linear expression})(\text{quotient}) + \text{remainder}$.

Generally, the linear expression is written in the form $(x - r)$, the quotient is represented by $q(x)$, and the remainder is represented by R.

$p(x) = (x - r)q(x) + \text{R}$

Consider this example. Determine the quotient of $(3x^2 + 10x + 1) \div (x + 4)$.

$$
\begin{array}{r}
3x - 2 \\
x + 4 \overline{\smash{\big)}\, 3x^2 + 10x + 1} \\
\underline{-(3x^2 + 12x)} \\
-2x + 1 \\
\underline{-(-2x - 8)} \\
9
\end{array}
$$

The quotient of $(3x^2 + 10x + 1) \div (x + 4)$ is $3x - 2 + \frac{9}{x + 4}$.

$\frac{3x^2 + 10x + 1}{x + 4} = 3x - 2 + \frac{9}{x + 4}$

$3x^2 + 10x + 1 = (x + 4)(3x - 2) + 9$

The **remainder theorem** states that when any polynomial function, $f(x)$, is divided by a linear expression of the form $(x - r)$, the remainder $R = f(r)$, or the value of the function when $x = r$. Remember, the factor theorem states that a polynomial has a linear expression as a factor if and only if the remainder is zero. Therefore, if $R = 0$, then $f(r) = 0$ and $(x - r)$ is a factor of $f(x)$.

In the previous example, using polynomial long division we determined $R = 9$; therefore, by the factor theorem, $(x + 4)$ is not a factor of $3x^2 + 10x + 1$. This can be confirmed using the remainder theorem.

When $f(x) = 3x^2 + 10x + 1$, to determine if $(x + 4)$ is a factor of $f(x)$, we must determine the value of $f(-4)$.

$f(-4) = 3(-4)^2 + 10(-4) + 1$
$f(-4) = 9$

This means the remainder of $f(x) = 9$, not 0; therefore, $(x + 4)$ is not a factor of $f(x) = 3x^2 + 10x + 1$.

You can also divide more complicated polynomials. For example, consider $(8x^2 + 3x + 6) \div (x^2 - x + 4)$. The process is similar.

$$
\begin{array}{r}
8 \;\; \text{R}\,(11x - 26) \\
x^2 - x + 4 \overline{\smash{\big)}\, 8x^2 + 3x + 6} \\
\underline{-(8x \;\; - 8x + 32)} \\
11x \;\; - 26 \\
8 + \frac{11x - 26}{x^2 - x + 4}
\end{array}
$$

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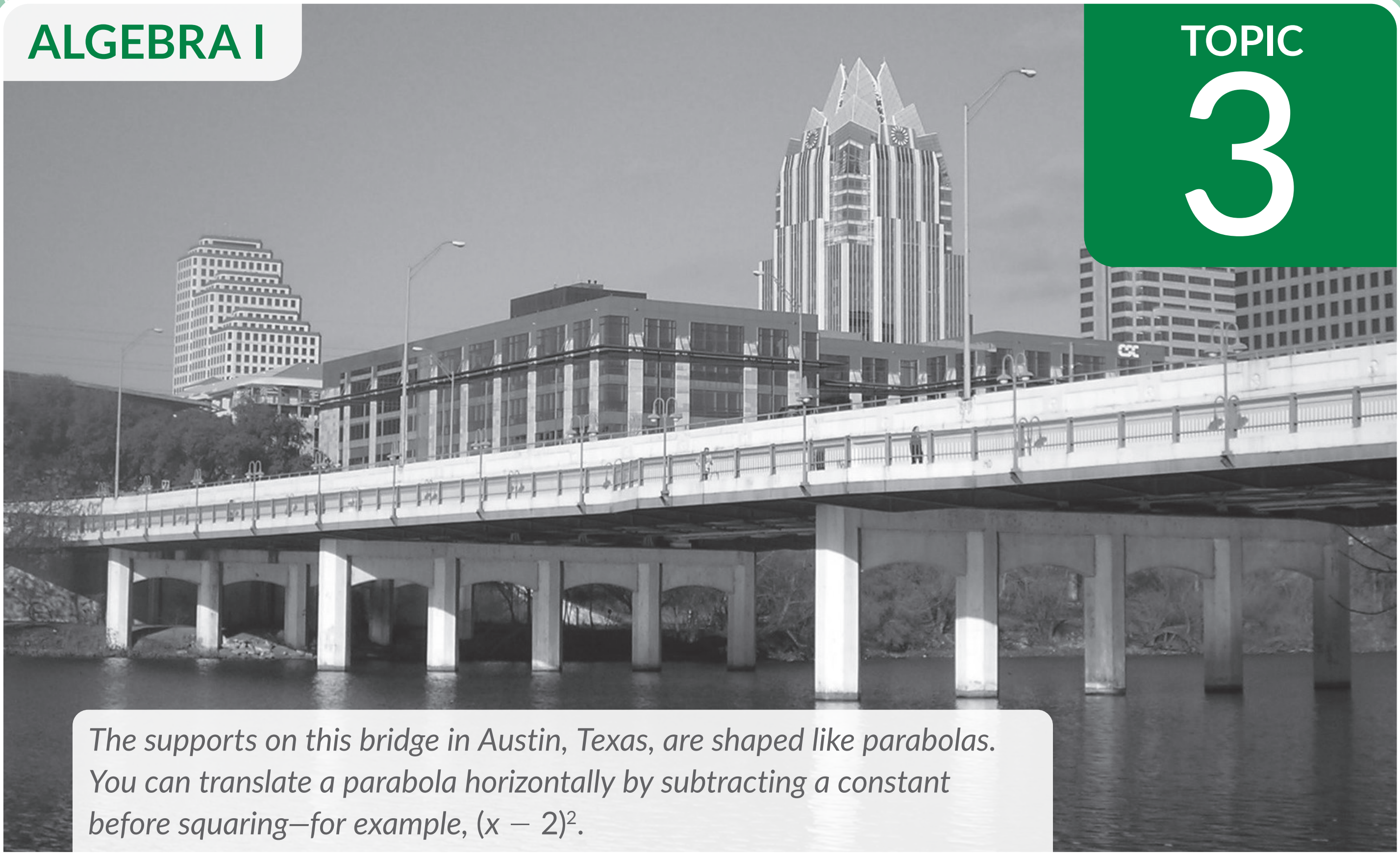
ALGEBRA I

TOPIC 3



The supports on this bridge in Austin, Texas, are shaped like parabolas. You can translate a parabola horizontally by subtracting a constant before squaring—for example, $(x - 2)^2$.

Solving Quadratic Equations

LESSON 1 Representing Solutions to Quadratic Equations . . . **827**

LESSON 2 Solutions to Quadratic Equations in Vertex Form . . **843**

LESSON 3 Factoring and Completing the Square **855**

LESSON 4 The Quadratic Formula **879**

LESSON 5 Using Quadratic Functions to Model Data **901**

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1 Representing Solutions to Quadratic Equations

OBJECTIVES

- Identify the zeros of a quadratic function, the roots of a quadratic equation, and the x-intercepts of a parabola using the equation of a quadratic function.
- Identify the double root of a quadratic equation as the two solutions of a quadratic equation at the minimum or maximum of the function.
- Write solutions of quadratic equations at specific output values using the axis of symmetry and the positive and negative square roots of the output value.
- Identify quadratic equations written as the difference of two perfect squares and rewrite these equations in factored form with a leading coefficient of 1.

NEW KEY TERMS

- principal square root
- double root
- zero product property

You have studied the graphs and equations for quadratic functions. How can you determine solutions of quadratic equations given different output values?

How can you use the structure of a parabola to understand the solutions of a quadratic equation?

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Getting Started

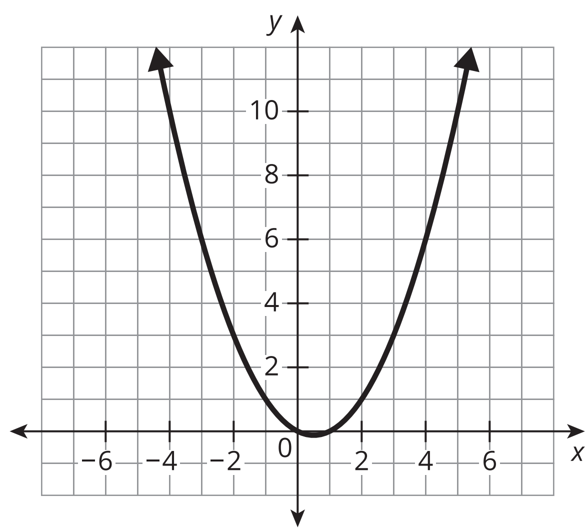
Shaking Hands

Consider one of the quadratic models you investigated in a previous topic.

Suppose that there is a monthly meeting at CIA headquarters for all employees. How many handshakes will it take for every employee at the meeting to shake the hand of every other employee at the meeting once?

You can model The Handshake Problem with the equation $f(n) = \frac{1}{2}n^2 - \frac{1}{2}n$ or $f(n) = \frac{1}{2}(n)(n - 1)$.

1. Determine the equation for the axis of symmetry. Indicate the axis of symmetry on the graph.



2. Draw a horizontal line at $y = 6$.

   a. Where does the line $y = 6$ intersect the graph of $f(n)$?

   b. What is the distance of each point from the axis of symmetry?

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c. What do the points of intersection mean in terms of this situation?

3. For each y, how many solutions does the equation $y = \frac{1}{2}n^2 - \frac{1}{2}n$ have?

4. Draw a horizontal line at $y = 0$ to determine the solutions to the equation $0 = \frac{1}{2}n^2 - \frac{1}{2}n$. How far is each solution away from the axis of symmetry?

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ACTIVITY 1.1

Solutions of a Quadratic Function

Remember . . .

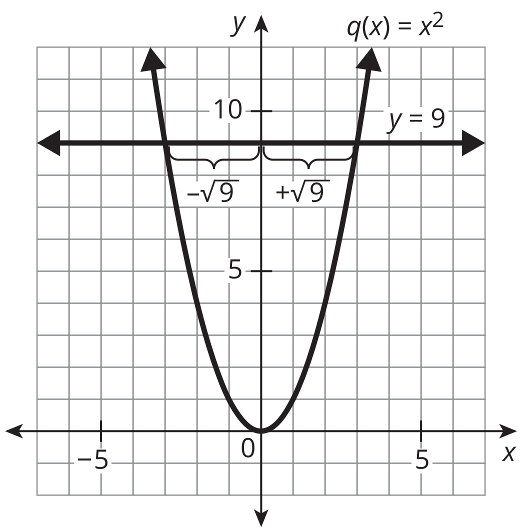
The square root property is $\sqrt{a^2} = \pm a$.

Recall that a quadratic function is a function of degree 2 because the greatest power for any of its terms is 2. This means that it has 2 zeros, or 2 solutions, at $y = 0$.

The two solutions of the quadratic parent function can be represented as square roots of numbers. Every positive number has two square roots, a positive square root (which is also called the **principal square root**) and a negative square root. To solve the equation $x^2 = 9$, you can take the square root of both sides of the equation.

$$\sqrt{x^2} = \pm\sqrt{9}$$
$$x = \pm 3$$

Solving $x^2 = 9$ on a graph means that you are looking for the points of intersection between $y = x^2$ and $y = 9$.



1. Consider the graph of the function $q(x) = x^2$ above.

    a. What is the equation for the axis of symmetry? Explain how you can use the function equation to determine your answer.

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b. Explain how the graph shows the two solutions for the function at $y = 9$ and their relationship to the axis of symmetry. Use the graph and the function equation to explain your answer.

c. Describe how you can determine the two solutions for the function at $y = 2$. Indicate the solutions on the graph.

d. Describe how you can determine the two solutions for the function at each y-value for $y \geq 0$.

The quadratic function $q(x) = x^2$ has two solutions at $y = 0$. Therefore, it has 2 zeros: $x = +\sqrt{0}$ and $x = -\sqrt{0}$. These two zeros of the function, or roots of the equation, are the same number, 0, so $y = x^2$ is said to have a **double root,** or **1 unique root**.

The root of an equation indicates where the graph of the equation crosses or touches the x-axis. A double root occurs when the graph just touches the x-axis but does not cross it.

The x-coordinates of the x-intercepts of a graph of a quadratic function are called the *zeros* of the quadratic function. The zeros are called the *roots* of the quadratic equation.

2. Use the graph and the function to explain why Fernando's equation is correct.

Fernando

The quadratic equation $y = x^2$ is symmetric about its axis of symmetry, $x = 0$. So, I can write the solution to the quadratic equation $y = x^2$, as $x = 0 \pm \sqrt{y}$.

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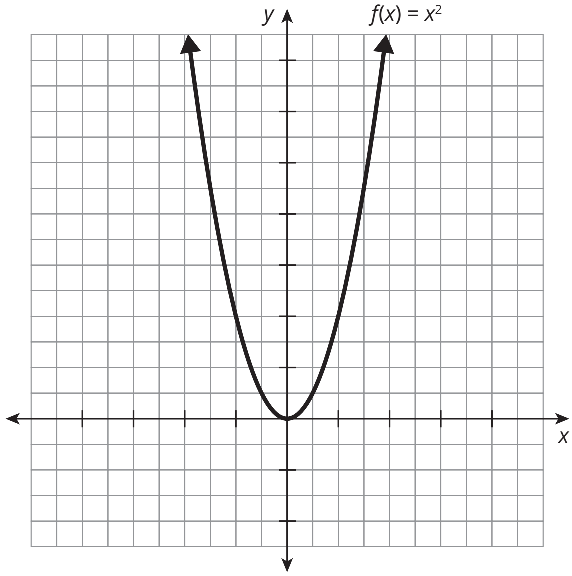
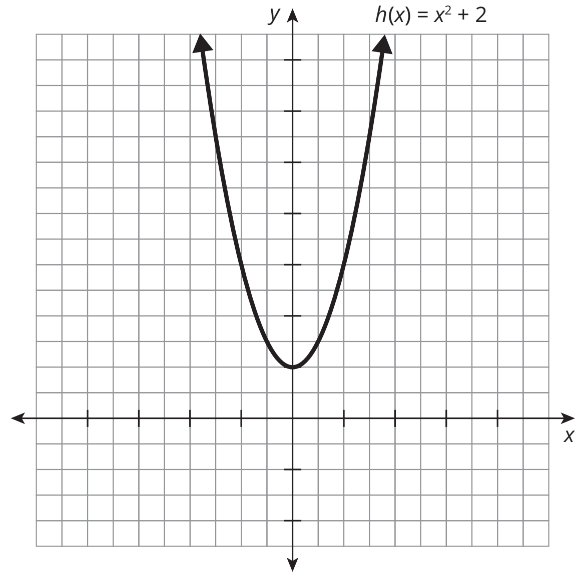
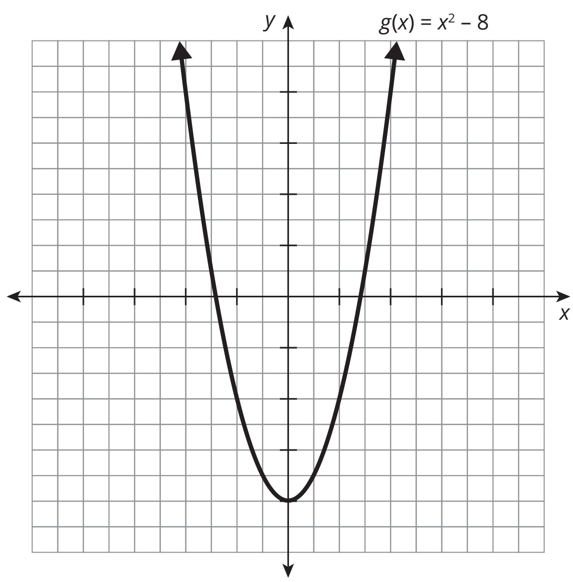
The graphs of three quadratic functions, $f(x)$, $h(x)$, and $g(x)$, are shown.







3. Use the graphs to identify the solutions to each equation. Then, determine the solutions algebraically and write the solutions in terms of their respective distances from the axis of symmetry.

a. $14 = x^2 + 2$

b. $x^2 = 10$

c. $-5 = x^2 - 8$

d. $19 = x^2 + 4$

e. $x^2 - 8 = 1$

f. $6 = x^2$

4. Consider the graphs of $f(x)$, $h(x)$, and $g(x)$. Which function has a double root? Explain your answer.

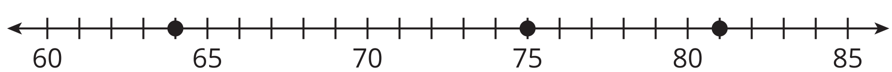
When you are solving quadratic equations, you may encounter solutions that are not perfect squares. You can either determine the approximate value of the radical or rewrite it in an equivalent radical form.

WORKED EXAMPLE

You can determine the approximate value of $\sqrt{75}$.

Determine the perfect square that is closest to, but less than, 75. Then, determine the perfect square that is closest to, but greater than, 75.

$$64 \le 75 \le 81$$



Determine the square roots of the perfect squares.

$$\sqrt{64} = 8 \qquad \sqrt{75} = ? \qquad \sqrt{81} = 9$$

Now that you know that $\sqrt{75}$ is between 8 and 9, you can test the squares of numbers between 8 and 9.

$$8.6^2 = 73.96 \qquad 8.7^2 = 75.69$$

Since 75 is closer to 75.69 than 73.96, 8.7 is the approximate square root of $\sqrt{75}$.

Ask Yourself . . .
Can you name all the perfect squares from 1^2 through 15^2?

WORKED EXAMPLE

You can use prime factors to rewrite $\sqrt{75}$ in an equivalent radical form.

First, rewrite the product of 75 to include any perfect square factors and then extract the square roots of those perfect squares.

$$\begin{aligned} \sqrt{75} &= \sqrt{3 \cdot 5 \cdot 5} \\ &= \sqrt{3 \cdot 5^2} \\ &= \sqrt{3} \cdot \sqrt{5^2} \\ &= 5\sqrt{3} \end{aligned}$$

Think About . . .
How could listing the prime factors of a radical expression help to extract square roots of perfect squares?

5. Estimate the value of each radical expression. Then, rewrite each radical by extracting all perfect squares, when possible.

a. $\sqrt{20}$

b. $\sqrt{18}$

c. $\sqrt{26}$

d. $\sqrt{116}$

6. Rewrite your answers from Question 3 by extracting perfect squares, when possible. Verify your rewritten answers using the graphs in Question 3.

ACTIVITY
1.2

Solutions from Standard Form to Factored Form

Recall that a quadratic function written in factored form is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$. In factored form, r_1 and r_2 represent the x-intercepts of the graph of the function.

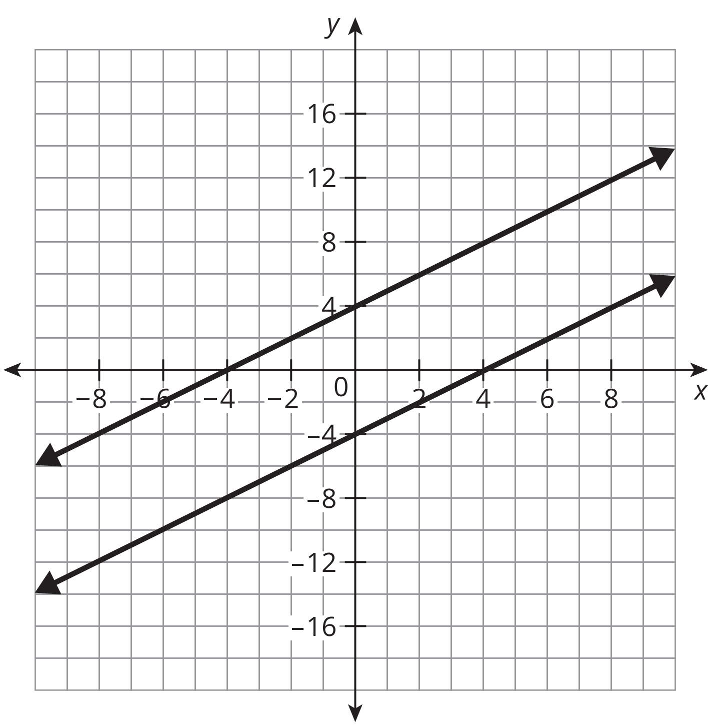
Think About . . .

Do you recognize the form of this quadratic?

1. Determine the zeros of the function $z(x) = x^2 - 16$. Then, write the function in factored form.

The function $z(x)$ in factored form is a quadratic function made up of two linear factors. Let's analyze the linear factors as separate linear functions, $g(x)$ and $h(x)$. Therefore, $z(x) = g(x) \cdot h(x)$.

2. Complete the table by writing the algebraic expressions to represent $g(x)$ and $h(x)$ and then determine the output values for the two linear factors and the quadratic product. Finally, sketch a graph of $z(x)$.



x	$g(x)$	$h(x)$	$z(x)$
			$x^2 - 16$
−4			
−2			
0			
2			
4			

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The **zero product property** states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

WORKED EXAMPLE

You can use the zero product property to identify the zeros of a function when the function is written in factored form.

$0 = x^2 - 16$

$0 = (x + 4)(x - 4)$	Rewrite the quadratic as linear factors.
$x - 4 = 0$ and $x + 4 = 0$	Apply the zero product property.
$x = 4$ $\quad\quad$ $x = -4$	Solve each equation for x.

3. Explain how the zeros of the linear function factors are related to the zeros of the quadratic function product.

The function $z(x) = x^2 - 16$ has an a-value of 1 and a b-value of 0. You can use a similar strategy to determine the zeros of a function when the leading coefficient is not 1, but the b-value is still 0.

WORKED EXAMPLE

You can determine the zeros of the function $f(x) = 9x^2 - 1$ by setting $f(x) = 0$ and using the properties of equality to solve for x.

$$9x^2 - 1 = 0$$
$$9x^2 = 1$$
$$x^2 = \frac{1}{9}$$
$$x = \pm\frac{1}{3}$$

You can then use the leading coefficient of 9 and the zeros at $\frac{1}{3}$ and $-\frac{1}{3}$ to rewrite the quadratic function in factored form.

$$f(x) = 9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)$$

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4. Consider the Worked Example.

   a. Explain why $\sqrt{\frac{1}{9}} = \pm\frac{1}{3}$.

   b. Use graphing technology to verify that $9x^2 - 1 = 9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)$.
   How can you tell from the graph that the two equations are equivalent?

Three students tried to rewrite the quadratic function $f(x) = 9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)$ as two linear factors using what they know about the difference of two squares.

Alexander

$9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right) =$
$(9x - 3)\left(x + \frac{1}{3}\right)$

Jacob

$9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right) =$
$(4.5x - 1.5)(4.5x + 1.5)$

Olivia

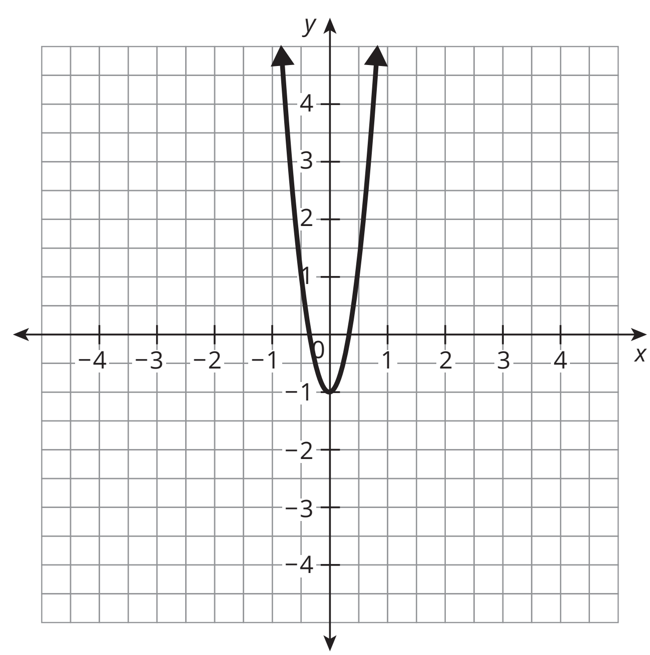
$9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right) =$
$(3x - 1)(3x + 1)$

5. Explain why Alexander and Jacob are incorrect and why Olivia is correct.

Ask Yourself . . .
Are these expressions still in factored form?

6. The graph of $f(x) = 9x^2 - 1$ is shown.

   a. Use Olivia's function, $f(x) = (3x - 1)(3x + 1)$, to sketch a graph of the linear factors. Then, use graphing technology to verify that $9x^2 - 1 = (3x - 1)(3x + 1)$.



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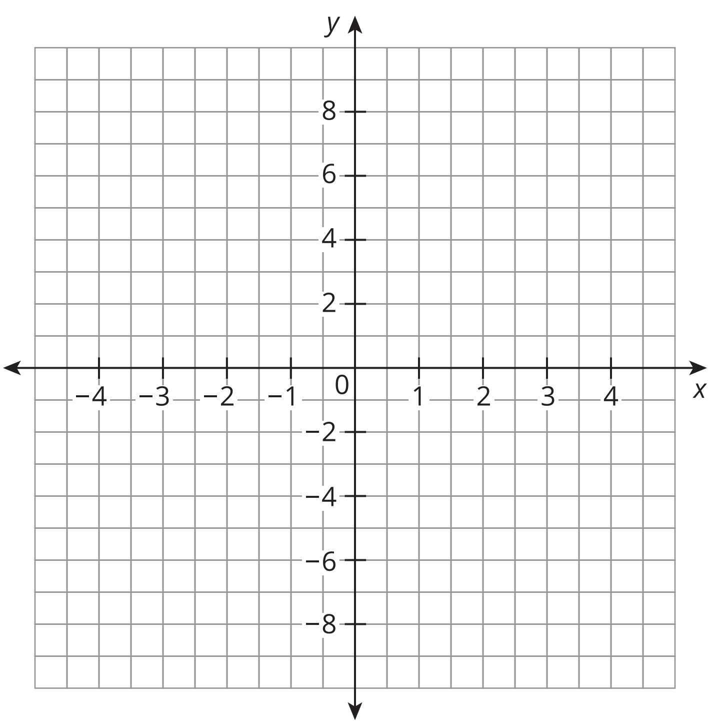
b. What is the relationship of the zeros of the function and its two linear factors?

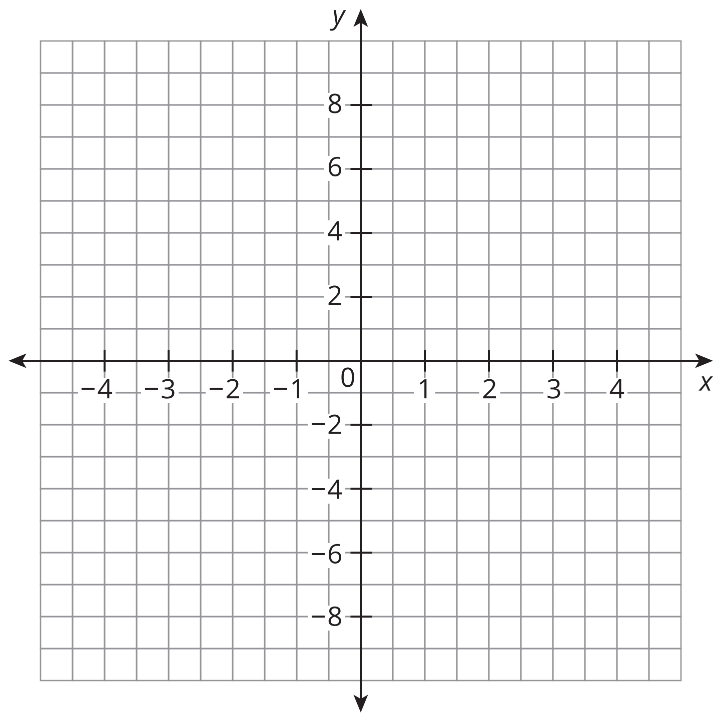
7. For each function:

- Sketch a graph. Label the axis of symmetry and the vertex.
- Use the properties of equality to identify the zeros and then write the zeros in terms of their respective distances from the line of symmetry.
- Use what you know about the difference of two squares to rewrite each quadratic as the product of two linear factors. Then, use the zero product property to verify the values of x, when $f(x) = 0$.
- Use graphing technology to verify that the product of the two linear factors is equivalent to the given function.

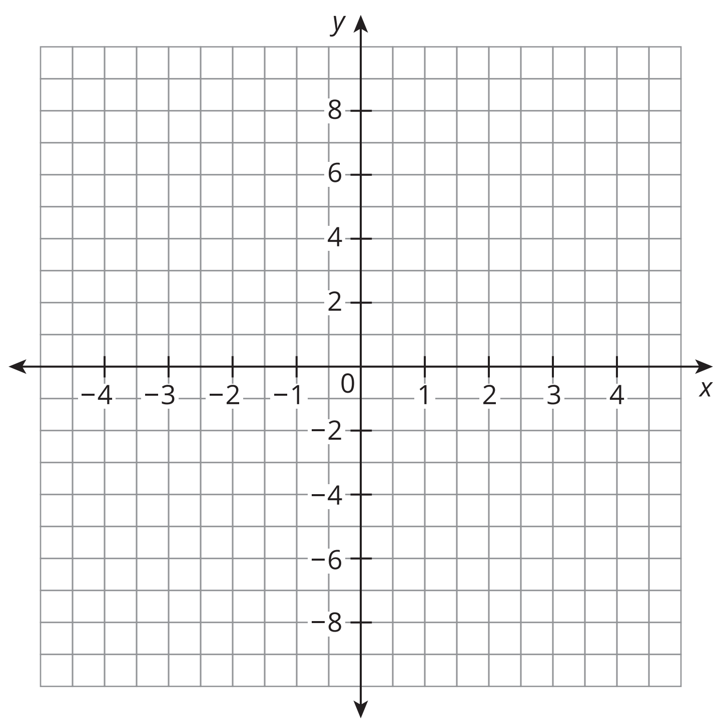


a. $f(x) = 4x^2 - 9$

b. $f(x) = x^2 - 2$



c. $f(x) = 25x^2 - 1$



Talk the Talk

The Difference of Squares

In this lesson you determined the zeros of quadratics written in the form $f(x) = ax^2 - c$.

1. Solve each equation.

   a. $x^2 - 25 = 0$

   b. $4x^2 - 1 = 0$

   c. $9x^2 - 2 = 0$

   d. $x^2 - 80 = 0$

2. Rewrite each quadratic function as two linear factors using what you know about the difference of two squares.

   a. $f(x) = x^2 - 49$

   b. $f(x) = \frac{4}{9}x^2 - 1$

   c. $f(x) = 16x^2 - 10$

   d. $f(x) = x^2 + 9$

3. Explain how to write any function of the form $f(x) = ax^2 - c$, where a and c are any real numbers, as two linear factors using what you know about the difference of two squares.

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Lesson 1 Assignment

Write

Complete each definition.

1. The zero product property states that if the product of two or more factors is equal to _________, then at least one factor must be equal to _________.

2. Every positive number has both a _________ square root and a _________ square root.

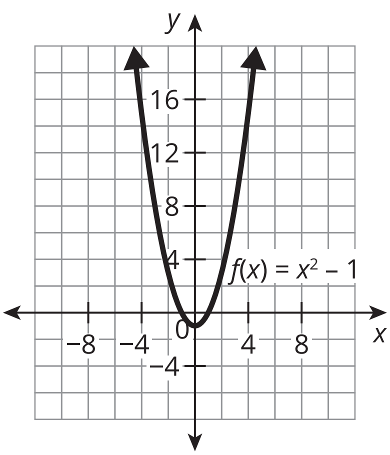
3. The function $f(x) = x^2$ has a ___________ at (0, 0).

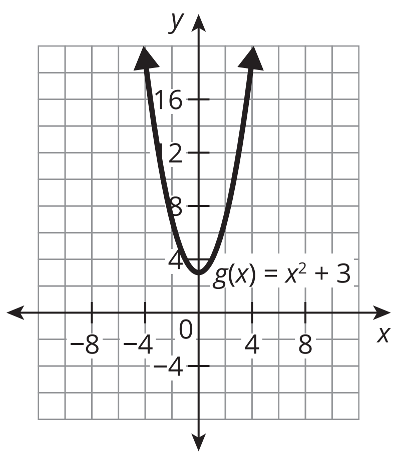
Remember

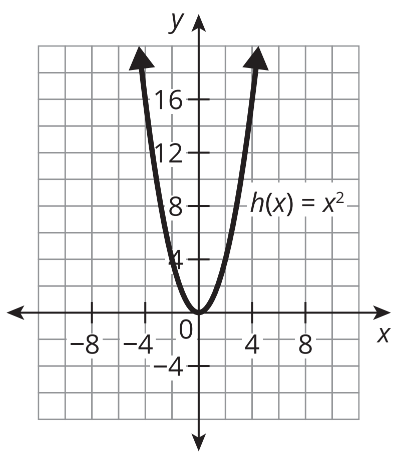
Any quadratic function of the form $f(x) = ax^2 - d$ can be rewritten as two linear factors in the form $(\sqrt{a}x - \sqrt{d})(\sqrt{a}x + \sqrt{d})$.

Practice

1. Determine the solutions for each equation. Identify the solutions on one of the graphs. Then, write the solutions in terms of their respective differences from the axis of symmetry.







a. $8 = x^2 + 3$

c. $2 = x^2 - 1$

b. $7 = x^2$

d. $x^2 = 11$

e. $x^2 + 9 = 13$

f. $14 = x^2 - 1$

Lesson 1 Assignment

2. Estimate the value of each radical expression. Then, rewrite each radical by extracting all perfect squares, when possible.

a. $\sqrt{21}$

b. $\sqrt{80}$

c. $\sqrt{63}$

d. $\sqrt{32}$

e. $\sqrt{98}$

f. $\sqrt{192}$

3. Rewrite each quadratic function as two linear factors using what you know about the difference of two squares.

a. $f(x) = 9x^2 - 16$

b. $f(x) = x^2 - 8$

c. $f(x) = 36x^2 - 1$

d. $f(x) = 25x^2 - 12$

Prepare

Describe the transformations to the graph of the parent function $f(x) = x^2$ given each equation.

1. $y = (x - 4)^2$

2. $y = \frac{1}{2}(x + 1)^2$

3. $y = -(10 + x)^2 - 3$

4. $y = (8 + x)^2 + 1$

2 Solutions to Quadratic Equations in Vertex Form

OBJECTIVES

- Identify solutions to and roots of quadratic equations given in the form $f(x) = (x - c)^2$.
- Identify solutions to and roots of quadratic equations given in the form $f(x) = a(x - c)^2$.
- Identify solutions to and roots of quadratic equations given in the form $f(x) = a(x - c)^2 + d$.
- Identify zeros of quadratic functions written in vertex form.

You have explored transformations of quadratic functions and vertex form.
How can you use vertex form and transformations to determine solutions to quadratic equations?

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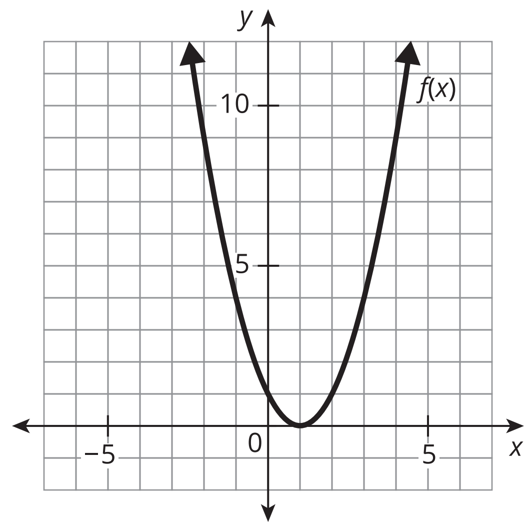
Getting Started

Slide, Slide, Slippity Slide

The coordinate plane shows the graph of the function $f(x) = (x - 1)^2$.

1. Describe the transformation applied to the parent function $f(x) = x^2$ that produces the graph of this function.



Daniela and Amir determined the zeros of the function $f(x) = (x - 1)^2$ algebraically in different ways.

Daniela

$$0 = (x - 1)^2$$

$$0 = (x - 1)(x - 1)$$

The zero product property says that one or both of the factors is equal to 0.

So, $x = 1$.

The equation has a double root at $x = 1$.

Amir

$$(x - 1)^2 = 0$$

$$\sqrt{(x - 1)^2} = \sqrt{0}$$

$$\pm(x - 1) = 0$$

$+(x - 1) = 0$	$-(x - 1) = 0$
$x = 1$	$-x + 1 = 0$
	$-x = -1$
	$x = 1$

The only unique solution for $y = 0$ is $x = 1$.

2. How can you use Daniela's or Amir's work to write solutions to the function in terms of their respective distances from the axis of symmetry?

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ACTIVITY 2.1

Solutions for Horizontal Translations

You have used graphs to solve equations. In this activity, you will use the graph of a quadratic equation to determine its solutions.

WORKED EXAMPLE

Consider the equation $(x - 1)^2 = 9$.

You can use the properties of equality to determine the solutions to an equation in this form.

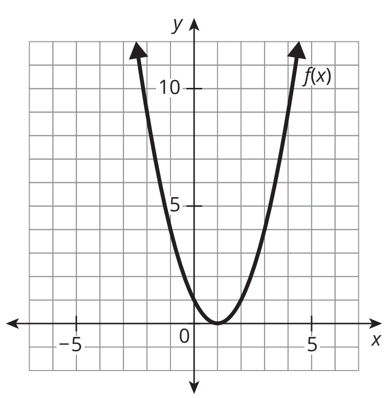
First, take the square root of both sides of the equation and then isolate x.

$$(x - 1)^2 = 9$$
$$\sqrt{(x - 1)^2} = \sqrt{9}$$
$$x - 1 = \pm 3$$
$$x = 1 \pm 3$$

Remember . . .
Solving $(x - 1)^2 = 9$ on a graph means locating where $y = (x - 1)^2$ intersects with $y = 9$.

1. Consider the graph of $y = (x - 1)^2$ in Getting Started.

   a. Graph the equation $y = 9$ on the same graph.



   b. Show the solutions on the graph. Interpret the solutions 1 ± 3 in terms of the axis of symmetry and the points on the parabola $y = (x - 1)^2$.

   c. What are the solutions to the equation $(x - 1)^2 = 9$?

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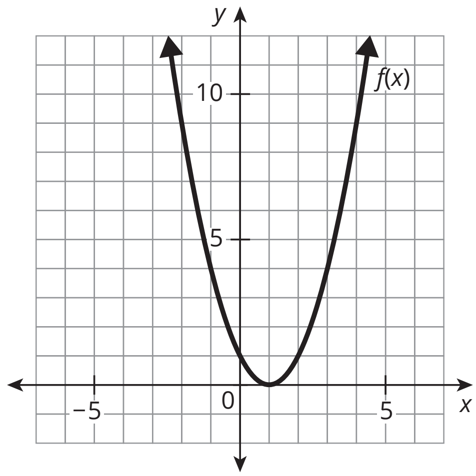
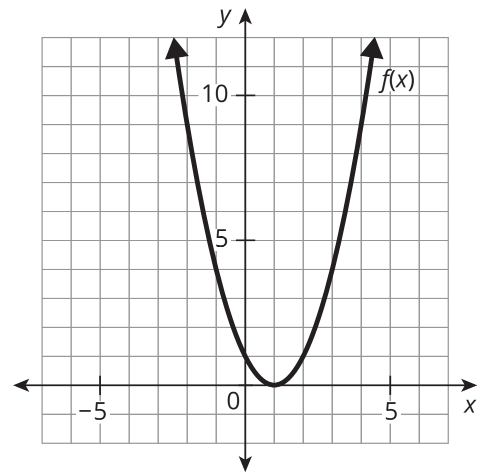
2. For each equation, show the solutions on the graph and interpret the solutions in terms of the axis of symmetry and the points on the parabola. Then, write the solutions.

a. $(x - 1)^2 = 4$



b. $(x - 1)^2 = 5$



3. Determine the exact and approximate solutions for each of the given equations.

a. $(r + 8)^2 = 83$

b. $(17 - d)^2 = 55$

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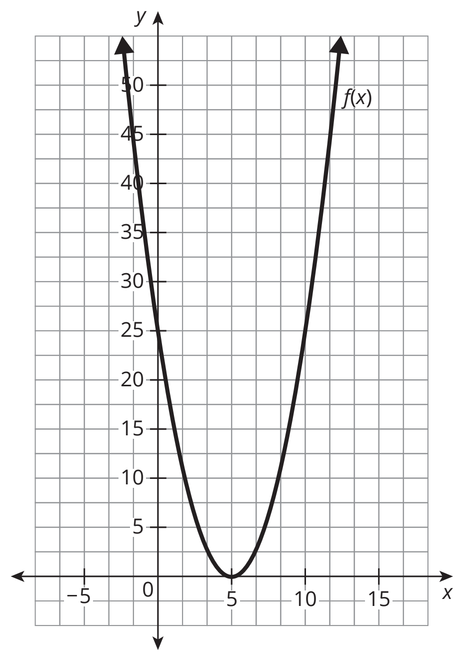
ACTIVITY
2.2 Solutions for Vertical Dilations

You have seen how to solve an equation for a quadratic function in the form $f(x) = (x - c)^2$, which represents a horizontal translation of the function. In this activity, you will consider quadratic equations with an additional vertical dilation. First, let's start with just a horizontal translation.

1. Consider the function $f(x) = (x - 5)^2$.



   a. Determine the solutions to $0 = (x - 5)^2$. Solve algebraically and label the solution on the graph.

   b. Interpret your solutions in terms of the axis of symmetry and the parabola $y = (x - 5)^2$.

   c. Describe the zeros of this function.

Now let's add a dilation factor.

2. Consider the function $g(x) = 2(x - 5)^2$.

   a. Write $g(x)$ in terms of $f(x)$ and describe the transformation.

   b. Sketch a graph on the same coordinate plane as $f(x)$.

   c. How have the zeros changed from $f(x)$ to $g(x)$?

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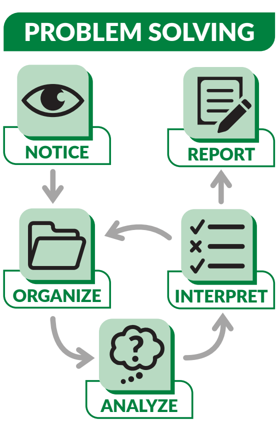
3. Koda formulated a conjecture about how the solutions of the transformed quadratic equation change from the original equation.

> The solutions of the original function are $x = 5 \pm \sqrt{y}$, so the solutions to the transformed equation will be $x = 5 \pm 2\sqrt{y}$.

Is Koda correct? If so, explain why. If not, describe the correct solutions for the transformed quadratic equation.

4. Make a conjecture. How does changing the *a*-value affect the solutions to the quadratic equations in this form?



5. Solve each quadratic equation. Give both exact and approximate solutions.

a. $(x - 4)^2 = 2$

b. $2(x - 1)^2 = 18$

c. $-2(x - 1)^2 = -18$

d. $4(x + 5)^2 = 21$

e. $-\frac{1}{2}(x + 8)^2 = -32$

f. $\frac{2}{3}(12 - x)^2 = 1$

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ACTIVITY 2.3 Solutions for Vertical Translations

You have determined solutions to quadratic equations given an equation in the form $f(x) = a(x - c)^2$. How can you solve a quadratic equation that also includes a vertical translation in the form $f(x) = a(x - c)^2 + d$?

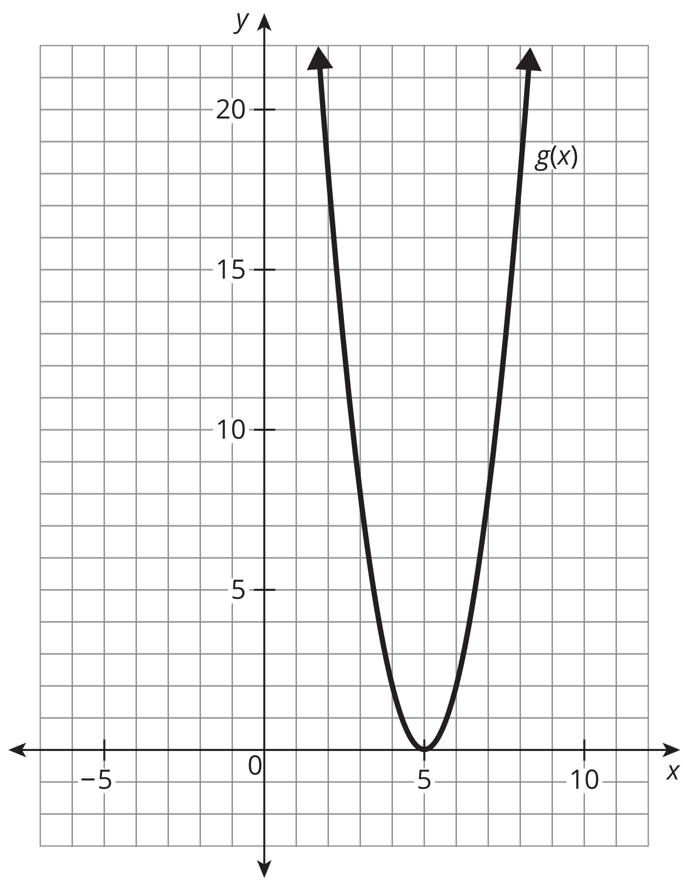
Remember . . .
A quadratic function in vertex form is written $f(x) = a(x - k)^2 + h$.

The graph of $g(x) = 2(x - 5)^2$ is shown. You know that the solution to the equation $0 = 2(x - 5)^2$ is $x = 5$.

1. Consider the function $h(x) = 2(x - 5)^2 - 1$.

   a. Write $h(x)$ in terms of $g(x)$ and describe the transformation.

   b. Sketch a graph of $h(x)$ on the same coordinate plane as $g(x)$.



2. Consider the equation $0 = 2(x - 5)^2 - 1$.

   a. Determine the solution algebraically and label the solution on the graph.

   b. Interpret the solutions in terms of the axis of symmetry and the parabola $y = 2(x - 5)^2 - 1$.

   c. Describe the zeros of this function.

Review the definition of **describe** in the Academic Glossary.

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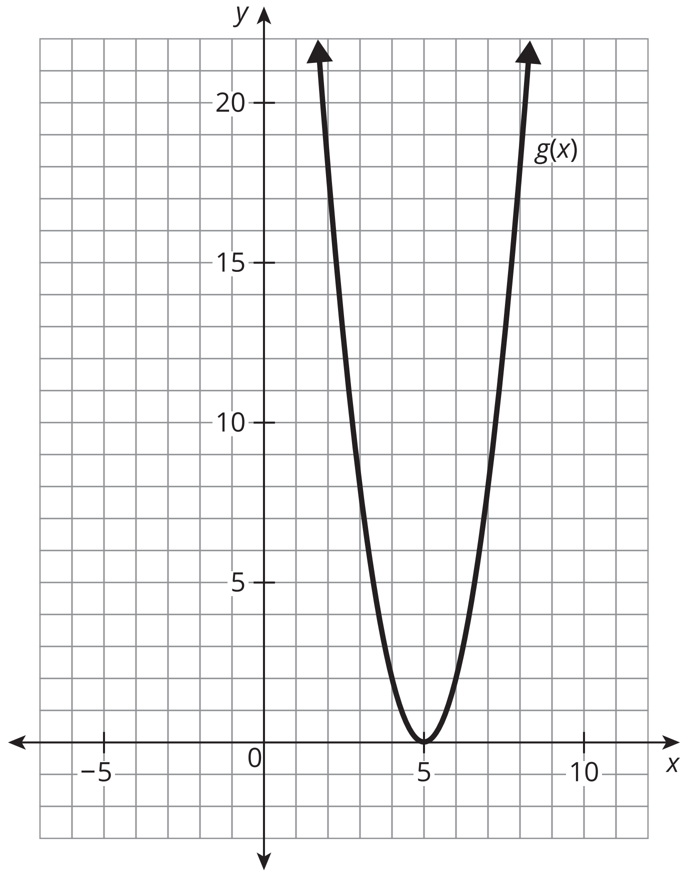
Now, let's investigate the effect of an equation in the form $f(x) = a(x - c)^2 + d$, where $a > 0$ and $d > 0$. Consider the function $j(x) = 2(x - 5)^2 + 1$ graphed, as shown.



Notice the graph of $j(x)$ does not cross the x-axis, which means there are no real zeros for this function.

3. Solve $0 = 2(x - 5)^2 + 1$ algebraically to show that x is not a real number.

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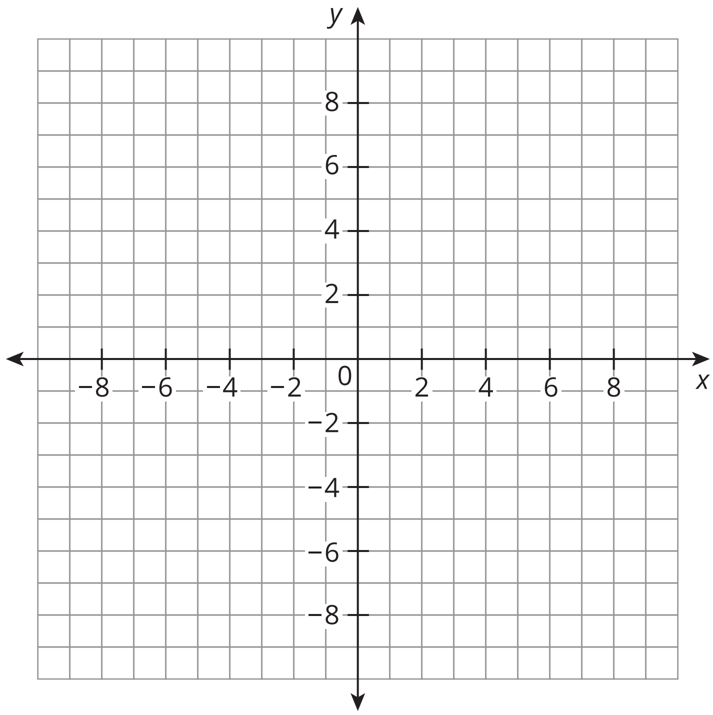
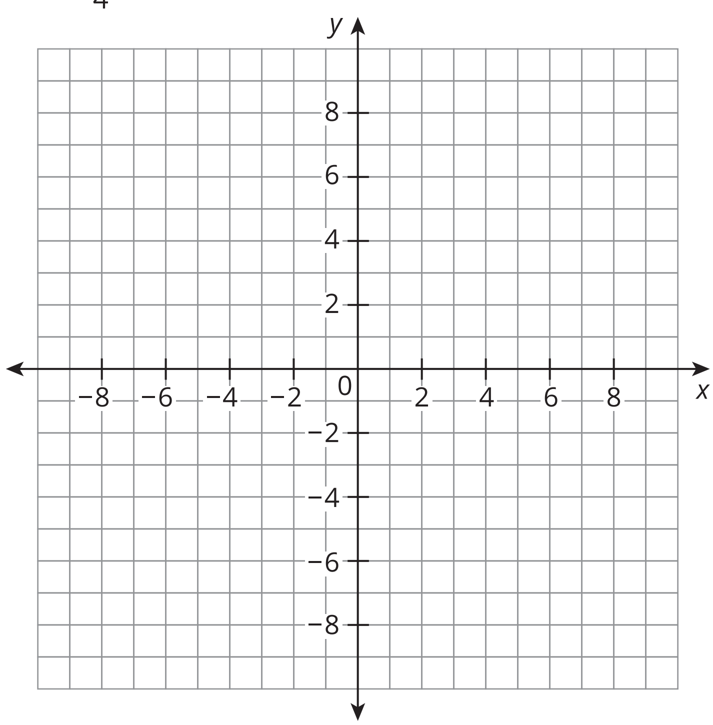
4. Sketch a graph of each quadratic function. Determine the types of zeros of each function. Solve algebraically and interpret on the graph in terms of the axis of symmetry and the points on the parabola.

A quadratic function can have 1 unique real zero, 2 real zeros, or no real zeros.

a. $f(x) = -3(x - 2)^2 + 4$



b. $f(x) = \frac{1}{4}(x + 5)^2 + 2$



Ask Yourself . . .
What is the relationship of the zeros of the function and the *x*-intercepts of the graph?

Talk the Talk

Spell It Out

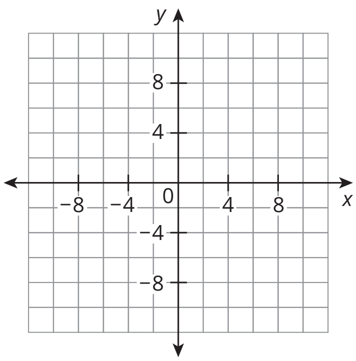
1. Describe the solution of any quadratic equation in the form $(x - c)^2 = 0$.

2. Describe the solution of any quadratic equation in the form $(x - c)^2 + d = 0$.

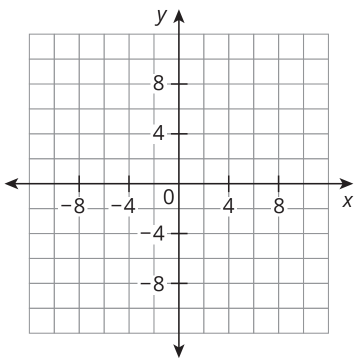
3. Describe the solution of any quadratic equation in the form $a(x - c)^2 + d = 0$.

4. Write an equation and sketch a graph that shows each number of zeros.

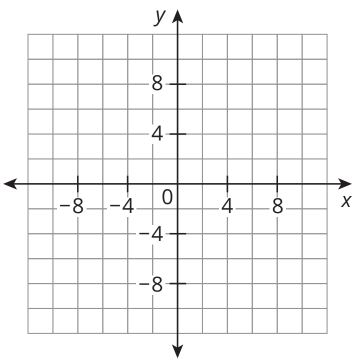
   a. 1 unique real zero



   b. 2 real zeros



   c. no real zeros



Lesson 2 Assignment

Write

Describe the number of possible real zeros for any quadratic function.

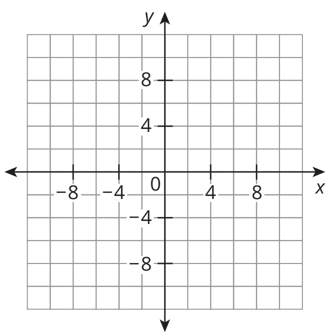
Remember

The solutions to a quadratic equation can be represented as the axis of symmetry plus or minus its distance to the parabola.

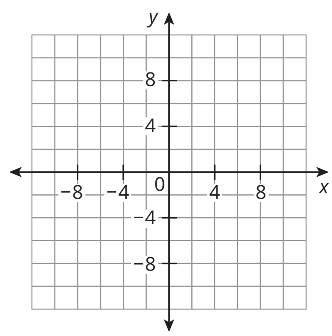
Practice

1. Sketch a graph of each quadratic function. Determine the zeros of each function and write each in terms of the axis of symmetry and its distance from the parabola.

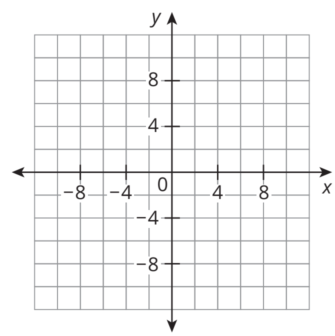
a. $f(x) = (x - 3)^2$



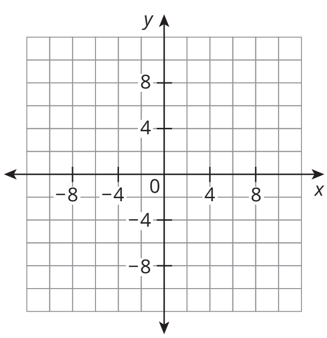
b. $f(x) = (x + 5)^2$



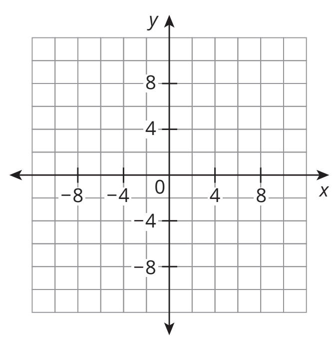
c. $f(x) = \left(x - \frac{1}{2}\right)^2$



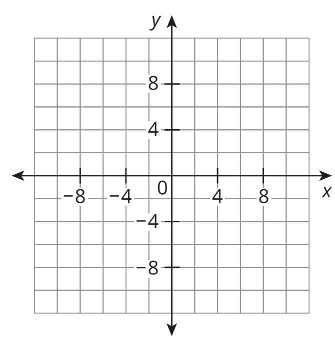
d. $f(x) = (x - 6)^2$



e. $f(x) = \left(x + \frac{15}{7}\right)^2$



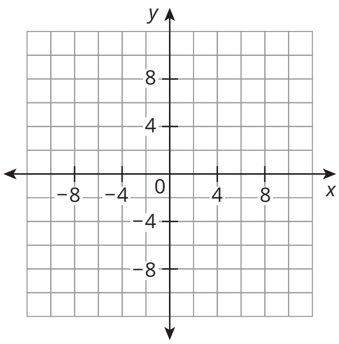
f. $f(x) = (x + 7)^2$



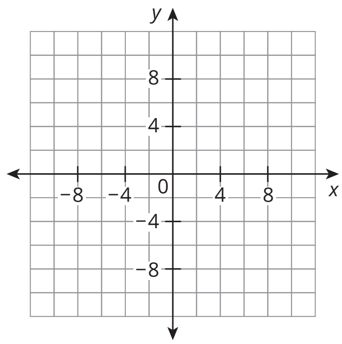
Lesson 2 Assignment

2. Sketch a graph of each quadratic function. Determine the zeros of each function and write each in terms of the axis of symmetry and its distance from the parabola.

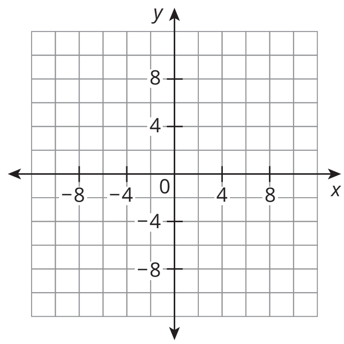
a. $f(x) = 2(x - 1)^2 - 1$



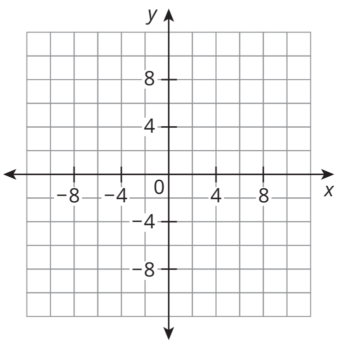
b. $f(x) = \frac{1}{2}(x + 2)^2 - 5$



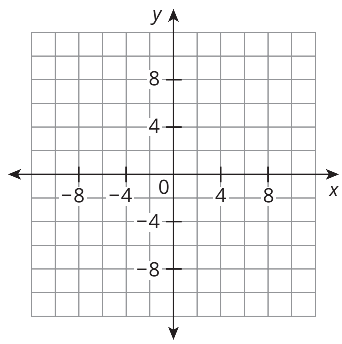
c. $f(x) = 4\left(x + \frac{1}{3}\right)^2 - 1$



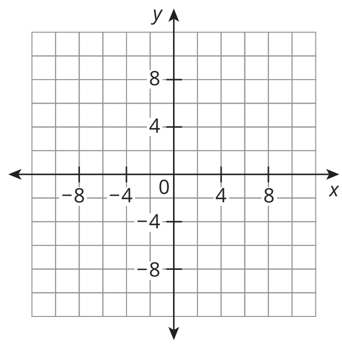
d. $f(x) = -3(x - 6)^2$



e. $f(x) = \frac{3}{4}(x + 5)^2 - \frac{2}{3}$



f. $f(x) = (x - 4)^2 - 2$



Prepare

Use the distributive property to determine each product.

1. $(x + 1)(x + 2)$

2. $(x + 4)(x - 5)$

3. $(2x - 3)(x - 4)$

4. $(x + 2)^2$

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3 Factoring and Completing the Square

OBJECTIVES

- Factor out the greatest common factor (GCF) of polynomials.
- Rewrite quadratic equations of the form $x^2 + bx$ in vertex form by completing the square.
- Factor quadratic trinomials to determine the roots of quadratic equations and to rewrite quadratic functions in forms that reveal different key characteristics.
- Demonstrate the reasoning behind the method of completing the square and use the method to determine the roots of quadratic equations of the form $ax^2 + bx + c$.

NEW KEY TERM

- completing the square

You have solved many different quadratic equations written as binomials.

How can you solve trinomial quadratic equations?

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Getting Started

LOL the GCF Again

In previous lessons, you multiplied two linear expressions to determine a quadratic expression. You have also rewritten quadratics in factored form.

You may remember that one way to factor an expression is to factor out the greatest common factor.

WORKED EXAMPLE

Consider the polynomial $3x + 15$. You can factor out the greatest common factor of the two terms, 3.

$$3x + 15 = 3x + 3(5)$$

$$= 3(x + 5)$$

$$3x + 15 = 3(x + 5)$$

1. Factor out the greatest common factor for each polynomial, when possible.

   a. $4x + 12$

   b. $x^2 - 5x$

   c. $3x^2 - 9x - 3$

   d. $-x - 7$

   e. $2x - 11$

   f. $5x^2 - 10x + 5$

ACTIVITY 3.1

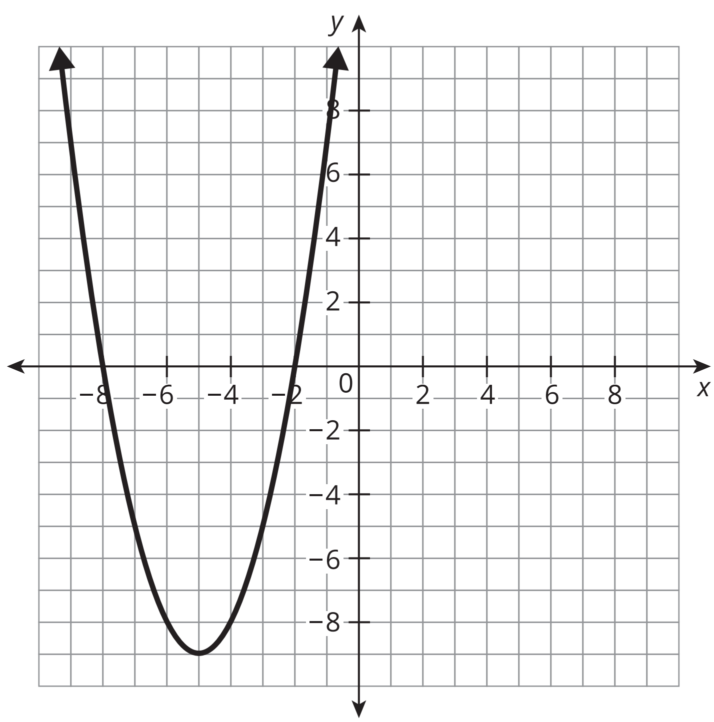
Factoring Trinomials

You have used special products—the difference of two squares and perfect square trinomials—to rewrite trinomials in factored form. In this activity, you will rewrite trinomials that are not special products in factored form.

1. Consider the equation $y = x^2 + 10x + 16$.

   a. Use the graph to identify the roots of the equation.

   b. Rewrite the original equation in factored form.



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Let's consider a strategy to factor a trinomial without graphing.

You can use a multiplication table to factor trinomials.

WORKED EXAMPLE

Factor the trinomial $x^2 + 10x + 16$.
Start by writing the leading term (x^2) and the constant term (16) in the table.

$\cdot$		
	x^2	
		16

Determine the two factors of the leading term and write them in the table.

$\cdot$	x	
x	x^2	
		16

Determine the factor pairs of the constant term. The factors of 16 are (1)(16), (2)(8), and (4)(4). Experiment with factors of the constant term to determine the pair whose sum is the coefficient of the middle term, 10.

$\cdot$	x	8
x	x^2	$8x$
2	$2x$	16

The sum of $2x$ and $8x$ is $10x$.
So, $x^2 + 10x + 16 = (x + 2)(x + 8)$.

2. Explain why the other factor pairs for $c = 16$ do not work.

3. Use the Worked Example to factor each trinomial.

a. $x^2 + 17x + 16$

$\cdot$		
	x^2	
		16

b. $x^2 + 6x - 16$

$\cdot$		
	x^2	
		-16

c. $x^2 - 6x - 16$

$\cdot$		
	x^2	
		-16

4. Factor each trinomial.

a. $x^2 + 5x - 24$

b. $x^2 - 3x - 28$

5. Consider the two examples shown.

Destiny

$2x^2 - 3x - 5$

$\cdot$	x	1
$2x$	$2x^2$	$2x$
-5	$-5x$	-5

$2x^2 - 3x - 5 = (2x - 5)(x + 1)$

Gabriela

$2x^2 + 3x - 5$

$\cdot$	x	-1
$2x$	$2x^2$	$-2x$
5	$5x$	-5

$2x^2 + 3x - 5 = (2x + 5)(x - 1)$

a. Compare the two given trinomials. What is the same and what is different about the values of *a*, *b*, and *c*?

b. Compare the factored form of each trinomial. What do you notice?

Remember . . .
The standard form of a quadratic equation is a trinomial in the form $y = ax^2 + bx + c$.

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6. Choose from the list to write the correct factored form for each trinomial.

a. $x^2 + 5x + 4 =$ ________________
$x^2 - 5x + 4 =$ ________________
$x^2 + 3x - 4 =$ ________________
$x^2 - 3x - 4 =$ ________________

- $(x + 1)(x - 4)$
- $(x + 1)(x + 4)$
- $(x - 1)(x + 4)$
- $(x - 1)(x - 4)$

b. $2x^2 + 7x + 3 =$ ________________
$2x^2 - 7x + 3 =$ ________________
$2x^2 + 5x - 3 =$ ________________
$2x^2 - 5x - 3 =$ ________________

- $(2x - 1)(x - 3)$
- $(2x - 1)(x + 3)$
- $(2x + 1)(x + 3)$
- $(2x + 1)(x - 3)$

c. $x^2 + 7x + 10 =$ ________________
$x^2 - 7x + 10 =$ ________________
$x^2 + 3x - 10 =$ ________________
$x^2 - 3x - 10 =$ ________________

- $(x - 2)(x + 5)$
- $(x + 2)(x + 5)$
- $(x - 2)(x - 5)$
- $(x + 2)(x - 5)$

7. Analyze the signs of each quadratic expression written in standard form and the operations in the binomial factors in Question 6. Then, complete each sentence with a phrase from the box.

the same	different	both positive	both negative
	one positive and one negative		

a. If the constant term is positive, then the operations in the binomial factors are ________________.

b. If the constant term is positive and the middle term is positive, then the operations in the binomial factors are ________________.

c. If the constant term is positive and the middle term is negative, then the operations in the binomial factors are ________________.

d. If the constant term is negative, then the operations in the binomial factors are ________________.

e. If the constant term is negative and the middle term is positive, then the operations in the binomial factors are
________________________________.

f. If the constant term is negative and the middle term is negative, then the operations in the binomial factors are
________________________________.

8. Factor each quadratic expression.

a. $x^2 + 8x + 15 =$ ________________

$x^2 - 8x + 15 =$ ________________

$x^2 + 2x - 15 =$ ________________

$x^2 - 2x - 15 =$ ________________

b. $x^2 + 10x + 24 =$ ________________

$x^2 - 10x + 24 =$ ________________

$x^2 + 2x - 24 =$ ________________

$x^2 - 2x - 24 =$ ________________

9. Hannah, Madison, and Ana were asked to factor the trinomial $15 + 2x - x^2$.

Hannah

$15 + 2x - x^2$

$(5 - x)(3 + x)$

Madison

$15 + 2x - x^2$

$(5 - x)(3 + x)$

$(x - 5)(x + 3)$

Ana

$15 + 2x - x^2$

$-x^2 + 2x + 15$

$-(x^2 - 2x - 15)$

$-(x - 5)(x + 3)$

Who's correct? Explain how the correct student(s) determined the factors. For the student(s) who is/are not correct, state why and make the correction.

Maria and Luis were working together to factor the trinomial $4x^2 + 22x + 24$. They first noticed that there was a greatest common factor and rewrote the trinomial as

$$2(2x^2 + 11x + 12).$$

Next, they considered the factor pairs for $2x^2$ and the factor pairs for 12.

$2x^2$: $(2x)(x)$
12: $(1)(12)$
$(2)(6)$
$(3)(4)$

Maria listed out all the possible combinations.

$2(2x + 1)(x + 12)$
$2(2x + 12)(x + 1)$

$2(2x + 2)(x + 6)$
$2(2x + 6)(x + 2)$

$2(2x + 3)(x + 4)$
$2(2x + 4)(x + 3)$

Luis immediately eliminated four out of the six possible combinations because the terms of one of the linear expressions contained common factors.

$2(2x + 1)(x + 12)$
$\cancel{2(2x + 12)(x + 1)}$

$\cancel{2(2x + 2)(x + 6)}$
$\cancel{2(2x + 6)(x + 2)}$

$2(2x + 3)(x + 4)$
$\cancel{2(2x + 4)(x + 3)}$

10. Explain Luis's reasoning. Then, circle the correct factored form of $4x^2 - 22x + 24$.

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ACTIVITY 3.2

Solving Quadratic Equations by Factoring

You have used properties of equality to solve equations in the forms shown.

$$y = x^2 + d$$
$$y = (x - c)^2$$
$$y = a(x - c)^2$$
$$y = a(x - c)^2 + d$$

Let's consider strategies to solve quadratics in the form $y = ax^2 + bx + c$ using the factoring strategies you just learned.

WORKED EXAMPLE

You can calculate the roots for the quadratic equation $x^2 - 4x = -3$.

$$x^2 - 4x = -3$$
$$x^2 - 4x + 3 = -3 + 3$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$

$(x - 3) = 0$	and	$(x - 1) = 0$
$x - 3 + 3 = 0 + 3$	and	$x - 1 + 1 = 0 + 1$
$x = 3$	and	$x = 1$

Think about . . .

What is the connection between the Worked Example and determining the roots from factored form, $y = a(x - r_1)(x - r_2)$?

1. Consider the Worked Example. Why is 3 added to both sides in the first step?

Remember . . .

The zero product property states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

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2. Determine each student's error, then solve each equation correctly.

Abby

$x^2 + 6x = 7$
$x(x + 6) = 7$
$x = 7$ and $x + 6 = 7$
$x = 1$

William

$x^2 + 5x + 6 = 6$
$(x + 2)(x + 3) = 6$
$x + 2 = 6$ and $x + 3 = 6$
$x = 4$ and $x = 3$

3. Use factoring to solve each quadratic equation, when possible.

a. $x^2 - 8x + 12 = 0$

b. $x^2 - 5x - 24 = 0$

c. $x^2 + 10x - 75 = 0$

d. $x^2 - 11x = 0$

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e. $x^2 + 8x = -7$

f. $x^2 - 5x = 13x - 81$

g. $\frac{2}{3}x^2 - \frac{5}{6}x = 0$

h. $f(x) = x^2 + 10x + 12$

Think about . . .

What efficiency strategies did you use to solve linear equations with fractional coefficients?

4. Describe the different strategies and reasoning that Angel and Huyen used to solve $4x^2 - 25 = 0$.

Angel

$$4x^2 - 25 = 0$$
$$4x^2 = 25$$
$$x^2 = \frac{25}{4}$$
$$x = \pm\sqrt{\frac{25}{4}}$$
$$x = \pm\frac{5}{2}$$

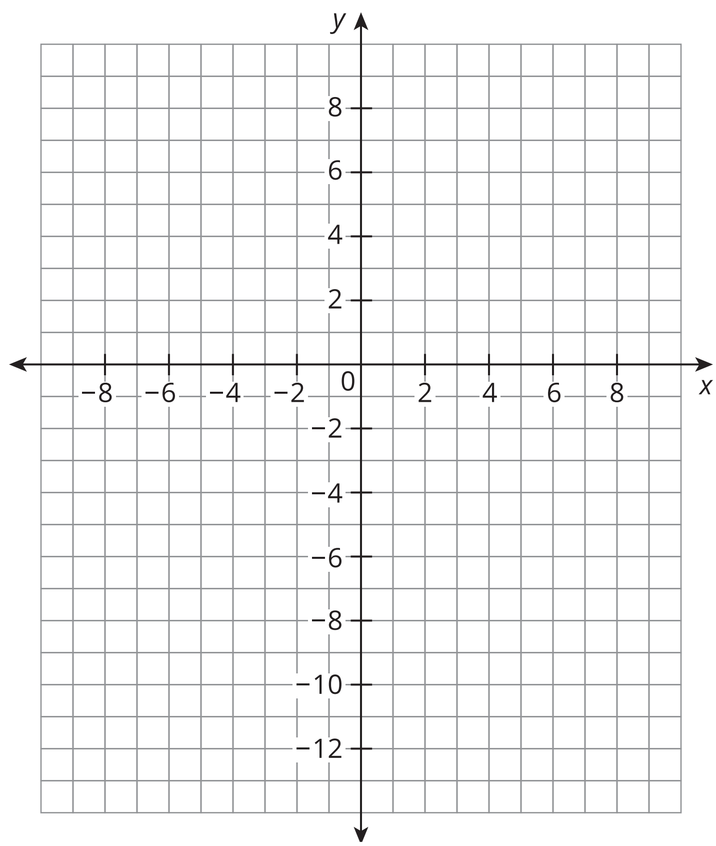
Huyen

$$4x^2 - 25 = 0$$
$$(2x - 5)(2x + 5) = 0$$
$$2x - 5 = 0 \text{ and } 2x + 5 = 0$$
$$2x = 5 \qquad 2x = -5$$
$$x = \frac{5}{2} \text{ and } \qquad x = -\frac{5}{2}$$

ACTIVITY
3.3 Completing the Square

When you cannot factor a quadratic function, does that mean it does not have zeros?

1. Consider the quadratic equation $y = x^2 + 10x + 12$.
   Use technology to graph the equation and then sketch it on the coordinate plane. Does this function have zeros? Explain your reasoning.



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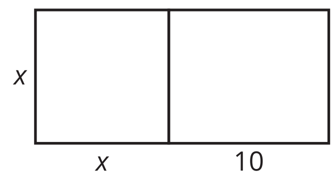
The quadratic function you graphed has zeros but cannot be factored, so you must consider another method for calculating its zeros. You can use your understanding of the relationship among the coefficients of a perfect square trinomial to construct a procedure to solve any quadratic equation.

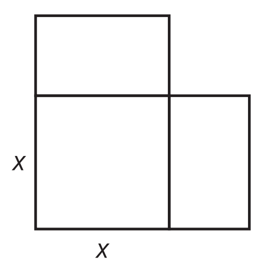
Previously, you factored trinomials of the form $a^2 + 2ab + b^2$ as the perfect square $(a + b)^2$. This knowledge can help you develop a procedure to solve any quadratic equation.

2. The expression $x^2 + 10x$ can be represented geometrically, as shown. Write the area of each rectangle within the diagram.



3. This figure can now be modified into the shape of a square by splitting the second rectangle in half and rearranging the pieces.

   a. Complete the side length labels for the split rectangle and write the area of each piece within the diagram.



Ask Yourself . . .
Why do you divide the second rectangle in half?

   b. Do the two figures represent the same expression? Explain your reasoning.

Review the definition of **represent** in the Academic Glossary.

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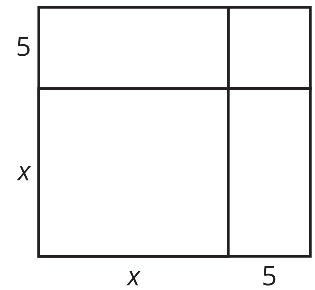
c. Complete the figure to form a square. Label the area of the piece you added.



d. Add the term *representing* the additional area to the original expression. What is the new expression?

e. Factor the new expression.

The process you just worked through is a method known as *completing the square*. **Completing the square** is a process for writing a quadratic expression in vertex form which then allows you to solve for the zeros.

4. Draw a model to complete the square for each expression. Then, factor the resulting trinomial.

a. $x^2 + 8x$

b. $x^2 + 5x$

5. Analyze your work in Question 4.

   a. Explain how to complete the square on an expression of the form $x^2 + bx$, where b is an integer.

   b. Describe how the coefficient of the middle term, b, is related to the constant term, c, in each trinomial you wrote in Question 4.

6. Use the descriptions you provided in Question 5 to determine the unknown second or third term to make each expression a perfect square trinomial. Then, write the expression as a binomial squared.

   a. $x^2 - 8x +$ ______________ $=$ ______________

   b. $x^2 + 5x +$ ______________ $=$ ______________

   c. $x^2 -$ ______________ $+ 100 =$ ______________

   d. $x^2 +$ ______________ $+ 144 =$ ______________

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ACTIVITY
3.4

Completing the Square to Determine Roots

So far, you have considered quadratic equations that can be rewritten by completing the square or factoring a trinomial.

You can use the completing the square method to determine the roots of a quadratic equation that cannot be factored.

WORKED EXAMPLE

Determine the roots of the equation $x^2 + 10x + 12 = 0$.

Isolate $x^2 + 10x$. You can complete the square and rewrite this as a perfect square trinomial.

$$x^2 + 10x + 12 - 12 = 0 - 12$$
$$x^2 + 10x = -12$$

Determine the constant term that would complete the square. Add this term to both sides of the equation.

$$x^2 + 10x + \_\_\_ = -12 + \_\_\_$$
$$x^2 + 10x + 25 = -12 + 25$$
$$x^2 + 10x + 25 = 13$$

Factor the left side of the equation.

$$(x + 5)^2 = 13$$

Determine the square root of each side of the equation.

$$\sqrt{(x + 5)^2} = \pm\sqrt{13}$$
$$x + 5 = \pm\sqrt{13}$$

Set the factor of the perfect square trinomial equal to each square root of the constant. Solve for x.

$x + 5 = \sqrt{13}$ and $x + 5 = -\sqrt{13}$

$x = -5 + \sqrt{13}$ and $x = -5 - \sqrt{13}$

$x \approx -1.39$ and $x \approx -8.61$

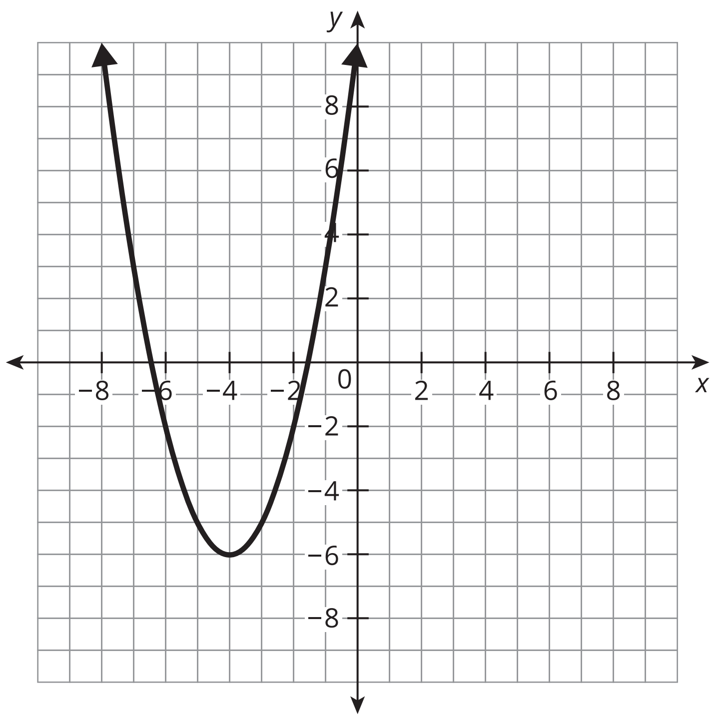
The roots are approximately (-1.39) and (-8.61).

Ask Yourself . . .
How was equality of the equation maintained through the completing the square process?

1. Consider the equation $y = x^2 + 8x + 10$.

   a. Use this method to determine the roots of the equation. Show your work.

   b. Use your work to label the zeros on the graph of the function $f(x) = x^2 + 8x + 10$.



2. Determine the roots of each equation by completing the square.

   a. $x^2 - 6x + 4 = 0$

   b. $x^2 - 12x + 6 = 0$

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ACTIVITY 3.5

Rewriting a Quadratic in Vertex Form

You can identify the axis of symmetry and the vertex of any quadratic function written in standard form by completing the square.

WORKED EXAMPLE

Consider the equation $y = ax^2 + bx + c$.

Step 1: $$y - c = ax^2 + bx$$

Step 2: $$y - c = a\left(x^2 + \frac{b}{a}x\right)$$

Step 3: $$y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right)$$

Step 4: $$y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

Step 5: $$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

Notice that the a-value was factored out before completing the square!

1. Explain why $a\left(\frac{b}{2a}\right)^2$ was added to the left side of the equation in Step 3.

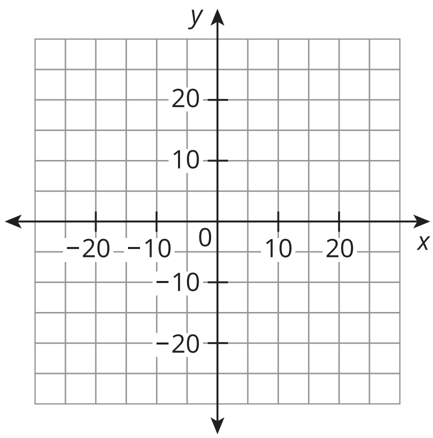
2. Given a quadratic function in the form $y = ax^2 + bx + c$:

   a. Identify the axis of symmetry.

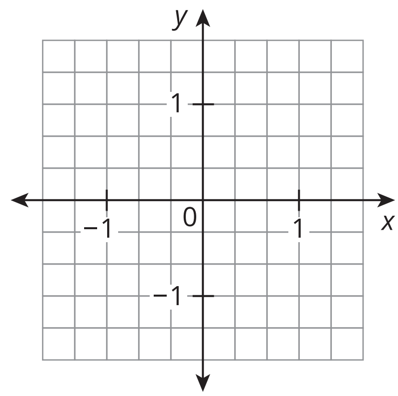
   b. Identify the location of the vertex.

3. Rewrite each quadratic equation in vertex form. Then, identify the zeros and sketch a graph of each function. Write the zeros in terms of the axis of symmetry and the parabola.

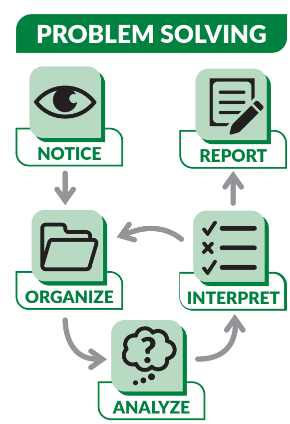
a. $y = x^2 + 8x - 9$



b. $y = 3x^2 + 2x - 1$



4. A ball is thrown straight up from 4 feet above the ground with a velocity of 32 feet per second. The height of the ball over time can be modeled with the function $h(t) = -16t^2 + 32t + 4$. What is the maximum height of the ball?



5. Malik is fencing in a rectangular plot outside of his back door so that he can let his dogs out to play. He has 60 feet of fencing and only needs to place it on three sides of the rectangular plot because the fourth side will be bound by his house. What dimensions should Malik use for the plot so that the maximum area is enclosed? What is the maximum area? Draw a diagram to support your work.

Ask Yourself . . .
Did you complete all the steps in the problem-solving model?

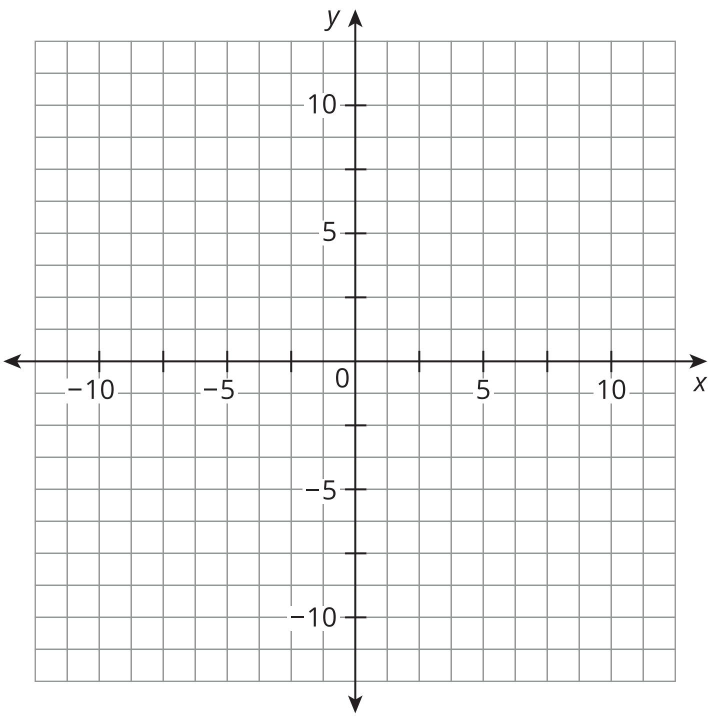
Talk the Talk

Play It Again

1. Consider the quadratic equation $y = x^2 - 4x - 5$.

   a. Rewrite the equation in factored form and vertex form.

   b. Graph the function. Identify the vertex, x- and y-intercepts, and the axis of symmetry. Then, explain how these are evident in each form of the equation.



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Lesson 3 Assignment

Write

Describe the process to solve a quadratic equation by factoring.

Remember

- Completing the square is a process for writing a quadratic expression in vertex form, which then allows you to solve for the zeros.
- Given a quadratic equation in the form $y = ax^2 + bx + c$, the vertex of the function is located at $x = \frac{-b}{2a}$ and $y = c - \frac{b^2}{4a}$.

Practice

1. Solve each equation.

   a. $0 = x^2 - 7x - 18$

   b. $x^2 + 10x = 39$

   c. $0 = x^2 - 10x + 12$

   d. $2x^2 + 4x = 0$

   e. $3x^2 - 22x + 7 = 0$

2. Determine the roots of the equation $y = x^2 + 9x + 3$. Check your solutions.

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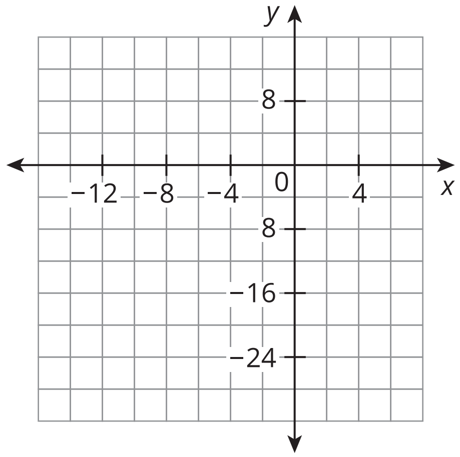
Lesson 3 Assignment

3. Consider the equation $y = 2x^2 + 10x - 8$.

 a. Graph the equation.



 b. Use your graph to estimate the solutions to the equation. Explain how you determined your answer.

 c. Two students completed the square to determine the solutions to this equation. Their work is shown. Who is correct? Explain your reasoning.

Lesson 3 Assignment

Student 1

$$y = 2x^2 + 10x - 8$$

$$2x^2 + 10x - 8 = 0$$

$$2x^2 + 10x = 8$$

$$2x^2 + 10x + 25 = 8 + 25$$

$$(2x + 5)^2 = 33$$

$$\sqrt{(2x + 5)^2} = \pm\sqrt{33}$$

$$2x + 5 = \pm\sqrt{33}$$

$$x = \frac{-5 \pm \sqrt{33}}{2}$$

$$x < -5.372 \text{ and } x < 0.372$$

Student 2

$$y = 2x^2 + 10x - 8$$

$$2x^2 + 10x - 8 = 0$$

$$\frac{2x^2 + 10x - 8}{2} = 0$$

$$x^2 + 5x = 4$$

$$x^2 + 5x + \frac{25}{4} = 4 + \frac{25}{4}$$

$$\left(x + \frac{5}{2}\right)^2 = \frac{41}{4}$$

$$\sqrt{\left(x + \frac{5}{2}\right)^2} = \pm\sqrt{\frac{41}{4}}$$

$$x + \frac{5}{2} = \pm\frac{\sqrt{41}}{2}$$

$$x = \frac{-5 \pm \sqrt{41}}{2}$$

$$x < -5.702 \text{ and } x < 0.702$$

d. Compare the solutions. Identify what the student who got the correct answer did that allowed them to correctly complete the square.

e. Write a statement about the value of the coefficient of the x^2-term before you can complete the square.

4. Determine the roots of the equation $y = 3x^2 + 24x - 6$. Check your solutions.

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Lesson 3 Assignment

5. Determine the roots and the location of the vertex of $y = x^2 + 20x + 36$. Write the zeros in terms of the axis of symmetry and the parabola.

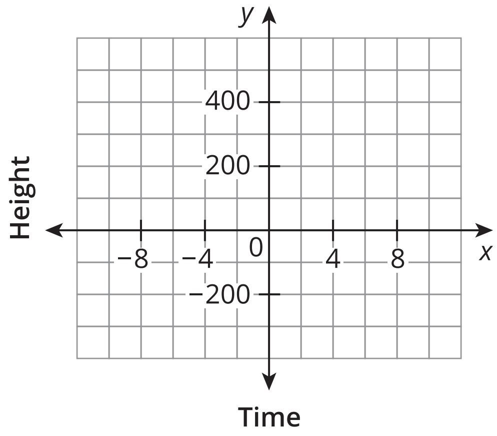
Prepare

A bucket of paint falls from the top of a skyscraper that is 564 feet tall.

1. Write a quadratic function to represent the height of the can over time.

2. Use technology to graph the function.

3. How many seconds will it take for the can of paint to hit the ground?



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4 The Quadratic Formula

OBJECTIVES

- Derive the quadratic formula from a quadratic equation written in standard form.
- Connect the quadratic formula to a graphical representation.
- Use the discriminant of the quadratic formula to determine the number of roots or zeros.
- Use the quadratic formula to determine roots and zeros.

NEW KEY TERMS

- quadratic formula
- discriminant

You know several strategies to solve quadratic equations, depending on the structure of the equation.

Is there a single strategy that will work to solve any quadratic equation?

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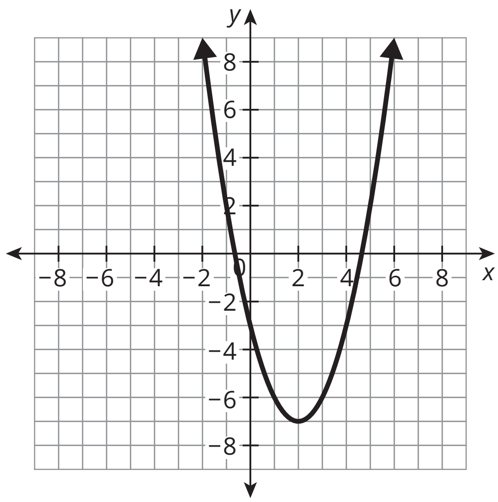
Getting Started

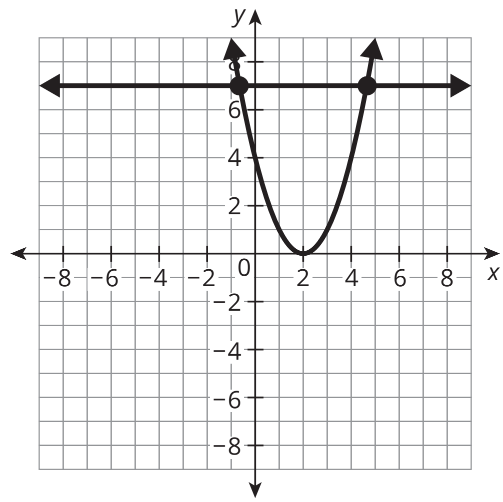
Really, They Aren't the Same

Consider each graph.

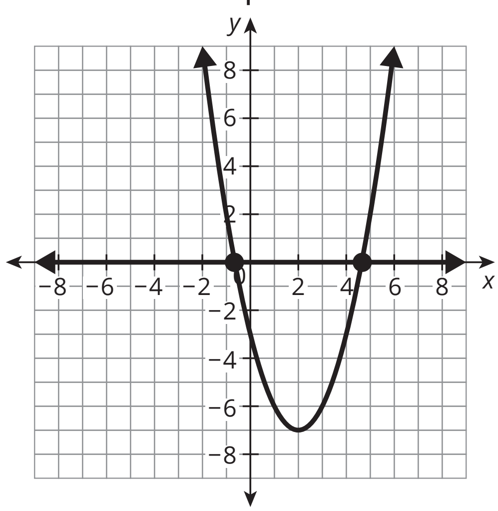
Graph A



Graph B



Graph C



1. Match each equation to its corresponding graph.

   a. $(x - 2)^2 - 7 = 0$

   b. $y = (x - 2)^2 - 7$

   c. $(x - 2)^2 = 7$

2. How do each of the graphs show solutions? How are the solutions related to the axis of symmetry?

ACTIVITY
4.1 Introducing the Quadratic Formula

In the previous lesson, you took the standard form of a quadratic equation, $y = ax^2 + bx + c$, and rewrote it in vertex form, $y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$, by completing the square to determine the vertex and axis of symmetry for graphing purposes.

Now, let's take the standard form of a quadratic equation, $y = ax^2 + bx + c$, and set $y = 0$ to determine the roots. You can complete the square to solve for the x-values when $y = 0$.

WORKED EXAMPLE

Write the equation in standard form with $y = 0$.	$ax^2 + bx + c = 0$
Complete the square.	$ax^2 + bx = -c$ $x^2 + \frac{b}{a}x = -\frac{c}{a}$ $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$ $\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$
Rewrite the right side of the equation.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$ $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$
Now that the equation is written in the form $(x - c)^2 = q$, the square root can be taken on each side. Extract the square roots. Solve for x. These are the roots for the quadratic equation in the standard form, $ax^2 + bx + c = 0$.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$ $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$

This approach can be taken one step further and rewritten as a single fraction.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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Think About . . .

So really, the quadratic formula is just taking the standard form of a quadratic equation and isolating, or solving for, x.

This equation is known as the *quadratic formula*. The **quadratic formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c represent real numbers and $a \neq 0$.

You can use the quadratic formula to determine the zeros of the function $f(x) = -4x^2 - 40x - 99$.

WORKED EXAMPLE

Rewrite the function as an equation to be solved for x when $y = 0$.	$-4x^2 - 40x - 99 = 0$
Determine the values of a, b, and c.	$a = -4, b = -40, c = -99$
Substitute the values into the quadratic formula.	$x = \frac{-(-40) \pm \sqrt{(-40)^2 - 4(-4)(-99)}}{2(-4)}$
Perform operations to rewrite the expression.	$x = \frac{40 \pm \sqrt{1600 - 1584}}{-8}$ $x = \frac{40 \pm \sqrt{16}}{-8}$ $x = \frac{40 \pm 4}{-8}$ $x = \frac{40 + 4}{-8}$ and $x = \frac{40 - 4}{-8}$ $x = \frac{44}{-8}$ and $x = \frac{36}{-8}$ $x = -5.5$ and $x = -4.5$
Interpret the solution.	The zeros of the function $f(x) = -4x^2 - 40x - 99$ are $x = -5.5$ and $x = -4.5$.

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The baseball team has a new promotional activity to encourage fans to attend games: launching free T-shirts! They can launch a T-shirt in the air with an initial velocity of 91 feet per second from $5\frac{1}{2}$ feet off the ground (the height of the team mascot).

A T-shirt's height can be modeled with the quadratic function $h(t) = -16t^2 + 91t + 5.5$, where t is the time in seconds and $h(t)$ is the height of the launched T-shirt in feet. They want to know how long it will take for a T-shirt to land back on the ground after being launched (if no fans grab it before then!).

Ask Yourself . . .
What would a sketch showing the height of the T-shirt over time look like?

1. Why does it make sense to use the quadratic formula to solve this problem?

2. Use the quadratic formula to determine how long it will take for a T-shirt to land back on the ground after being launched.

Ask Yourself . . .
Do you think an exact solution or an approximate solution is more appropriate for this context?

3. Classify your solutions as rational or irrational.

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ACTIVITY 4.2

Interpreting the Quadratic Formula Graphically

You used the quadratic formula to solve a quadratic equation. Let's connect the quadratic formula to the graph. Remember, the quadratic formula can be written to show two roots.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \qquad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

How do these roots, $x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$ relate to the graph?

1. What does the first term of each root represent on the graph?

2. The second term of each root represents the distance the root lies from the axis of symmetry. Why is the second term in each root the same except for the sign?

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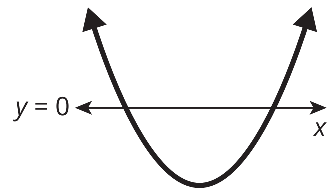
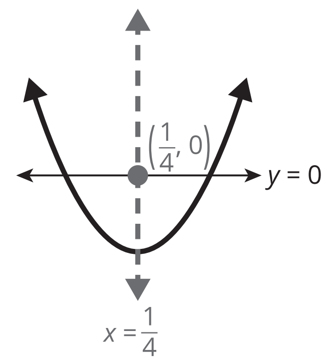
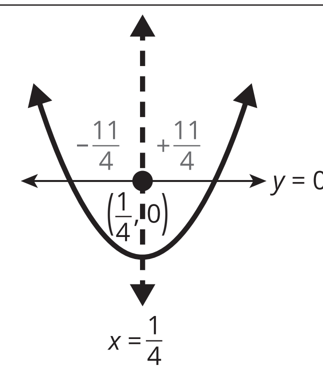
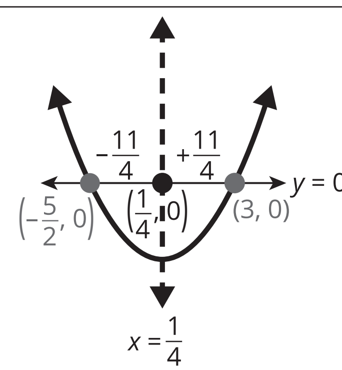
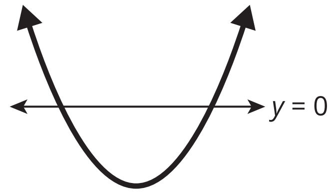
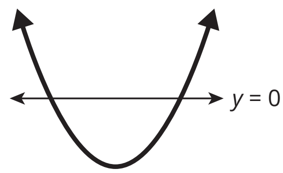
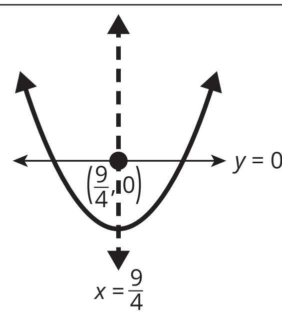
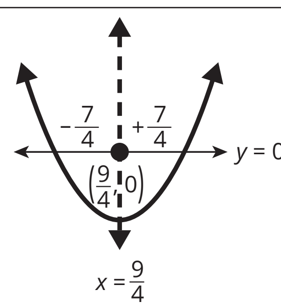
Let's analyze how the structure of the quadratic formula is evident in the graphical representation of the zeros of a quadratic function.

WORKED EXAMPLE

Consider this graphical representation to determine the real roots of the quadratic equation $y = 2x^2 - x - 15$.

<table>
<tr><th>Steps</th><th>Graph</th></tr>
<tr><td>Set y equal to zero and identify the values of a, b, and c.
$0 = 2x^2 - x - 15$
$a = 2 \quad b = -1 \quad c = -15; \quad a > 0$</td><td>
</td></tr>
<tr><td>Identify the axis of symmetry and label the point where it intersects $y = 0$.
$x = \frac{-(-1)}{2(2)} = \frac{1}{4}$</td><td>
</td></tr>
<tr><td>Identify the distance from the axis of symmetry to the parabola along $y = 0$.
$+\frac{\sqrt{b^2 - 4ac}}{2a} =$
$\frac{\sqrt{(-1)^2 - 4(2)(-15)}}{2(2)} = \frac{\sqrt{121}}{4} = \frac{11}{4}$
$-\frac{\sqrt{b^2 - 4ac}}{2a} = -\frac{11}{4}$</td><td>
</td></tr>
<tr><td>Identify the roots and label them on the graph.
$\frac{1}{4} + \frac{11}{4} = \frac{12}{4} = 3$
$\frac{1}{4} - \frac{11}{4} = -\frac{10}{4} = -\frac{5}{2}$</td><td>
</td></tr>
<tr><td colspan="2">The real roots of $y = 2x^2 - x - 15$ are $x = 3$ and $x = -\frac{5}{2}$.</td></tr>
</table>

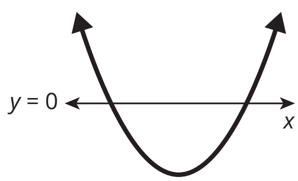
3. Repeat the process to determine the real roots of the equation $y = 2x^2 - 9x + 4$.

Steps	Graph
a. Let $y = 0$ and identify the values of *a*, *b*, and *c*.	
b. Identify the axis of symmetry and label the point where it intersects the *x*-axis.	
c. Identify the distance from the axis of symmetry to the parabola along $y = 0$ and label the distance on both sides.	
d. Identify the roots and label them on the graph.	
e. Summarize.	

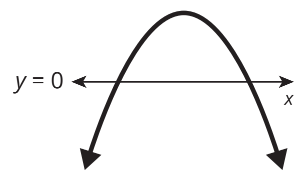
The graphs of quadratic equations are parabolas that have either an absolute maximum or an absolute minimum. A quadratic equation with two real roots crosses the *x*-axis in two places. A quadratic equation with a double real root, or one unique real root, touches the *x*-axis but does not cross it.

Quadratic Equations

With Two Real Roots

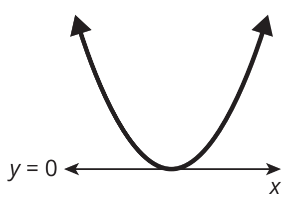


$$y = ax^2 + bx + c$$
$$a > 0$$

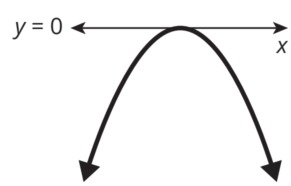


$$y = ax^2 + bx + c$$
$$a < 0$$

With Double Real Roots



$$y = ax^2 + bx + c$$
$$a > 0$$



$$y = ax^2 + bx + c$$
$$a < 0$$

4. Draw and label the following components on each graph in terms of the equation $y = ax^2 + bx + c$.

   a. vertex

   b. axis of symmetry

   c. intersection of the *x*-axis and line of symmetry

   d. the distance represented by the expression $+\frac{\sqrt{b^2 - 4ac}}{2a}$

   e. the distance represented by the expression $-\frac{\sqrt{b^2 - 4ac}}{2a}$

   f. each root

ACTIVITY
4.3 Using the Quadratic Formula

Let's analyze the structure of the quadratic formula and examine common student mistakes. Consider Michael's work.

1. Michael is determining the exact zeros for $f(x) = x^2 - 14x + 19$. His work is shown.

   a. Identify the error Michael made when determining the zeros.

   b. Determine the correct zeros of the function.

Michael

$f(x) = x^2 - 14x + 19$

$a = 1, b = -14, c = 19$

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(19)}}{2(1)}$$

$$x = \frac{14 \pm \sqrt{196 - 76}}{2}$$

$$x = \frac{14 \pm \sqrt{120}}{2}$$

$$x = \frac{14 \pm \sqrt{30 \cdot 4}}{2}$$

$$x = \frac{14 \pm 2\sqrt{30}}{2}$$

$$x = 7 \pm 2\sqrt{30}$$

"Leave the solutions in exact form" means not to estimate any radical values with rounded decimals.

2. Use the quadratic formula to determine the zeros for each function given. Leave the solutions in exact form and classify them as rational or irrational.

   a. $f(x) = -2x^2 - 3x + 7$

   b. $r(x) = -3x^2 + 19x - 7$

3. Olivia is solving the quadratic equation $x^2 - 7x - 8 = 3$. Her work is shown.

Olivia

$x^2 - 7x - 8 = 3$

$a = 1, b = -7, c = -8$

$x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-8)}}{2(1)}$

$x = \dfrac{7 \pm \sqrt{49 + 32}}{2}$

$x = \dfrac{7 \pm \sqrt{81}}{2}$

$x = \dfrac{7 \pm 9}{2}$

$x = \dfrac{7 + 9}{2}$ or $x = \dfrac{7 - 9}{2}$

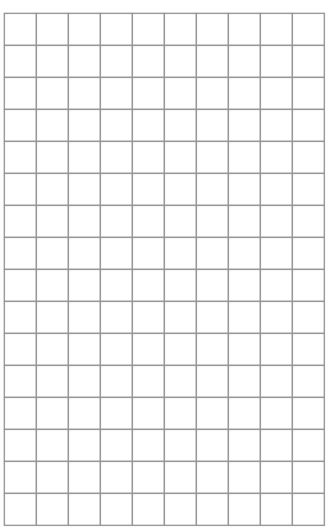
$x = \dfrac{16}{2} = 8$ or $x = \dfrac{-2}{2} = -1$

The roots are 8 and -1.

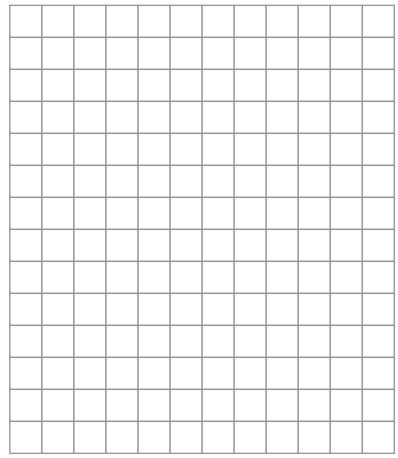
a. Identify Olivia's error.

b. Use the quadratic formula correctly to determine the solution to Olivia's quadratic equation. Classify the solutions as rational or irrational.

c. Use technology to graph each side of the original quadratic equation $x^2 - 7x - 8 = 3$. Sketch your graph. Then, interpret the meaning of the intersection points.



d. Next, rewrite the given quadratic equation so that one side of the equation is equal to zero. Use technology to graph each side of the quadratic equation. Sketch your graph. Interpret the meaning of the intersection points.



Think about . . .
What does it mean to determine the solutions to a quadratic equation? What does it mean to determine the roots of a quadratic equation?

e. Compare the *x*-values of the intersection points from part (c), the *x*-values of the intersection points in part (d), and the solutions using the quadratic formula. What do you notice?

4. Use the quadratic formula to determine the zeros for each function. Round the solutions to the nearest hundredth and classify them as either rational or irrational.

   a. $f(x) = 2x^2 + 10x - 1.02$

   b. $h(x) = 3x^2 - 11x - 2$

Ask Yourself . . .
How can you use graphs to support your reasoning?

5. Reflect on the different quadratic equations you have solved so far in this lesson.

   a. How many real roots does each quadratic equation in this lesson have?

   b. Do all quadratic equations have two real roots? Explain why or why not.

   c. Do you think that a quadratic function could have no real roots? Explain why or why not.

   d. Could a quadratic equation have more than two real roots? Explain why or why not.

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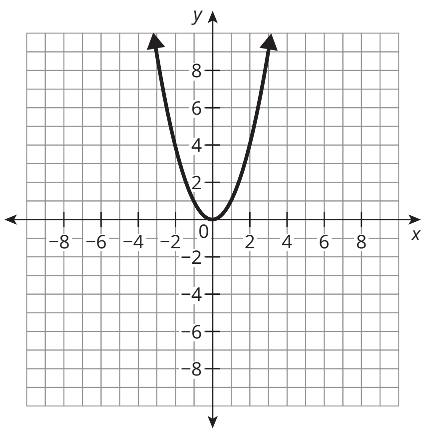
ACTIVITY 4.4

Making Sense of the Discriminant

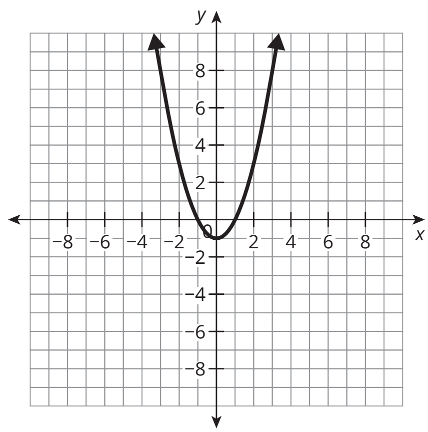
A quadratic function can have one unique real zero, two real zeros, or at times, no real zeros. Let's investigate how the quadratic formula can inform you about different types of zeros.

Consider three quadratic equations and their graphs.

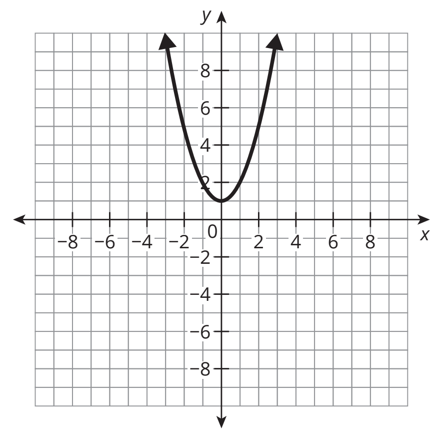
$y = x^2$



$y = x^2 - 1$



$y = x^2 + 1$



Think about . . .

You have analyzed many graphs of quadratic equations. What do you know about the roots of a quadratic equation that touches, but does not intersect, the x-axis or intersects the x-axis at two points?

1. Use the quadratic formula to solve each quadratic equation. Show your work.

2. What do you notice about the relationship between the number of real roots, the graph, and the results of substituting the values of a, b, and c into the radicand of the quadratic formula?

Because this portion of the formula "discriminates" the number of real zeros, or roots, it is called the **discriminant.**

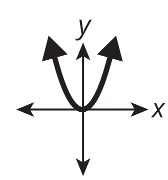
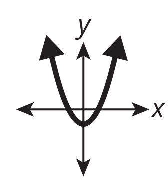
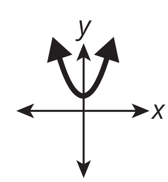
3. Using the discriminant, write an inequality to describe when a quadratic function has each solution.

   a. no real roots/zeros

   b. one unique real root/zero

   c. two unique real roots/zeros

The table shown summarizes the types of solutions for any quadratic equation or function.

		Interpretation of the Solutions		
Equation/ Function	**Solutions**	**Number of Unique Real Zeros**	**Number of x-Intercepts**	**Sketch**
$f(x) = x^2$	$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(0)}}{2(1)}$ $= \frac{0 \pm \sqrt{0}}{2}$ $= 0 \pm \sqrt{0}$	1	1	
$g(x) = x^2 - 1$	$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(-1)}}{2(1)}$ $= \frac{0 \pm \sqrt{4}}{2}$ $= 0 \pm 1$	2	2	
$h(x) = x^2 + 1$	$x = \frac{-0 \pm \sqrt{0^2 - (4)(1)(1)}}{2(1)}$ $= \frac{0 \pm \sqrt{-4}}{2}$	0	0	

Every quadratic equation with real coefficients has either 2 real roots or 0 real roots. However, when a graph of a quadratic equation has 1 x-intercept, the equation *still* has 2 real roots. In this case, the 2 real roots are considered a double root.

4. Use the discriminant to determine the number of real roots for each equation. Then, solve for the roots/zeros.

   a. $y = 2x^2 + 12x - 2$

   b. $0 = 2x^2 + 12x + 20$

   c. $y = x^2 + 12x + 36$

   d. $y = 3x^2 + 7x - 20$

   e. $y = 4x^2 - 9$

   f. $0 = 9x^2 + 12x + 4$

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Talk the Talk

Show Me the Ways

1. Determine the real roots of the quadratic equation $y = 2x^2 + 4x - 6$ using the four methods you learned in this topic.

Factoring	Completing the Square
Using the quadratic formula	Graphing

Ask Yourself . . .
Which tool or strategy is most efficient?

Lesson 4 Assignment

Write

How can you determine the types of solutions when using the quadratic formula?

Remember

The **quadratic formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c represent real numbers and $a \neq 0$.

On the graph of a quadratic function, $\frac{\pm \sqrt{b^2 - 4ac}}{2a}$ is the distance from $\left(-\frac{b}{2a}, 0\right)$ to each root.

Practice

The formula shown can be used to calculate the distance, s, an object travels in t seconds. In this formula, u represents the initial velocity, and a represents a constant acceleration. Use this formula to answer each question.

$$S = ut + \frac{1}{2}at^2$$

1. Javier is driving 48 miles per hour and is starting to merge onto the highway, so he must increase his speed. He accelerates at a rate of 7 miles per hour for several seconds.

   a. Substitute the initial velocity and constant acceleration into the formula to write an equation to represent the distance Javier travels.

   b. Use the quadratic formula to determine the roots of the equation. What do the roots represent in the context of the problem situation? Explain your reasoning.

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Lesson 4 Assignment

2. Daniela is driving her car 32 miles per hour when she passes Koda's house. She then accelerates at a rate of 3 miles per hour for several minutes until she passes the movie theater. Daniela knows that the movie theater is 2.9 miles from Koda's house.

   a. Substitute the initial velocity, constant acceleration, and distance into the formula to write an equation representing the distance Daniela travels.

   b. Use the quadratic formula to determine the roots of the equation you wrote in part (a). What do the roots represent in the context of the problem situation? Explain your reasoning.

3. Use the discriminant to determine the number of real roots for each equation. Then, solve the quadratic equations with real roots.

   a. $4x^2 + 8x - 12 = 0$

   b. $x^2 + 2x - 10 = 0$

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Lesson 4 Assignment

c. $9x^2 - 12x + 4 = 0$

d. $3x^2 - 4 = 0$

e. $3x^2 + 2x - 2 = 0$

f. $x^2 - 3x + 5 = 0$

Prepare

1. Determine a linear regression model that best models the data.

x	y
1	32
2	35
3	34
4	35
5	39
6	38
7	40
8	42
9	41

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5 Using Quadratic Functions to Model Data

OBJECTIVES

- Use a quadratic function to model data.
- Interpret characteristics of a quadratic function in terms of a problem situation.
- Use graphs of quadratic functions to make estimations and predictions.

You know how to model data with regression.

How can you determine whether a quadratic regression model may best model the data?

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Getting Started

That Might Be a Bad Idea. . .

A 12-ounce can of soda was put into a freezer. The table shows the volume of the soda in the can, measured at different temperatures.

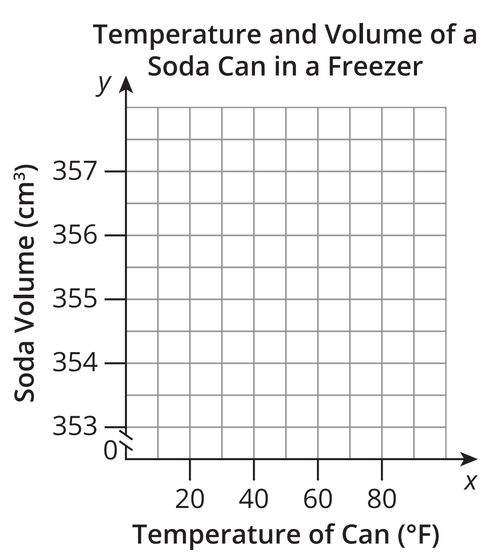
The first step of the modeling process is to notice and wonder. What do you notice about the data? Is there a question it brings to mind that you wonder about?

Temperature of Can (°F)	Soda Volume cm^3
68.0	355.51
50.0	354.98
42.8	354.89
39.2	354.88
35.6	354.89
32.0	354.93
23.0	355.13
14.0	355.54

1. Describe the data distribution.

2. Create a scatterplot of the data. Sketch the plot of points on the coordinate plane shown.

The second step of the modeling process is to organize and mathematize. The scatterplot is a way to organize the data.



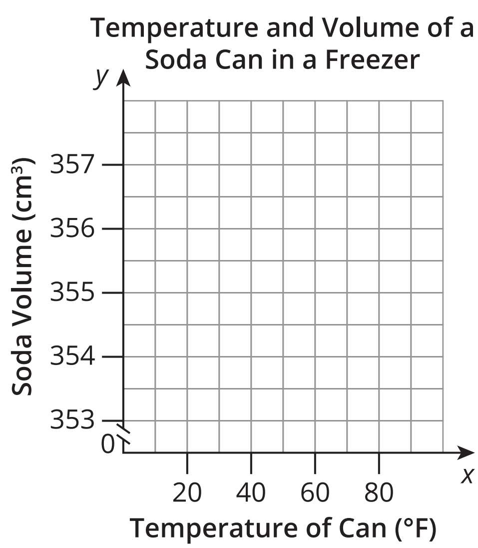
ACTIVITY 5.1

Using Quadratic Functions to Model Data

Let's continue to analyze the data and make some estimations and predictions about the volume of soda at different temperatures.

1. Use technology to calculate the regression model that best models the data in the previous activity. Sketch the graph of the regression model on the coordinate plane on which you created your scatterplot. Explain why the regression model best represents the data.

You can mathematize the data by modeling it with an appropriate regression model.



2. State the domain and range of your function. How do they compare to the domain and range of this problem situation?

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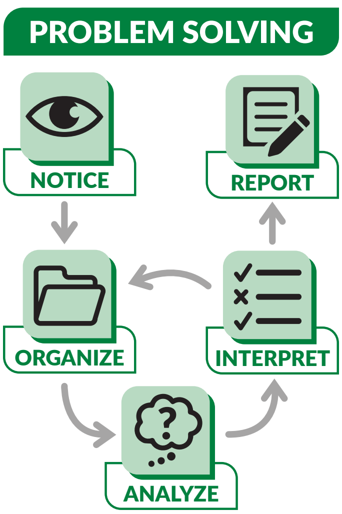
The third step of the modeling process is to predict and analyze, and the fourth step is to test and interpret. These questions focus on these two steps of the process.

3. Use the regression model to answer each question.

   a. Determine the y-intercept and interpret its meaning in terms of this problem situation.

   b. Determine the x-intercepts and interpret the meaning of each in terms of this problem situation.



4. Estimate the volume of the soda can when the temperature is:

   a. 20°F.

   b. 60°F.

5. Estimate the temperature when the volume is 356 cm^3.

6. Write a summary of the problem situation, your model of the solution, and any limitations of your model.

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ACTIVITY 5.2 Analyzing a Quadratic Model

A local police department is offering special classes for interested high school students this summer. Destiny decides to enroll in an introductory forensic science class. On the first day, Dr. Jackson tells Destiny's class that crime scenes often involve speeding vehicles which leave skid marks on the road as evidence. Taking into account the road surface, weather conditions, the percent grade of the road, and vehicle type, they use the function

$$f(s) = 0.034s^2 + 0.96s - 26.6$$

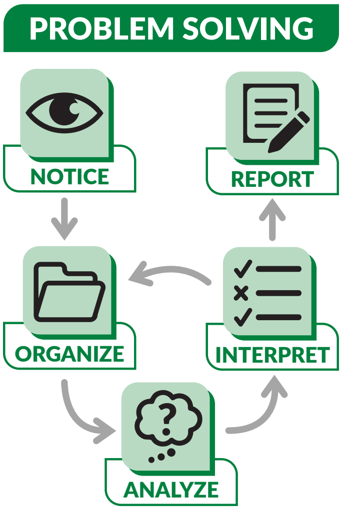
to determine the length in feet of skid marks left by a vehicle based on its speed, s, in miles per hour.

1. Complete the table based on $f(s)$. Label the column titles with the independent and dependent quantities and their units.

25	
30	
45	
55	
60	
75	
90	
100	
110	

2. According to the table, what are the domain and range for the problem situation?

3. Graph the table values and sketch the graph of $f(s)$ on the grid. Label the axes.



Ask Yourself . . .
How do these data differ from the data in the table?

During another class period, Dr. Jackson takes Destiny's class to a mock crime scene to collect evidence.

4. One piece of evidence is a skid mark that is 300 feet long.

   a. Use the graph to estimate the speed of the vehicle that created this skid mark. Explain your process.

   b. Use the equation to estimate the speed of the vehicle that created this skid mark. Show your work.

   c. Does the exact speed seem reasonable for the problem situation?

5. At a second mock crime scene the skid mark is 800 feet long.

   a. Use the equation to predict the speed of the vehicle that created this skid mark. Show your work.

   b. Does the exact speed seem reasonable for the problem situation?

6. Write a report about the length of skid marks left by vehicles and vehicle speeds. Discuss possible factors that would affect the length of the skid marks left by a vehicle and what effect these factors would have on the graph of $f(s)$.

Talk the Talk

Celebrate the Centenarians!

Using data from the US Census Bureau, the table lists the number of Americans, in thousands, who lived to be over 100 years old for the specified years.

Year	Number of Americans (thousands)
1980	32
1990	37
2000	50
2010	53
2020	80

1. Does the data appear to be linear or quadratic?

2. Use graphing technology to determine the regression model of best fit and calculate the correlation coefficient.

3. Estimate how many Americans were over 100 years old in 2005.

4. Predict the number of Americans that will live to be over 100 years old in the year 2030.

5. Predict when there will be 90,000 Americans that will live to be over 100 years old.

Ask Yourself . . .
How can you use quadratic regression models in everyday life?

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Lesson 5 Assignment

Write

How would you determine if a quadratic function is the best model for a set of data?

Remember

You can use quadratic regression models to represent real-world situations.

Practice

1. The table shows the percent of public schools with internet access from 1994 to 2005. (The largest growth years are shown in the table.)

Year	Percent of Public Schools with Internet Access
1994	35
1995	50
1996	65
1997	78
1998	89
1999	95
2000	98
2001	99
2002	99
2003	100
2005	100

   a. Predict whether a linear or quadratic regression model will best fit the data. Explain your reasoning.

   b. Create a scatterplot of the data.

   c. Does your scatterplot change or support your answer to part (a)? Explain your reasoning.

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Lesson 5 Assignment

2. Fernando thinks a quadratic regression model is the best fit the data.

   a. Calculate the quadratic regression model of the data. Do you agree with Fernando? Explain your reasoning.

   b. Year 2004 is missing from the data. Calculate the percent of public schools with internet access in 2004. Does your answer make sense in terms of the problem situation? Explain your reasoning.

   c. Calculate the percent of public schools with internet access in 2020. Does your answer make sense in terms of the problem situation? Explain your reasoning.

   d. What are the x-intercepts and what do they mean in terms of the problem situation?

Lesson 5 Assignment

e. In what year does the percent of public schools with internet access begin to decline? Explain how you determined your answer.

f. Do you think it is likely that the percent of public schools with internet access will decline? Explain your reasoning.

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Name: ____________________________

Solving Quadratic Equations

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents I am **building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill.**

I can demonstrate an understanding of the standards in the *Solving Quadratic Equations* topic by:

TOPIC 3: *Solving Quadratic Equations*	Beginning of Topic	Middle of Topic	End of Topic
solving quadratic equations by the most efficient method: inspection, properties of equality and square roots, graphically, factoring, completing the square, or using the quadratic formula.	◯	◯	◯
recognizing special products, including the difference of two squares, and using them to factor polynomials.	◯	◯	◯
factoring trinomials, when possible, including perfect square trinomials.	◯	◯	◯
completing the square to rewrite a quadratic expression in vertex form.	◯	◯	◯
explaining that the quadratic formula is derived by completing the square of the standard form of a quadratic equation $y = ax^2 + bx + c$.	◯	◯	◯
understanding that the quadratic formula, $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$, represents axis of symmetry plus or minus the distance to the parabola.	◯	◯	◯
using technology, determining the quadratic regression model that provides a reasonable fit to data to estimate solutions and making predictions for real-world problems.	◯	◯	◯

continued on the next page

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TOPIC 3 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Solving Quadratic Equations* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 3 SUMMARY

Solving Quadratic Equations Summary

NEW KEY TERMS

- principal square root
- roots [raíces]
- double root [raíz doble]
- zero product property [propiedad del producto cero]
- completing the square
- quadratic formula [fórmula cuadrática]
- discriminant [discriminante]

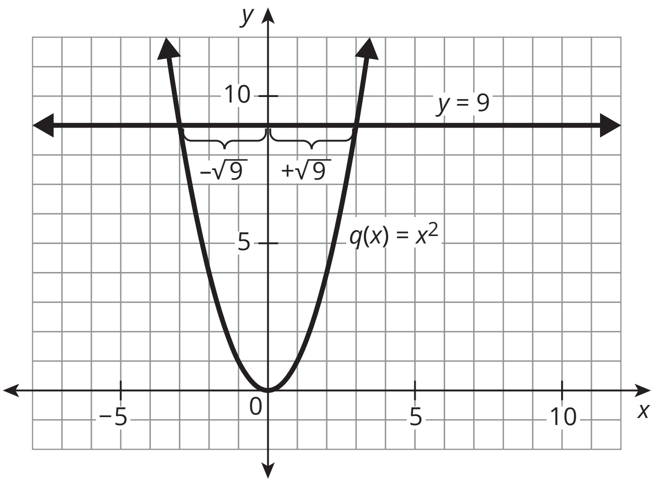
LESSON 1 Representing Solutions to Quadratic Equations

A quadratic function is a function of degree 2 because the greatest power for any of its terms is 2. This means that it has at most 2 zeros, or at most 2 solutions, at $y = 0$.

The two solutions of a quadratic function can be represented as square roots of numbers. Every positive number has two square roots, a positive square root, which is also called the **principal square root**, and a negative square root. To solve the equation $x^2 = 9$, take the square root of both sides of the equation.

$$\sqrt{x^2} = \sqrt{9}$$
$$x = \pm 3$$

You can solve $x^2 = 9$ on a graph by finding the points of intersection between $y = x^2$ and $y = 9$. The solutions are both 3 units from the axis of symmetry, $x = 0$.



The x-intercepts of a graph of a quadratic function are called the *zeros* of the quadratic function. The zeros are called the **roots** of the quadratic equation.

The quadratic function $q(x) = x^2$ has two solutions at $y = 0$; therefore, it has 2 zeros: $x = +\sqrt{0}$ and $x = -\sqrt{0}$. These two zeros of the function, or roots of the equation, are the same number, 0, so the function $q(x) = x^2$ is said to have a **double root**.

When you encounter solutions that are not perfect squares, you can either determine the approximate value of the radical or rewrite it in an equivalent radical form.

To approximate a square root, determine the perfect square that is closest to, but less than, the given value and the perfect square that is closest to, but

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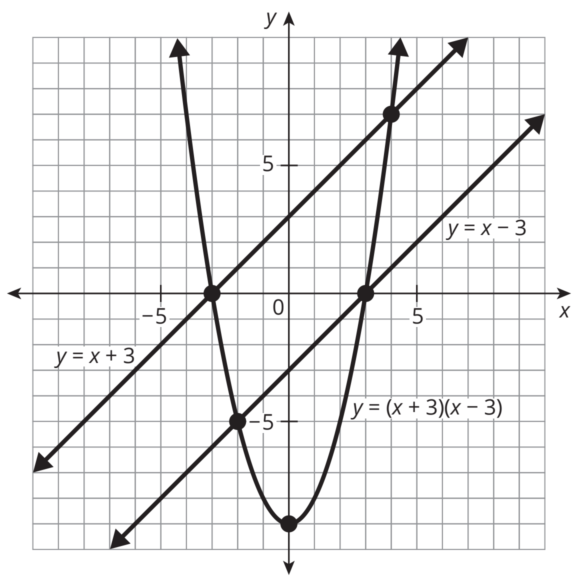
greater than, the given value. You can use these square roots to approximate the square root of the given number.

For example, the approximate value of $\sqrt{40}$ falls between $\sqrt{36}$, or 6, and $\sqrt{49}$, or 7. Since $6.3^2 = 39.69$ and $6.4^2 = 40.96$, the approximate value of $\sqrt{40}$ is 6.3.

To rewrite a square root in equivalent radical form, first rewrite the product of the radicand to include any perfect square factors. Then, extract the square roots of those perfect squares.

$$\begin{aligned}\sqrt{27} &= \sqrt{9 \cdot 3}\\ &= \sqrt{9} \cdot \sqrt{3}\\ &= 3\sqrt{3}\end{aligned}$$

A quadratic function written in factored form is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$. In factored form, r_1 and r_2 represent the x-intercepts of the graph of the function. The x-intercepts of the graph of the quadratic function $f(x) = ax^2 + bx + c$ and the zeros of the function are the same as the roots of the equation $ax^2 + bx + c = 0$.



To write a quadratic function in factored form, first determine the zeros of the function $f(x) = x^2 - 9$, set the trinomial expression equal to 0, and solve for x.

$$\begin{aligned}0 &= x^2 - 9\\ 9 &= x^2\\ \sqrt{9} &= \sqrt{x^2}\\ \pm 3 &= x\end{aligned}$$

You can then use the zeros to write the function in factored form, $f(x) = (x + 3)(x - 3)$.

The **zero product property** states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero. You can see from the graph that the zeros of the function $f(x) = x^2 - 9$ occur where either $y = x + 3$ or $y = x - 3$ are zero.

LESSON 2 Solutions to Quadratic Equations in Vertex Form

The solutions to any quadratic equation are located on the parabola, equidistant from the axis of symmetry.

A quadratic function in vertex form $f(x) = a(x - h)^2 + k$ is translated horizontally h units, dilated vertically by the factor a, and translated vertically k units.

For the equation $y = (x - c)^2$, the solutions can be represented by $c \pm \sqrt{y}$.
For the equation $y = a(x - c)^2$, the solutions can be represented by $c \pm \sqrt{\frac{y}{a}}$.
For the equation $y = a(x - c)^2 + d$, the solutions can be represented by $c \pm \sqrt{\frac{y - d}{a}}$.

LESSON 3 Factoring and Completing the Square

You can factor trinomials by rewriting them as the product of two linear expressions.

For example, to factor the trinomial $x^2 + 10x + 16$, determine the factor pairs of the constant term. The factors of 16 are (1)(16), (2)(8), and (4)(4). Then, determine the pair whose sum is the coefficient of the middle term, 10.

	x	8
x	x^2	$8x$
2	$2x$	16

The sum of $2x$ and $8x$ is $10x$. So, $x^2 + 10x + 16 = (x + 2)(x + 8)$.

You can use factoring and the zero product property to solve quadratics in the form $y = ax^2 + bx + c$.

For example, you can solve the quadratic equation $x^2 - 4x = -3$.

$$\begin{aligned} x^2 - 4x &= -3 \\ x^2 - 4x + 3 &= -3 + 3 \\ x^2 - 4x + 3 &= 0 \\ (x - 3)(x - 1) &= 0 \end{aligned}$$

$$\begin{array}{rcr} (x - 3) = 0 & \text{or} & (x - 1) = 0 \\ x - 3 + 3 = 0 + 3 & \text{or} & x - 1 + 1 = 0 + 1 \\ x = 3 & \text{or} & x = 1 \end{array}$$

For a quadratic function that has zeros but cannot be factored, there is another method for solving the quadratic equation. **Completing the square** is a process for writing a quadratic expression in vertex form, which then allows you to solve for the zeros.

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For example, you can calculate the roots of the equation $x^2 - 4x + 2 = 0$.

Isolate $x^2 - 4x$.	$x^2 - 4x + 2 - 2 = 0 - 2$ $x^2 - 4x = -2$
Complete the square and rewrite this as a perfect square trinomial.	
Determine the constant term that would complete the square.	$x^2 - 4x + ? = -2 + ?$
Add this term to both sides of the equation.	$x^2 - 4x + 4 = -2 + 4$ $x^2 - 4x + 4 = 2$
Factor the left side of the equation.	$(x - 2)^2 = 2$
Determine the square root of each side of the equation.	$\sqrt{(x - 2)^2} = \sqrt{2}$ $(x - 2) = \pm\sqrt{2}$
Set the factor of the perfect square trinomial equal to each square root of the constant and solve for x.	$x - 2 = \sqrt{2}$ or $x - 2 = -\sqrt{2}$ $x = 2 + \sqrt{2}$ or $x = 2 - \sqrt{2}$ $x \approx 3.41$ or $x \approx 0.59$

The roots are approximately 3.41 and 0.59.

Completing the square can also be used to identify the axis of symmetry and the vertex of any quadratic function written in standard form.

When a function is written in standard form, $ax^2 + bx + c$, the axis of symmetry is $x = -\frac{b}{2a}$.

Given a quadratic equation in the form $y = ax^2 + bx + c$, the vertex of the function is located at $x = -\frac{b}{2a}$ and $y = c - \frac{b^2}{4a}$.

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LESSON
4

The Quadratic Formula

You can use the **quadratic formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c represent real numbers and $a \neq 0$.

For example, given the function $f(x) = 2x^2 - 4x - 3$ we can identify the values of a, b, and c.

$$a = 2; b = -4; c = -3$$

Then, we use the quadratic formula to solve.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{16 + 24}}{4}$$

$$x = \frac{4 \pm \sqrt{40}}{4}$$

$$x \approx \frac{4 + 6.325}{4} \approx 2.581 \quad \text{or}$$

$$x \approx \frac{4 - 6.325}{4} \approx -0.581$$

The roots are approximately 2.581 and (−0.581).

A quadratic function can have one real zero, two real zeros, or at times, no real zeros.

You can use the part of the quadratic formula underneath the square root symbol to identify the number of real zeros or roots. Because this portion of the formula "discriminates" the number of real zeros or roots, it is called the **discriminant**.

When the discriminant is positive, the quadratic has two real roots. If the discriminant is negative, the quadratic has no real roots. When the discriminant is 0, the quadratic has a double real root.

You can also use the discriminant to describe the nature of the roots. If the discriminant is a perfect square, then the roots are rational. If the discriminant is not a perfect square, then the roots are irrational.

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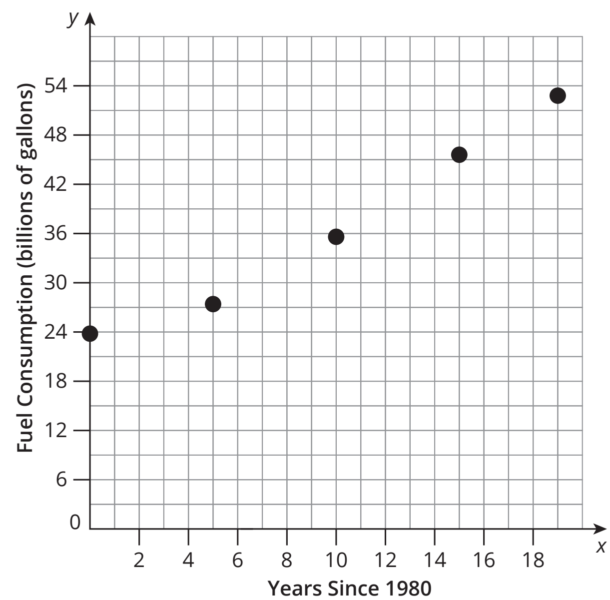
LESSON 5

Using Quadratic Functions to Model Data

Quadratic regression models can be used to represent real-world situations and make estimations and predictions.

For example, as vans, trucks, and SUVs have increased in popularity, the fuel consumption of these types of vehicles has also increased.

Years Since 1980	Fuel Consumption (billions of gallons)
0	23.8
5	27.4
10	35.6
15	45.6
19	52.8



The quadratic regression model that best fits the data is $y = 0.0407x^2 + 0.809x + 23.3$. The r^2-value for the quadratic regression fit is 0.996. Just as with linear and exponential regressions, the model can be used to make predictions for the data.

For example, you can predict the fuel consumption in the year 2020 by substituting $x = 40$ into the regression model.

$$y = 0.0407(40)^2 + 0.809(40) + 23.3$$
$$y \approx 121$$

In 2020, fuel consumption was about 121 billion gallons.

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