MODULE 1

Extending Linear Relationships

ALGEBRA II

TOPIC 1

Absolute value functions are V-shaped, like this skein of geese. (Did you know that a group of geese is called a skein in the sky, but a gaggle on the ground?)

Absolute Value Functions and Equations

INTRODUCTION LESSON

Introduction to the Problem-Solving Model and Learning Resources

OBJECTIVES

- Establish a community of learners.
- Discover learning resources available for this course.
- Describe and apply the problem-solving model through written explanations.

In previous math classes, you have analyzed patterns and relationships, learned about numbers and operations in base ten and fractions, measurement and data, and geometry.

What resources are available in this course to help you extend your mathematical thinking?

Getting Started

You Already Know A Lot

Each lesson in this book begins with a Getting Started that gives you the opportunity to use what you know about the world and what you have learned in previous math classes. You know a lot from a variety of learning experiences.

Think back to how you learn something new.

1. List three different skills that you recently learned. Then, describe why you wanted to learn that skill and the strategies that you used.

New Skill	Motivation to Learn the New Skill	Strategies I Used to Learn This Skill

Ask Yourself . . .
How do your strategies change based on what you are learning and what you already know about it?

Think About . . .
Listening well, cooperating with others, and appreciating different perspectives are essential life skills.

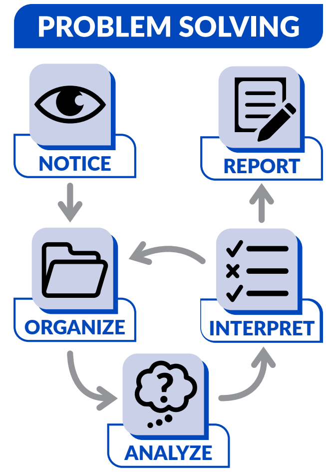
One learning strategy is to talk with your peers. In this course, you will work with your classmates to solve problems, discuss strategies, and learn together.

Compare and discuss your list with a classmate.

2. Which strategies do you have in common? Which strategies does your classmate have that you did not think of on your own?

Be prepared to share your list of learning strategies with the class.

ACTIVITY 1.1 Learning Resources



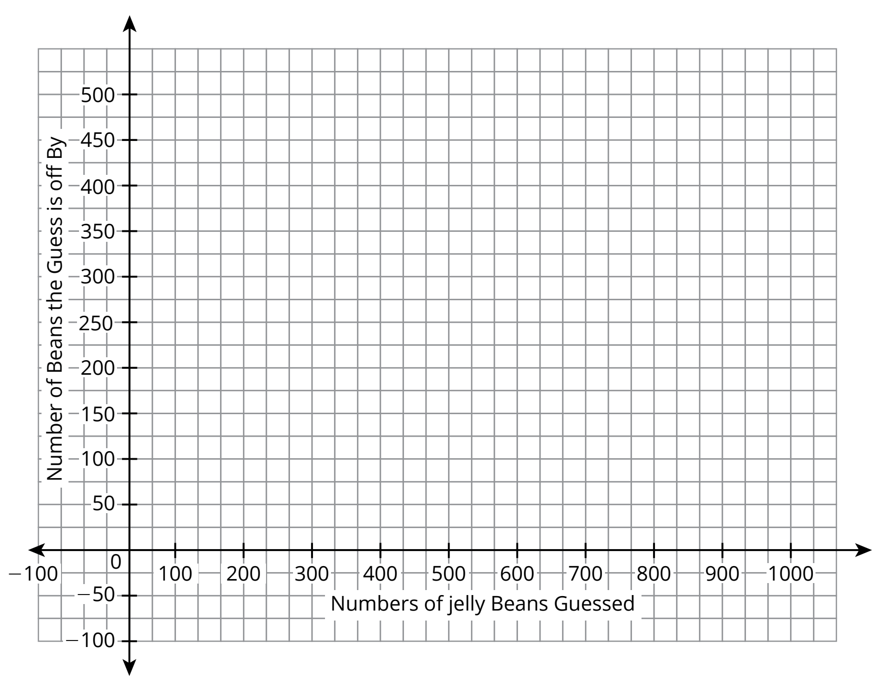
In this course, you will learn new math concepts by exploring and investigating ideas, reading, writing, and talking to your classmates. You will even learn by making mistakes with concepts you haven't mastered yet.

Let's practice exploring and investigating. You do not need to answer the question yet. You will solve the question as you work through the problem-solving model.

Ms. Chen judges the annual Jelly Bean Challenge at the summer fair. Every year, she encourages the citizens in her town to guess the number of jelly beans in a jar. She records each guess along with how far it is from the actual number of jelly beans in a table.

The table shows the first 6 guesses Ms. Chen records at the summer fair. Use the table to create a graph of the situation. How many jelly beans are in the jar? **Explain your reasoning**.

Number of Jelly Beans Guessed	Number of Jelly Beans the Guess is Off By
330	220
680	130
475	75
528	22
825	275
585	35



The Academic Glossary provides definitions of terms you will see throughout the course as you think, reason, and communicate your ideas. The Math Glossary provides the definitions of key terms in each lesson.

The Academic Glossary is your guide as you engage with the kind of thinking you do as you are learning the content.

1. Locate the phrase **Explain your reasoning** in the Academic Glossary in the Course Guide. What questions should you ask yourself as you determine the number of jelly beans in the jar?

2. What is a related word or phrase for **explain your reasoning**?

The problem-solving model provides a structure to help you become a better problem-solver.

Notice and Wonder

NOTICE

The first step in modeling a situation mathematically is to understand the problem, gather information, notice patterns, and formulate mathematical questions about what you notice.

Read through the questions to ask yourself for the first step of the problem-solving model. Use the *Problem-Solving Model Graphic Organizer* to answer the questions and show your thinking.

3. What do you notice about the table and graph?

4. Why do you think the first step of the problem-solving model is important? How will it help you when you solve problems?

Organize and Mathematize

ORGANIZE

The second step in the problem-solving model is to devise a plan. When devising a plan, you will organize your information and begin to represent it using mathematical notation.

Read through the questions to ask yourself for the second step of the problem-solving model. Use the *Problem-Solving Model Graphic Organizer* to answer the questions and show your thinking.

5. Describe your strategy to determine the number of jelly beans in the jar.

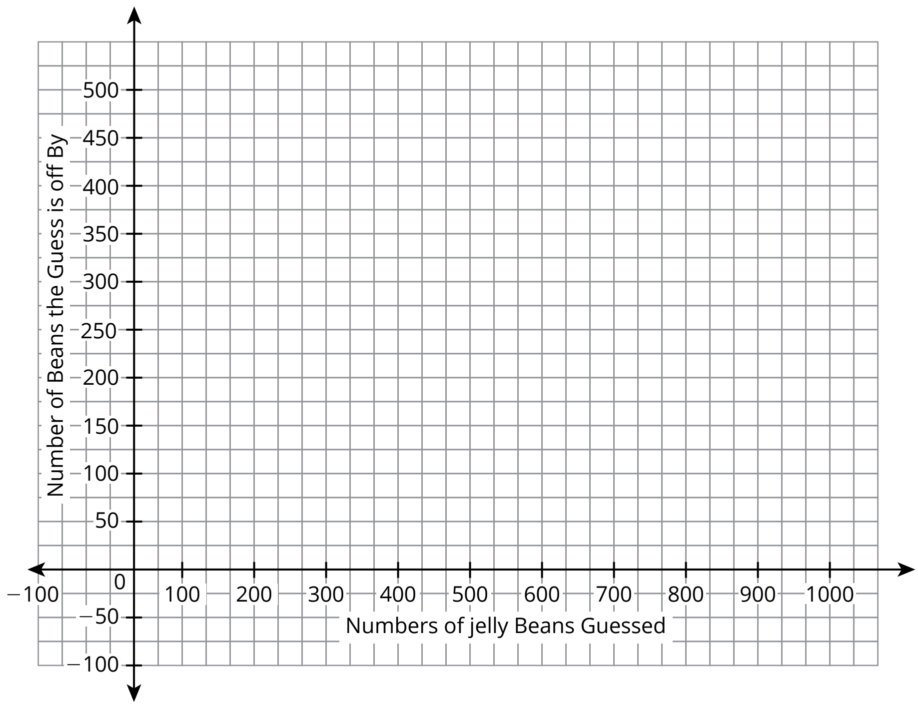
6. Why do you think the second step of the problem-solving model process is important? How can you use the questions to ask yourself to help develop a strategy to solve the problem?

Predict and Analyze

The third step in the modeling process involves carrying out your plan. As you carry out the plan you will complete operations, analyze mathematical results, make predictions, and extend patterns.

Read through the questions to ask yourself for the third step of the problem-solving model. Use the *Problem-Solving Model Graphic Organizer* to answer the questions and show your thinking.

7. Use your strategy to determine the number of jelly beans in the jar.



8. How can the questions to ask for the third step help you communicate your mathematical thinking?

Test and Interpret

The fourth step in the modeling process is to look back, interpret your results, and test your mathematical predictions in the real world. When your predictions are incorrect, or your results are not reasonable, you can revisit your mathematical work and make adjustments—or start all over!

Read through the questions to ask yourself for the fourth step of the problem-solving model. Use the *Problem-Solving Model Graphic Organizer* to answer the questions and show your thinking.

9. Ms. Chen uses a different jar. Can you apply the same reasoning to determine the number of jelly beans in the new jar?

10. How do the questions for the fourth step of the problem-solving model help you evaluate the reasonableness of your solution?

Report

The final step in the modeling process is to share your results.

11. How does listening to others help you learn mathematics?

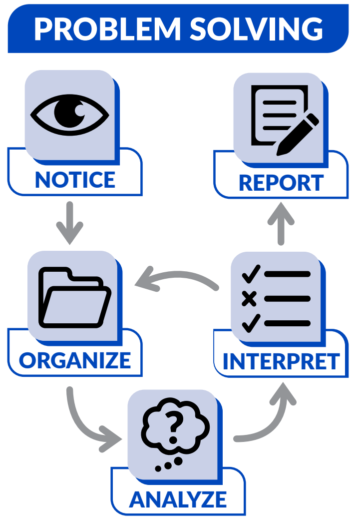
Locate the TEKS mathematical process standards in the Course Guide.

12. Which TEKS mathematical process standard(s) did you use to create a graph that represents the height of the liquid in the cup as the volume changes in Mario's cup?

Talk the Talk

You will see this symbol throughout the course to remind you to use the problem-solving model.

The Problem-Solving Model



In this lesson, you used a problem-solving model to solve a real-world problem. The basic steps of the problem-solving model are summarized in the diagram.

Summarize what is involved in each phase of this problem-solving model.

Notice and Wonder

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Organize and Mathematize

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Predict and Analyze

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Test and Interpret

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Report

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1 Defining Absolute Value Functions

OBJECTIVES

- Graph absolute value functions and identify their key attributes.
- Use graphs to solve absolute value equations.
- Verbally describe how absolute value functions can represent real-world situations.

KEY TERMS

- absolute value function

You know how to use graphs to solve linear, quadratic, and exponential equations.
How can you use a graph to solve an absolute value equation?

Getting Started

Absolute value is indicated with vertical bars: $|-4|$ is read as "the absolute value of -4."

Distance Is Always Positive

Recall the absolute value of a number is its distance from zero on the number line.

1. Follow your teacher's instructions to model each absolute value expression on the *x*-axis of a classroom coordinate plane. Rewrite each expression without the absolute value symbol.

   a. $|-2|$

   b. $|2|$

   c. $|1 - 2|$

   d. $|-3 - (-5)|$

   e. $|-2 \cdot 3|$

   f. $|0 \cdot 4|$

   g. $\left|\frac{12}{-3}\right|$

   h. $|8 \div (-4)|$

2. Write your observations about the absolute value expressions you and your classmates modeled on the number line.

3. Provide counterexamples to show why Catalina's statement is incorrect.

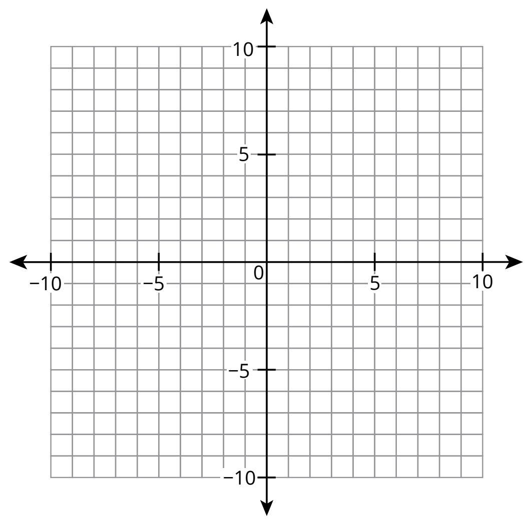
> **Catalina**
> Absolute values are always positive. So, $|a| = -a$ is not possible.

ACTIVITY 1.1 Graphs of Absolute Value Functions

Follow your teacher's instructions to model the functions $f(x) = x$ and $g(x) = |x|$ on the classroom coordinate plane with your classmates.

1. Record the coordinates of the plotted points for $f(x) = x$ in the table.

x	$f(x) = x$	$g(x) = \|x\|$
−9		
−6		
−4		
−1		
0		
3		
5		
8		



2. Change all the plotted points to model the function $g(x) = |x|$. In the table, record the coordinates of the new points for $g(x) = |x|$.

3. Describe how the points move from the graph of $f(x) = x$ to the graph of $g(x) = |x|$.

Remember . . .
The *domain* is the set of all input values in a relation. The *range* is the set of output values in a relation.

4. Graph the function $g(x) = |x|$. Use the graph to identify the key attributes of $g(x)$. Write the domain and range of the function as inequalities.

   a. Domain

   b. Range

   c. x- and y- intercepts

   d. Symmetry

   e. Vertex

   f. Maximum

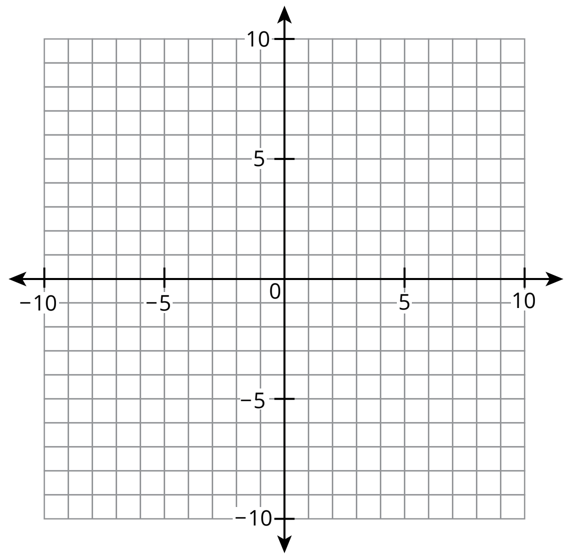
   g. Minimum

You graphed the parent *absolute value function*. An **absolute value function** is a function in the form of $g(x) = |x|$.

Next, consider the function $h(x) = -x$. Model this function on the classroom coordinate plane with your classmates.

5. Record the coordinates of the plotted points for $h(x) = -x$ in the table.

x	$h(x) = -x$	$j(x) = \|-x\|$
−9		
−6		
−4		
−1		
0		
3		
5		
8		

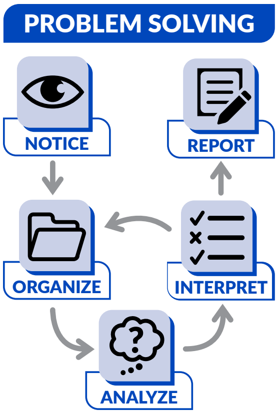


6. Change all the plotted points to model the function $j(x) = |-x|$. In the table, record the coordinates of the new points for $j(x) = |-x|$.

7. Describe how the points move from the graph of $h(x) = -x$ to the graph of $j(x) = |-x|$.

Use a straightedge to be precise when you graph.

8. Graph the function $j(x) = |-x|$. Compare this function with the function $g(x) = |x|$.



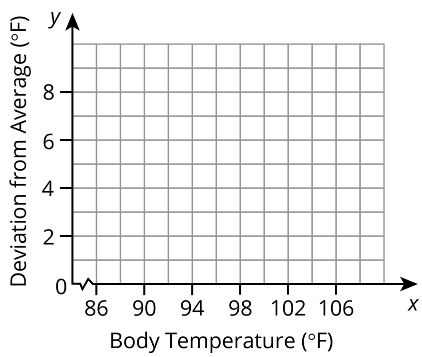
Use the problem-solving model whenever you see this icon.

ACTIVITY 1.2 Real-World Application of an Absolute Value Function

You can model a situation using an absolute value function.

It has been common knowledge that the average human internal body temperature is 98.6°F. But recently scientists have begun to revise this average downward: to around 97.5°F.

1. Sketch a graph that models the relationship between a person's internal body temperature, x, and its deviation from the average, y. **Explain how you constructed your sketch.** Then write an absolute value equation to represent the situation and the graph.



2. Consider the function you wrote to represent the situation.

   a. Write the domain and range of the function as inequalities.

   b. Compare the domain and range of the situation with the domain and range of the function. Explain your reasoning.

3. What does the vertex of your graph represent in terms of the situation?

4. Write the equation for the axis of symmetry for your function and describe what it means in terms of the situation.

5. Use the axis of symmetry to identify intervals of increase and decrease on the graph. Explain what these intervals mean in terms of the situation.

6. Although people's body temperatures vary depending on many factors, a deviation, y, of around 2.9°F from the average indicates a serious medical issue. Hypothermia can start at around 2.9°F less than the average, and a fever can start around 2.9°F more than the average.

   a. Graph the equation $y = 2.9$ on the coordinate plane in Question 1.

   b. What two equations can you write, without absolute values, to show the approximate internal body temperatures represented by fever and hypothermia? Explain your reasoning.

   c. Use the graph to write the solutions to the equations you wrote in part (b). Show your work.

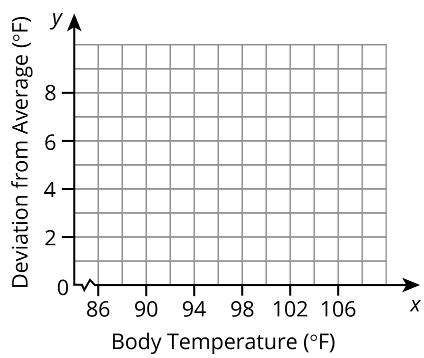
Talk the Talk

Hot Dogs

Dogs have an average internal body temperature of around 101.75°F.

1. Formulate an absolute value function to model the relationship between a dog's internal body temperature, x, and its deviation from the average.

2. Use graphing technology to sketch a graph of the function.



3. Use inequalities to write the domain and range of the function. Then, compare the domain and range of the situation with the domain and range of the function.

4. Use your sketch or graphing technology to identify additional key attributes of the function.

   a. Vertex

   b. Axis of symmetry

   c. Maximum or minimum value

   d. *x*- and *y*-intercepts

5. What is the body temperature of a dog who's temperature deviates 2.5 degrees from the average temperature?

Lesson 1 Assignment

Write

Describe how you can use a graph to solve an absolute value equation.

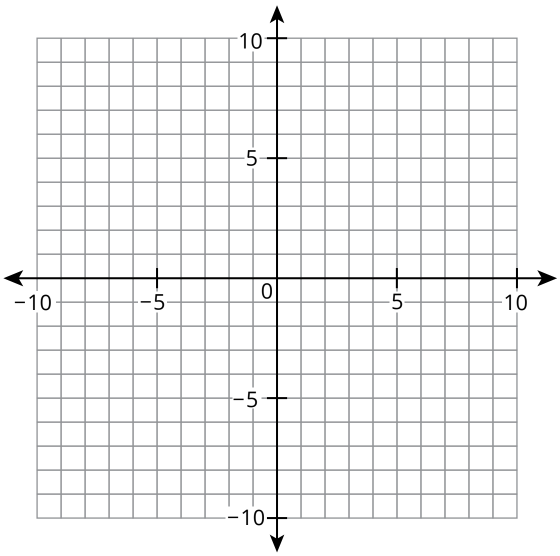
Remember

The parent absolute value function, $y = |x|$, is a V-shaped graph.

Practice

1. Graph each absolute value function. Write the domain and range using inequalities and identify the vertex, intercepts, axis of symmetry, and any maximum or minimum values.

   a. $f(x) = \frac{1}{2}|x|$



   Domain:

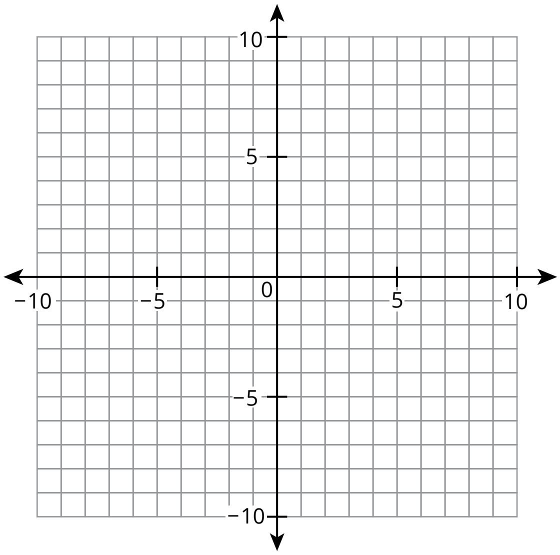
   Range:

   Axis of symmetry:

   Maximum:

   Minimum:

   b. $g(x) = -|x| + 3$



   Domain:

   Range:

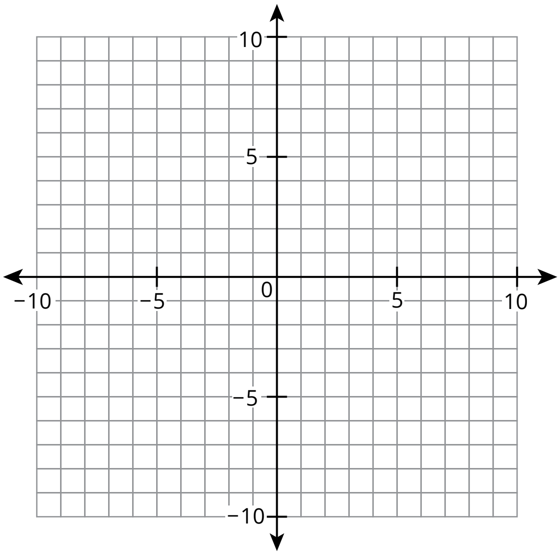
   Axis of symmetry:

   Maximum:

   Minimum:

Lesson 1 Assignment

c. $h(x) = |x - 4| + 2$



Domain:

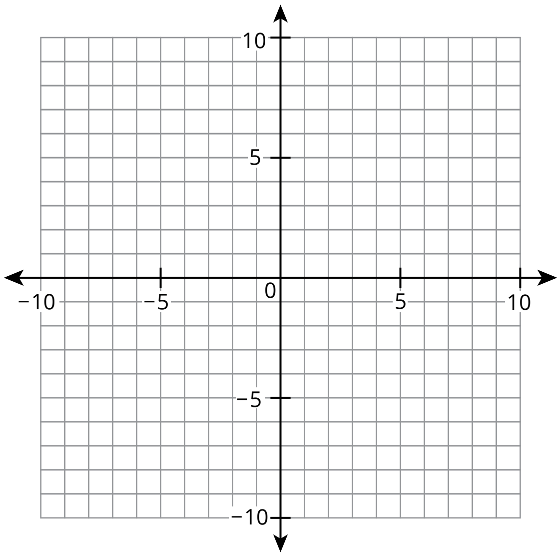
Range:

Axis of symmetry:

Maximum:

Minimum:

d. $h(x) = -|x + 3| - 1$



Domain:

Range:

Axis of symmetry:

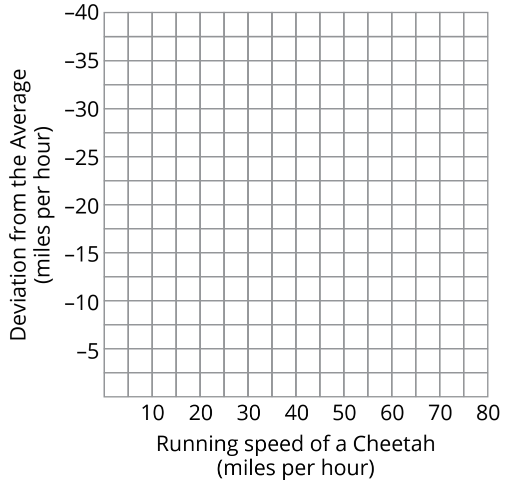
Maximum:

Minimum:

2. A cheetah can run between 70 and 75 miles per hour! However, the average running speed of a cheetah is 40 miles per hour.

   a. Formulate an absolute value function to model the relationships between a cheetah's running speed, x, and its deviation from the average speed.

b. Use technology to sketch a graph of the function.



c. Use inequalities to write the domain and range of the function. How do the domain and range of the problem situation compare to the domain and range of the function?

d. Use the graph to determine the speed the cheetah is running when the cheetah's speed deviates 25 miles per hour from the average speed.

Lesson 1 Assignment

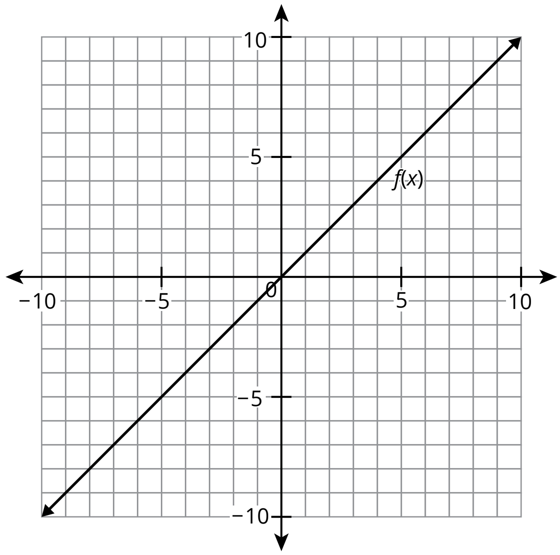
Prepare

The graph of the linear parent function, $f(x) = x$, is shown. Graph each transformation.

1. $g(x) = f(x) + 5$

2. $h(x) = 2 \cdot f(x) - 3$

3. $j(x) = \frac{1}{2} \cdot f(x - 1)$



2 Transformations of Absolute Value Functions

OBJECTIVES

- Experiment with transformations of absolute value functions using technology.
- Graph absolute value functions and transformations of absolute value functions.
- Determine the effect of replacing the parent absolute value function $f(x) = |x|$ with $f(x) + d$, $af(x)$, $f(bx,)$ and $f(x - c)$ for different values of a, b, c, and d.
- Distinguish between function transformations that occur outside the function and inside the argument of the function and verbally describe the differences.

KEY TERMS

- reflection
- line of reflection
- argument of a function

You know how to transform linear functions. How can you define absolute value functions and show transformations of this function type?

Is there another way to compare values?

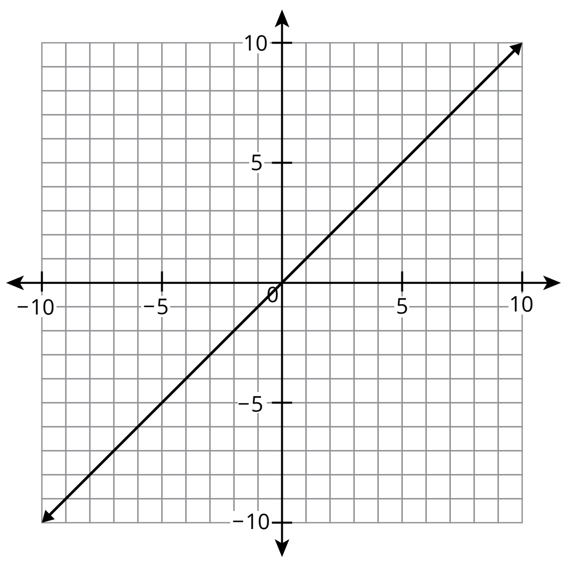
Getting Started

Back to Basics

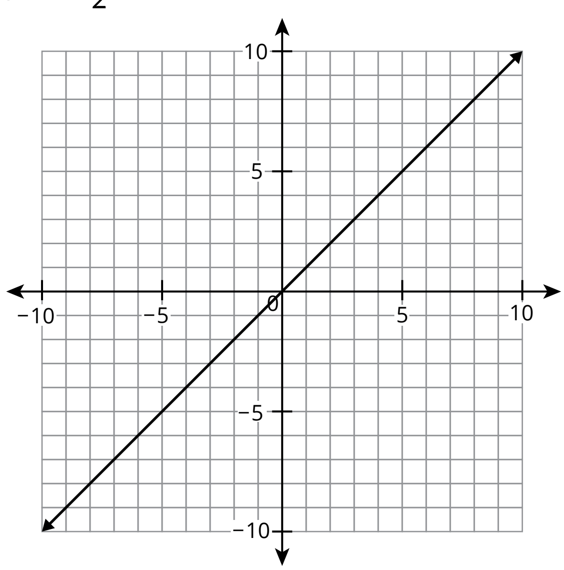
In previous courses, you have transformed linear, quadratic, and exponential functions using transformation form, $y = af(b(x - c)) + d$.

The equation of the parent linear function $f(x) = x$ is given along with the equation for the transformed function $g(x)$. Describe the transformation(s) performed on $f(x)$ to produce $g(x)$. Then graph $g(x)$.

1. $f(x) = x$
   $g(x) = 2f(x - 1)$

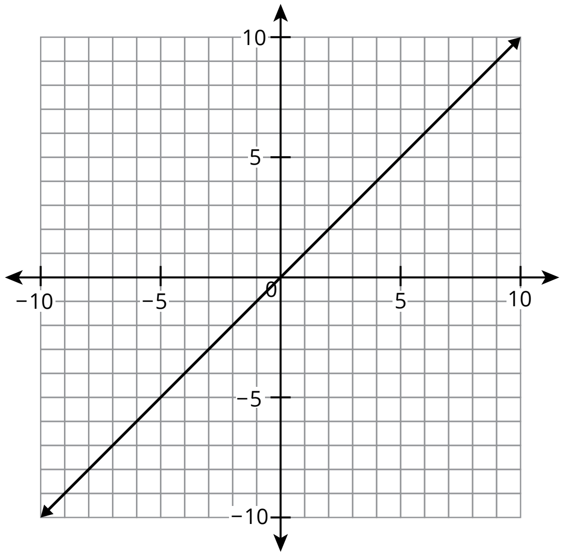


2. $f(x) = x$
   $g(x) = \frac{1}{2}f(x) + 3$



3. $f(x) = x$

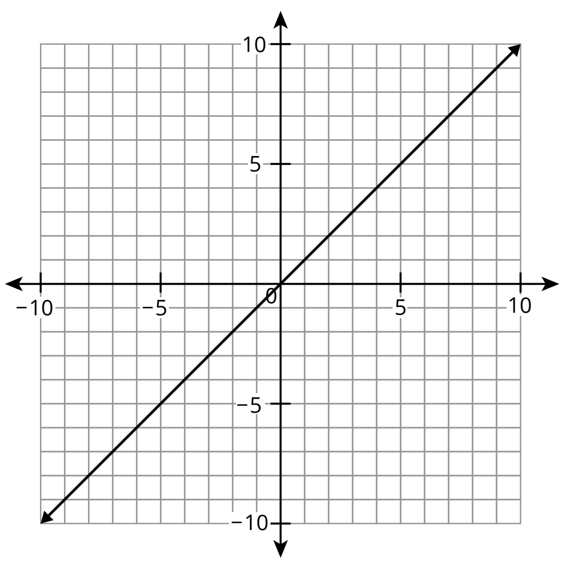
$g(x) = f(3x) - 4$



4. $f(x) = x$

$g(x) = -f(x - 2) - 1$

5.



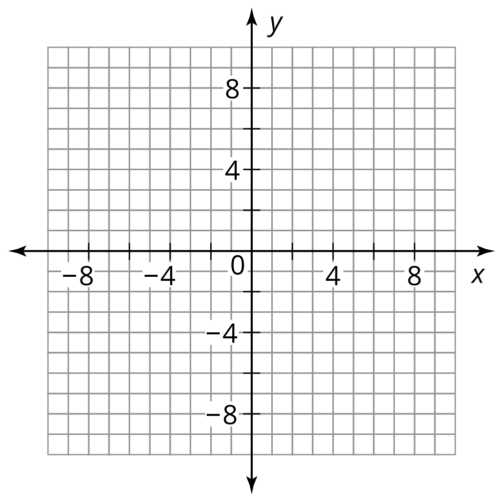
ACTIVITY 2.1

Transformations Outside the Function

Consider the three absolute value functions shown.

$f(x) = |x|$ $\qquad$ $c(x) = |x| + 3$ $\qquad$ $d(x) = |x| - 3$

1. Use technology to graph each function. Then, sketch and label the graph of each function.



2. Write the functions $c(x)$ and $d(x)$ in terms of the parent function $f(x)$. Then describe the transformations of each function.

3. Describe the similarities and differences between the three graphs. How do these similarities and differences relate to the equations of the functions $f(x)$, $c(x)$, and $d(x)$?

Recall that a function $t(x)$ of the form $t(x) = f(x) + d$ is a vertical translation of the function $f(x)$. The value $|d|$ describes how many units up or down the graph of the original function is translated.

4. Describe each graph in relation to the parent function $f(x) = |x|$. Then use coordinate notation to represent the vertical translation.

   a. $g(x) = f(x) + d$ when $d > 0$

   b. $g(x) = f(x) + d$ when $d < 0$

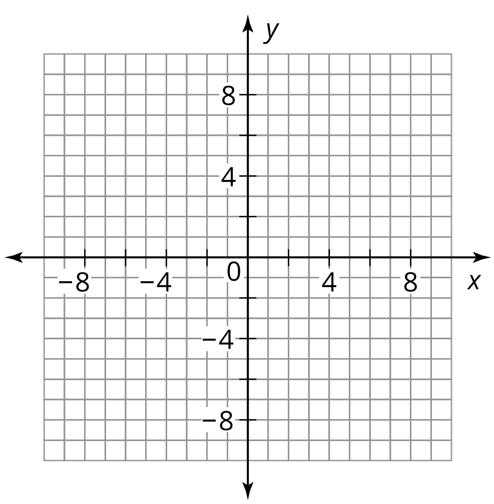
   c. Each point (x, y) on the graph of $f(x)$ becomes the point ______________ on $g(x)$.

Consider these absolute value functions.

$$f(x) = |x| \qquad k(x) = \frac{1}{2}|x|$$

$$j(x) = 2|x| \qquad p(x) = -|x|$$

5. Use technology to graph each function. Then, sketch and label the graph of each function.



6. Write the functions $j(x)$, $k(x)$, and $p(x)$ in terms of the parent function $f(x)$. Then describe the transformations of each function.

Recall that a function $t(x)$ of the form $g(x) = a \cdot f(x)$ is a vertical dilation of the function $f(x)$. The a-value describes the vertical dilation of the graph of the original function.

7. Describe each graph in relation to the parent function $f(x) = |x|$. Then use coordinate notation to represent the vertical dilation.

Ask Yourself . . .
How else can you represent this information?

a. $g(x) = a \cdot f(x)$ when $a > 1$

b. $g(x) = a \cdot f(x)$ when $a < 0$

c. $g(x) = a \cdot f(x)$ when $0 < a < 1$

d. Each point (x, y) on the graph of $f(x)$ becomes the point ________________ on $g(x)$.

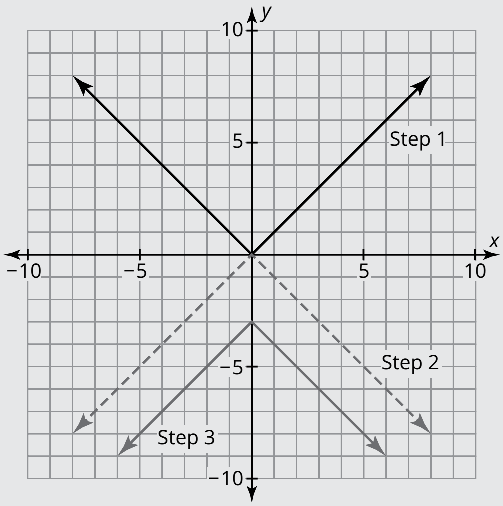
Remember . . .
A **reflection** of a graph is the mirror image of the graph about a line of reflection.

Remember . . .
A **line of reflection** is the line that the graph is reflected across. A horizontal line of reflection affects the y-coordinates.

You know that changing the a-value of a function to its opposite reflects the function across a horizontal line. But the *line of reflection* for the function might be different depending on how you write the transformation and the order the transformations are applied.

8. James and Ricardo each sketched a graph of the function $b(x) = -|x| - 3$ using different strategies. Write the step-by-step reasoning used by each student.

James

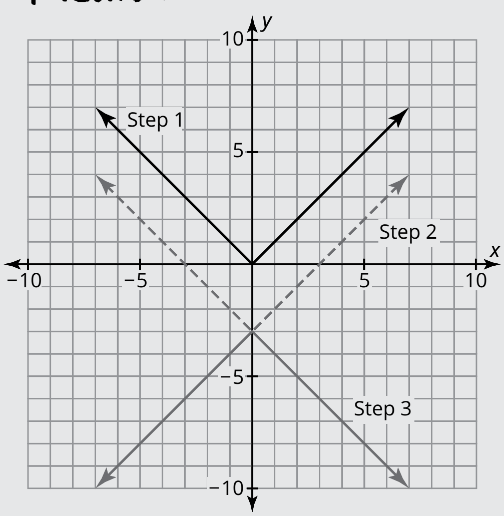


Step 1:

Step 2:

Step 3:

Ricardo



Step 1:

Step 2:

Step 3:

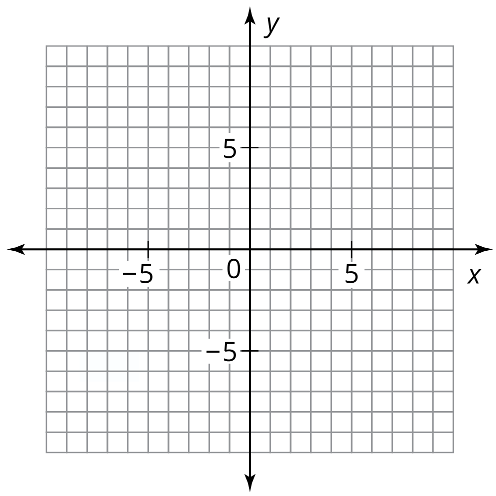
9. Explain how changing the order of the transformations affects the line of reflection.

Given the function $f(x) = |x|$. Use the coordinate plane shown to answer Questions 10 through 14.

10. Consider the function $g(x) = 2f(x) + 1$.

    a. Use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of $g(x)$.

    b. Graph and label $g(x)$ on the coordinate plane shown.



11. Consider the function $h(x) = -2f(x) + 1$.

    a. Use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of $s(x)$.

    b. Graph and label $h(x)$ on the same coordinate plane shown.

12. Describe the graph of $h(x)$ in terms of $r(x)$.

13. Consider the function $-g(x)$.

    a. Use coordinate notation to describe how each point (x, y) on the graph of $g(x)$ becomes a point on the graph of $-g(x)$.

    b. Graph and label $-g(x)$ on the coordinate plane shown.

14. Describe the graph of $-g(x)$ in terms of $g(x)$.

15. How do the a-value and d-value affect the minimum and maximum values of the function?

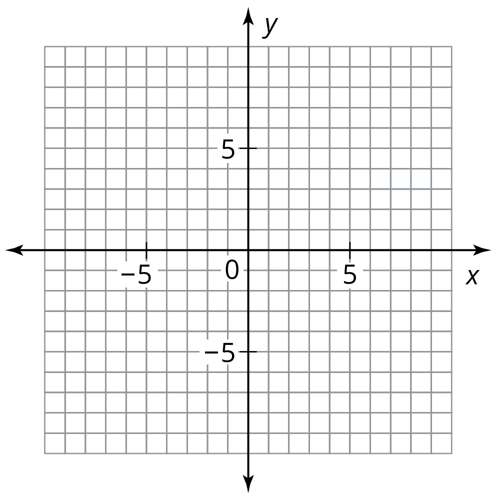
ACTIVITY
2.2

Transformations Inside the Function

Consider these absolute value functions.

$f(x) = |x|$ $\qquad$ $m(x) = |x - 2|$ $\qquad$ $n(x) = |x + 2|$

1. Use technology to graph each function. Then, sketch and label the graph of each function. Describe how $m(x)$ and $n(x)$ relate to $f(x)$.



A function $t(x)$ of the form $t(x) = f(x - c)$ is a horizontal translation of the function $f(x)$. The value $|c|$ describes the number of units the graph of $f(x)$ is translated right or left. If $c > 0$, the graph is translated to the right. If $c < 0$, the graph is translated to the left.

2. Write the functions $m(x)$ and $n(x)$ in terms of the parent function $f(x)$. Describe how changing the c-value in the functions $m(x)$ and $n(x)$ horizontally translated the function $f(x)$.

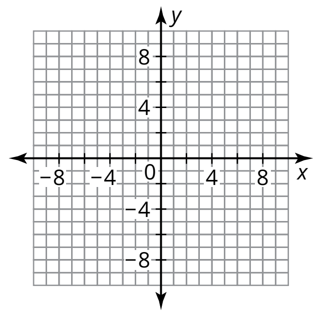
Remember . . .
The expression $x + c$ is the same as $x - (-c)$.

3. Use coordinate notation to show how each point (x, y) on the graph of $f(x)$ becomes a point on a graph that has been horizontally translated.

Consider these absolute value functions.

$$f(x) = |x| \qquad k(x) = \left|\frac{1}{2}x\right|$$
$$j(x) = |2x| \qquad p(x) = |-x|$$

4. Use technology to graph each function. Then, sketch and label the graph of each function.



5. Write the functions $j(x)$, $k(x)$, and $p(x)$ in terms of the parent function $f(x)$. Then describe the transformations of each function.

Recall that a function $t(x)$ of the form $t(x) = f(b \cdot x)$ is a horizontal dilation of the function $f(x)$. The b-value describes the horizontal dilation of the graph of the original function.

6. Describe each graph in relation to the parent function $f(x) = |x|$. Then use coordinate notation to represent the horizontal dilation.

   a. $g(x) = f(b \cdot x)$ when $b > 1$

   b. $g(x) = f(b \cdot x)$ when $b < 0$

   c. $g(x) = f(b \cdot x)$ when $0 < b < 1$

   d. Each point (x, y) on the graph of $f(x)$ becomes the point

   ____________ on $g(x)$.

Ask Yourself . . .
How does changing the b-value compare to changing the a-value?

ACTIVITY 2.3 Combining Transformations of Absolute Value Functions

The **argument of a function** is the expression inside the parentheses. For $y = f(x - c)$ the expression $x - c$ is the argument of the function.

When a function is transformed by changing the *a*- or *d*-values or both, these changes are said to occur "outside the function." These values affect the output to a function, *y*. When the *b*- or *c*-values are changed, this changes the *argument of the function*. A change to the argument of a function is said to happen "inside the function." These values affect the input to a function, *x*.

outside the function

$$g(x) = a \cdot f(b(x - c)) + d$$

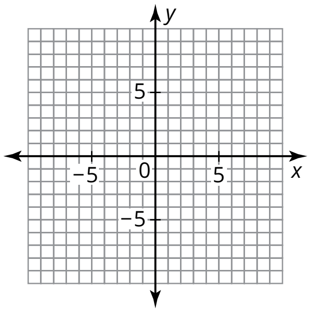
inside the function

1. Use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of $g(x)$.

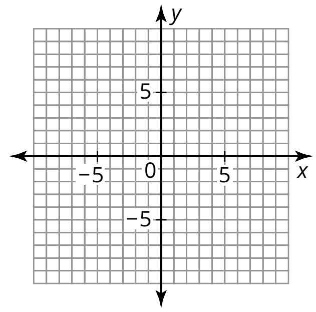
The ordered pair $(x, |x|)$ describes any point on the graph of the parent absolute value function $f(x) = |x|$. For a transformation of the function, any point on the graph of the new function can be written as $\left(\frac{1}{b}x + c,\ a|(b(x - c))| + d\right)$.

2. Given the parent absolute value function $f(x) = |x|$. Consider each transformation. Describe how the transformations affected $f(x)$. Then use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of the transformed function. Finally, sketch a graph of each new function.

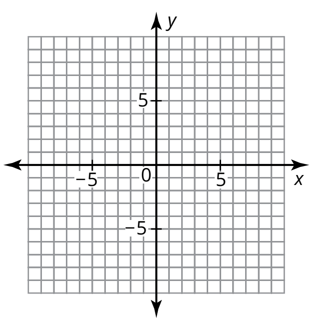
a. $m(x) = 2f(x - 1)$



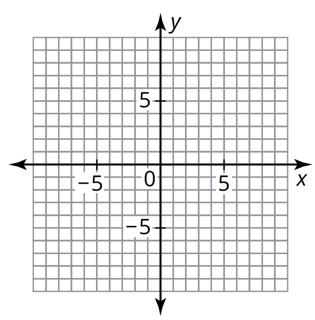
b. $n(x) = \frac{1}{2}f(x + 2) - 2$



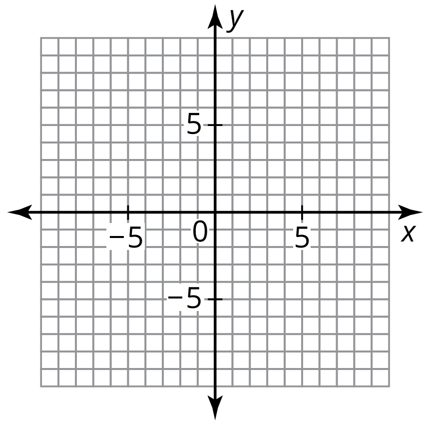
c. $r(x) = 2f(x + 3) + 1$



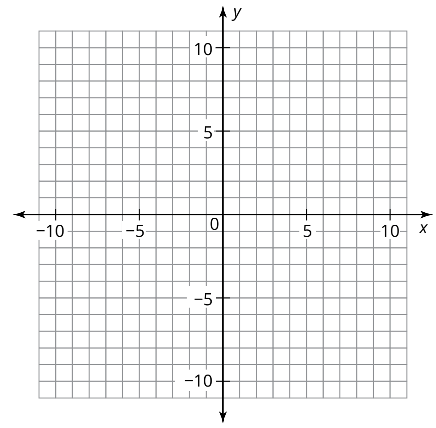
d. $s(x) = -2f(x + 3) + 1$



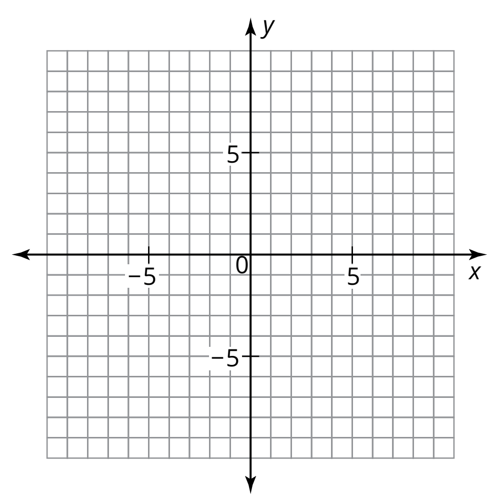
e. $w(x) = f(2(x - 1)) - 3$



f. $v(x) = \left(-\frac{1}{2}f(x - 1)\right) + 3$



3. Graph $-r(x)$ on the same coordinate plane as $r(x)$ in Question 2 part (c). Describe the similarities and differences between the graph of $s(x)$ in Question 2 part (d). and the graph of $-r(x)$.



ACTIVITY
2.4

Writing Equations in Transformation Form

1. Consider the function, $f(x) = |x|$. Write the function in transformation function form in terms of the transformations described, then write an equivalent equation.

Transformation	Transformation Function Form	Equation
a. Reflection across the x-axis		
b. Horizontal translation of 2 units to the left and a vertical translation of 3 units up		
c. Vertical stretch of 2 units and a reflection across the line $y = 0$		
d. Vertical dilation of 2 units and a reflection across the line $y = 3$		
e. Horizontal translation of 3 units to the right, a vertical translation down 2 units, and a vertical dilation of $\frac{1}{2}$		
f. Vertical compression by a factor of 4		
g. Vertical stretch by a factor of 4		
h. Horizontal compression by a factor of 3		

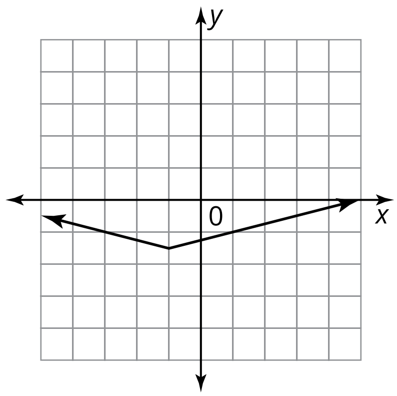
Talk the Talk

a, *b*, *c*, and *d*

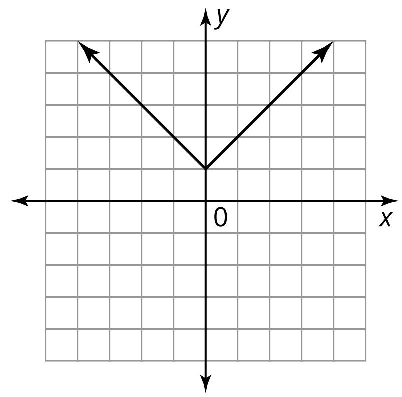
The function $f(x) = a|b(x - c)| + d$ is graphed with varying values for a, b, c, and d.

1. Match the given values of a, b, c, and d with the graph of the function with corresponding values. Explain your reasoning.

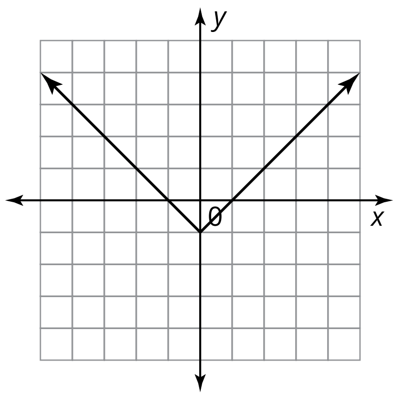
Graph A



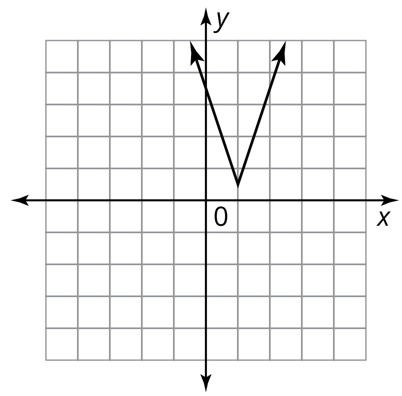
Graph B



Graph C



Graph D



a. $a = 1$, $c = 0$, and $d > 0$

b. $a = 1$, $c = 0$, and $d < 0$

c. $a > 1$, $c > 0$, and $d > 0$

d. $b > 1$, $c < 0$, and $d < 0$

2. Complete the table by describing the graph of each function as a transformation of the parent function $f(x) = |x|$.

Function Form	Equation Information	Description of Transformation
$f(x) = \|x\| + d$	$d < 0$	
	$d > 0$	
$f(x) = a\|x\|$	$a < 0$	
	$0 < \|a\| < 1$	
	$\|a\| > 1$	
$f(x) = \|x - c\|$	$c < 0$	
	$c > 0$	
$f(x) = \|bx\|$	$b < 0$	
	$0 < \|b\| < 1$	
	$\|b\| > 1$	

3. Determine whether each statement is true or false. If the statement is false, rewrite the statement as true.

   a. In the transformation function form $g(x) = af(b(x - c)) + d$, the a-value vertically stretches or compresses $f(x)$, the c-value translates $f(x)$ horizontally, the b-value horizontally stretches or compresses $f(x)$, and the d-value translates the function $f(x)$ vertically.

   b. Key attributes of the parent absolute value function include a domain and range of all real numbers.

   c. The domain of absolute value functions is not affected by translations or dilations.

   d. Vertical translations do not affect the range of absolute value functions.

   e. Horizontal translations do not affect the range of absolute value functions.

   f. Vertical dilations do not affect the range of absolute value functions.

Lesson 2 Assignment

Write

Given a parent function $y = f(x)$ and a function written in transformation form $g(x) = a \cdot fb((x–c)) + d$, describe how the transformations that are inside a function affect a graph differently than those on the outside of the function.

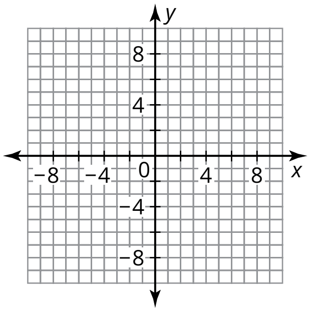
Remember

The parent absolute value function is $f(x) = |x|$.
The transformed function $y = f(x) + d$ shows a vertical translation of the function.
The transformed function $y = af(x)$ shows a vertical dilation of the function when $a > 0$ and when $a < 0$ it shows a vertical dilation and reflection across the x-axis.
The transformed function $y = f(x - c)$ shows a horizontal translation of the function.
The transformed function $y = f(bx)$ shows a horizontal dilation of the function. If $b < 0$ the function is reflected across the y-axis.

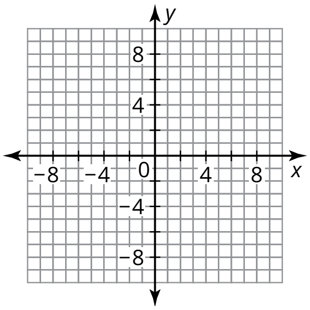
Practice

Given the parent function $f(x) = |x|$. Consider each transformation. Describe how the transformations affected $f(x)$. Then, use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph the transformed function. Finally, sketch a graph of each new function.

1. $g(x) = \frac{1}{3}f(x) - 2$

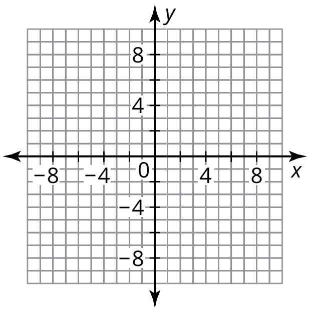


2. $j(x) = f(2(x + 1)) + 4$

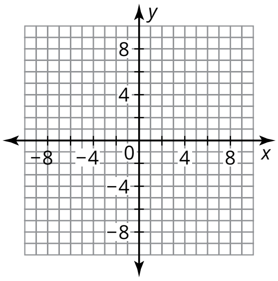


Lesson 2 Assignment

3. $m(x) = -\frac{1}{2} f(x - 3) - 1$



4. $p(x) = -f\left(-\frac{1}{2}(x + 4)\right) + 3$



Prepare

Simplify each expression.

1. $|9 + (-4)|$

2. $|-1 - 5|$

3. $|4 \cdot (-6)|$

4. $|0 \div (-2)|$

3 Absolute Value Equations and Inequalities

OBJECTIVES

- Understand and solve absolute value equations.
- Solve and graph absolute value inequalities on number lines.
- Graph absolute value functions and use the graph to determine and verbally describe solutions.

KEY TERMS

- absolute value equation
- absolute value inequality
- equivalent compound inequality

You know what the graphs of absolute value functions look like. How can you use what you know about graphs and linear equations to solve absolute value equations and inequalities?

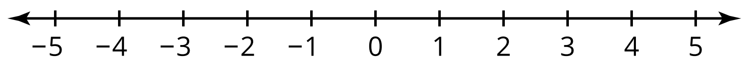
Getting Started

Opposites Attract? Absolutely!

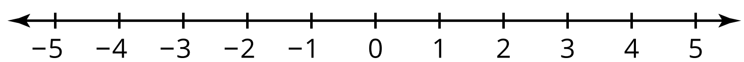
You can solve many absolute value equations using inspection.

1. Graph the solution set of each equation on the number line given.

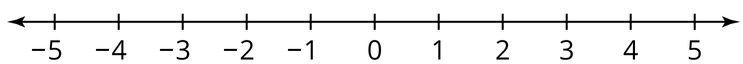
   a. $|x| = 5$

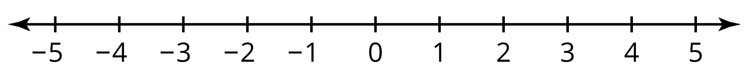
   b. $|x| = 2$

   c. $|x| = -3$

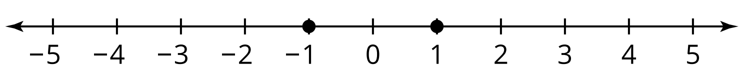
   

   d. $|x| = 0$

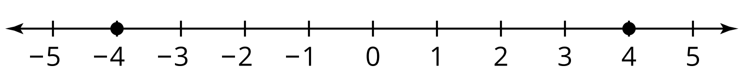
   

2. Write the absolute value equation for each solution set graphed.

   a.

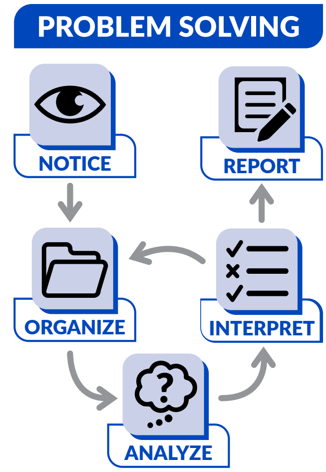
   

   b.

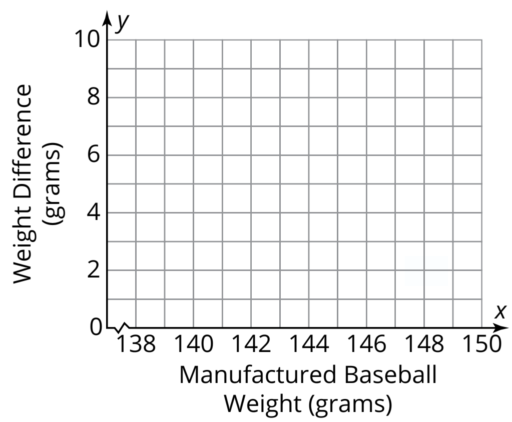
ACTIVITY 3.1

Creating an Absolute Value Function from a Situation



The official rules of baseball state that all baseballs used during professional games must be within a specified range of weights. The baseball manufacturer sets the target weight of the balls at 145.045 grams on its machines to ensure the baseballs they manufacture stay within the official rules.

1. Sketch a graph that models the relationship between a manufactured baseball's weight, x, and its distance from the target weight, y. Explain how you constructed your sketch.



Ask Yourself . . .
How is the function transformed from the parent function $f(x) = |x|$?

2. Write an absolute value equation to represent the situation and the graph.

3. The specified weight allows for a difference of 3.295 grams in the actual weight of a ball and the target weight. Since the weight must be within a distance of 3.295 grams from the target weight, $y = 3.295$.

   a. Graph the equation $y = 3.295$ on the coordinate plane in Question 1.

   b. What two equations can you write, without absolute values, to show the least acceptable weight and the greatest acceptable weight of a baseball? Explain your reasoning.

   c. Use the graph to write the solutions to the equations you wrote in part (b). Explain your reasoning.

ACTIVITY
3.2 Solving Absolute Value Equations

The two equations you wrote can be represented by the **absolute value equation** $|w - 145.045| = 3.295$. To solve any absolute value equation, recall the definition of absolute value.

WORKED EXAMPLE

Consider this absolute value equation.

$$|a| = 6$$

There are two points that are 6 units away from zero on the number line: one to the right of zero, and one to the left of zero.

$$+(a) = 6 \quad \text{or} \quad -(a) = 6$$
$$a = 6 \quad \text{or} \quad a = -6$$

Now consider the case where $a = x - 1$.

$$|x - 1| = 6$$

If you know that $|a| = 6$ can be written as two separate equations, you can rewrite any absolute value equation.

$$+(a) = 6 \quad \text{or} \quad -(a) = 6$$
$$+(x - 1) = 6 \quad \text{or} \quad -(x - 1) = 6$$

1. How do you know the expressions $+(a)$ and $-(a)$ represent opposite distances?

2. Linh and Jackson continued to solve the absolute value equation $|x - 1| = 6$ in different ways. Compare their strategies and then determine the solutions to the equation.

Linh

$(x - 1) = 6$ or $(x - 1) = -6$

Jackson

$x - 1 = 6$ or $-x + 1 = 6$

3. Solve each absolute value equation. Show your work.

a. $|x + 7| = 3$

b. $|3x + 7| = -8$

c. $|x - 9| = 12$

d. $|2x + 3| = 0$

Ask Yourself . . .
Before you solve each equation, think about the number of solutions each equation may have. You may be able to save yourself some work—and time!

4. Lauren, Nakota, Alejandra, and Hailey each solved the equation $|x| - 4 = 5$.

Lauren

$|x| - 4 = 5$

$x - 4 = 5$ $\quad$ $-(x) - 4 = 5$

$x = 9$ $\quad$ $-x = 9$

$x = -9$

Nakota

$|x| - 4 = 5$

$|x| = 9$

$x = 9$ $\quad$ $-(x) = 9$

$x = -9$

Alejandra

$|x| - 4 = 5$

$(x) - 4 = 5$ $\quad$ $-[(x) - 4] = 5$

$x - 4 = 5$ $\quad$ $-x + 4 = 5$

$x = 9$ $\quad$ $-x = 1$

$x = -1$

Hailey

$|x| - 4 = 5$

$(x) - 4 = 5$ $\quad$ $-(x) - 4 = -5$

$x = 9$ $\quad$ $-x - 4 = -5$

$-x = -1$

$x = 1$

a. Explain how Alejandra and Hailey incorrectly rewrote the absolute value equation as two separate equations.

b. Explain the difference in the strategies that Lauren and Nakota used.

5. Solve each absolute value equation.

a. $|x| + 16 = 32$

b. $23 = |x - 8| + 6$

Ask Yourself . . .
Consider isolating the absolute value part of the equation before you rewrite it as two equations.

c. $3|x - 2| = 12$

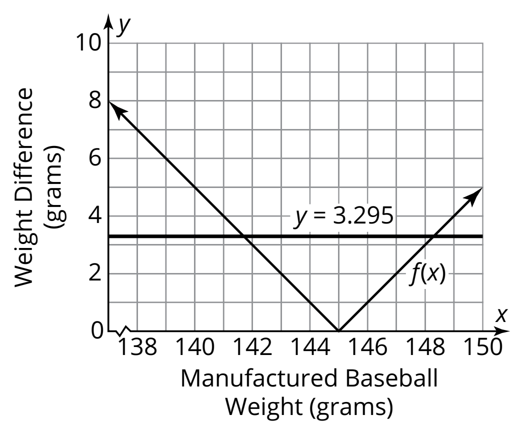
d. $35 = 5|x + 6| - 10$

ACTIVITY 3.3 Absolute Value Inequalities

You determined the absolute value equation $|w - 145.045| = 3.295$ to identify the most and least a baseball could weigh and still be within the specifications. The manufacturer wants to determine all of the acceptable weights that the baseball could be and still fit within the specifications. You can write an **absolute value inequality** to represent this problem situation.

1. Write an absolute value inequality to represent all baseball weights that are within the specifications.

2. Use the graph to determine whether the weight of each given baseball is acceptable. Substitute each value in the inequality to verify your answer.



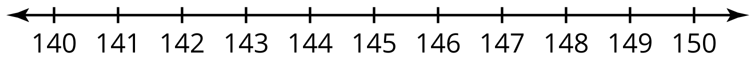
a. 147 grams

b. 140.8 grams

c. 148.34 grams

d. 141.75 grams

3. Use the graph on the coordinate plane to graph the inequality on the number line showing all the acceptable weights. Explain the process you used.



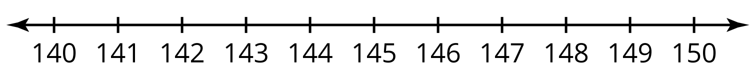
4. Complete the inequality to describe all the acceptable weights, where w is the baseball's weight.

$$\_\_\_\_\_\_\_\_\_\_\_\_ \leq w \leq \_\_\_\_\_\_\_\_\_\_\_\_$$

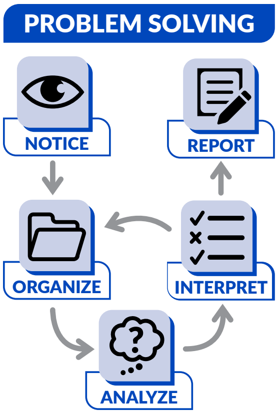
5. Joey has the job of disposing of all baseballs that are not within the acceptable weight limits.

   a. Write an absolute value inequality to represent the weights of baseballs that Joey can dispose of.

   b. Graph the inequality on the number line. Explain the process you used.



   c. Write the inequalities that describe all of the baseballs that are not within the acceptable weight limits.



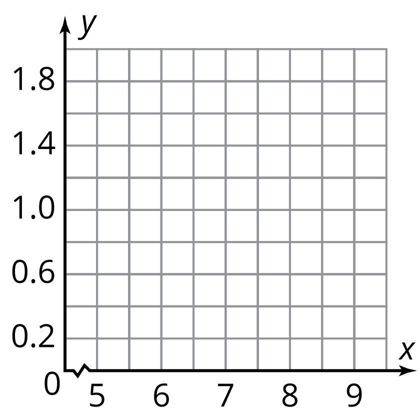
ACTIVITY 3.4 Solving Problems with Absolute Value Functions

In Little League Baseball, the diameter of the ball is slightly smaller than that of a professional baseball.

1. For Little League baseballs, the manufacturer sets the target diameter to be 7.47 centimeters. The specified diameter allows for a difference of 1.27 centimeters.

   a. Write an absolute value equation to represent the relationship between a manufactured baseball's diameter, *d*, and its distance from the target diameter.

   b. Sketch the graph of the absolute value function, f(*d*), on the coordinate plane. Label the *x*- and *y*-axis.



c. Use your graph to estimate the diameters of all the Little League baseballs that fit within the specifications. Explain how you determined your answer.

d. Algebraically determine the diameters of all the baseballs that fit within the specification. Write your answer as an inequality.

2. The manufacturer knows that the closer the diameter of the baseball is to the target, the more likely it is to be sold. The manufacturer decides to keep only the baseballs that are less than 0.75 centimeter from the target diameter.

Ask Yourself . . .
What tools or strategies can you use to solve this problem?

a. Algebraically determine which baseballs will not fall within the new specified limits and will not be kept. Write your answer as an inequality.

b. How can you use your graph to determine whether you are correct?

ACTIVITY 3.5 Absolute Value and Compound Inequalities

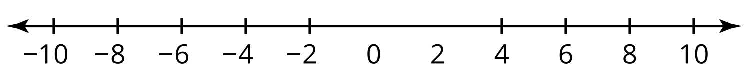
Notice that the equivalent compound inequalities do not contain absolute values.

Absolute value inequalities can take four different forms, as shown in the table. To solve an absolute value inequality, you can first write it as an **equivalent compound inequality**.

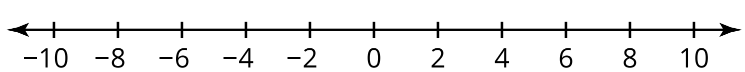
Absolute Value Inequality	Equivalent Compound Inequality
$\lvert ax + b \rvert < c$	$-c < ax + b < c$
$\lvert ax + b \rvert \leq c$	$-c \leq ax + b \leq c$
$\lvert ax + b \rvert < c$	$ax + b < -c$ or $ax + b > c$
$\lvert ax + b \rvert \geq c$	$ax + b \leq -c$ or $ax + b \geq c$

1. Solve the absolute value inequality by rewriting it as an equivalent compound inequality. Then graph the solution set on the number line.

   a. $\lvert x + 3 \rvert < 4$

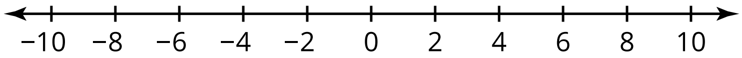


   b. $6 \leq \lvert 2x - 4 \rvert$

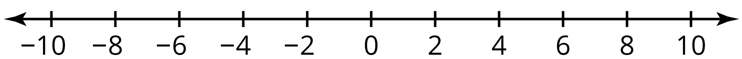


Remember . . .
As a final step, don't forget to check your solution.

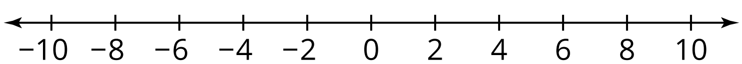
c. $|-5x + 8| + 2 < 25$



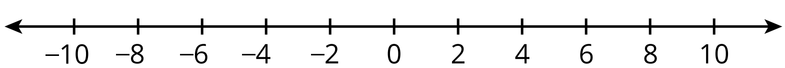
d. $|x + 5| > -1$



e. $|x + 5| < -1$



f. $2|x - 3| < 8$



Talk the Talk

Absolute Pursuit

1. Solve each absolute value equation.

   a. $|3x + 5| = 7$

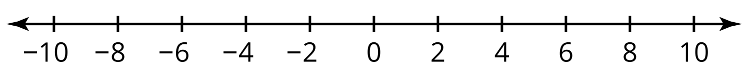
   b. $0 = |2x - 4|$

   c. $6|4x + 3| = -30$

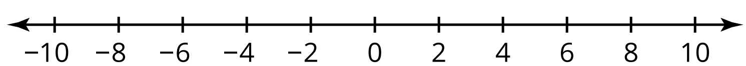
   d. $2|3x - 9| + 5 = 23$

2. Solve each absolute value inequality. Then, graph your solution set on the number line.

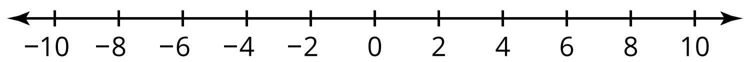
   a. $|2x + 3| < 7$



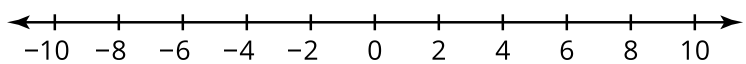
   b. $|3x + 1| \geq 8$



c. $24 \geq 2|5x - 2|$



d. $3|x - 4| + 3 > 15$



Lesson 3 Assignment

Write

Describe the similarities and differences between solving an absolute value equation and an absolute value inequality.

Remember

You can rewrite any absolute value equation as two equations to solve. If $|x| = c$, where c is any real number, then $+(x) = c$ or $-(x) = c$.

Absolute value inequalities can take four different forms. To solve an absolute value inequality, you can first write it as an equivalent compound inequality.

Absolute Value Inequality	Equivalent Compound Inequality
$\|ax + b\| < c$	$-c < ax + b < c$
$\|ax + b\| \leq c$	$-c \leq ax + b \leq c$
$\|ax + b\| > c$	$ax + b < -c$ or $ax + b > c$
$\|ax + b\| \geq c$	$ax + b \leq -c$ or $ax + b \geq c$

Practice

1. The Billingsly Cookie Company is trying to come up with a cookie that is low in fat but still has good taste. The company decides on a target fat content of 5 grams per cookie. In order to be labeled low-fat, a difference of 1.8 grams per cookie is acceptable.

   a. Write an expression that represents the difference between the fat in a cookie from the new recipe and the target fat content. Use f to represent the amount of fat in a cookie from the new recipe.

   b. Write an absolute value inequality to represent the restrictions on the difference in the amount of fat.

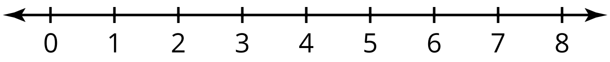
Lesson 3 Assignment

c. One of the bakers creates a cookie recipe that has 6.5 grams of fat per cookie. Is this recipe acceptable? Explain your reasoning.

d. Another baker comes up with a cookie recipe that has 2.9 grams of fat per cookie. Is this recipe acceptable? Explain your reasoning

e. Algebraically determine the greatest and least number of grams of fat a cookie can contain and still fall within the required specifications. Write your answer as an inequality.

f. Sketch the graph of the absolute value inequality from part (b).



Lesson 3 Assignment

2. Solve each abolsute value equation.

   a. $|x| + 8 = 15$

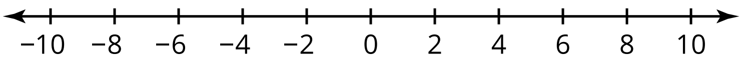
   b. $|x + 5| = -15$

   c. $|3x - 7| = 10$

   d. $3|x + 4| = 27$

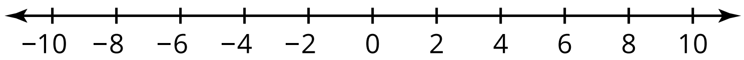
3. Solve each absolute value inequality and graph the solution.

   a. $|x + 4| \leq 5$

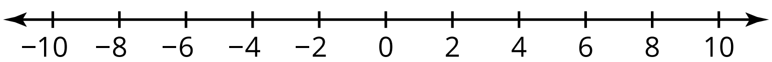


Lesson 3 Assignment

b. $|3x - 1| > 14$



c. $|x - 9| > -1$



Prepare

1. $\begin{cases} 2x + 3y = 8 \\ x = -2 \end{cases}$

2. $\begin{cases} -6x + \frac{1}{2}y = 4 \\ y = 4 \end{cases}$

3. $\begin{cases} 5x - y = 17 \\ x = y + 1 \end{cases}$

TOPIC 1 SELF-REFLECTION

Name: ______________________________

Absolute Value Functions and Equations

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me,** 2 represents **I am building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Absolute Value Functions and Equations* topic by:

TOPIC 1: *Absolute Value Functions and Equations*	Beginning of Topic	Middle of Topic	End of Topic
Graphing an absolute value function.	○	○	○
Describing the key attributes of the graph of an absolute value function.	○	○	○
Transforming an absolute value function noting the results of changing the *a*-, *b*-, c-, and *d*-values.	○	○	○
Formulating an absolute value equation.	○	○	○
Solving absolute value equations and inequalities using a graph of the properties of equality.	○	○	○

continued on the next page

TOPIC 1 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Absolute Value Functions and Equations* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

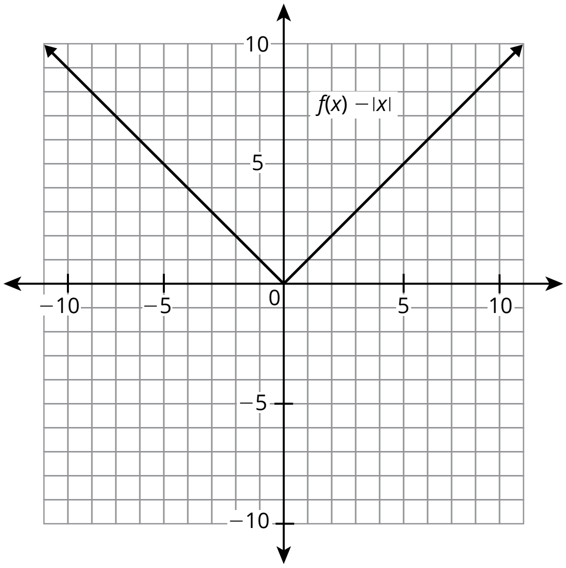
3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 1 SUMMARY

Absolute Value Functions and Equations

LESSON 1 Defining Absolute Value Functions

The graph of the parent **absolute value function**, $f(x) = |x|$, is in a *V* shape. It is the combination of the graphs of $f(x) = x$ and $f(x) = -x$.



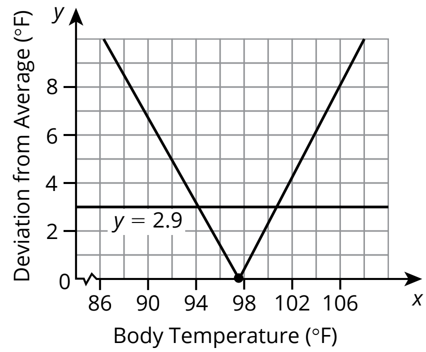
You can model many real-world situations with an absolute value function. For example, the equation $y = |x - 97.5|$ represents the average human internal body temperature. A deviation, *y*, of around 2.9°F from the average indicates a serious medical issue. Hypothermia can start at around 2.9°F less than the average, and a fever can start around 2.9°F more than the average.

You can determine when a fever and hypothermia begin by solving the equation $|x - 97.5| = 2.9$ with a graph.

First, graph $y = |x - 97.5|$ and $y = 2.9$.

KEY TERMS

- absolute value function [función de valor absoluto]
- reflection [reflexión]
- line of reflection [línea de reflexión]
- argument of a function [argumento de una función]
- absolute value equation [ecuación de valor absoluto]
- absolute value inequality [desigualdad de valor absoluto]
- equivalent compound inequality [desigualdad compuesta equivalente]



The points of intersection of the graphs of the absolute value function and a constant function are the solutions to the equation.

So, hypothermia begins at 94.6°F and a fever begins at 100.4°F.

LESSON 2 Transforming Absolute Value Functions

The graph of the absolute value function $f(x) = |x|$ is in a *V* shape. It is the combination of the graphs of $f(x) = x$ and $f(x) = -x$.

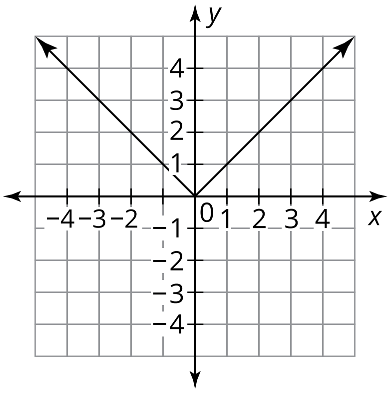
For the parent function $f(x) = |x|$, the transformed function $gx = f(x) + d$ is a vertical translation of the function $f(x)$. For $d > 0$, the graph vertically shifts up. For $d < 0$, the graph vertically shifts down. The amount of shift is given by $|d|$.

For the parent function $f(x) = |x|$, the transformed function $g(x) = a \cdot f(x)$ is a vertical dilation of the function $f(x)$ For $|a| > 1$, the graph vertically stretches by a factor of a units. For $0 < |a| < 1$, the graph vertically compresses by a factor of a units. For $a < 0$, the graph reflects across a horizontal line. A **reflection** of a graph is the mirror image of the graph about a line of reflection. A **line of reflection** is the line that the graph is reflected across. A horizontal line of reflection affects the *y*-coordinates.

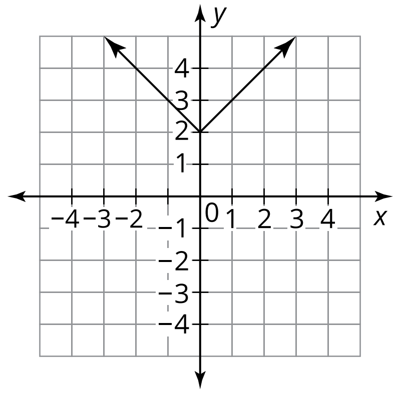
For the parent function $f(x) = |x|$, the transformed function $g(x)= f(x - c)$ is a horizontal translation of the function $f(x)$. For $c > 0$, the graph horizontally shifts right. For $c < 0$, the graph horizontally shifts left. The amount of shift is given by $|c|$.

For the parent function $f(x) = |x|$, the transformed function $g(x) = f(bx)$ is a horizontal dilation of the function $f(x)$. For $|b| > 1$, the graph is horizontally compressed by a factor of $\frac{1}{|b|}$. For $0 < |b| < 1$, the graph will be horizontally stretched by a factor of $\frac{1}{|b|}$ units. For $b < 0$, the graph also reflects across the *y*-axis.

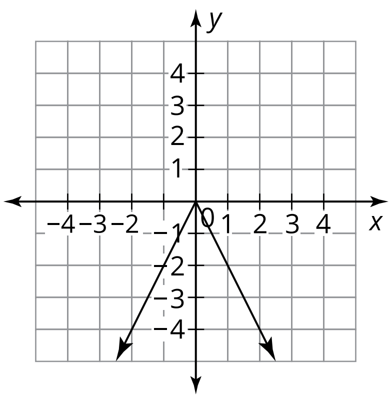
$f(x) = |x|$



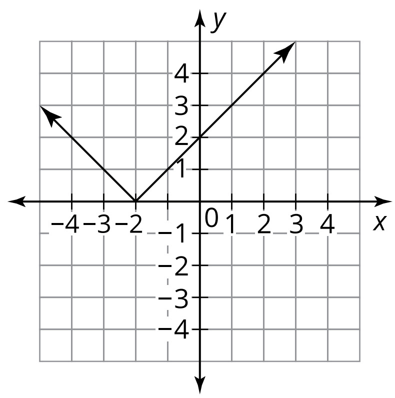
$y = |x| + 2$



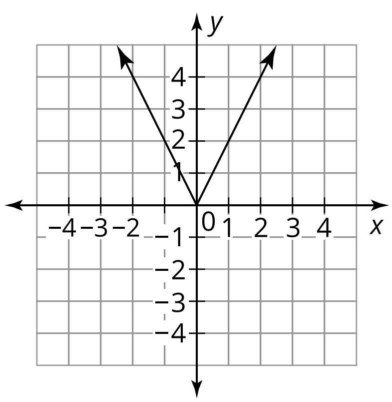
$f(x) = -2|x|$



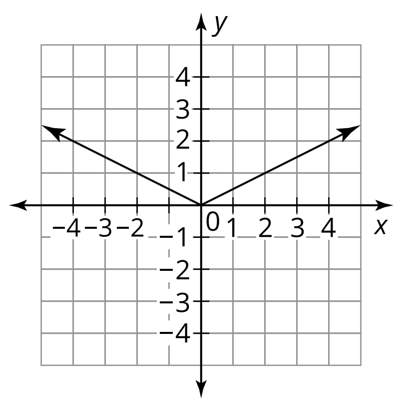
$f(x) = |x - (-2)|$



$f(x) = |2x|$



$f(x) = \left|\frac{1}{2}x\right|$



LESSON 3 Absolute Value Equations and Inequalities

To solve an **absolute value equation**, write the positive and negative equations that the absolute value equation represents. Then, solve each equation.

$$|5x - 4| = 21$$

$$+(5x - 4) = 21 \qquad -(5x - 4) = 21$$

$$5x - 4 = 21 \qquad 5x - 4 = -21$$

$$5x - 4 + 4 = 21 + 4 \qquad 5x - 4 + 4 = -21 + 4$$

$$5x = 25 \qquad 5x = -17$$

$$\frac{5x}{5} = \frac{25}{5} \qquad \frac{5x}{5} = \frac{(-17)}{5}$$

$$x = 5 \qquad x = -3\frac{2}{5}$$

If there is a range of solutions that satisfy a problem situation, you can write an **absolute value inequality**. To evaluate for a specific value, substitute the value for the variable.

For example, suppose a swimmer who wants to compete on a city swim club should be able to swim the 100-meter freestyle in 54.24 seconds plus or minus 1.43 seconds. Can a swimmer with a time of 53.15 seconds qualify for the team?

$$|t - 54.24| \leq 1.43$$

$$|53.15 - 54.24| \leq 1.43$$

$$|-1.09| \leq 1.43$$

$$1.09 \leq 1.43$$

The swimmer qualifies because his time is less than 1.43 seconds from the base time.

Absolute value inequalities can take four different forms with the absolute value expression compared to a value, *c*. To solve an absolute value inequality, you must first write it as an **equivalent compound inequality**. "Less than" inequalities will be conjunctions and "greater than" inequalities will be disjunctions.

Absolute Value Inequality	Equivalent Compound Inequality
$\|ax + b\| < c$	$-c < ax + b < c$
$\|ax + b\| \leq c$	$-c \leq ax + b \leq c$
$\|ax + b\| > c$	$ax + b < -c$ or $ax + b > c$
$\|ax + b\| \geq c$	$ax + b \leq -c$ or $ax + b \geq c$

TOPIC 2

You can determine the solution to a system of linear equations by determining the point where the lines intersect.

Applications of Linear Relationships

1 Solving Systems of Equations

OBJECTIVES

- Solve systems of two linear equations.
- Write and solve systems of three linear equations in three variables by using substitution.
- Write and solve systems of three linear equations in three variables by using Gaussian elimination.

KEY TERM

- Gaussian elimination

You know how to solve a system of linear equations in two variables using graphs, linear combinations, and substitution. How can you use similar methods to solve systems of equations involving three linear equations?

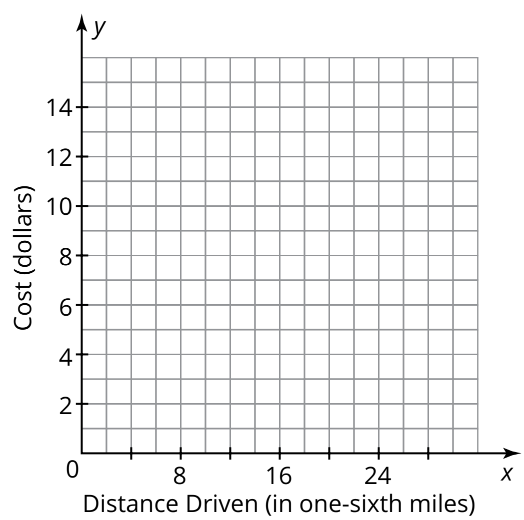
Getting Started

Which Fare Is Fair?

You would like to take a taxi to the airport. There are two local taxi companies. Cab Company A charges $2.60 plus $0.20 per one-sixth of a mile driven. Cab Company B charges $5.00 plus $0.10 per one-sixth of a mile driven.

1. Write a system of two linear equations in two variables to represent this problem situation. Be sure to define your variables.

2. Graph the system of linear equations.



Remember . . .
The solution to a system of linear equations occurs when the values of the variables satisfy all of the linear equations.

3. Estimate the solution to the system of linear equations. Justify your reasoning.

4. Solve the system of linear equations algebraically.

5. What does the solution mean in terms of the problem situation?

6. Suppose that Cab Company B decides to increase its fare to $0.20 per one-sixth mile driven. Write a new system of equations to reflect the increased fare. When will the cost of using the two taxi companies be equal for the same number of miles? Explain your reasoning.

Think About . . .
What are some different methods you have learned to solve a system of linear equations in two variables algebraically?

7. Think about the graphs of different systems of two linear equations.

   a. Describe the different ways in which the two graphs can intersect, and provide a sketch of each case.

   b. How does this relate to the number of solutions to a system of two linear equations?

ACTIVITY
1.1

Solving Systems of Three Linear Equations

You can solve systems of three linear equations in three variables by using substitution. First, solve one equation for a variable and then substitute that expression into the other two equations. This reduces the system of three equations in three variables to a system of two equations in two variables, which you can then solve using any method.

1. As a fundraising event, a club sold tickets to a special viewing of a classic movie. The club sold all 800 seats in the school's auditorium. The tickets were three different prices based on the age of the ticket holder: \$2.50 for children under 12 years old, \$3.50 for youth between 12 and 18 years old, and \$5.00 for adults. The total amount of money taken in was \$2937.50, and there were 4 times as many youth tickets as children's tickets sold.

    a. Write a system of three linear equations in three variables to represent this situation. Be sure to define your variables.

    b. Which equation from your system is solved for a single variable? Use substitution to create a system of two equations in two variables.

WORKED EXAMPLE

Solve the previous system of equations.

$$\begin{cases} 5c + a = 800 \\ 16.5c + 5a = 2937.5 \end{cases}$$

1. Multiply the first equation by −5.

$$-5(5c + a = 800)$$
$$-25c - 5a = -4000$$

2. Add the resulting equation to the second equation.

$$\begin{array}{r} -25c - 5a = -4000 \\ +\ 16.5c + 5a = 2937.5 \\ \hline -8.5c = -1062.5 \end{array}$$

3. Solve for the remaining variable.

$$c = 125$$

2. Use substitution to explain what the solution represents in terms of the problem situation.

3. Joey and Brianna were asked to solve this system of three linear equations using substitution. Analyze their methods.

$$\begin{cases} x + y + z = -2 \\ 2x - 3y + 2z = -14 \\ 4x + 3y - z = 5 \end{cases}$$

Joey

$x + y + z = -2$

$x = -2 - y - z$

$$\begin{aligned} 2x - 3y + 2z &= -14 \\ 2(-2 - y - z) - 3y + 2z &= -14 \\ -4 - 2y - 2z - 3y + 2z &= -14 \\ -4 - 5y &= -14 \\ -5y &= -10 \end{aligned}$$

$$\begin{aligned} 4x + 3y - z &= 5 \\ 4(-2 - y - z) + 3y - z &= 5 \\ -8 - 4y - 4z + 3y - z &= 5 \\ -8 - y - 5z &= 5 \\ y + 5z &= -13 \end{aligned}$$

Then I solved the system $\begin{cases} -5y = -10 \\ y + 5z = -13. \end{cases}$

Brianna

$4x + 3y - z = 5$

$4x + 3y - 5 = z$

$$\begin{aligned} 2x - 3y + 2z &= -14 \\ 2x - 3y + 2(4x + 3y - 5) &= -14 \\ 2x - 3y + 8x + 6y - 10 &= -14 \\ 10x + 3y - 10 &= -14 \\ 10x + 3y &= -4 \end{aligned}$$

$$\begin{aligned} x + y + z &= -2 \\ x + y + (4x + 3y - 5) &= -2 \\ 5x + 4y - 5 &= -2 \\ 5x + 4y &= 3 \end{aligned}$$

Then I solved the system $\begin{cases} 10x + 3y = -4 \\ 5x + 4y = 3 \end{cases}$.

a. Describe the similarities and differences in Joey's and Brianna's methods.

b. Demonstrate that Joey's method and Brianna's method will both yield the same solution. Explain your reasoning.

Ask Yourself . . .
Did you justify your mathematical reasoning?

c. Could this system have been solved using a different substitution? How do you select which variable to solve for? Explain your reasoning.

4. During the movie fundraising event, the concession stand at the auditorium sells popcorn, fruit, and water. The price of a box of popcorn is $0.50 more than the price of a piece of fruit, and the price of two waters is $0.50 less than the price of three pieces of fruit. You order two boxes of popcorn, one piece of fruit, and three waters, and the total comes to $7.75.

   a. Write a system of three linear equations in three variables to represent this situation. Be sure to define your variables.

   b. Calculate the price of each item. Use substitution to solve the system of three linear equations in three variables.

ACTIVITY 1.2

Gaussian Elimination

Previously you have used linear combinations to solve systems of linear equations in two variables. When solving linear equations in three variables a method called *Gaussian elimination* is used.

Gaussian elimination is a method for solving linear systems of equations, named after the mathematician Carl Friedrich Gauss. It involves using linear combinations of the equations in the system to isolate one variable per equation.

The goal of Gaussian elimination is to eliminate all variables except one in each equation.

To do this, look for ways to:

1. swap one equation with another
2. multiply one equation by a non-zero constant on both sides
3. replace one equation with a multiple of another equation that is then added to the original equation

WORKED EXAMPLE

Consider the system $\begin{cases} 2x + 2y + 3z = 3 \\ x + 3y + 2z = 5. \\ 3x + y + z = 5 \end{cases}$

You can use the second equation to eliminate the x in the other two equations.

1. Multiply the second equation by -2 and add the result to the first equation.

$$-2(x + 3y + 2z = 5) \rightarrow \begin{array}{r} -2x - 6y - 4z = -10 \\ +2x + 2y + 3z = 3 \\ \hline -4y - z = -7 \end{array}$$

2. Replace the first equation in the system with the new equation.

$$\begin{cases} -4y - z = -7 \\ x + 3y + 2z = 5 \\ 3x + y + z = 4 \end{cases}$$

3. Multiply the second equation by -3 and add the result to the third equation.

$$-3(x + 3y + 2z = 5) \rightarrow \begin{array}{r} -3x - 9y - 6z = -15 \\ +3x + y + z = 4 \\ \hline -8y - 5z = -11 \end{array}$$

4. Replace the third equation in the system with the new equation.

$$\begin{cases} -4y - z = -7 \\ x + 3y + 2z = 5 \\ -8y - 5z = -11 \end{cases}$$

You have now eliminated the x-variable in all equations except the second equation. Continue in this same manner to isolate y and z.

1. Continue to solve the system of three linear equations in three variables from the worked example using Gaussian elimination.

    a. Swap the first and second equations so that the equation with the x variable is on top.

    b. Multiply the second equation by -2 and add the result to the third equation.

c. Multiply the resulting equation by $-\frac{1}{3}$. Replace the third equation.

d. Add the third equation to the second equation. Replace the second equation.

e. Multiply the second equation by $-\frac{1}{4}$.

f. Multiply the second equation by -3 and add the result to the first equation. Replace the first equation.

g. Multiply the third equation by -2 and add the result to the first equation. Replace the first equation.

2. Gracie claims that the system in Question 1 could have been solved more efficiently by starting with a different approach.

Gracie

I multiplied the second equation by -5 and added it to the third equation.

$$\begin{cases} x + 3y + 2z = 5 \\ -4y - z = -7 \\ -8y - 5z = -11 \end{cases}$$

$$-5(-4y - z = -7) \quad \longrightarrow \quad \begin{array}{r} 20y + 5z = 35 \\ -8y - 5z = -11 \\ \hline 12y \qquad = 24 \end{array}$$

a. How is Gracie's method different from the method used in Question 1?

b. Complete Gracie's method. Describe your steps.

c. How does this solution compare to the solution you got in Question 1?

d. Compare the two methods. What can you determine about the order in which equations are combined when using Gaussian elimination?

There is no one correct order in which to perform the steps to Gaussian elimination. The idea is to keep performing linear combinations until each equation contains a different isolated variable.

3. Nicky, Nia, and Miguel are each asked to use Gaussian elimination to solve a system of three linear equations in three variables, but they each have a different idea of how to start the problem.

$$\begin{cases} -2x + y + 2z = -3 \\ -3x - y + 2z = -11 \\ 2x + y - z = 8 \end{cases}$$

Nicky

I would add equation 1 to equation 3, and replace equation 3. This would eliminate the x variable.

$$\begin{array}{r} -2x + y + 2z = -3 \\ 2x + y - z = 8 \\ \hline 2y + z = 5 \end{array}$$

$$\begin{cases} -2x + y + 2z = -3 \\ -3x - y + 2z = -11 \\ 2y + z = 5 \end{cases}$$

Nia

I would add equation 1 to equation 2 and replace equation 2. This would eliminate the y variable.

$$\begin{array}{r} -2x + y + 2z = -3 \\ -3x - y + 2z = -11 \\ \hline -5x + 4z = -14 \end{array}$$

$$\begin{cases} -2x + y + 2z = -3 \\ -5x + 4z = -14 \\ 2x + y - z = 8 \end{cases}$$

Miguel

I would subtract equation 2 from equation 1 and replace equation 2. This would eliminate the z variable.

$$\begin{array}{r} -2x + y + 2z = -3 \\ 3x + y - 2z = 11 \\ \hline x + 2y = 8 \end{array}$$

$$\begin{cases} -2x + y + 2z = -3 \\ x + 2y = 8 \\ 2x + y - z = 8 \end{cases}$$

a. Describe the similarities and differences among all three methods to start the problem.

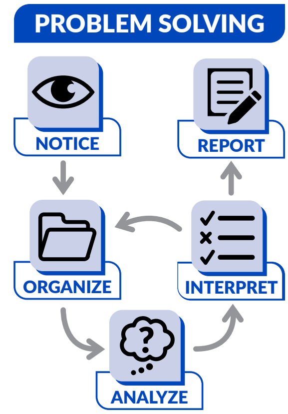
b. Whose starting method is correct? Explain your reasoning.

c. Is there a different way to begin the problem than the ones that were mentioned? Justify your answer.

You can choose any of the methods you want that Nicky, Nia, or Miguel used.

4. Solve the system of three linear equations in three variables from Question 3 using Gaussian elimination.

ACTIVITY
1.3 Loose Change



1. Lucia has 12 coins in her coin collection. The mix of Seated Liberty quarters, Buffalo Head nickels, and Barber dimes add up to a total value of $2.00. She has three times as many Seated Liberty quarters as Buffalo Head nickels.

   a. Write a system of three linear equations in three variables to represent this problem situation. Be sure to define your variables.

   b. How many of each coin does Colleen have in her pocket? Solve this system using Gaussian elimination.

Ask Yourself . . .
Did you make a plan to solve the problem?

2. In 19th century Texas, the Republic of Texas issued its own currency called Texas Treasury Notes to finance government operations. These notes came in denominations of \$1, \$5, and \$10. A coin and currency collector has 25 Texas Treasury Notes in total. The total note value of his collection is \$100. The collector has five times as many \$1 notes as \$5 notes.

   a. Write a system of three linear equations to represent this problem situation. Be sure to define your variables.

   b. How many of each note does the collector have?

3. A small manufacturing company produces three types of pants: jeans, cargo, and casual. In one week, the company produces a total of 1,200 pants. The cost to produce each pair of jeans is \$10, cargo is \$15, and casual is \$25. The total production cost for the week is \$21,000. Additionally, the number of pairs of casual pants produced is equal to half the combined total of jeans and cargo pants. Use Gaussian elimination or substitution to determine how many pairs of each type of pants the company produced?

Talk the Talk

You Can Have It Your Way

You can use either substitution or Gaussian elimination to solve systems of three linear equations in three variables.

1. List an advantage and a disadvantage to the substitution method.

2. List an advantage and a disadvantage to the Gaussian elimination method.

3. For each system, determine whether you would prefer to use substitution or Gaussian elimination. Justify your reasoning. Perform the first step.

a. $\begin{cases} 5x + 3y + 2z = -3 \\ 4x - 3y = 4 \\ 2x - 3y + 4z = -14 \end{cases}$

b. $\begin{cases} x + y + 2z = -7 \\ 3x + 4y - 2z = -10 \\ x - y = 7 \end{cases}$

4. Solve the system using whichever method you prefer.

$$\begin{cases} 2x + 3y - 4z = -16 \\ 2y + z = -1 \\ 3x - 4z = -9 \end{cases}$$

Lesson 1 Assignment

Write

Define Gaussian elimination in your own words.

Remember

A system of equations can be solved graphically or algebraically. When solving a system of three linear equations algebraically, you can use substitution or Gaussian elimination.

Practice

1. Solve each system of equations using Gaussian elimination.

a. $\begin{cases} x + y + z = 6 \\ 2x - y + 3z = 14 \\ x - 2y + z = 0 \end{cases}$

b. $\begin{cases} 2x + 3y + z = 11 \\ 4x - 2y + 5z = 15 \\ 3x + y - z = 2 \end{cases}$

c. $\begin{cases} 4x + 3y + 2z = 25 \\ 5x - y + z = 11 \\ -2x + 4y - z = 4 \end{cases}$

2. You are helping a friend with her flower garden. She decided to add some clematis vines, rose bushes, and peony plants to her garden. The price of a clematis vine is \$3 more than twice a peony plant. The price of a rose bush is \$5 more than a clematis vine. She decided to buy 3 clematis plants, 4 rose bushes and 2 peony plants for a total of \$233.

   a. Write a system of three linear equations in three variables to represent this situation. Define your variables.

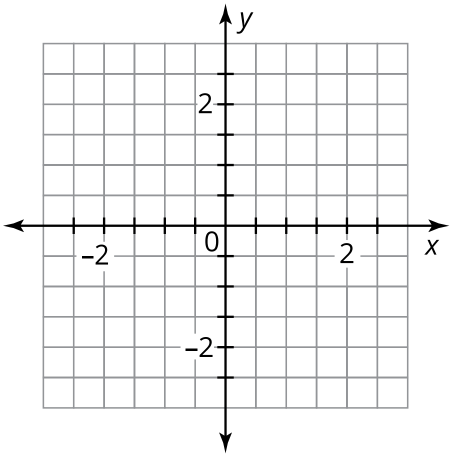
b. Calculate the price of each item. Use substitution to solve the system of three linear equations in three variables.

Lesson 1 Assignment

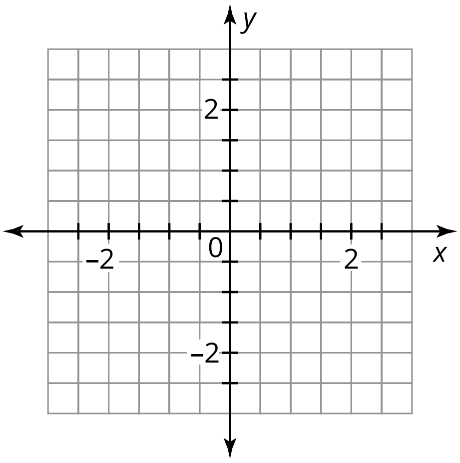
Prepare

Graph each linear inequality.

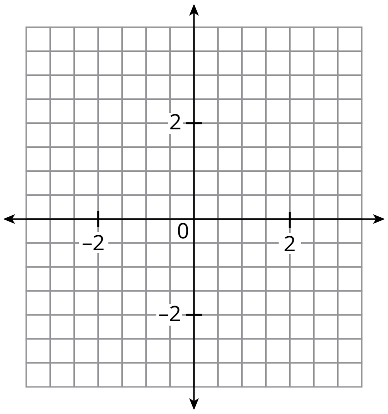
1. $y > x - 1$



2. $y \leq -\frac{1}{2}x + 2$



3. $x + y \geq 3$



2 Optimization

OBJECTIVES

- Write systems of linear inequalities in two variables.
- Solve systems of two or more linear inequalities in two variables.
- Determine possible solutions in the solution set of systems of two or more linear inequalities in two variables.
- Determine constraints from a problem situation.
- Use the vertices of a solution region to calculate maximum or minimum values.

KEY TERMS

- solution set of a system of inequalities
- linear programming

You have used the intersection of linear equations on a coordinate plane to solve a system of linear equations.
How can you use the intersection of linear inequalities on a coordinate plane to determine the greatest or least values that a function can take?

Getting Started

Remember . . .
A boundary line, determined by the inequality, divides the plane into two half-planes, and the inequality symbol indicates which half-plane contains all the solutions.

Maximize Your Earnings

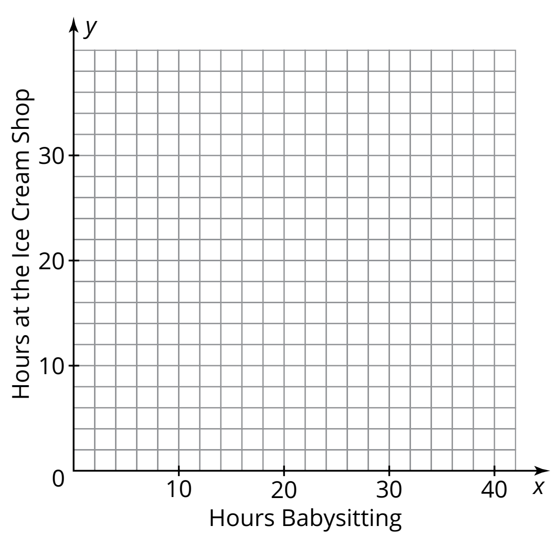
Recall that you can graph a linear inequality in two variables on a coordinate plane by determining the boundary line and shading the appropriate half-plane.

1. Juliana works two jobs and is trying to maximize her weekly earnings. Her babysitting job pays $10 per hour, and her job at an ice cream shop pays $7.50 per hour. Suppose she needs to earn at least $90 to cover her weekly expenses.

   a. Write a linear inequality to represent the total amount of money Juliana needs to earn at her jobs. Be sure to define your variables.

   b. Graph the linear inequality on the coordinate plane.

Think About . . .
Will the boundary line be a solid or a dashed line?



c. Juliana's friend Xavier suggests that she work 18 hours at her babysitting job and 12 hours at the ice cream shop. If Juliana follows Xavier's advice, will she earn enough to cover her weekly expenses? Explain your reasoning.

d. Write this solution as an ordered pair. Where does it fall on the graph?

e. Determine another possible solution to this problem situation. Justify your reasoning.

f. Why is the graph restricted to the first quadrant?

g. Write two inequalities that represent this restriction.

Think About . . .
Which values are reasonable for the variables in the problem situation?

h. Use the inequalities you wrote in parts (a) and (g) to write a system of inequalities that represents the problem situation.

ACTIVITY 2.1

Systems of Linear Inequalities

You wrote a system of linear inequalities to represent the amount of money Juliana needs to earn at her jobs. How does adding another inequality to the system affect the solution?

1. Suppose that Juliana can work a maximum of 20 total hours per week.

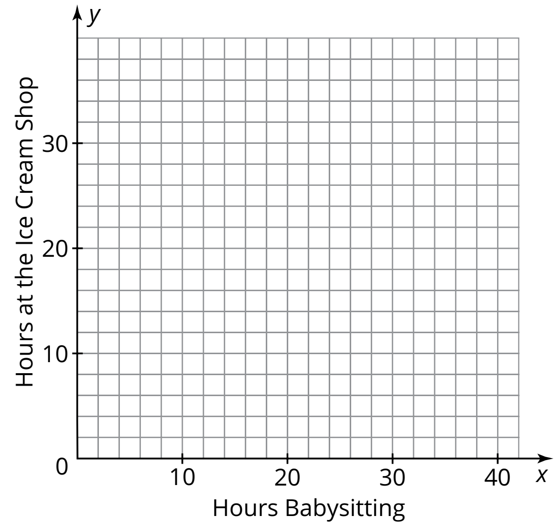
   a. Write a linear inequality to represent the total number of hours that Juliana could work per week.

   b. Is Xavier's suggestion of 18 hours at her babysitting job and 12 hours at the ice cream shop still reasonable? Justify your reasoning.

Remember . . .
In a system of linear inequalities, the inequalities are known as constraints because the values of the expressions are "constrained" to lie within a certain region on the graph.

   c. Rewrite the system of inequalities to include the new inequality from part (a).

d. Graph the system of linear inequalities on one coordinate plane.



e. Suppose that Juliana is only able to babysit for 6 hours. How many hours must she work at the ice cream shop in order to meet her earnings requirement? Justify your reasoning.

f. Write your solution as an ordered pair, and plot it on the graph in part (d). Where does it fall?

g. Ben suggests that Juliana also work 10 hours at the ice cream shop. Is this solution reasonable? How does it compare to your answer in part (e)?

h. Write Ben's solution as an ordered pair and plot it on the graph in part (d). Where does it fall?

i. Compare these solutions with the ordered pair (18, 12) suggested by Xavier in part (b). What do you notice?

Ask Yourself . . .
What patterns do you notice?

The **solution set of a system of inequalities** is the intersection of the solutions to each inequality. Every point in the intersection satisfies all inequalities in the system.

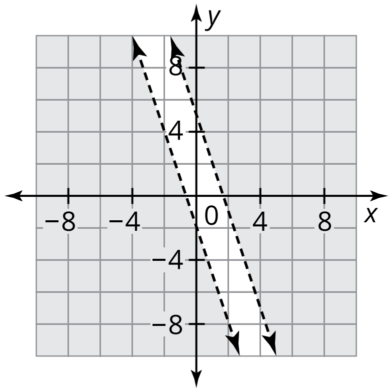
2. Select one ordered pair that represents a point inside the overlapping region. Verify that it is a solution to the system of linear inequalities.

3. Select one ordered pair that represents a point outside the overlapping region. Verify that it is not a solution to the system of linear inequalities.

4. Antonio and Omar were each asked to graph the system of linear inequalities shown.

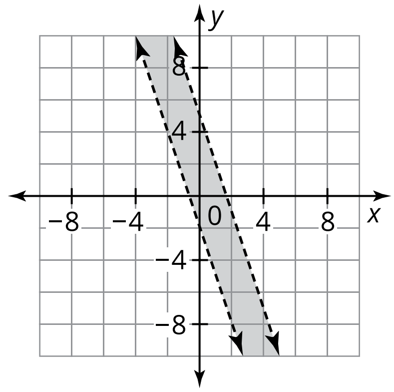
$$\begin{cases} y > -3x + 5 \\ y < -3x - 2 \end{cases}$$

Antonio



The two inequalities do not overlap, and therefore the system has no solution.

Omar



The solution to the system of inequalities falls in between the two lines.

a. Who is correct? Justify your reasoning.

b. Select an ordered pair from each region of the graph to algebraically verify your response.

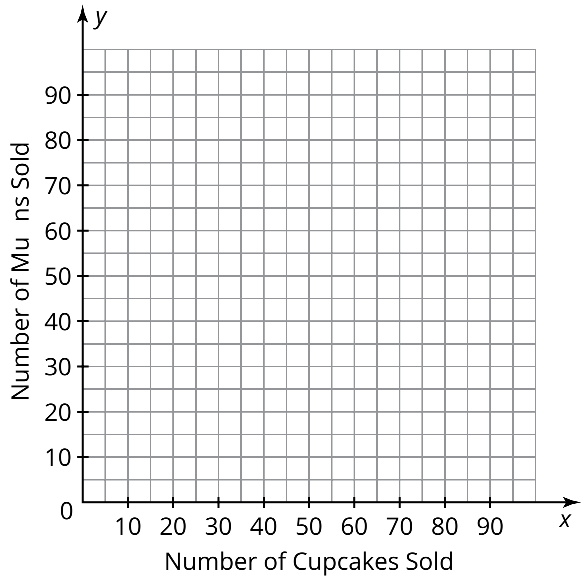
5. Why are the lines in Question 4 dashed?

6. Will two parallel lines always produce a system of linear inequalities with no solution?

7. A bakery that specializes in cupcakes and muffins sells cupcakes for \$2.00 each and muffins for \$1.25 each. The bakery can bake a total of 8 dozen treats each day and needs to make \$100 per day.

   a. Write a system of linear inequalities to represent this problem situation. Be sure to define your variables.

b. Graph the system of linear inequalities on the coordinate plane.



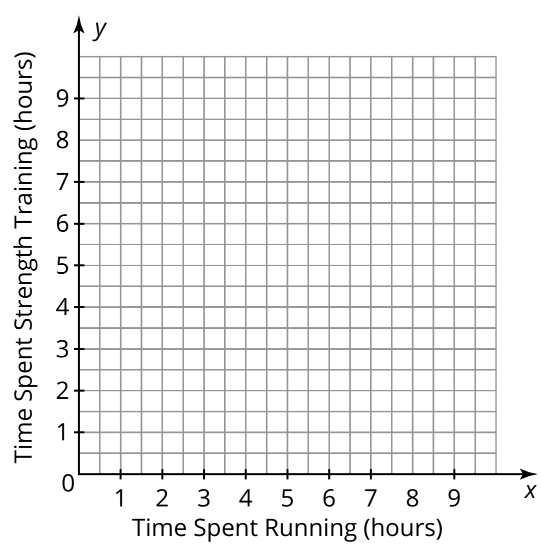
c. Identify one possible solution. Is this reasonable within the context of the problem situation?

ACTIVITY 2.2 Linear Programming

James is a distance runner who wants to design the most effective workout plan to prepare for his next race. To adequately prepare for the race, James plans to do a combination of running and strength training. He must run between 3 and 6 hours per week and wants to devote at least twice as much time to running as to strength training. James can spend no more than 8 hours working out each week.

1. Write a system of inequalities to represent the constraints of this problem situation. Be sure to define your unknowns.

2. Graph the system of inequalities on the coordinate plane shown. Shade the region that represents the solution set.



You can use the intersection points to determine the optimal solution to a system of inequalities through a process called *linear programming*. In **linear programming**, the vertices of the solution region of the system of linear inequalities are substituted into a given equation to determine the maximum or minimum value.

3. Label the vertices of the shaded region.

4. Running burns 600 calories per hour, while strength training burns 250 calories per hour. Write an equation to represent this relationship. Be sure to define your variables.

5. Use the vertices of the solution region to determine how many hours James should devote to each type of exercise in order to maximize the number of calories burned.

Ask Yourself . . .
How can you use linear programming in everyday life?

6. A sponsor wants James to advertise their company by wearing its brand name on his workout gear. They will pay him \$15 for each hour he wears the brand name while running, and \$20 for each hour he wears the brand name while strength training. Determine how many hours James should devote to each type of exercise in order to maximize his earnings.

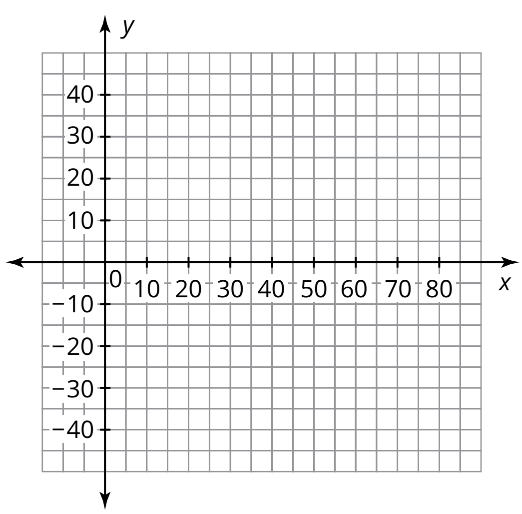
Talk the Talk

Making Bling for Cha-Ching

1. Catalina uses glass beads to make bracelets and necklaces. It takes her 30 minutes to make a bracelet and 45 minutes to make a necklace. She works at most 40 hours a week. She wants to make at least 30 bracelets. The profit from a bracelet is $10, and the profit from a necklace is $18.

   a. Write a system of linear inequalities in two variables to model the situation. Be sure to define your variables.

   b. Graph the system of linear inequalities on the coordinate plane.



   c. Identify one possible solution. Is this reasonable within the context of the problem situation?

d. Determine the solution that maximizes the profit.

2. Compare the solution of a system of linear inequalities and the solution to an equation calculated by linear programming.

Lesson 2 Assignment

Write

Define each term using your own words.

1. solution to a system of inequalities
2. linear programming

Remember

The solution to a system of linear inequalities is the intersection of the shaded regions for each linear inequality in the system. In linear programming, the vertices of the solution region of the system of linear inequalities are substituted into a given equation to find the maximum or minimum value.

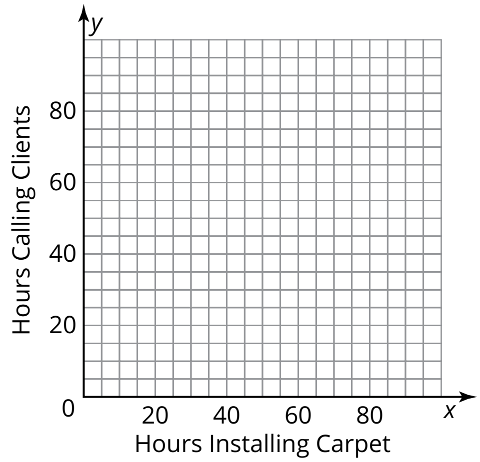
Practice

1. Ricardo works for a flooring company as a carpet installer. His primary responsibility is to install carpet. His secondary responsibility is to call prospective clients in order to earn the company's future business. At least 25% of his total number of work hours must be spent making these calls. According to his contract, he can work a maximum of 40 hours per week installing carpet and a maximum of 25 hours per week calling prospective clients. Ricardo is allowed a maximum of 60 total work hours per week. Ricardo is trying to determine the number of hours he should spend installing carpet and calling clients in order to maximize his income.

    a. Write a system of inequalities to represent the constraints of the problem situation. Define the unknowns in the inequalities.

Lesson 2 Assignment

b. Graph the inequalities on a coordinate plane. Shade the region that represents the solution set.



c. Identify one possible solution. Is this reasonable within the context of the problem situation?

d. Ricardo earns $25 for every hour he works installing carpet and $20 for every hour he works calling prospective clients. Determine the number of hours Ricardo should work installing carpet and the number of hours he should work calling clients in order to maximize his weekly income.

Lesson 2 Assignment

e. According to the flooring company's accountant, the company earns \$40 in profit from new business for every hour Ricardo spends calling prospective clients. The company also makes a profit of \$34 for every hour Ricardo spends installing carpets. Determine the number of hours the company should have Ricardo install carpets and the number of hours they should have Ricardo calling clients each week in order to maximize their weekly profit.

Lesson 2 Assignment

Prepare

Rewrite each expression with the fewest possible terms.

1. $(2x + 7y - 3z) + (-x + 4y + 6z)$

2. $(10x - 4y - 9z) - (3x - 5y + z)$

3. $\frac{1}{2}(-8x + 5y + 12z)$

3 Solving Systems with Matrices

OBJECTIVES

- Determine the dimensions of a matrix.
- Identify specific matrix elements.
- Determine the inverse of a matrix.
- Use matrices to write and solve systems of equations.

KEY TERMS

- matrix
- dimensions
- square matrix
- matrix element
- matrix multiplication
- identity matrix
- multiplicative inverse of a matrix
- matrix equation
- coefficient matrix
- variable matrix
- constant matrix

You have solved a system of three linear equations algebraically using substitution or Gaussian elimination.
How can you use matrices to write and solve a system of linear equations?

Getting Started

Track What Sold

The high school lacrosse team just completed their winter fundraisers. They sold \$20, \$25, and \$50 gasoline cards and \$20, \$25, and \$50 grocery cards during the months of November and December. Two parents volunteered to keep track of the total sales.

Mr. Lopez's record for the fundraisers is as follows:

> November: The team sold 230 gas cards for \$20, 110 gas cards for \$25, and 550 gas cards for \$50. The team sold 112 grocery cards for \$20, 89 grocery cards for \$25, and 95 grocery cards for \$50.
>
> December: The team sold 496 gas cards for \$20, 850 gas cards for \$25, and 348 gas cards for \$50. The team sold 129 grocery cards for \$20, 233 grocery cards for \$25, and 308 grocery cards for \$50.

Ms. Tanaka's record for the fundraisers is as follows:

$$\begin{array}{l} \\ \\ \text{\$20 cards} \\ \text{\$25 cards} \\ \text{\$50 cards} \end{array} \begin{array}{c} \text{November} \\ \begin{array}{cc} \text{Gas} & \text{Grocery} \end{array} \\ \begin{bmatrix} 230 & 112 \\ 110 & 89 \\ 550 & 95 \end{bmatrix} \end{array} + \begin{array}{l} \\ \\ \text{\$20 cards} \\ \text{\$25 cards} \\ \text{\$50 cards} \end{array} \begin{array}{c} \text{December} \\ \begin{array}{cc} \text{Gas} & \text{Grocery} \end{array} \\ \begin{bmatrix} 496 & 129 \\ 850 & 233 \\ 348 & 308 \end{bmatrix} \end{array}$$

1. Do Mr. Lopez's record and Ms. Tanaka's record give you the same information?

2. Which report is easier to read?

3. Calculate the total amount of money collected selling $25 grocery cards.

4. Calculate the total amount of money collected selling $50 gas cards.

5. Out of the 6 types of cards sold, which card sold the least?

6. Which report did you use to answer Questions 3, 4, and 5? Why?

7. Is there another way to record the fundraisers using Ms. Tanaka's method? Explain your reasoning.

ACTIVITY 3.1 Matrices and Their Inverses

Systems of equations can become cumbersome to solve by hand, particularly when the system contains three or more variables. There is another method that you can use to solve systems of linear equations using technology. It involves a mathematical object called a *matrix*. To better understand how technology uses a "matrix" to solve a system of equations, it is best to begin with a brief overview of the structure and operations of a matrix.

Rows are horizontal, while columns are vertical.

A **matrix** (plural **matrices**) is an array of numbers composed of rows and columns. A matrix is usually designated by a capital letter. The **dimensions** of a matrix are its number of rows and its number of columns. A **square matrix** is a matrix that has an equal number of rows and columns.

This matrix is an $n \times m$ matrix, because it has n rows and m columns.

$$\begin{bmatrix} a_{11} & a_{12} \dots a_{1m} \\ a_{21} & a_{22} \dots a_{2m} \\ \vdots & \vdots \quad \vdots \quad \vdots \\ a_{n1} & a_{n2} \dots a_{nm} \end{bmatrix}$$

Each number in the matrix is known as a **matrix element**. Each matrix element is labeled using the notation a_{ij}, where a_{ij} means the number in the ith row and the jth column.

The store in your town just got a shipment of a top-selling running shoe. They are available in sizes 9 through 13 and in widths *C*, *D*, *E*, and *EE*. The manager of the store keeps track of the inventory according to the number of pairs of shoes in each size and width using the matrix shown.

$$A = \begin{array}{c} \\ C \\ D \\ E \\ EE \end{array} \begin{array}{c} \begin{array}{ccccc} 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{bmatrix} 3 & 5 & 3 & 5 & 0 \\ 5 & 7 & 6 & 4 & 3 \\ 6 & 5 & 7 & 7 & 2 \\ 2 & 0 & 8 & 5 & 2 \end{bmatrix} \end{array}$$

1. Consider matrix *A*.

   a. Describe what each row and each column represents in terms of the problem situation.

   b. What are the dimensions of this matrix?

c. Determine the number that is in each location. Describe the meaning of the matrix element in terms of the problem situation.

- The element a_{34}

- The element a_{45}

- The element a_{21}

2. Another shoe store in a neighboring town carries the same types of shoes and represents its inventory using matrix B.

$$B = \begin{array}{c} \\ C \\ D \\ E \\ EE \\ EEE \end{array} \begin{array}{c} \begin{array}{cccc} 10 & 11 & 12 & 13 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 4 & 4 & 6 \\ 1 & 5 & 5 & 7 \\ 2 & 3 & 3 & 5 \\ 2 & 2 & 2 & 6 \end{bmatrix} \end{array}$$

a. What are the dimensions of matrix B?

b. What matrix element represents the number of size 10E shoes? Write your answer in matrix notation.

c. How is b_{42} different from b_{24}?

Each value in matrix A is the number of times that sprinter finished in that place. $A_{11} = 3$, which means Lauren finished in first place 3 times.

Coach Flores' team is at their first track meet. The finishing places and number of medals earned for four of her sprinters are displayed in matrix A. Points are awarded according to the scoring matrix B.

$$A = \begin{array}{r} \\ \text{Lauren} \\ \text{Nakota} \\ \text{Alejandra} \\ \text{Hailey} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{1st} & \text{2nd} & \text{3rd} & \text{4th} & \text{5th} \end{array} \\ \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \qquad B = \begin{array}{r} \\ \text{1st} \\ \text{2nd} \\ \text{3rd} \\ \text{4th} \\ \text{5th} \end{array} \begin{array}{c} \text{Points} \\ \begin{bmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 2 \end{bmatrix} \end{array}$$

3. Determine each sprinter's score for the meet.

4. Describe the method you used to calculate each sprinter's score.

5. Record the sprinters' overall scores as a 4×1 matrix. Label the rows and columns.

When you determined the matrix representing the overall scores for each sprinter, you actually performed a process called *matrix multiplication*.

In **matrix multiplication**, an element a_{pq} of the product matrix is determined by multiplying each element in row p of the first matrix by an element from column q in the second matrix and calculating the sum of the products. In order to multiply matrices, the number of columns in the first matrix must be the same as the number of rows in the second matrix.

Sprinter's Matrix **Scoring Matrix** **Results Matrix**

$$A = \begin{array}{l} \\ \text{Lauren} \\ \text{Nakota} \\ \text{Alejandra} \\ \text{Hailey} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{1st} & \text{2nd} & \text{3rd} & \text{4th} & \text{5th} \end{array} \\ \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \cdot \begin{array}{r} \\ \text{1st} \\ \text{2nd} \\ \text{3rd} \\ \text{4th} \\ \text{5th} \end{array} \begin{array}{c} \text{Points} \\ \begin{bmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 2 \end{bmatrix} \end{array} = \begin{array}{l} \\ \text{Lauren} \\ \text{Nakota} \\ \text{Alejandra} \\ \text{Hailey} \end{array} \begin{array}{c} \text{Points} \\ \begin{bmatrix} 38 \\ 24 \\ 34 \\ 34 \end{bmatrix} \end{array}$$

6. Record the dimensions for each of the matrices in the table.

Dimensions of Sprinter's Matrix	Dimensions of Scoring Matrix	Dimensions of Results Matrix

Think About . . .
How does the number of columns in the sprinter's matrix compare to the number rows in the scoring matrix?

7. Joey made an observation about the dimensions of the product matrix.

Joey

sprinters $\underbrace{\text{by medals • medals by points}}_{\text{results}}$ = sprinters × points

Write a generalization about the dimensions of the product matrix, based on Joey's statement.

A product matrix AB is defined if the number of columns in matrix A is the same as the number of rows in matrix B so that multiplication can be performed.

8. Consider the matrices $A = \begin{bmatrix} 1 & 0 & -5 & 3 \\ -2 & 1 & 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ 0 & 1 \\ -1 & 5 \\ 2 & 0 \end{bmatrix}$.

   a. Is the product matrix AB defined? Explain your reasoning.

   b. Predict the dimensions of the product matrix AB. Explain your reasoning.

The **identity matrix**, I, is a square matrix whose elements are 0s and 1s. The 1s are arranged diagonally from upper left to lower right as shown.

Recall that for any real number n, the multiplicative inverse is $\frac{1}{n}$ because the product of n and $\frac{1}{n}$ is the multiplicative identity, or 1. Similarly, when a matrix is multiplied by its inverse matrix, their product is the *identity matrix*.

The **multiplicative inverse of a matrix** (or just inverse) of A is designated as A^{-1}, and is a matrix such that $A \cdot A^{-1} = I$ or the identity matrix. Non-square matrices do not have inverses.

9. Jackson used his knowledge of multiplicative inverses to make an assumption about the inverse of matrix R. Use matrix multiplication to determine why Jackson's conclusion is incorrect.

Jackson

$$R = \begin{bmatrix} \frac{2}{3} & 1 \\ 6 & -1 \end{bmatrix}$$

Matrix S is the inverse of matrix R because each matrix element in matrix S is the reciprocal of the corresponding matrix element in matrix R.

$$S = \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{6} & -1 \end{bmatrix}$$

$I_1 = [1]$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. Use technology to determine whether the two matrices are inverses. Explain your reasoning.

a. $C = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

b. $G = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ $H = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & 0 \end{bmatrix}$

ACTIVITY

3.2 Solving Matrix Equations

Matrices can be used to solve a system of equations. A system can be written as a **matrix equation**, or an equation with matrices.

Consider a system of three linear equations in three variables, written in standard form: $Ax + By + Cz = D$. The system can be written as a matrix equation $A \cdot X = B$, by writing it as the product of a *coefficient matrix*, A, and a *variable matrix*, X, equal to a *constant matrix*, B.

A **coefficient matrix** is a square matrix that consists of each coefficient of each equation in the system of equations, in order, when they are written in standard form. A **variable matrix** is a matrix in one column that represents all of the variables in the system of equations. A **constant matrix** is a matrix in one column that represents each of the constants in the system of equations.

In a system with n equations, the coefficient matrix will be an $n \times n$ matrix, the variable matrix will be an $n \times 1$ matrix, and the constant matrix will be an $n \times 1$ matrix.

WORKED EXAMPLE

Consider the system of three linear equations in three variables:

$$\begin{cases} 2x - y - 2z = 3 \\ 3x + y - 2z = 11 \\ -2x - y + z = -8. \end{cases}$$

Each equation is written in standard form $Ax + By + Cz = D$.

In the matrix equation $A \cdot X = B$, A is a 3×3 matrix consisting of each coefficient, X is a 3×1 matrix representing all of the variables in the system, and B is a 3×1 matrix consisting of each constant term.

$$\begin{bmatrix} 2 & -1 & -2 \\ 3 & 1 & -2 \\ 2 & -1 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix}$$

$$\text{Coefficient Matrix} \longrightarrow A \cdot \underset{\substack{\uparrow \\ \text{Variable Matrix}}}{X} = B \longleftarrow \text{Constant Matrix}$$

1. Verify that the matrix equation is an equivalent representation of the system of equations.

   a. Use matrix multiplication to calculate $A \cdot X$.

   b. Write the system of equations that the matrix equation represents. Justify your reasoning.

   c. Is the matrix equation an equivalent representation of the system of equations?

2. Write each system of equations as a matrix equation.

a. $\begin{cases} x + 2y - 2z = -8 \\ 2x - y + z = -1 \\ -x + 3y + 2z = 13 \end{cases}$

b. $\begin{cases} 3x + 2y + 2z = 4 \\ 5x = -4y \\ -x + 2y - z = 12 \end{cases}$

3. Describe the error in Nicky's method.

Is each equation of the form $Ax + By + Cz = D$?

Nicky's

$$\begin{cases} 2x - 4y = 5z - 1 \\ -x = 5y - 3z \\ 3x - 5y - 3z = 14 \end{cases}$$

$$\begin{bmatrix} 2 & -4 & 5 \\ -1 & 5 & -3 \\ 3 & -5 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 14 \end{bmatrix}$$

You can use technology with matrices as a tool when solving matrix equations.

WORKED EXAMPLE

Consider the system of equations $\begin{cases} 2x - y - 2z = 3 \\ 3x + y - 2z = 11 \\ -2x - y + z = -8 \end{cases}$

Step 1: Write the system as a matrix equation.

$$\begin{bmatrix} 2 & -1 & -2 \\ 3 & 1 & -2 \\ 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix}.$$

$$A \cdot X = B$$

Step 2: Calculate the inverse of A using your choice of technolgy.

$$A^{-1} = \begin{bmatrix} 1 & 3 & 4 \\ -1 & -2 & -2 \\ 1 & 4 & 5 \end{bmatrix}$$

Step 3: Use technology to multiply the inverse by matrix B to calculate the solution to the system.

$$\begin{bmatrix} 1 & 3 & 4 \\ -1 & -2 & -2 \\ 1 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

The solution to the matrix equation is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, or (2, 3, –1).

4. Nia is asked to solve a system of three linear equations using technology with matrices and she gets an error message letting him know it is a noninvertible matrix.

Nia

$$\begin{cases} x + 2y - z = 1 \\ 2x - 2y - z = 2 \\ 3x + 6y - 3z = 3 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -1 \\ 3 & 6 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A \cdot X = B$$
$$X = A^{-1} \cdot B$$

Why might Nia have seen the error message?

When any row of a matrix is a multiple of another row, or any column is a multiple of another column, the inverse of the matrix does not exist. When the coefficient matrix is noninvertible, the system is not able to be solved. In this case, the system will either have many solutions or no solution.

When using technology, you will get a similar error message for both cases, because in both cases, the inverse of the coefficient matrix does not exist. You can use the constant matrix to differentiate between a system with many solutions and a system with no solution.

WORKED EXAMPLE

Many Solutions

$$\begin{cases} x + 5y - 2z = 3 \\ 2x + 10y - 4z = 6 \\ x + y + 3z = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & 2 \\ 2 & 10 & -4 \\ 1 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}$$

If each variable in one equation is a multiple of another equation, and the constants are a multiple by the same factor, the system will have many solutions.

No Solution

$$\begin{cases} x + 5y - 2z = 3 \\ 2x + 10y - 4z = 7 \\ x + y + 3z = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & 2 \\ 2 & 10 & -4 \\ 1 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ 1 \end{bmatrix}$$

If each variable in one equation is a multiple of another equation, and the constants are not a multiple by the same factor, the system will have no solution.

5. Determine whether the system of equations in Question 4 has no solution or many solutions.

6. Two of the equations in the system of three linear equations are given.

$$\begin{cases} 5x - 3y + z = -4 \\ x + 2y - 3z = 0 \end{cases}$$

a. Write a third equation that would produce a system with no solution. Explain your reasoning.

b. Write a third equation that would produce a system with many solutions.

7. Write each system of equations as a matrix equation. Then calculate the solution to each system of linear equations by using technology with matrices.

a. $\begin{cases} 2x - 3y = 7 \\ y + z = -5 \\ x + 2y + 4z = -17 \end{cases}$

b. $\begin{cases} 5x + y + 3z = 9 \\ -x - 2y - z = -16 \\ 2x + 4y + 2z = -30 \end{cases}$

c. $\begin{cases} x - 4y + 3z = -7 \\ 2x + 3y - 5z = 19 \\ 4x + y - z = 17 \end{cases}$

d. $\begin{cases} 2x - 3z = 4 \\ 2x + y - 5z = -1 \\ 3y - 4z = 2 \end{cases}$

Talk the Talk

Show Us Your Stuff

As you learned in a previous lesson, you can model a real-world situation using a system of linear equations. Now you have another tool to use to help you solve the system.

1. Your school's Key Club decided to sell fruit baskets to raise money for a local charity. The club sold a total of 80 fruit baskets. There were three different types of fruit baskets. Small fruit baskets sold for $15.75 each, medium fruit baskets sold for $25 each, and large fruit baskets sold for $32.50 each. The Key Club took in a total of $2086.25, and they sold twice as many large baskets as small baskets.

    a. Formulate a system of three linear equations in three variables to represent this problem situation. Be sure to define your variables.

    b. Solve the system of three linear equations using technology with matrices. Write your answer in terms of the problem situation.

c. Describe the solution in terms of the problem situation.

2. Is it always easier to solve a system of linear equations using matrices? Explain your reasoning.

Lesson 3 Assignment

Write

Rewrite the definition for the following terms in your own words.

1. multiplicative identity matrix
2. multiplicative inverse of a square matrix
3. matrix equation
4. variable matrix
5. constant matrix
6. coefficient matrix

Remember

When a matrix is multiplied by its inverse matrix, their product is the identity matrix. Non-square matrices do not have inverses. Matrices can be used to solve a system of equations by writing the system as a matrix equation. Technology can be used to determine the inverse of a matrix and to solve matrix equations.

Practice

1. Use matrix multiplication to determine whether the two matrices are inverses. Explain your reasoning.

a. $M = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ $N = \begin{bmatrix} -1 & 0 & 0 \\ -2 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Lesson 3 Assignment

2. Consider the system of three linear equations in three variables

$$\begin{cases} 2x + 2z = 2 \\ 5x + 3y = 4 \\ 3y - 4z = 4 \end{cases}$$

a. Write a matrix equation for the system in the form $A \cdot X = B$.

b. Use technology to determine A^{-1}.

c. Solve the matrix equation.

3. Maria is helping with a school fundraiser by selling fruit baskets. Basket A contains 3 apples, 2 pears, and 4 oranges and sells for \$9.65. Basket B contains 4 apples, 3 pears, and 3 oranges and sells for \$10.70. Basket C contains 2 apples, 2 pears, and 2 oranges and sells for \$6.30. What is the cost of each apple, pear and orange?

a. Write a system of equations to represent the scenario. Define your variables.

b. Write the system of equations as a matrix equation.

c. Calculate the solution to the system of linear equations by using technology with matrices.

4. A middle school theater department sells tickets for their upcoming production. A child's ticket costs \$3.50, a student ticket costs \$5, and an adult ticket costs \$8.50. They sell the same number of student tickets as adult tickets. They sold a total of 82 tickets and the total income from ticket sales was \$495.

   a. Write a system of three linear equations in three variables to represent this scenario. Define your variables.

b. Write the system of equations as a matrix equation.

c. How many of each ticket type did the theater department sell? Calculate the solution to the system of linear equations using technology with matrices. Round decimals to four decimal places.

Prepare

Solve each equation.

1. $2x - 5 = 97$

2. $\frac{1}{3}x + 40 = 280$

3. $-4x - 10 = -26$

4 Inverses of Linear Functions

OBJECTIVES

- Determine the inverse of a given situation using words.
- Determine the inverse of a function numerically using a table, an equation, and a graph, and explain the process orally.
- Determine whether given functions are one-to-one functions.
- Identify function types that are always, sometimes, or never one-to-one functions.

KEY TERMS

- inverse of a function
- one-to-one function

You know that a function takes a set of inputs and maps them to a set of outputs.
What happens when the outputs and inputs are reversed?

Getting Started

Inside an Enigma

One method of creating a code is called a substitution cipher. For a substitution cipher, you can take each letter of the alphabet in numeric order and assign it to a different number using a mathematical rule.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

For example, if the cipher were written as $x + 4$, then L, which is currently assigned to 12, would be assigned to $12 + 4 = 16$, or P, in the code. The letter Y, which is currently assigned to 25, would be assigned to 3, or C, in the code. The word "inverse" would have the code "mrzivwi."

Cipher: $x + 4$

Word	Code
inverse	mrzivwi

1. Write a substitution cipher rule. Then write a short note to a classmate in code. Give your rule and coded note to a classmate to decode.

2. Use mathematical notation to write the rule you used to decode your classmate's note.

3. Why is it important that the substitution cipher rule be a function?

4. Compare the inputs and outputs of the cipher rule you created and the rule used to decode your note. What do you notice?

ACTIVITY 4.1

The Inverse of a Function

Miguel is planning a trip to Poland. Before he leaves, he wants to exchange his money for the Polish zloty, the official currency of Poland. The exchange rate as of January 2025 is 4 zloty per 1 U.S. dollar.

1. Complete the table of values to show the currency conversion for U.S. dollars to Albanian Lek.

U.S. Currency (dollars)	Polish Currency (zloty)
100	
250	
400	
650	
1000	

2. Write an equation to represent the number of zloty in terms of the number of U.S. dollars.

Suppose at the end of his trip, Miguel needs to convert any remaining zloty to dollars. This situation is the *inverse* of the original situation.

3. What are the independent and dependent quantities of the inverse of the problem situation? How do these quantities compare to the quantities in Question 1?

4. Complete the table of values to show the inverse of the problem situation.

Polish Currency (zloty)	U.S. Currency (dollars)

5. Compare the tables in Questions 1 and 4. What do you notice?

6. Use the table to write an equation for the inverse of the problem situation. Does this equation represent a function? Explain your answer.

Recall that a function takes an input value, performs some operation(s) on this value, and creates an output value. The **inverse of a function** takes the output value, performs some operation(s) on this value, and arrives back at the original function's input value. In other words, an inverse of a function "undoes" the function.

WORKED EXAMPLE

Given a function, $f(x)$, you can determine the inverse algebraically by following these steps.

$$f(x) = 2x + 3$$

Step 1: Replace the function $f(x)$ with another variable, generally y.

$$y = 2x + 3$$

Step 2: Switch the x and y variables in the equation.

$$x = 2y + 3$$

Step 3: Solve for y.

$$x - 3 = 2y$$

$$\frac{1}{2}x - \frac{3}{2} = y$$

$$y = \frac{1}{2}x - \frac{3}{2}$$

7. Use function notation to represent the number of zloty $f(x)$ in terms of the number of U.S. dollars, x. Then complete the steps shown in the Worked Example to represent the number of U.S. dollars in terms of the number of zloty. Compare the inverse to the equation you wrote in Question 6. What do you notice?

ACTIVITY 4.2 Graphing Inverses of Functions

In the previous activity, you wrote the inverse of a linear function using algebra. Let's consider how to show the inverse of a function using its graph.

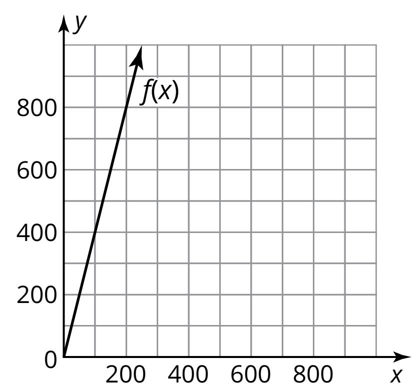
Given a function, $f(x)$, you can determine the inverse of a function graphically by following these steps.

Step 1: Copy the coordinate plane and graph $f(x)$ and the line $y = x$ onto patty paper.

Step 2: Heavily trace the graph of $f(x)$ with a pencil.

Step 3: Reflect the patty paper across the line $y = x$, and rub the paper so that the image of the graph of its inverse appears.

1. Consider the graph of the function $f(x) = 4x$ from the previous activity. Complete the steps in the Worked Example to graph the inverse using patty paper.



How do the algebraic process and the graphical process to determine an inverse compare?

2. Compare the image you created and the graph of the inverse.

   a. What do you notice about the image and the graph of the inverse?

   b. What does this tell you about the graph of a function and its inverse and about the line $y = x$?

3. For each function and a given point on the graph of the function, determine the corresponding point on the graph of the inverse of the function.

   a. Given that (3, 2) is a point on the graph of $g(x)$, what is the corresponding point on the graph of the inverse of $g(x)$?

   b. Given that (–1, 0) is a point on the graph of $h(x)$, what is the corresponding point on the graph of the inverse of $h(x)$?

   c. Given that (a, b) is a point on the graph of $f(x)$, what is the corresponding point on the graph of the inverse of $f(x)$?

ACTIVITY
4.3

One-to-One Functions

In this activity, you will determine the inverse of a function using multiple representations.

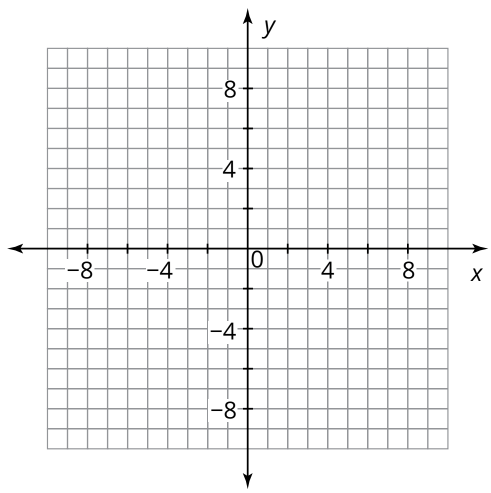
1. For each given function, determine the inverse using each representation.
   - Complete a table of values for the function and its inverse.
   - Sketch the graph of the function using a solid line. Then sketch the inverse of the function on the same coordinate plane using a dashed line.
   - Write an equation for the inverse.
   - Determine whether the function is a one-to-one function. Explain your reasoning.

A function is a **one-to-one function** if both the function and its inverse are functions.

Remember. . .
Use a straightedge to draw your lines.

a. $f(x) = 3x - 6$

x	$f(x)$
−2	
−1	
0	
1	
2	

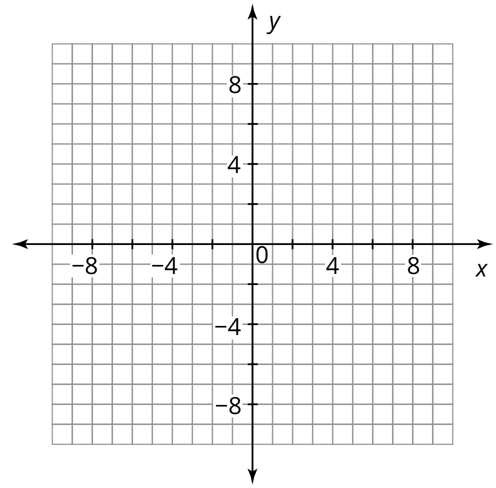


Inverse of $f(x)$	
x	y
	−2
	−1
	0
	1
	2

Inverse Equation:

b. $g(x) = -x + 4$

x	$g(x)$
-2	
-1	
0	
1	
2	

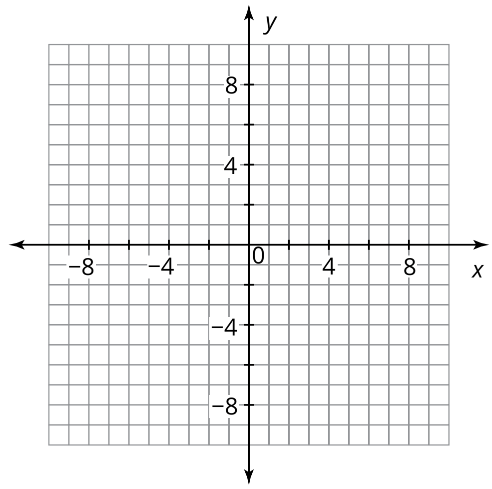


Inverse of $g(x)$	
x	y
	-2
	-1
	0
	1
	2

Inverse Equation:

c. $h(x) = 2$

x	$h(x)$
−2	
−1	
0	
1	
2	

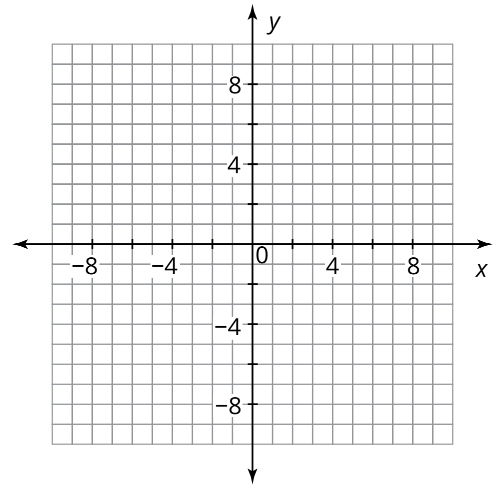


Inverse of $h(x)$	
x	y
	−2
	−1
	0
	1
	2

Inverse Equation:

d. $r(x) = |x|$

x	$r(x)$
−2	
−1	
0	
1	
2	

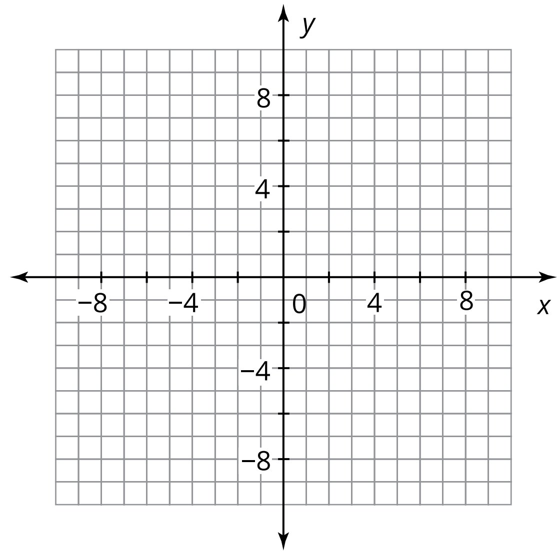


Inverse of $r(x)$	
x	y
	−2
	−1
	0
	1
	2

Inverse Equation:

e. $s(x) = x^2 + 4$

$s(x)$	
x	y
−2	
−1	
0	
1	
2	

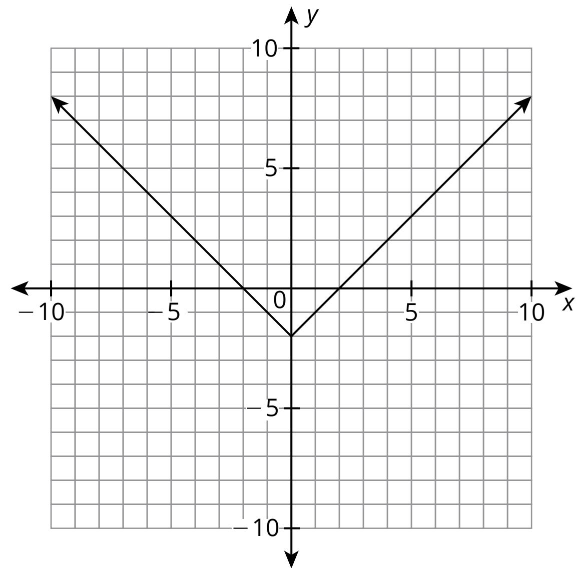


Inverse of $s(x)$	
x	y
	−2
	−1
	0
	1
	2

Inverse Equation:

f. $v(x) = |x| - 2$

$v(x)$	
x	y
−2	
−1	
0	
1	
2	



Inverse of $v(x)$	
x	y
	−2
	−1
	0
	1
	2

Inverse Equation:

2. How can you determine whether the inverse of a linear function exists?

3. Can a linear function and its inverse be the same function? If so, provide an example. If not, explain why not.

Ask Yourself . . .
How can you organize and record your mathematical ideas?

4. Complete the graphic organizer on the next page. Write the definition for the inverse of a function. Then describe how to determine the inverse of a function algebraically, graphically, and numerically.

   For a one-to-one function $f(x)$, the notation for its inverse is $f^{-1}(x)$.

   The notation for inverse, $f^{-1}(x)$, does not mean the same thing as x^{-1}. The expression x^{-1} can be rewritten as $\frac{1}{x}$; however, $f^{-1}(x)$ cannot be rewritten, because it is only used as notation. In other words, $f^{-1}(x) \neq \frac{1}{f(x)}$.

5. Which of the inverse equations from Question 1 can be written with the inverse notation $f^{-1}(x)$?

Graphic Organizer

Definition

Algebraic Description

INVERSES OF FUNCTIONS

Graphical Description

Numeric Description (Table of Values)

Talk the Talk

1-2-1

In this lesson, you determined the inverses of linear functions. You also determined whether the inverses were also functions.

Recall that a function is a one-to-one function if both the function and its inverse are functions.

1. Xavier and Lucia are working on a homework assignment for which they must identify all functions that are one-to-one functions. Xavier says that all linear functions are one-to-one functions, so they don't even need to look at the linear functions. Lucia disagrees and says that not all linear functions are one-to-one functions. Who is correct? Explain how you determined which student is correct.

2. Complete each sentence with *always*, *sometimes*, or *never*.

   a. A linear function is ____________________ a one-to-one function.

   b. An exponential function is ____________________ a one-to-one function.

   c. A quadratic function is ____________________ a one-to-one function.

   d. An absolute value function is ____________________ a one-to-one function.

Lesson 4 Assignment

Write

Describe how to use a graph to prove two relationships are inverses of each other.

Remember

An inverse of a function "undoes" the function. The inverse of a function is determined by replacing $f(x)$ with y, switching the x and y variables, and solving for y. A one-to-one function is a function in which its inverse is also a function.

Practice

1. Clothing and shoe sizes typically vary from country to country. Catalina is going to be spending a year in Italy and plans on shopping for dresses while she's there. While investigating the differences in sizing, she determines that to change the U.S. dress size to an Italian dress size, she must add 12 to the U.S. dress size and then double the sum.

    a. Complete the table of values to show the dress size conversion for U.S. dresses to Italian dresses.

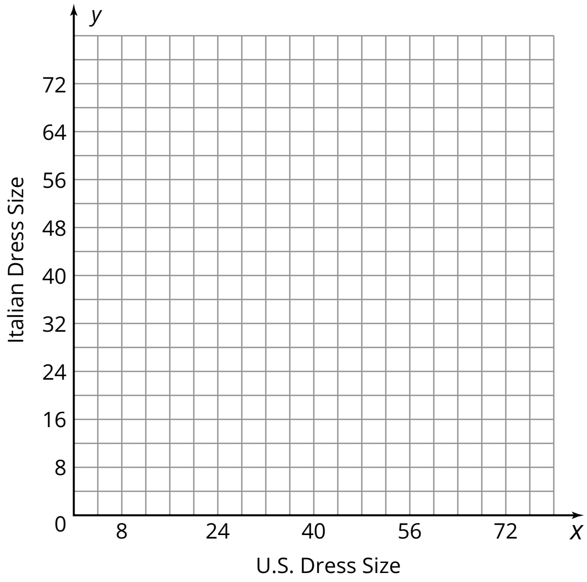
    b. Write a function, $f(x)$, to represent the Italian dress size in terms of x, the U.S. dress size.

    c. Determine the inverse of this problem situation using words.

U.S. Dress Size	Italian Dress Size
4	
6	
10	
14	
18	

d. Determine the inverse of the function algebraically. What does the inverse function represent in terms of the problem situation?

e. Sketch the graph of the original function. Then, sketch the inverse on the same graph.



2. Determine the inverse of each function. Is the inverse also a function? Explain why or why not.

a. $f(x) = 0.6x - 2$

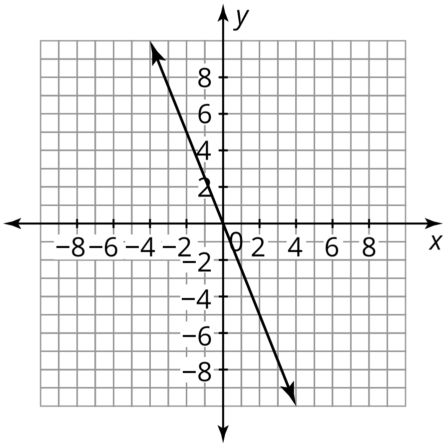
b. $g(x) = \frac{8}{3}x + 12$

c. $h(x) = -2x^2 + 8$

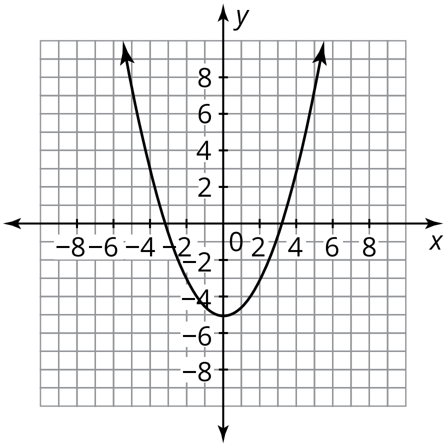
3. Determine whether the functions $j(x) = 3 - 1.5x$ and $k(x) = -\frac{2}{3}x + 2$ are inverses. If so, explain how you know. If not, determine each function's inverse.

4. Sketch the inverse of the given function on the same graph as the function. Is the inverse also a function? Explain why or why not.

a.



b.



Lesson 4 Assignment

Prepare

Factor each expression.

1. $x^2 + 7x + 12$

2. $2x^2 - 7x - 15$

3. $x^2 - 49$

4. Determine the value of b that makes the quadratic expression factorable.

   a. $2x^2 + bx + 3 = (2x + 1)(x + 3)$

   b. $6x^2 - 5x + b = (3x + 2)(2x - 3)$

   c. $bx^2 - 10x + 8 = (3x - 4)(x - 2)$

TOPIC 2 SELF-REFLECTION

Name: ____________________

Applications of Linear Relationships

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me,** 2 represents **I am building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Applications of Linear Relationships* topic by:

TOPIC 2: *Applications of Linear Relationships*	Beginning of Topic	Middle of Topic	End of Topic
defining a system of equations and solutions of a system.	○	○	○
solving a system of three linear equations in three variables using substitution, linear combinations, and technology using matrices.	○	○	○
explaining why some systems have no solutions or infinitely many solutions.	○	○	○
using Gaussian elimination to solve a system of three equations in three variables algebraically.	○	○	○
modeling a real-world scenario with a system of equations or inequalities.	○	○	○
interpreting solutions to a system of equations or inequalities in terms of a context.	○	○	○
graphing a system of linear inequalities on a coordinate plane and locating points that satisfy the system.	○	○	○
representing a system of linear equations as a matrix equation.	○	○	○
solving a system of three linear equations in three variables using technology with matrices.	○	○	○

continued on the next page

TOPIC 2 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Applications of Linear Relationships* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 2 SUMMARY

Applications of Linear Relationships

LESSON 1 Solving Systems of Equations

Recall that a system of two linear equations can be solved algebraically by using either the substitution method or the linear combinations method. Systems of two linear equations can have one solution, no solution, or an infinite number of solutions.

To solve a system of three linear equations using substitution, the first step is to solve for one variable in one of the equations. Then substitute this expression for that variable in the other two equations. The two new equations will then have only two unknown variables and can be solved using either substitution or linear combinations.

The goal of **Gaussian elimination** is to use linear combinations to isolate one variable for each equation. When using this method, you can:

- swap the positions of two equations.
- multiply an equation by a nonzero constant.
- add one equation to the multiple of another.

For example, you can solve the system $\begin{cases} x + 5y - 6z = 24 \\ -x - 4y + 5z = -21 \\ 5x - 4y + 2z = 21 \end{cases}$ using Gaussian elimination.

Add the first and second equation and replace the second equation.

$$\begin{array}{r} x + 5y - 6z = 24 \\ -x - 4y + 5z = -21 \\ \hline y - z = 3 \end{array} \rightarrow \begin{cases} x + 5y - 6z = 24 \\ y - z = 3 \\ 5x - 4y + 2z = 21 \end{cases}$$

Multiply the first equation by -5 and add it to the third equation. Replace the third equation.

$$\begin{array}{r} -5x - 25y + 30z = -120 \\ 5x - 4y + 2z = 21 \\ \hline -29y + 32z = -99 \end{array} \rightarrow \begin{cases} x + 5y - 6z = 24 \\ y - z = 3 \\ -29y + 32z = -99 \end{cases}$$

KEY TERMS

- Gaussian elimination [eliminación gaussian]
- solution set of a system of linear inequalities [solución de un sistema de desigualdades lineales]
- linear programming [programación lineal]
- matrix [matriz]
- dimensions [dimensiones]
- square matrix
- matrix element [elemento de una matriz]
- matrix multiplication [multiplicación de matrices]
- identity matrix [matriz de identidad]
- multiplicative inverse of a matrix [inversa multiplicativa de una matriz]
- matrix equation [ecuación matricial]
- coefficient matrix [matriz de coeficientes]
- variable matrix [matriz de variables]
- constant matrix [matriz constante]
- inverse of a function [inversa de una función]
- one-to-one function

Multiply the second equation by 29 and add it to the third equation. Replace the third equation.

$$\begin{aligned} 29y - 29z &= 87 \\ -29y + 32z &= -99 \\ \hline 3z &= -12 \end{aligned} \quad \rightarrow \quad \begin{cases} x + 5y - 6z = 24 \\ y - z = 3 \\ 3z = -12 \end{cases}$$

Multiply the third equation by $\frac{1}{3}$ and add it to the second equation. Replace the second equation.

$$\begin{aligned} y - z &= 3 \\ z &= -4 \\ \hline y &= -1 \end{aligned} \quad \rightarrow \quad \begin{aligned} x + 5y - 6z &= 24 \\ y &= -1 \\ 3z &= -12 \end{aligned}$$

Multiply the second equation by -5 and add it to the first equation. Replace the first equation.

$$\begin{aligned} x + 5y - 6z &= 24 \\ -5y \qquad &= 5 \\ \hline x \qquad - 6z &= 29 \end{aligned} \quad \rightarrow \quad \begin{cases} x - 6z = 29 \\ y = -1 \\ 3z = -12 \end{cases}$$

Multiply the third equation by 2 and add it to the first equation. Replace the first equation.

$$\begin{aligned} x - 6z &= 29 \\ 6z &= -24 \\ \hline x \qquad &= 5 \end{aligned} \quad \rightarrow \quad \begin{cases} x = 5 \\ y = -1 \\ 3z = -12 \end{cases}$$

Multiply the third equation by $\frac{1}{3}$. Replace the third equation.

$$\begin{aligned} \tfrac{1}{3}(3z &= -12) \\ z &= -4 \end{aligned} \quad \rightarrow \quad \begin{cases} x = 5 \\ y = -1 \\ z = -4 \end{cases}$$

The solution to the system is $(x, y, z) = (5, -1, -4)$.

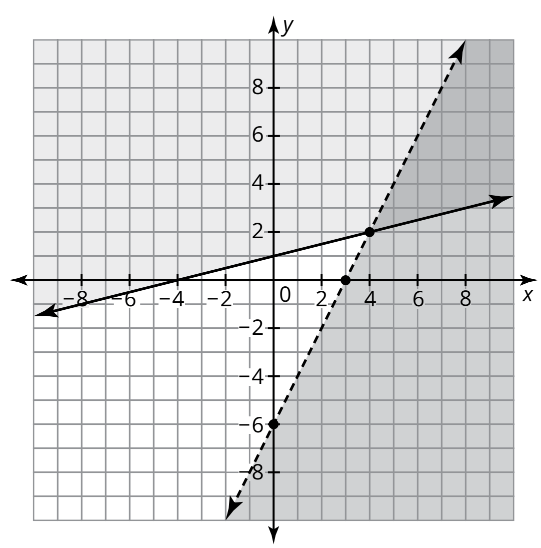
LESSON 2 Optimization

The **solution set of a system of inequalities** is the intersection of the solutions to each inequality. Every point in the intersection satisfies all inequalities in the system.

For example, consider the system:

$$\begin{cases} y < 2x - 6 \\ y \geq \frac{1}{4}x + 1 \end{cases}.$$

Graph the system of linear inequalities on one coordinate plane.



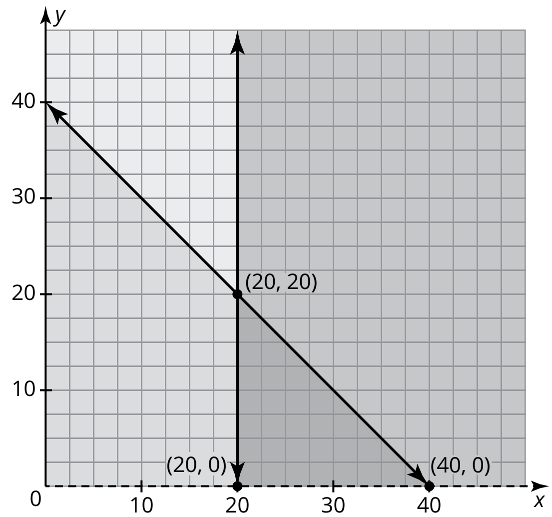
The solution to the system of linear inequalities is the overlapping region of the solutions to each inequality in the system. The points (6, 4) and (8, 8) are both solutions to the system of linear inequalities.

In **linear programming**, the vertices of the solution region of the system of linear inequalities are substituted into an equation to find the maximum or minimum value.

For example, consider the situation in which Lauren works as a lifeguard and as a tutor over the summer. She can work no more than 40 hours each week. Her lifeguarding job requires that she work at least 20 hours each week. If Lauren earns $15 for each hour she lifeguards and $25 for each hour she tutors, how many hours should she work at each job to maximize her earnings?

The situation can be modeled by a system of linear inequalities. Let x represent the number of hours lifeguarding and let y represent the number of hours tutoring.

$$\begin{cases} x + y \leq 40 \\ x \geq 20 \\ x > 0 \\ y > 0 \end{cases}$$



The vertices of the solution region are (20, 0), (20, 20), and (40, 0).

Substitute the coordinates into an equation representing Lauren's total earnings for both jobs.

$15x + 25y = E$
$15(20) + 25(0) = 300$
$15(20) + 20(20) = 700$
$15(40) + 25(0) = 600$

Lauren should work 20 hours as a lifeguard and 20 hours as a tutor to maximize her earnings, which would be $700.

LESSON 3 Solving Systems with Matrices

The **identity matrix**, I, is a square matrix such that for any matrix A, $AI = A$. The elements of the identity matrix are all zeros except for a diagonal from the upper left to the lower right where the elements are all ones. The **multiplicative inverse of a matrix**, A, labeled as A^{-1}, is a matrix such that when matrix A is multiplied by it the result is the identity matrix. In symbols, $A \cdot A^{-1} = I$. You can use technology to find the inverse of a square matrix. Not every square matrix has an inverse, and non-square matrices do not have inverses.

You can use matrix multiplication to verify that matrix J^{-1} is the inverse of matrix J.

$$J = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \qquad J^{-1} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} (-2)\left(-\frac{3}{2}\right) + 1(-2) & (-2)\left(\frac{1}{2}\right) + 1(1) \\ (-4)\left(-\frac{3}{2}\right) + 3(-2) & (-4)\left(\frac{1}{2}\right) + 3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 - 2 & -1 + 1 \\ 6 - 6 & -2 + 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since the product matrix $J \cdot J^{-1}$ yields the identity matrix, J^{-1} is the inverse of matrix J.

A system of linear equations can be written as a **matrix equation**, in the form $A \cdot X = B$, where A is the coefficient matrix, X is the variable matrix, and B is the constant matrix. A **coefficient matrix** is a square matrix that consists of each coefficient of each equation in the system of equations, in order, when they are written in standard form. A **variable matrix** is a matrix in one column that represents all of the variables in the system of equations. A **constant matrix** is a matrix in one column that represents each of the constants in the system of equations.

In a system with n equations, the coefficient matrix will be an $n \times n$ matrix, the variable matrix will be an $n \times 1$ matrix, and the constant matrix will be an $n \times 1$ matrix.

For example, the system of equations $\begin{cases} 2x + 8y + 3z = -21 \\ 7x - 5y + z = 72 \\ -6x - 2y + 3z = 17 \end{cases}$ can be written as the matrix equation shown.

$$\underbrace{\begin{bmatrix} 2 & 8 & 3 \\ 7 & -5 & 1 \\ -6 & -2 & 3 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -21 \\ 72 \\ 17 \end{bmatrix}}_{B}$$

To solve the matrix equation $A \cdot X = B$, multiply both sides by the inverse of matrix A, or A^{-1}. You can use technology with matrices to calculate the solution.

$$A^{-1} = \begin{bmatrix} \frac{13}{374} & \frac{15}{187} & -\frac{23}{374} \\ \frac{27}{374} & -\frac{12}{187} & -\frac{19}{374} \\ \frac{2}{17} & \frac{2}{17} & \frac{3}{17} \end{bmatrix}$$

$$X \quad = \qquad A^{-1} \qquad \cdot \quad B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{13}{374} & \frac{15}{187} & -\frac{23}{374} \\ \frac{27}{374} & -\frac{12}{187} & -\frac{19}{374} \\ \frac{2}{17} & \frac{2}{17} & \frac{3}{17} \end{bmatrix} \cdot \begin{bmatrix} -21 \\ 72 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix}$$

The solution to the system is $(4, -7, 9)$.

LESSON 4 Inverses of Linear Functions

A function takes an input value, performs some operation(s) on this value, and creates an output value. The inverse of a function takes the output value, performs some operation(s) on this value, and arrives back at the original function's input value. In other words, an **inverse of a function** is a function that "undoes" another function.

Given a function $f(x)$, you can determine the inverse function algebraically by following these steps:

Step 1: Replace $f(x)$ with y.

Step 2: Switch the x and y variables.

Step 3: Solve for y.

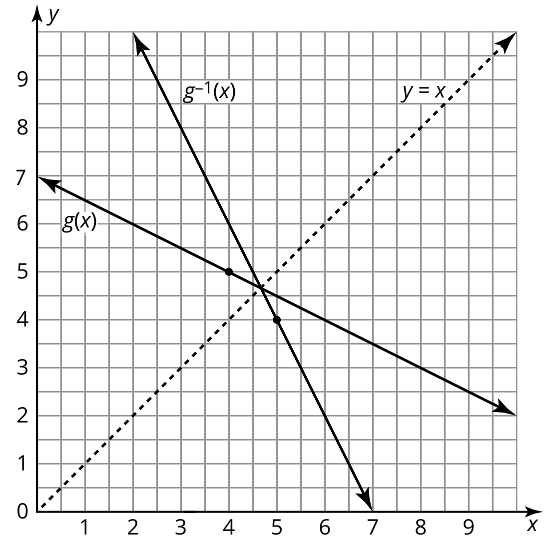
Step 4: If y is a function, replace y with $f^{-1}(x)$.

The graph of the inverse of a function is a reflection of the function across the line $y = x$.

For example, given that $(4, 5)$ is a point on the graph of $g(x)$, the corresponding point on the graph of $g^{-1}(x)$ is $(5, 4)$.

A function is a **one-to-one function** when both the function and its inverse are functions.

A linear and exponential function are always one-to-one functions, but a quadratic function is not.



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