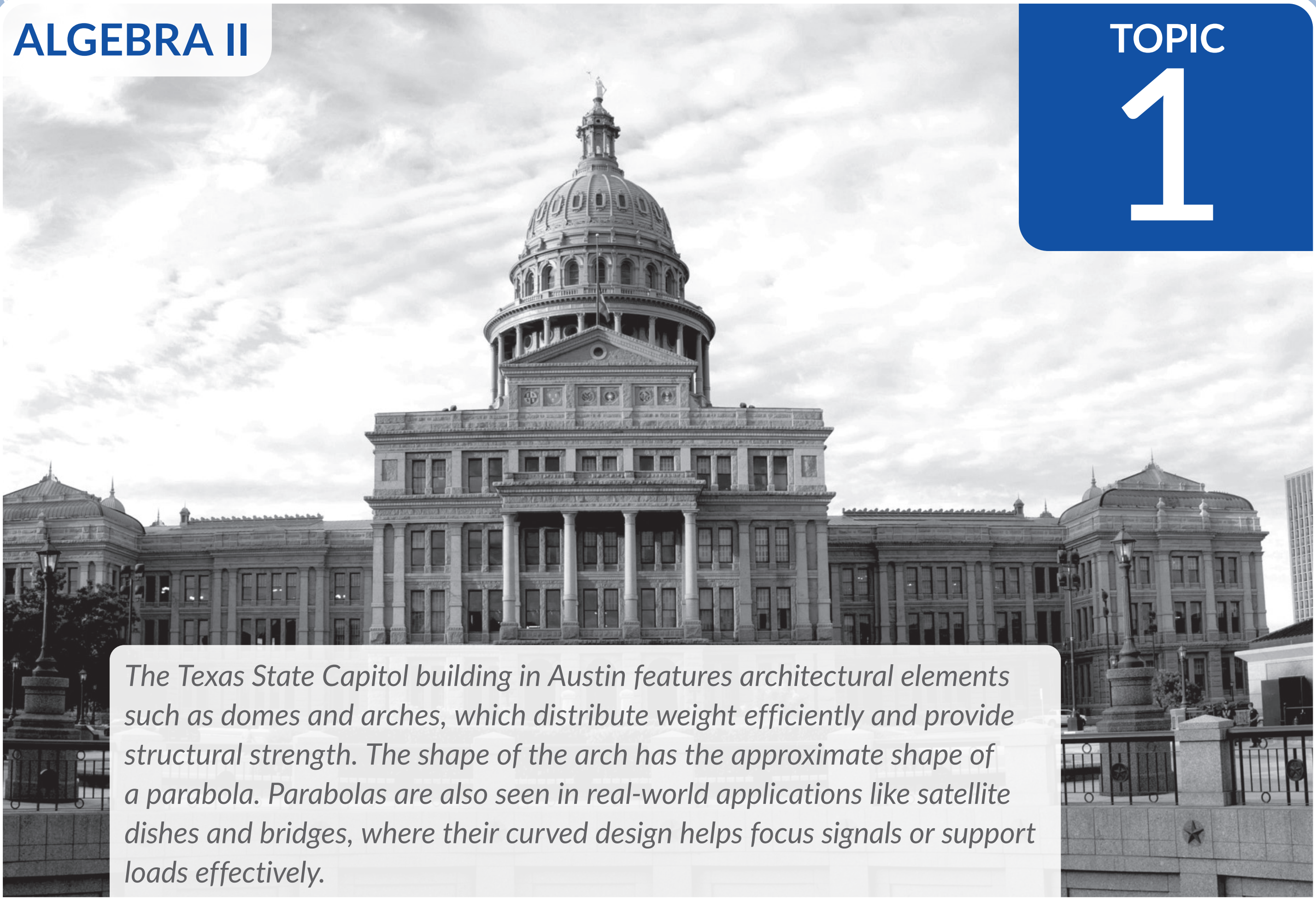
MODULE 2

Exploring Quadratic Functions

ALGEBRA II

TOPIC 1



The Texas State Capitol building in Austin features architectural elements such as domes and arches, which distribute weight efficiently and provide structural strength. The shape of the arch has the approximate shape of a parabola. Parabolas are also seen in real-world applications like satellite dishes and bridges, where their curved design helps focus signals or support loads effectively.

Quadratic Functions and Equations

1 Forms of Quadratic Functions

OBJECTIVES

- Match a quadratic function with its corresponding graph.
- Identify and verbally describe key attributes of quadratic functions based on the form of the function.
- Analyze the different forms of quadratic functions.
- Rewrite a quadratic function given in standard form in vertex form.

KEY TERMS

- standard form of a quadratic function
- factored form of a quadratic function
- vertex form of a quadratic function
- concavity of a parabola

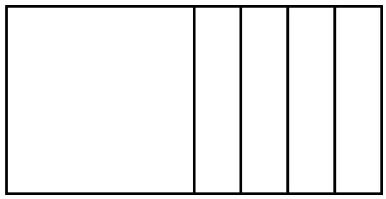
You have explored quadratic functions and identified their key attributes.

How can you use the different forms of a quadratic function to identify key attributes?

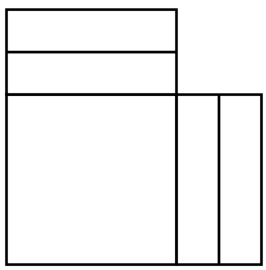
Getting Started

Square It!

One way to represent an expression is with algebra tiles.
The expression $x^2 + 4x$ is shown.



You can adjust the tiles into the shape of a square by splitting the x tiles into two groups.



1. Do the figures represent the same expression? Explain your reasoning.

2. Complete the square by filling in the blank space with square 1 tiles.

3. Add the term representing the additional area to the original expression. What is the new expression?

4. Factor the expression.

The process you just worked through is known as *completing the square*. In this lesson, you will use this process to rewrite a quadratic in standard form in vertex form.

5. Use algebra tiles or draw a model to complete the square for each quadratic expression. Write your answer in factored and standard form.

   a. $x^2 + 6x$

   b. $x^2 + 8x$

6. Analyze your work in Question 5.

   a. Explain how to complete the square on an expression of the form $x^2 + bx$, where b is an integer.

   b. Describe how the coefficient of the middle term, b, is related to the constant term, c, in each trinomial you wrote in Question 5.

7. Use the descriptions you provided in Question 6 to determine the unknown second or third term to make each expression a perfect square trinomial. Then, write the expression as a binomial squared.

   a. $x^2 - 8x +$ ______________ = ______________

   b. $x^2 + 5x +$ ______________ = ______________

   c. $x^2 -$ ______________ $+ 100 =$ ______________

   d. $x^2 +$ ______________ $+ 144 =$ ______________

ACTIVITY 1.1 Forms of Quadratic Functions

Recall that quadratic functions can be written in different forms.

Standard form: $f(x) = ax^2 + bx + c$, where a does not equal 0

Factored form: $f(x) = (ax - r_1)(x - r_2)$, where a does not equal 0

Vertex form: $f(x) = a(x - h)^2 + k$, where a does not equal 0

The graphs of quadratic functions can be described using key attributes x-intercept(s), y-intercept, vertex, axis of symmetry, and concavity.

Concavity of a parabola describes the direction the parabola opens. A parabola is concave up if it opens upward; a parabola is concave down if it opens downward.

1. The form of a quadratic function reveals different key attributes. State the attributes you can determine from each form.

   a. Standard form

   b. Factored form

   c. Vertex form

2. Isabella and Harper were asked to determine the vertex of two different quadratic functions, each written in a different form. Analyze their calculations.

Isabella

$f(x) = 2x^2 + 12x + 10$

The quadratic function is in standard form. So I know the axis of symmetry is $x = \frac{-b}{2a}$.

$$x = \frac{-12}{2(2)}$$
$$= -3$$

Now that I know the axis of symmetry, I can substitute that value into the function to determine the y-coordinate of the vertex.

$$f(-3) = 2(-3)^2 + 1(2-3) + 10$$
$$= 2(9) - 36 + 10$$
$$= 18 - 36 + 10$$
$$= -8$$

Therefore, the vertex is $(-3, -8)$.

Harper

$g(x) = \frac{1}{2}(x + 3)(x - 7)$

The form of the function tells me the x-intercepts are -3 and 7.
I also know the x-coordinate of the vertex is directly in the middle of the x-intercepts. So, all I have to do is calculate their average.

$$x = \frac{-3 + 7}{2}$$
$$= \frac{4}{2} = 2$$

Now that I know the x-coordinate of the vertex, I can substitute that value into the function to determine the y-coordinate.

$$g(2) = \frac{1}{2}(2 + 3)(2 - 7)$$
$$= \frac{1}{2}(5)(-5)$$
$$= -12.5$$

Therefore, the vertex is $(2, -12.5)$.

a. How are these methods similar? How are they different?

b. What must Harper do to use Isabella's method?

c. What must Isabella do to use Harper's method?

3. Cut out each quadratic equation and graph located at the end of the lesson.

   a. Tape each quadratic equation to its corresponding graph.

Do not use technology for this activity. Instead, use the information you can determine by analyzing each equation and each graph to determine a match.

   b. Explain the method(s) you used to match each equation with its graph.

Remember . . .
The shape of the graph of a quadratic function is called a parabola.

4. Analyze the three tables located at the end of the lesson. Tape each function and its corresponding graph from Question 3 in the "Graphs and Their Functions" section of the appropriate table. Then, explain how you can determine each key attributes based on the form of the given function.

Think About . . .
whether the function has a maximum or minimum and where it is located

ACTIVITY 1.2 From Standard to Vertex Form

You can identify the axis of symmetry and the vertex of any quadratic function written in standard form by completing the square.

Notice that the a-value was factored out before completing the square!

WORKED EXAMPLE

Consider the equation $y = ax^2 + bx + c$.

Step 1: $y - c = ax^2 + bx$

Step 2: $y - c = a\left(x^2 + \frac{b}{a}x\right)$

Step 3: $y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right)$

Step 4: $y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$

Step 5: $y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$

1. Explain why $a\left(\frac{b}{2a}\right)^2$ was added to the left side of the equation in Step 3.

2. Given a quadratic function in the form $y = ax^2 + bx + c$:

   a. Identify the axis of symmetry.

   b. Identify the location of the vertex.

c. Identify the location of the x-intercepts.

3. Rewrite each quadratic equation in vertex form. Then, identify the axis of symmetry, vertex and direction of opening for each parabola.

a. $y = x^2 + 8x - 9$

Axis of symmetry:

Vertex:

Direction of opening:

b. $y = 3x^2 + 2x - 1$

Axis of symmetry:

Vertex:

Direction of opening:

c. $y = -x^2 + 6x - 5$

Axis of symmetry:

Vertex:

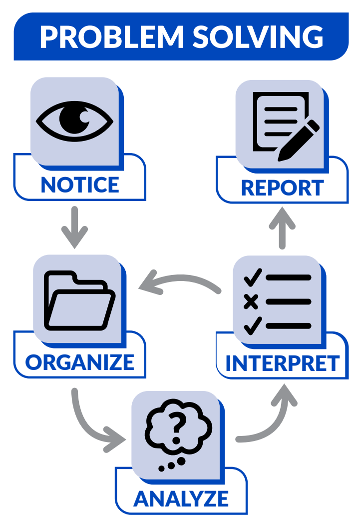
Direction of opening:

d. $y = -2x^2 + 8x - 7$

Axis of symmetry:

Vertex:

Direction of opening:



4. A ball is thrown straight up from 3 feet above the ground with a velocity of 40 feet per second. The height of the ball over time can be modeled with the function $h(t) = -16t^2 + 32t + 4$. Rewrite the function in vertex form to determine the maximum height of the ball.

Ask Yourself . . .
Did you complete all the steps in the problem-solving model?

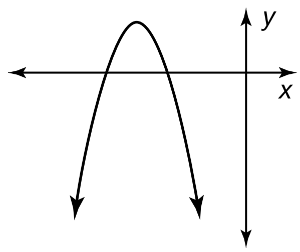
ACTIVITY 1.3

Modeling a Quadratic Graph

Think About . . .
What information is given by each function and the relative position of its graph?

1. Analyze each graph. Then, circle the function(s) which could model the graph. Use key attributes of the function to describe the reasoning you used to eliminate or choose each function.

   a.



$f_1(x) = -2(x + 1)(x + 4)$

$f_2(x) = -\frac{1}{3}x^2 - 3x - 6$

$f_3(x) = 2(x + 1)(x + 4)$

$f_4(x) = 2x^2 - 8 .9$

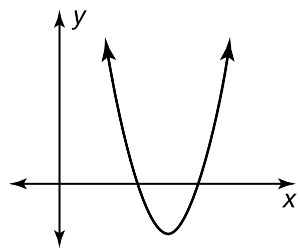
$f_5(x) = 2(x - 1)(x - 4)$

$f_6(x) = -(x - 6)^2 + 3$

$f_7(x) = -3(x + 2)(x - 3)$

$f_8(x) = -(x + 6)^2 + 3$

b.



$f_1(x) = 2(x - 75)^2 - 92$

$f_2(x) = (x - 8)(x + 2)$

$f_3(x) = 8x^2 - 88x + 240$

$f_4(x) = -3(x - 1)(x - 5)$

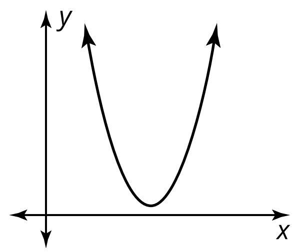
$f_5(x) = -2(x - 75)^2 - 92$

$f_6(x) = x^2 + 6x - 2$

$f_7(x) = 2(x + 4)^2 - 2$

$f_8(x) = (x + 1)(x + 3)$

c.



$f_1(x) = 3(x + 1)(x - 5)$

$f_2(x) = 2(x + 6)^2 - 5$

$f_3(x) = 4x^2 - 400x + 10{,}010$

$f_4(x) = 3(x + 1)(x + 5)$

$f_5(x) = 2(x - 6)^2 + 5$

$f_6(x) = x^2 + 2x - 5$

2. Consider the two functions shown from Question 1. Identify the form of the function given, and then write the function in the other two forms, if possible. If it is not possible, explain why.

a. $f_1(x) = -2(x + 1)(x + 4)$

b. $f_5(x) = 2(x - 6)^2 + 5$

Talk the Talk

Get Squared Away

1. Identify the form of each quadratic function. Then, identify the key attributes of each function.

   a. $y = 4(x - 2)(x + 1)$

   b. $y = 3(x - 1)^2 + 4$

   c. $y = -x^2 + 4x$

2. Rewrite each quadratic function in vertex form. Identify the axis of symmetry, vertex, and concavity.

   a. $y = x^2 - 10x + 16$

   b. $y = -3x^2 + 6x + 2$

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So you can cut out the Algebra Tiles on the next page

Quadratic Equations Cutouts

a. $f(x) = 2(x + 1)(x + 5)$	**b.** $f(x) = \frac{1}{3}x^2 + \pi x + 6.4$	**c.** $f(x) = -2.5(x - 3)(x - 3)$
d. $f(x) = (x - 1)^2$	**e.** $f(x) = 2(x - 1)(x - 5)$	**f.** $f(x) = x^2 + 12x - 1$
g. $f(x) = -(x + 4)^2 - 2$	**h.** $f(x) = -5x^2 - x + 21$	**i.** $f(x) = -(x + 2)^2 - 4$

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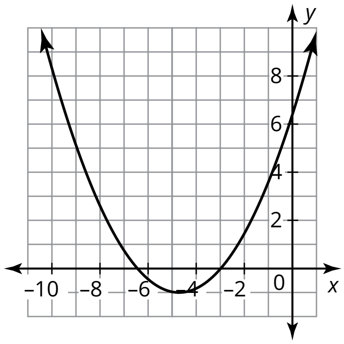
Graph Cutouts

A.



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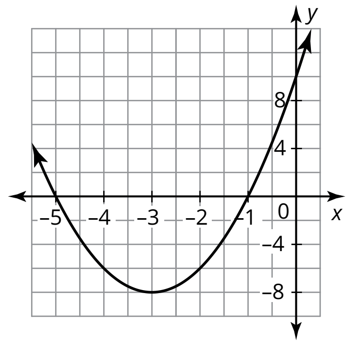

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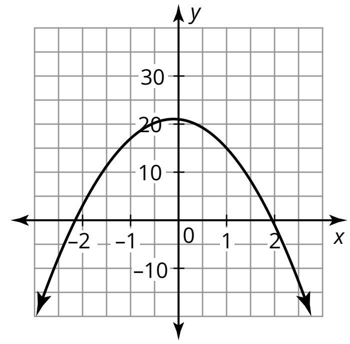

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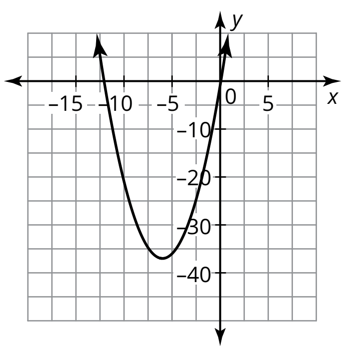

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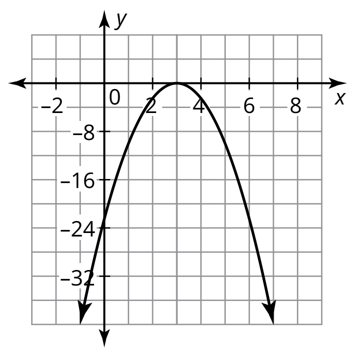

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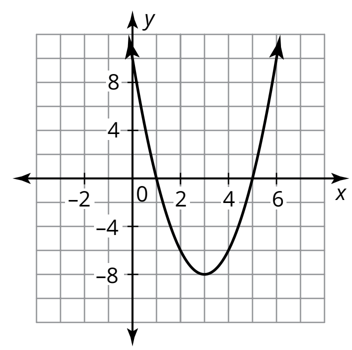
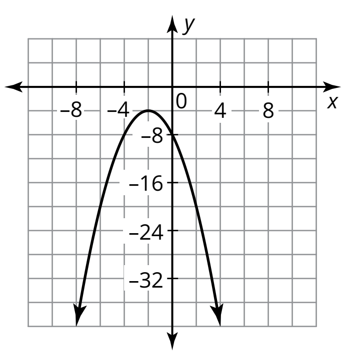
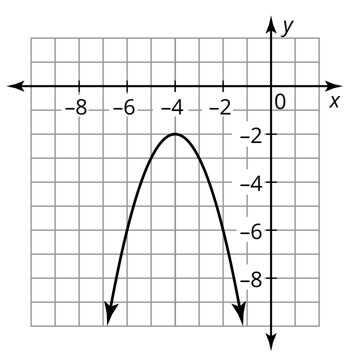


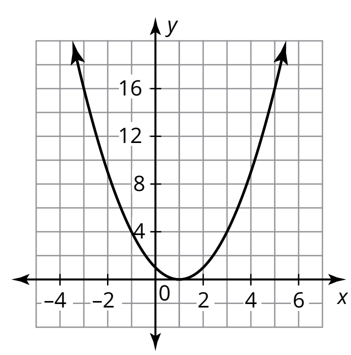
G.



H.



I.



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Graphs and Their Functions

<table>
<tr><th colspan="6">Standard Form
$f(x) = ax^2 + bx + c$, where $a \neq 0$</th></tr>
<tr><td colspan="6">Graphs and Their Functions</td></tr>
<tr><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td></tr>
<tr><td colspan="6">Methods to Determine Key Attributes</td></tr>
<tr><td colspan="2">Axis of Symmetry</td><td colspan="2">x-intercept(s)</td><td colspan="2">Concavity</td></tr>
<tr><td colspan="3">Vertex</td><td colspan="3">y-intercept</td></tr>
</table>

Factored Form

$f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$

Graphs and Their Functions

Methods to Determine Key Attributes

Axis of Symmetry

***x*-intercept(s)**

Concavity

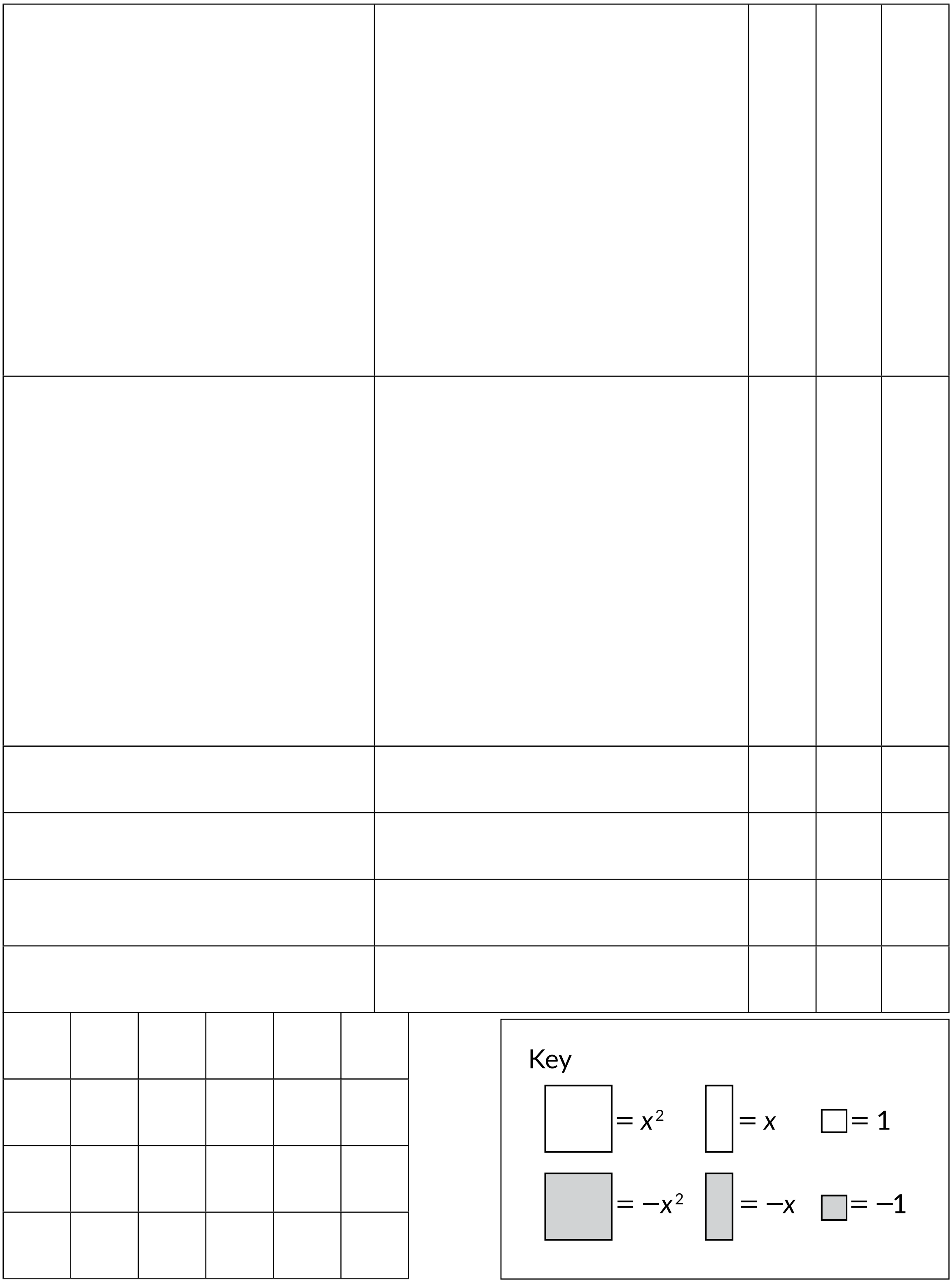
Vertex

***y*-intercept**

Vertex Form $f(x) = a(x - h)^2 + k$, where $a \neq 0$		
Graphs and Their Functions		
Methods to Determine Key Attributes		
Axis of Symmetry	***x*-intercept(s)**	**Concavity**

Vertex	***y*-intercept**

Algebra Tiles



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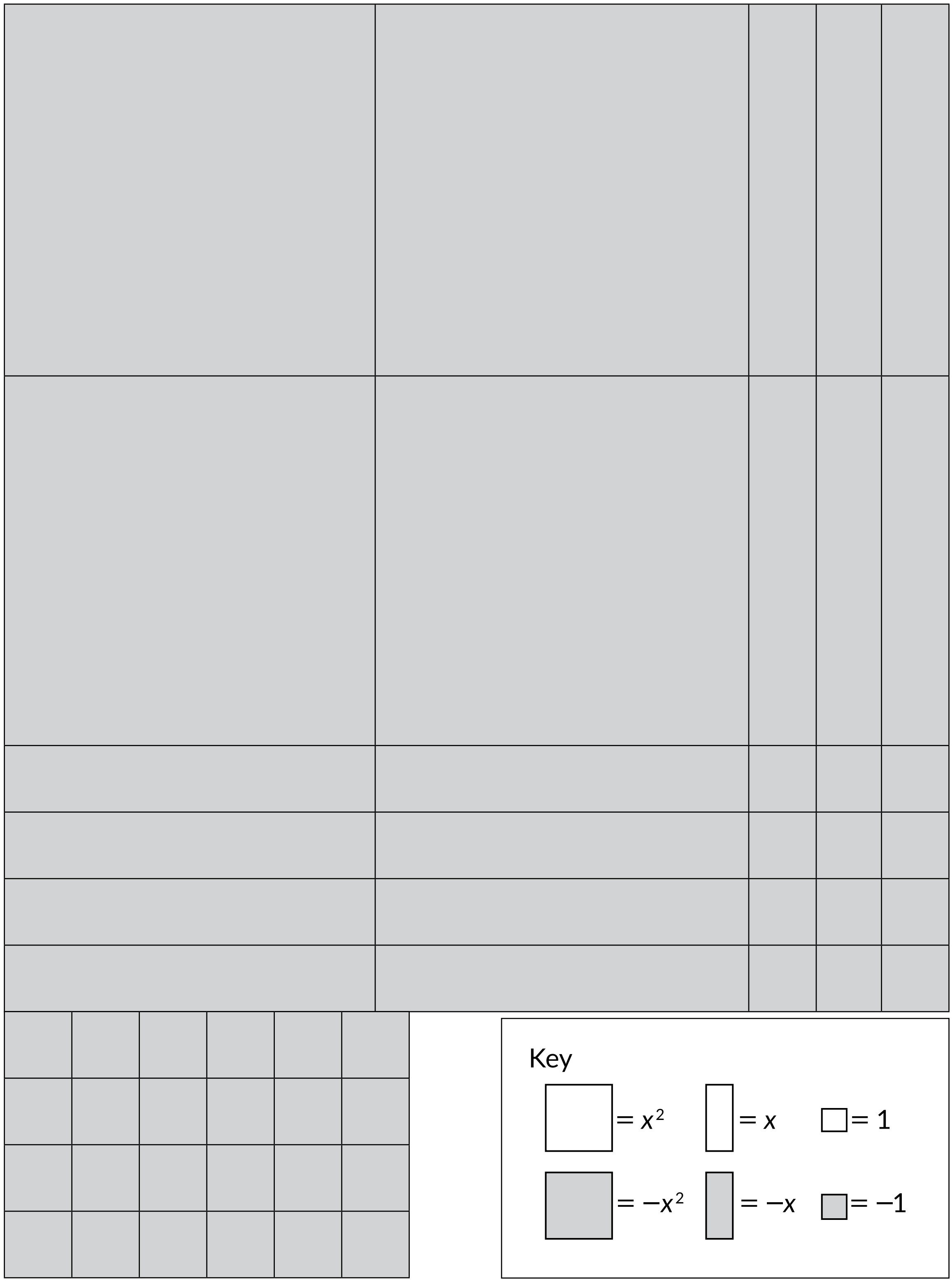
Algebra Tiles



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Lesson 1 Assignment

Write

1. Write a quadratic function for each form and state whether the form helps determine the *x*-intercepts, the *y*-intercept, or the vertex of the graph.
   a. Standard form
   b. Factored form
   c. Vertex form

2. Describe how to determine the concavity of a parabola.

Remember

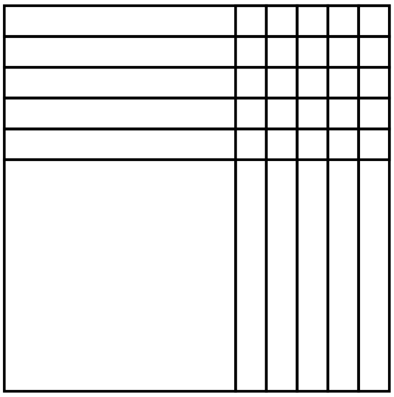
The form of a quadratic function—standard, factored, or vertex—reveals different key attributes, such as the *x*-intercept(s), *y*-intercept, vertex, axis of symmetry, and concavity up or down.

You can rewrite a quadratic function given in standard form in vertex form by completing the square.

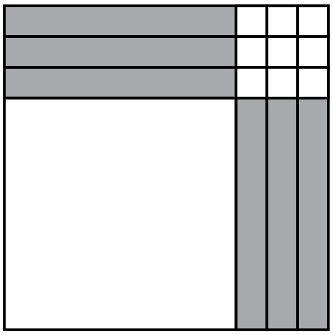
Practice

1. Use algebra tiles or draw a model to complete the squares for each given expression. Write your answer in factored and standard form.

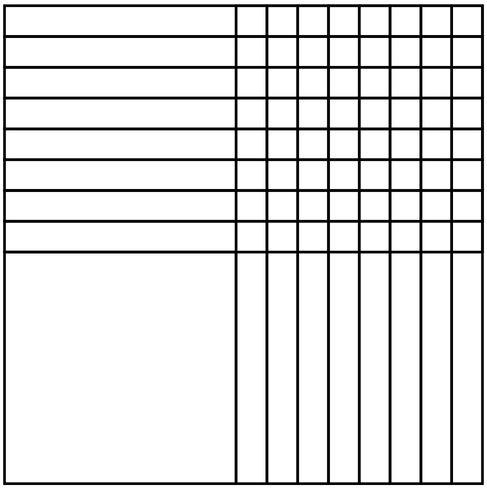
   a. $x^2 + 10x$



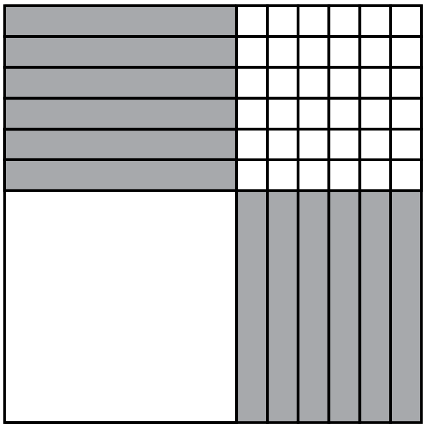
   b. $x^2 - 6x$



c. $x^2 + 16x$



d. $x^2 - 12x$



2. Given $y = 2(x - 3)(x + 1)$. Identify its form and its key attributes.

3. Rewrite $y = 2x^2 - 8x + 5$ in vertex form. Identify its axis of symmetry, vertex, and concavity.

Lesson 1 Assignment

4. A company is designing a rectangular billboard, where the length is x meters and the width is $(30 - x)$ meters. The area $A(x)$ of the billboard can be modeled by the quadratic function $A(x) = x(30 - x)$.

   a. Rewrite the function in vertex form by completing the square.

   b. What is the maximum area of the billboard? At what length x does this maximum area occur?

Lesson 1 Assignment

Prepare

Identify the axis of symmetry of each quadratic function.

1. $f(x) = -4(x - 3)^2 + 2$

2. $g(x) = 2(x + 5)(x - 7)$

3. $h(x) = 4x^2 + 6x + 1$

2 Writing Quadratic Functions

OBJECTIVES

- Determine how many points are necessary to create a unique quadratic equation.
- Verbally explain your process for deriving a quadratic equation given three points using a system of equations.
- Write quadratic functions to represent problem situations.

You have explored the key attributes of different forms of quadratic functions.

How can you use these attributes to write a quadratic equation to model the graph of a parabola, even if you only know certain points on the graph?

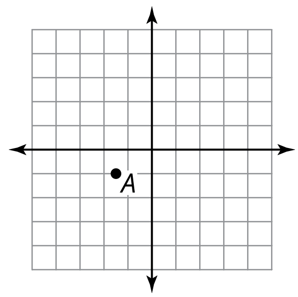
Getting Started

Be My One and Only

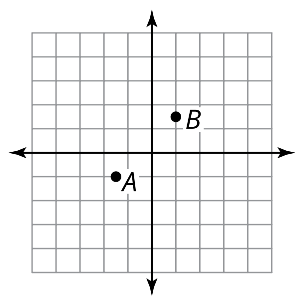
Consider the family of linear functions.

1. Use the given point(s) to sketch possible solutions.

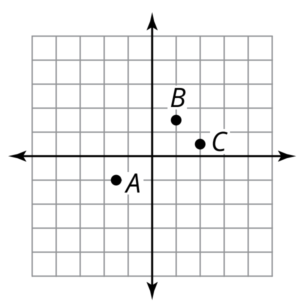
    a. How many lines can you draw through point *A*?



    b. How many lines can you draw through both points *A* and *B*?



    c. How many lines can you draw through all points *A*, *B*, and *C*?

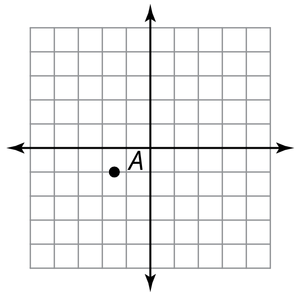


2. What is the minimum number of points you need to draw a unique line?

Consider the family of quadratic functions.

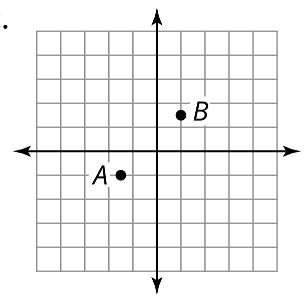
3. Use the given point(s) to sketch possible solutions.

a. How many parabolas can you draw through point A?

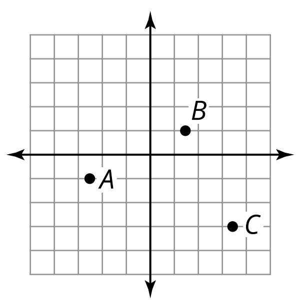


b. How many parabolas can you draw through both points ***A*** and ***B***?

c.



d. How many parabolas can you draw through all points A, B, and C?



ACTIVITY 2.1

Writing a Unique Quadratic Function

You have used properties of linear functions to write linear equations. In this activity, you will use properties of quadratic functions to write quadratic equations in various forms.

1. Diego and Sofia each wrote a quadratic equation with a vertex of (4, 8). Analyze each student's work. Describe the similarities and differences in their equations and determine who is correct.

Diego

$y = a(x - h)^2 + k$

$y = a(x - 4)^2 + 8$

$y = -\frac{1}{2}(x - 4)^2 + 8$

Sofia

$y = a(x - h)^2 + k$

$y = a(x - 4)^2 + 8$

$y = (x - 4)^2 + 8$

You can write a unique quadratic function given a vertex and a point on the parabola.

WORKED EXAMPLE

Write the quadratic function given the vertex (5, 2) and the point (4, 9).

Substitute the given values into the vertex form of the function. $f(x) = a(x - h)^2 + k$
$9 = a(4 - 5)^2 + 2$

Then, solve for *a*.

$$9 = a(-1)^2 + 2$$
$$9 = 1a + 2$$
$$7 = 1a$$
$$7 = a$$

Finally, substitute the *a*-value into the function. $f(x) = 7(x - 5)^2 + 2$

You can write a unique quadratic function given the roots and a point on the parabola.

WORKED EXAMPLE

Write a quadratic function given the roots (–2, 0) and (4, 0), and the point (1, 6).

Substitute the given values into the factored form of the function. $f(x) = a(x - r_1)(x - r_2)$
$6 = a(1 - (-2))(1 - 4)$

Then, solve for *a*.

$$6 = a(1 + 2)(1 - 4)$$
$$6 = a(3)(-3)$$
$$6 = -9a$$
$$-\frac{2}{3} = a$$

Finally, substitute the *a*-value into the function. $f(x) = -\frac{2}{3}(x + 2)(x - 4)$

Remember . . .
The roots of a quartic equation are the values of the variable that make the equation equal to zero.

Think About . . .
What does it mean for a point to be a maximum or minimum point?

2. Determine which form of a quadratic function would be most efficient to write the function using the given information. Write *standard form*, *factored form*, *vertex form*, or *none* in the space provided.

   a. minimum point (6, –75)
   y-intercept (0, 15) ____________________

   b. points (2, 0), (8, 0), and (4, 6) ____________________

   c. points (100, 75), (450, 75), and (150, 95) ____________________

   d. points (3, 3), (4, 3), and (5, 3) ____________________

   e. *x*-intercepts (7.9, 0) and (–7.9, 0)
   point (–4, –4) ____________________

   f. roots (3, 0) and (12, 0)
   point (10, 2) ____________________

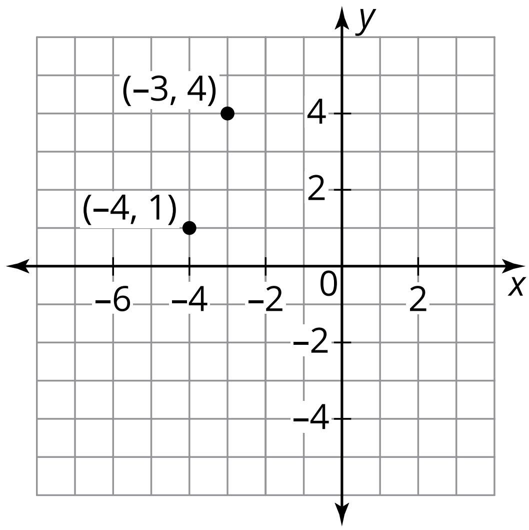
Ask Yourself . . .
What patterns do you notice?

   g. Max hits a baseball off a tee that is 3 feet high. The ball reaches a maximum height of 20 feet when it is 15 feet from the tee. ____________________

3. Write a quadratic equation in vertex form that includes the given points. If it is not possible to write a function, state why not.

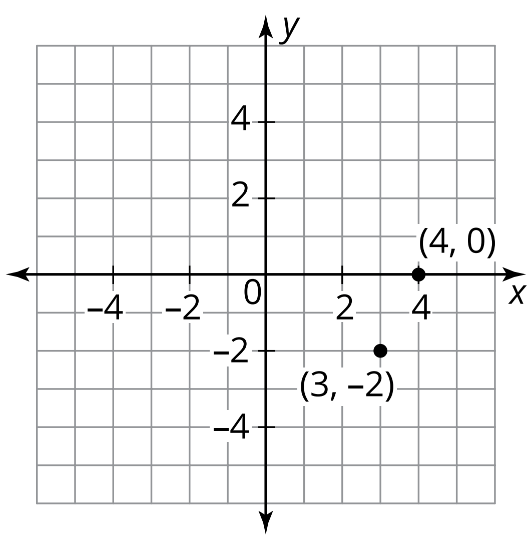
a. Given: vertex (–3, 4); point (–4, 1)

$f(x) =$ ______________________



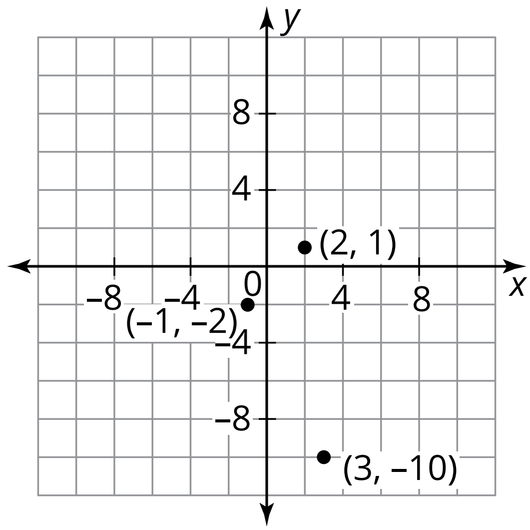
b. Given: vertex (3, –2); one of two *x*-intercepts (4, 0)

$f(x) =$ ______________________



c. Given: points (2, 1), (–1, –2), (3, –10)

$f(x) =$ ____________________________



4. Chloe says that she can write a unique quadratic function given only two points. Is she correct? Explain your reasoning.

ACTIVITY 2.2 Using Algebra to Write a Quadratic Equation

In a previous course, you used technology to determine a function that best models a data set. What if you do not have technology? What if you don't have technology?

You know that you need a minimum of 3 non-linear points to create a unique parabola. To create an equation to represent the parabola, you can use a system of equations.

WORKED EXAMPLE

Consider the three points given in Question 3 part (c) in the previous activity: A (2, 1), B (–1, –2), and C (3, –10).

To write an equation in standard form to represent a parabola that passes through three given points, begin by substituting the x- and y-values of each point into $y = ax^2 + bx + c$.

Point A:	$1 = a(2)^2 + b(2) + c$ $1 = 4a + 2b + c$	Equation A: $1 = 4a + 2b + c$
Point B:	$-2 = a(-1)^2 + b(-1) + c$ $-2 = a - b + c$	Equation B: $-2 = a - b + c$
Point C:	$-10 = a(3)^2 + b(3) + c$ $-10 = 9a + 3b + c$	Equation C: $-10 = 9a + 3b + c$

Now, use linear combinations and substitution to solve for a, b, and c.

STEP 1: Subtract Equation B from A:

$$\begin{array}{r} 1 = 4a + 2b + c \\ -(-2 = a - b + c) \\ \hline 3 = 3a + 3b \end{array}$$

STEP 2: Subtract Equation B from C:

$$\begin{array}{r} -10 = 9a + 3b + c \\ -(-2 = a - b + c) \\ \hline -8 = 8a + 4b \end{array}$$

STEP 3: Solve the equation from Step 1 in terms of a.

$$\begin{aligned} 3 &= 3a + 3b \\ 3 - 3b &= 3a \\ 1 - b &= a \end{aligned}$$

STEP 4: Substitute the value for *a* into the equation from Step 2.	$-8 = 8(1 - b) + 4b$ $-8 = 8 - 4b$ $16 = 4b$ $4 = b$
STEP 5: Substitute the value for *b* into the equation from Step 3.	$a = 1 - (4)$ $a = -3$
STEP 6: Substitute the values for *a* and *b* into Equation *A*.	$1 = 4a + 2b + c$ $1 = 4(-3) + 2(4) + c$ $1 = -12 + 8 + c$ $1 = -4 + c$ $5 = c$
STEP 7: Substitute the values for *a*, *b*, and *c* into the standard form of a quadratic.	$y = -3x^2 + 4x + 5$

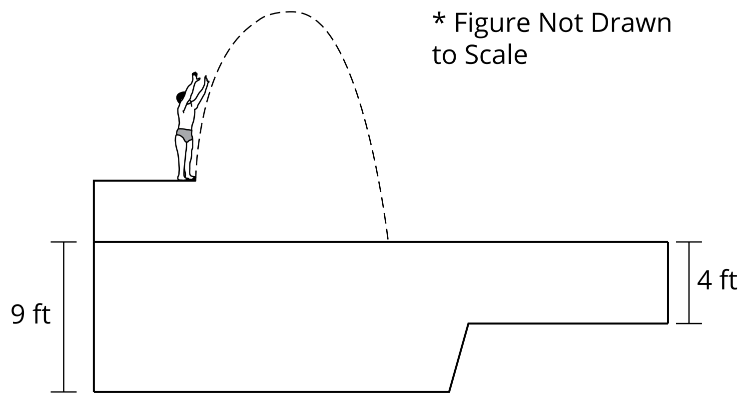
1. Use the Worked Example to write a quadratic equation that passes through the points (–1, 5), (0, 3), and (3, 9).

2. Isabella says she can solve the system of equations in the Worked Example using Gaussian elimination instead of substitution and linear combinations. Ethan says he can solve the system of equations in the Worked Example using technology with matrices instead of substitution and linear combinations. Who's correct?

A homeowners association has hired a pool-building company to create the community pool for their new development of homes. The rectangular pool is to have one section with a 4-foot depth, and another section with a 9-foot depth. The pool will also have a diving board. By law, the regulated depth of water necessary to have a diving board is 9 feet. The homeowners association would like to have the majority of the pool have a 4-foot depth in order to accommodate a large number of young children.

The diving board will be 3 feet above the edge of the pool's surface and extend 5 feet into the pool. After doing some research, the pool building company determined that the average diver would be 5 feet in the air when they are 8 feet from the edge of the pool, and 6 feet in the air when they are 10 feet from the edge of the pool.

3. Use the dive model shown to write an equation, and then determine the minimum length of the 9-foot-depth section of the pool. Explain your reasoning.



Talk the Talk

From Points to Parabolas

1. Determine a quadratic function with vertex at (2, –3) and (4, 1) a point on the parabola.

2. Write an equation in standard form to represent a parabola that passes through the points (–1, 6), (0, 2), and (2, –2)

A baseball player needs one more home run to advance to the next round of the home run derby. On the last pitch, he takes a swing and makes contact. Initially, he hits the ball at 5 feet above the ground. At 32 feet from home plate, his ball was 23.7 feet in the air, and at 220 feet from home plate, his ball was 70 feet in the air.

3. Consider the function that represents the relationship between the height of the ball and its distance from home plate.

   a. Write a quadratic function using the three points described in the problem situation.

   b. What was the maximum height of the player's baseball?

Lesson 2 Assignment

Write

Describe the process for writing a quadratic function when given three points.

Remember

Given three points, you can determine a unique quadratic function algebraically by writing and solving a system of equations.

Practice

1. Write a quadratic equation for the parabola that passes through the point (2, −3) with roots (−6, 0) and (4, 0).

2. Mitzu shoots an arrow from an initial height of 2 meters. The arrow reaches its maximum height of 20 meters after it has flown a distance of 60 meters.

   a. Write a quadratic function to represent the height of the arrow as a function of its distance.

   b. Determine the height of the arrow after it has flown a distance of 100 meters.

Lesson 2 Assignment

3. Use your knowledge of reference points to write an equation for the quadratic function that has a vertex at (4, –3) and passes through (6, –1).

4. Write a quadratic equation that passes through the points (–2, 8), (1, 14), and (0, 10).

Lesson 2 Assignment

Prepare

Factor each expression.

1. $9x^2 - 25y^2$

2. $x^2 - 14x + 49$

3. $16x^2 + 24x + 9$

3 Solving Quadratic Equations

OBJECTIVES

- Factor quadratic trinomials to determine the roots of quadratic equations and to rewrite quadratic functions in forms that reveal different key attributes.
- Complete the square to determine the roots of quadratic equations of the form $ax^2 + bx + c$.
- Use the quadratic formula to determine roots and zeros.
- Explain in written form when various methods for determining quadratic roots and zeros are most appropriate or efficient.

KEY TERM

- quadratic formula

You have analyzed the different structures of quadratic equations.

How can the structure of a quadratic equation help you determine a solution strategy? Is there a single strategy that works to solve any quadratic equation?

Getting Started

Remember . . .

Some example properties of inequality include:

- addition property of equality
- subraction property of equality
- multiplication property of equality
- division property of equality
- square root property of equality

Grassroots

You know how to use the properties of equality to solve equations in the forms shown.

$$y = x^2 + d$$
$$y = (x - c)^2$$
$$y = a(x - c)^2$$
$$y = a(x - c)^2 + d$$

1. Use properties of equality to solve each equation. State the property used in each step of your solution.

   a. $27 = x^2 - 9$

   b. $(x + 3)^2 = 121$

   c. $48 = 3(x - 1)^2$

   d. $\frac{1}{2}(x + 5)^2 - 18 = 0$

2. Describe the strategy Ethan used to solve part (a) in Question 1.

Ethan

$$27 = x^2 - 9$$
$$27 - 27 = x^2 - 9 - 27$$
$$0 = x^2 - 36$$
$$0 = (x - 6)(x + 6)$$

$(x - 6) = 0$ and $(x + 6) = 0$

$x = 6$ and $x = -6$

ACTIVITY 3.1

Solving Quadratic Equations by Factoring

Let's consider strategies to solve quadratics in the form $y = ax^2 + bx + c$ using the factoring strategies you have learned.

WORKED EXAMPLE

You can use factoring to calculate the roots for the quadratic equation $2x^2 - 8x = -6$.

$$2x^2 - 8x = -6$$
$$2x^2 - 8x + 6 = -6 + 6$$
$$2(x^2 - 4x + 3) = 0$$
$$2(x - 3)(x - 1) = 0$$

$$(x - 3) = 0 \quad \text{and} \quad (x - 1) = 0$$

$$x - 3 + 3 = 0 + 3 \quad \text{and} \quad x - 1 + 1 = 0 + 1$$

$$x = 3 \quad \text{and} \quad x = 1$$

Remember . . .
The zero product property states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

Think About . . .
What is the connection between the Worked Example and determining the roots from factored form, $y = a(x - r_1)(x - r_2)$?

1. Why is 6 added to both sides in the first step of the Worked Example?

2. Determine each student's error and then solve each equation correctly.

Isabella

$x^2 + 6x = 7$

$x(x + 6) = 7$

$x = 7$ and $x + 6 = 7$

$x = 1$

Samuel

$x^2 + 5x + 6 = 6$

$(x + 2)(x + 3) = 6$

$x + 2 = 6$ and $x + 3 = 6$

$x = 4$ and $x = 3$

3. Use factoring to solve each quadratic equation, if possible.

a. $x^2 - 8x + 12 = 0$

b. $x^2 + 8x = -7$

c. $\frac{2}{3}x^2 - \frac{5}{6}x = 0$

d. $f(x) = x^2 + 10x + 12$

Think About . . .

What efficiency strategies did you use to solve linear equations with fractional coefficients?

4. Describe the different strategies and reasoning that Diego and Harper used to solve $4x^2 - 25 = 0$.

Diego

$$4x^2 - 25 = 0$$
$$4x^2 = 25$$
$$x^2 = \frac{25}{4}$$
$$x = \pm\sqrt{\frac{25}{4}}$$
$$x = \pm\frac{5}{2}$$

Harper

$$4x^2 - 25 = 0$$
$$(2x - 5)(2x + 5) = 0$$
$$2x - 5 = 0 \text{ and } 2x + 5 = 0$$
$$2x = 5 \qquad 2x = -5$$
$$x = \frac{5}{2} \text{ and } x = -\frac{5}{2}$$

ACTIVITY 3.2

Completing the Square to Determine Roots

You can use the completing the square method to determine the roots of a quadratic equation that cannot be factored.

WORKED EXAMPLE

Complete the square to determine the roots of the equation $2x^2 + 20x + 24 = 0$.

Isolate $2x^2 + 20x$.	$2x^2 + 20x + 24 - 24 = 0 - 24$ $2x^2 + 20x = -24$
Factor out the leading coefficient	$2(x^2 + 20x) = -24$
Divide both sides by 2	$x^2 + 10x = -12$
Determine the constant term that would complete the square. Add this term to both sides of the equation. Multiply the constant by 2.	$x^2 + 10x + \_\_\_ = -12 + \_\_\_$ $x^2 + 10x + 25 = -12 + 25$ $x^2 + 10x + 25 = 13$
Rewrite the left side as a perfect square.	$(x + 5)^2 = 13$
Take the square root of each side of the equation.	$\sqrt{(x + 5)^2} = \sqrt{13}$ $x + 5 = \pm\sqrt{13}$
Set the factor of the perfect square trinomial equal to each square root of the constant. Then solve for x.	$x + 5 = \sqrt{13}$ $x = -5 + \sqrt{13}$ $x \approx -1.39$ $x + 5 = -\sqrt{13}$ $x = -5 - \sqrt{13}$ $x \approx -8.61$

The roots are approximately -1.39 and -8.61.

Ask Yourself . . .
How was the equality of the equation maintained through the completing the square process?

1. Complete the square to determine the roots of each equation.

   a. $x^2 - 6x + 4 = 0$

   b. $3x^2 - 36x + 18 = 0$

2. A ball is thrown straight up from 4 feet above the ground with a velocity of 32 feet per second. The height of the ball over time can be modeled with the function $h(t) = -16t^2 + 32t + 4$. What is the maximum height of the ball?

3. Victoria is fencing in a rectangular plot outside of her back door so that she can let her dogs out to play. She has 60 feet of fencing and only needs to place it on three sides of the rectangular plot because the fourth side will be bound by her house. What dimensions should Victoria use for the plot so that the maximum area is enclosed? What is the maximum area? Draw a diagram to support your work.

ACTIVITY
3.3 Using the Quadratic Formula

The **quadratic formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c represent real numbers and $a \neq 0$.

WORKED EXAMPLE

You can use the quadratic formula to determine the zeros of the function $f(x) = -4x^2 - 40x - 99$.

Rewrite the function as an equation to be solved for x when $y = 0$.

$$-4x^2 - 40x - 99 = 0$$

Determine the values of a, b, and c.

$$a = -4,\ b = -40,\ c = -99$$

Substitute the values into the quadratic formula.

$$x = \frac{-(-40) \pm \sqrt{(-40)^2 - 4(-4)(-99)}}{2(-4)}$$

Perform operations to rewrite the expression.

$$x = \frac{40 \pm \sqrt{1600 - 1584}}{-8}$$

$$x = \frac{40 \pm \sqrt{16}}{-8}$$

$$x = \frac{40 \pm 4}{-8}$$

$$x = \frac{40 + 4}{-8} \quad \text{and} \quad x = \frac{40 - 4}{-8}$$

$$x = \frac{44}{-8} \quad \text{and} \quad x = \frac{36}{-8}$$

$$x = -5.5 \quad \text{and} \quad x = -4.5$$

The zeros of the function $f(x) = -4x^2 - 40x - 99$ are $x = -5.5$ and $x = -4.5$.

A baseball team has a new promotional activity to encourage fans to attend games: launching free T-shirts! They can launch a T-shirt in the air with an initial velocity of 91 feet per second from $5\frac{1}{2}$ feet off the ground (the height of the team mascot).

A T-shirt's height can be modeled with the quadratic function $h(t) = -16t^2 + 91t + 5.5$, where t is the time in seconds and $h(t)$ is the height of the launched T-shirt in feet. They want to know how long it will take for a T-shirt to land back on the ground after being launched (if no fans grab it before then!).

Ask Yourself . . .
What would a sketch showing the height of the T-shirt over time look like?

1. Why does it make sense to use the quadratic formula to solve this problem?

Ask Yourself . . .
Do you think an exact solution or approximate solution is more appropriate for this context?

2. Use the quadratic formula to determine how long it will take for a T-shirt to land back on the ground after being launched.

3. Sofia is solving the quadratic equation $x^2 - 7x - 8 = 3$. Her work is shown.

   a. Identify Sofia's error.

   b. Determine the solution to Sofia's quadratic equation.

Sofia

$x^2 - 7x - 8 = 3$

$a = 1, b = -7, c = -8$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-8)}}{2(1)}$$

$$x = \frac{7 \pm \sqrt{49 + 32}}{2}$$

$$x = \frac{7 \pm \sqrt{81}}{2}$$

$$x = \frac{7 \pm 9}{2}$$

$$x = \frac{7 + 9}{2} \text{ or } x = \frac{7 - 9}{2}$$

$$x = \frac{16}{2} = 8 \text{ or } x = \frac{-2}{2} = -1$$

The roots are 8 and −1.

4. Use the quadratic formula to determine the zeros for each function.
   Round the solutions to the nearest hundredth.

   a. $f(x) = 2x^2 + 10x - 1.02$

   b. $h(x) = 3x^2 + 11x - 2$

Talk the Talk

Show Me the Ways

1. Determine the real roots of the quadratic equation $y = 2x^2 + 4x - 6$ using each method.

Factoring

Completing the Square

$y = 2x^2 + 4x - 6$

Using the Quadratic Formula

Graphing

Lesson 3 Assignment

Write

Describe when you would use each method to solve a quadratic equation.

1. Factoring
2. Complete the square
3. Quadratic formula

Remember

You can solve quadratic equations using factoring, completing the square, the quadratic formula, and graphing.

Practice

1. Solve each equation.

a. $0 = x^2 - 7x - 18$

b. $x^2 + 10x = 39$

c. $0 = x^2 - 10x + 12$

d. $2x^2 + 4x = 0$

e. $3x^2 - 22x + 7 = 0$

Lesson 3 Assignment

2. Determine the roots of each equation. Check your solutions.

   a. $y = x^2 + 9x + 3$

   b. $y = 3x^2 + 24x - 6$

3. Daniel is driving 48 miles per hour and is speeding up to merge onto the highway. He gradually accelerates at a rate of 7 miles per hour for several seconds. The formula $s = ut + \frac{1}{2}at^2$ can be used to calculate the distance, s, an object travels in t seconds. In this formula, u represents the initial velocity, and a represents a constant acceleration.

   a. Substitute the initial velocity and constant acceleration into the formula to write an equation to represent the distance Daniel travels.

   b. Use the quadratic formula to determine the roots of the equation. What do the roots represent in the context of the problem situation? Explain your reasoning.

Lesson 3 Assignment

Prepare

Rewrite each square root expression in simplest form, when possible.

1. $\sqrt{48}$

2. $\sqrt{120}$

3. $\sqrt{-225}$

4 Imaginary and Complex Numbers

OBJECTIVES

- Rewrite expressions involving negative roots using *i*.
- Rewrite expressions involving imaginary numbers.
- Understand properties of the set of complex numbers.
- Determine and explain in written form the sets to which numbers belong.
- Calculate complex roots of quadratic equations and complex zeros of quadratic functions.
- Interpret complex roots of quadratic equations and complex zeros of quadratic functions.
- Determine and write an explanation of whether a function has complex solutions from a graph and from an equation in radical form.
- Determine the number of roots of a quadratic equation from a graph and from an equation in radical form.
- Add, subtract, and multiply complex numbers.

KEY TERMS

- the number *i*
- imaginary zeros
- imaginary roots
- complex numbers
- real part of a complex number
- imaginary part of a complex number
- imaginary numbers
- pure imaginary number
- Fundamental Theorem of Algebra

You have solved quadratic equations with real number solutions.

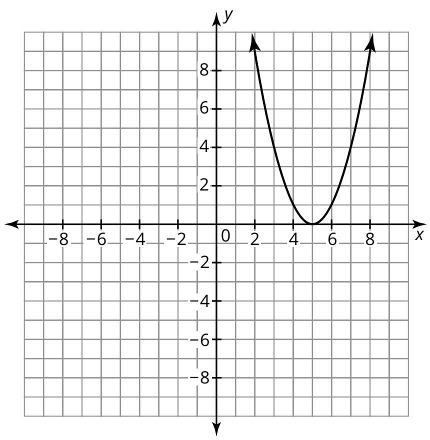
How can you solve a quadratic equation that has a solution that is not real?

Getting Started

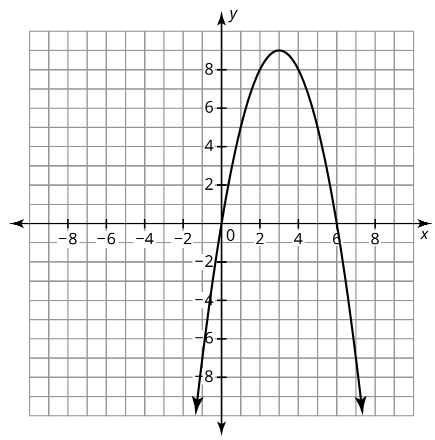
Did I Cross the Line?

1. Consider the quadratic functions and their graphs shown.

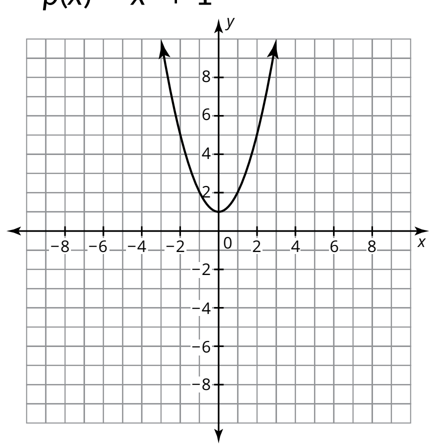
$f(x) = x^2 - 10x + 25$



$c(x) = -x^2 + 6x$



$p(x) = x^2 + 1$

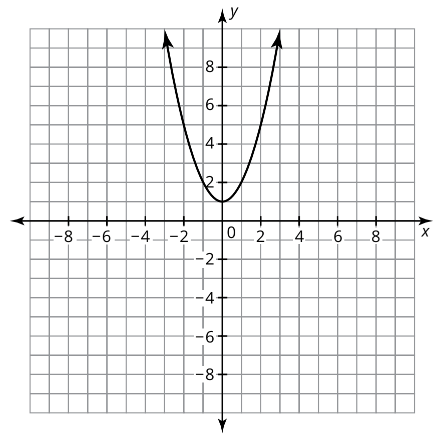


a. List all the key attributes you know about each function. Be sure to include the number of zeros, the x-intercept(s), the y-intercept, the axis of symmetry, and the vertex.

b. Compare the three functions. What do they have in common? What is different about the functions?

ACTIVITY 4.1 The Number *i*

Consider the function $p(x) = x^2 + 1$ and its graph from the Getting Started.



Emma and Noah determined the zeros of the function.

Emma

$x^2 + 1 = 0$

$x^2 = -1$

$x = \pm\sqrt{-1}$

Noah

$x^2 + 1 = 0$

$x^2 = -1$

$x = \pm 1$

1. What did Noah do wrong? Use the graph to justify your answer.

2. Consider Emma's solution. Does the solution fall within the real number system? Explain your reasoning.

In order to calculate the square of any real number, there must be some way to calculate the square root of a negative number. That is, there must be a number such that, when it is squared, it is equal to a negative number.
For this reason, mathematicians defined what is called *the number i*.
The number *i* is a number such that $i^2 = -1$. The number i is also called the imaginary identity.

3. If $i^2 = -1$ then determine the value of each expression.

   a. i

   b. i^3

   c. i^4

The number i is similar to the number π: even though they are both numbers, each is special enough that it gets its very own symbol.

4. Recall the function $p(x) = x^2 + 1$. Write the zeros of the function in terms of i.

Functions and equations that have solutions requiring i have **imaginary zeros, imaginary roots,** and imaginary solutions.

5. How can you tell from the graph of a quadratic equation whether or not it has real solutions or imaginary solutions?

6. Do you think you can determine the imaginary solutions by examining the graph? Explain your reasoning.

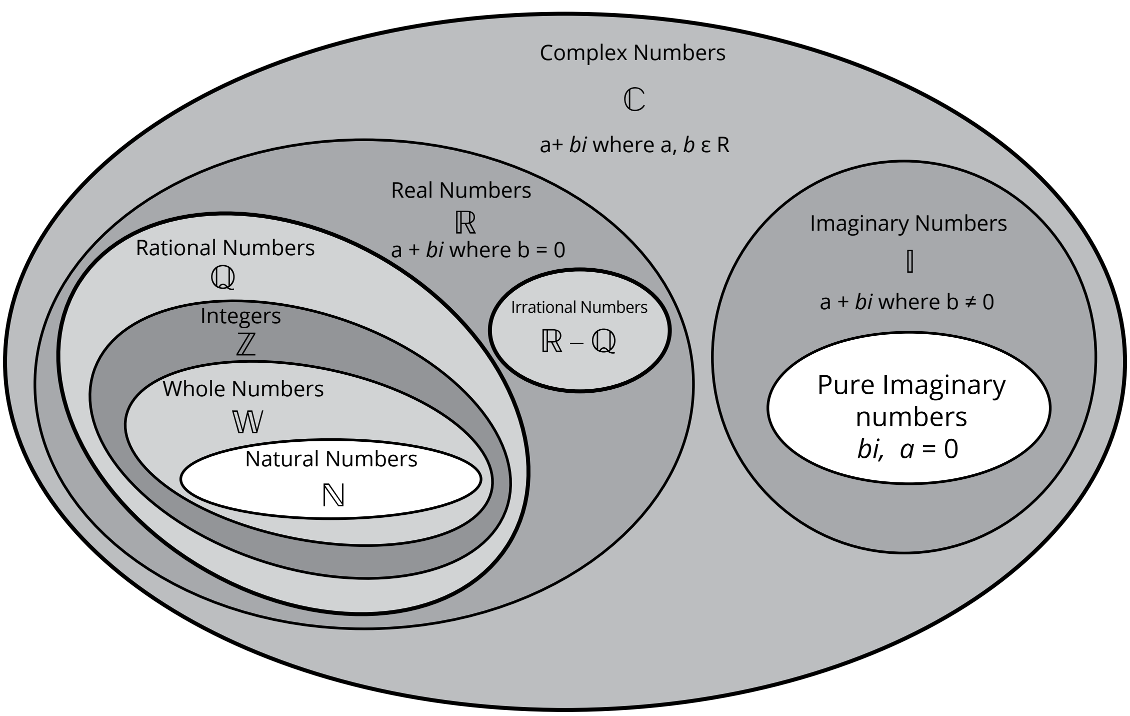
ACTIVITY 4.2

The Complex Number System

The set of **complex numbers** is the set of all numbers written in the form *a* + *bi*, where *a* and *b* are real numbers. The term *a* is called the **real part of a complex number**, and the term *bi* is called the **imaginary part of a complex number**. The set of complex numbers is represented by the notation C.

The set of **imaginary numbers** is the set of all numbers written in the form *a* + *bi*, where *a* and *b* are real numbers and *b* is not equal to 0. The set of imaginary numbers is represented by the notation I. A **pure imaginary number** is a number of the form *a* + *bi*, where *a* is equal to 0 and *b* is not equal to 0.

The ∈ symbol means "an element of". Therefore, "$a, b \in \mathbb{R}$" means that the values for *a* and *b* are elements of the set of real numbers.



1. Complete each statement with *always*, *sometimes*, or *never*.

   a. If a number is an imaginary number, then it is ____________ a complex number.

   b. If a number is a complex number, then it is ____________ an imaginary number.

   c. If a number is a real number, then it is ____________ a complex number.

   d. If a number is a real number, then it is ____________ an imaginary number.

   e. If a number is a complex number, then it is ____________ a real number.

2. Use the diagram on the previous page to list *all* number sets that describe each given number.

   a. 3

   b. $\sqrt{7}$

   c. $3i$

   d. $5.\overline{45}$

   e. $\frac{7}{8}$

   f. $6 - i$

WORKED EXAMPLE

You can use the number *i* to rewrite $\sqrt{-25}$.

Factor out –1. $\sqrt{-25} = \sqrt{(-1)(25)}$

Rewrite the radical expression. $= \sqrt{-1} \cdot \sqrt{25}$

Apply the square root on $\sqrt{25}$. $= 5\sqrt{-1}$

Rewrite $\sqrt{-1}$ as *i*. $= 5i$

So, $\sqrt{-25}$ can be rewritten as $5i$.

3. Rewrite each expression using *i*.

   a. $\sqrt{-4}$

   b. $\sqrt{-12}$

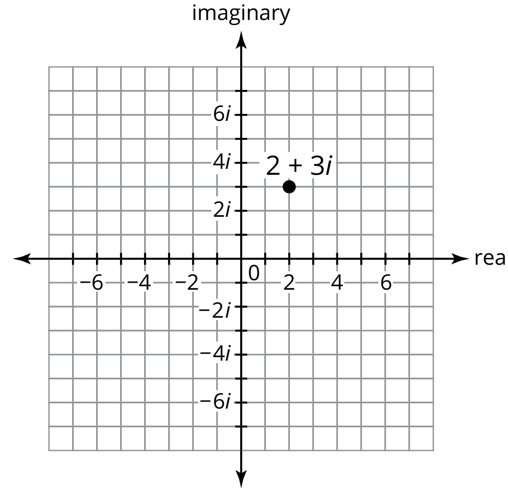
   c. $5 + \sqrt{-50}$

   d. $\dfrac{6 - \sqrt{-8}}{2}$

ACTIVITY 4.3 Add, Subtract, and Multiply Complex Numbers

A complex number can be represented on the complex plane. The horizontal axis of the complex plane contains all the real numbers, and the vertical axis contains all of the imaginary numbers.

The complex number $2 + 3i$ can be represented as shown.



1. Plot and label each of these complex numbers on the complex plane shown.

   a. $5 - 3i$

   b. $-3 + 4i$

   c. $8 + 6i$

   d. $-6i$

   e. -7

   f. i

   g. $5 + 3\sqrt{-1}$

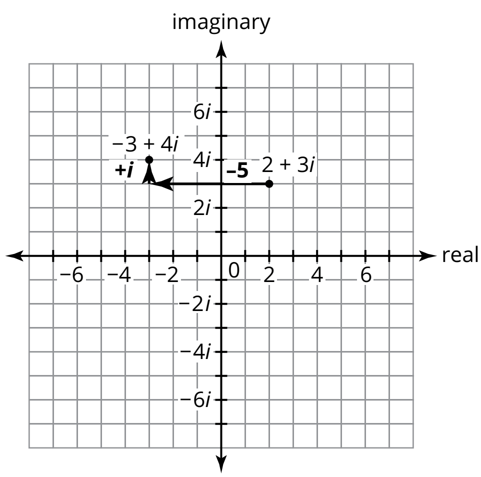
   h. $-3\sqrt{-1} + 3$

   i. $-4\sqrt{-1} - 3$

2. Describe the process you can use to plot a complex number $a + bi$ on the complex plane.

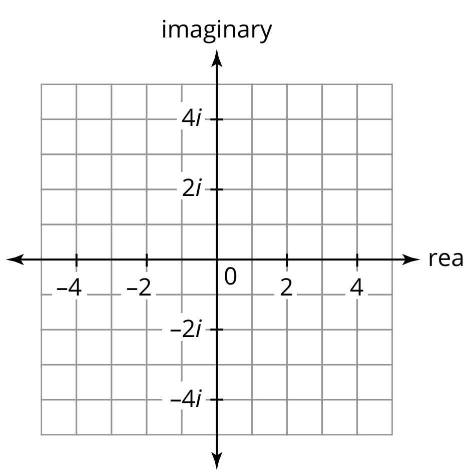
Let's consider the sum of two complex numbers.

The complex plane shows the sum of $(2 + 3i) + (-5 + i)$.



3. Suppose you start at the point $(-5 + i)$ and add $(2 + 3i)$.

   a. Show how to determine the sum on the complex plane.



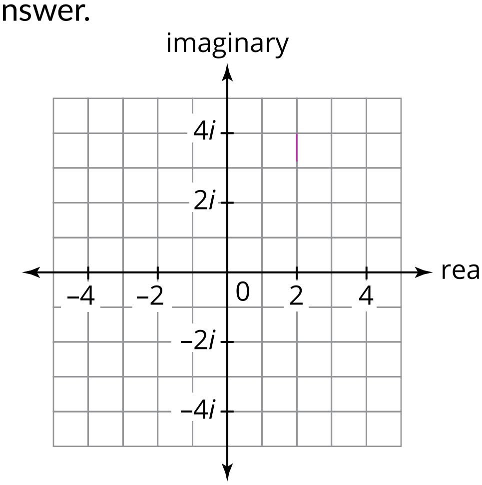
b. What do you notice about the sum? What does it suggest about the commutative property and complex numbers?

4. Explain why Ava's reasoning is correct. Support your answer with an example.

> **Ava**
> To add two complex numbers, you can add their real parts separately, then add their imaginary parts separately, and then combine the results into the form a + bi.

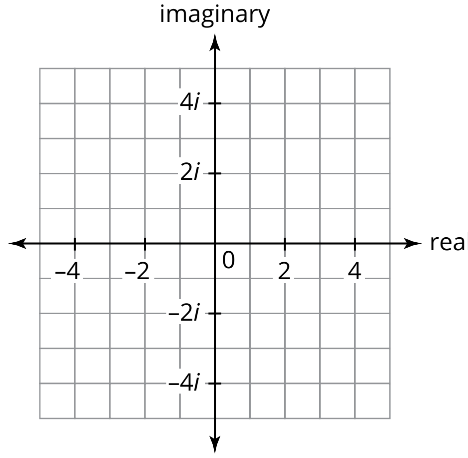
Remember . . .
Subtracting a number is the same as adding its negative.

5. Is $(-3 + 4i) - (-5 + i)$ equal to $2 + 3i$? Use the complex plane to confirm your answer.



6. Suppose you start at the point 2, or $2 + 0i$, and add $3i + (-5 + i)$.

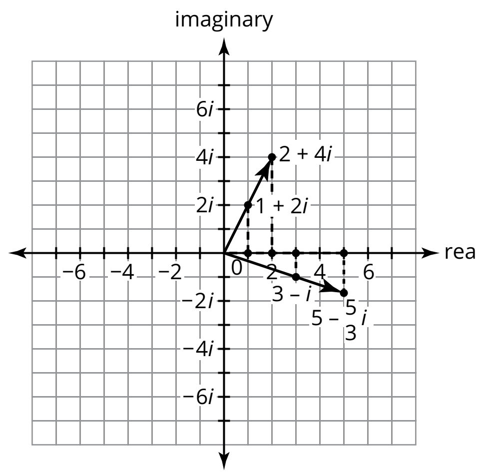
   a. Show how to determine the sum on the complex plane.



   b. Compare the sum with $(2 + 3i) + (-5 + i)$. What do you notice? What does it suggest about the associative property and complex numbers?

You can multiply complex numbers by real numbers. Multiplying by a real number, k, results in a dilation of the point on the complex plane: $k(a + bi) = ka + kbi$.

7. Consider the complex numbers plotted and labeled. Four different products can be represented. Complete each product and explain your answers.



a. (__________)$(1 + 2i) = (2 + 4i)$

b. (__________)$(2 + 4i) = (1 + 2i)$

c. (__________)$(3 - i) = \left(5 - \frac{5}{3}i\right)$

d. (__________)$\left(5 - \frac{5}{3}i\right) = (3 - i)$

8. What do these products suggest about the distributive property and complex numbers?

ACTIVITY 4.4

Operations with Complex Numbers

When operating with complex numbers involving i, combine like terms by treating i as a variable (even though it is a constant).

Remember . . .

The value of i^2 is -1.

1. Perform each operation. Show your work.

   a. $(7 + 9i) + (8 - i)$

   b. $(3 + 2i) - (1 - 6i)$

   c. $4i + 3 - 6 + i - 1$

2. Determine each product.

   a. $5i(3 - 2i)$

   b. $(5 + 3i)(2 - 3i)$

c. $(2 + i)(2 - i)$

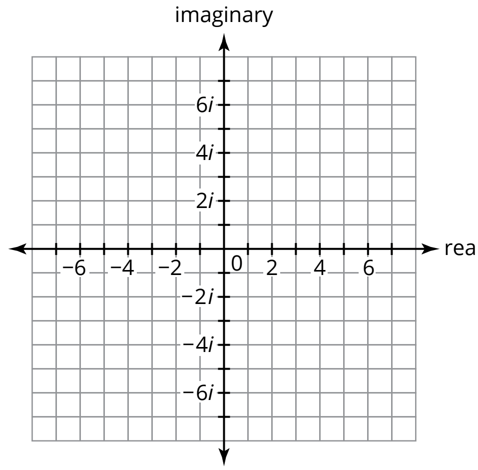
d. $\left(\frac{1}{2} + i\right)\left(\frac{1}{2} - i\right)$

e. $(3 + 2i)(3 - 2i)$

f. $(1 - 3i)(1 + 3i)$

g. What do you notice about each product for parts c through f?

3. What effect does multiplying a complex number by i have on the location of the point on the complex plane? Use examples to justify your thinking.



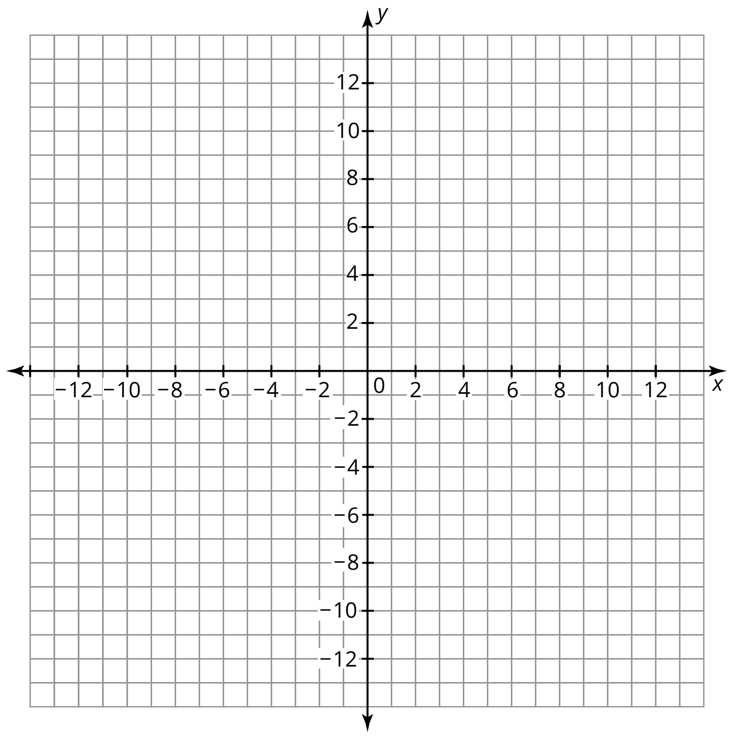
ACTIVITY 4.5

Solving Quadratics with Imaginary Zeros

Consider each quadratic function.

$f(x) = x^2 + 9$ $\quad$ $g(x) = 4(x - 2)^2 + 3$ $\quad$ $h(x) = x^2 - 2x + 2$

1. Use technology to sketch each function on the coordinate plane. Label each function on your graph.



2. What do the solutions of the three functions have in common?

3. Consider the function $f(x) = x^2 + 9$.

   a. Describe the structure of the function. How does it compare to the function $q(x) = x^2 - 9$?

b. Consider an equation written in the form $ax^2 + c = 0$.
Complete the table to show when the solutions of a function are real or imaginary.

	c is positive	c is negative
a is positive		
a is negative		

c. If the factored form of a difference of squares is $a^2 - b^2 = (a - b)(a + b)$, what is the factored form of a sum of squares, $a^2 + b^2$?

d. What are the solutions to $f(x) = x^2 + 9$?

4. Consider the function $g(x) = (x - 2)^2 + 1$.

a. How can you use the structure of the function to determine whether its zeros are real or imaginary?

Remember . . .
Will the graph of the function pass through the x-axis?

b. What is the solution to $(x - 2)^2 + 1 = 0$?

5. Consider the function $h(x) = x^2 - 2x + 2$.

a. What is the most efficient method to determine the zeros? Explain your reasoning.

Remember . . .
The discriminant of the function is $b^2 - 4ac$.

b. Determine the discriminant of the function. How can you use the discriminant to know whether the solutions are real or imaginary?

c. What is the solution to $x^2 - 2x + 2 = 0$?

d. Describe how you can check your solutions.

You can apply the distributive property to determine whether an equation in factored form is equivalent to an equation in standard form.

WORKED EXAMPLE

Consider the expression $(x - i)(x + i)$.

Multiply binomials.	$(x - i)(x + i) = x^2 + xi - xi - i^2$
Group like terms.	$= x^2 + (xi - xi) - i^2$
Combine like terms.	$= x^2 - i^2$
Use the powers of i to rewrite i^2 as -1.	$= x^2 - (-1)$
Rewrite.	$= x^2 + 1$

So, $(x - i)(x + i) = x^2 + 1$.

Recall that a quadratic function in factored form is written in the form $y = a(x - r_1)(x - r_2)$.

6. Use your answer to Question 5 part (c) to write the function $h(x) = x^2 - 2x + 2$ in factored form.

7. Explain why the function you wrote in factored form is equivalent to $h(x) = x^2 - 2x + 2$.

8. Use any method to determine the zeros of each function.

   a. $f(x) = -x^2 - 8x - 18$

b. $g(x) = 2x^2 - 2x + 3$

c. $h(x) = -3x^2 - 4x - 4$

9. Solve each quadratic equation.

a. $2 = x^2 - 3x + 5$

b. $-4 = 3x^2 - 10x - 12$

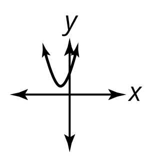
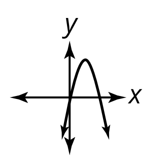
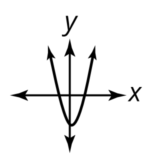
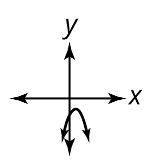
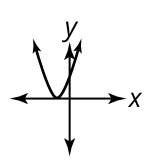
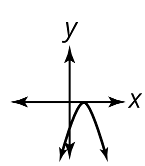
c. $8 = -2x^2 - 5x + 3$

ACTIVITY 4.6

Beyond Imagination

The **Fundamental Theorem of Algebra** states that any polynomial equation of degree *n* must have exactly *n* complex roots or solutions. Any root may be a multiple root.

1. Complete the table to determine the number of real and imaginary roots for different quadratic equations.

Location of Vertex	Concavity	Sketch	Number of *x*-Intercepts	Number and Type of Roots
Above the *x*-axis	Up		0	2 imaginary roots
	Down			
Below the *x*-axis	Up			
	Down			
On the *x*-axis	Up			
	Down			

2. Elijah says that any quadratic equation has only one of these 3 types of solutions:

- 2 unique real number solutions
- 2 equal real number solutions (a double root)
- 1 real and 1 imaginary solution

Liam says that any quadratic equation has only one of these 3 types of solutions:

- 2 unique real number solutions
- 2 equal real number solutions (a double root)
- 2 imaginary solutions

Mateo says that any quadratic equation has only one of these 4 types of solutions:

- 2 unique real number solutions
- 2 equal real number solutions (a double root)
- 2 imaginary solutions
- 1 real and 1 imaginary solution

Who's correct? Explain your reasoning.

3. Explain why it is not possible for a quadratic equation to have 2 equal imaginary solutions (double imaginary root).

Talk the Talk

Complex Problems, Complex Solutions

1. Perform each operation. Show your work.

   a. $(4 + 3i) + (5 - 8i)$

   b. $(12 - 9i) - (-3 + 7i)$

   c. $(-3 - 5i)(2 + i)$

2. Solve the quadratic equation $-4 = 3x^2 + 7x + 9$.

Lesson 4 Assignment

Write

Match each definition to the corresponding term.

1. the set of all numbers written in the form $a + bi$, where a and b are real numbers
2. the set of all numbers written in the form $a + bi$, where a and b are real numbers and b is not equal to 0
3. the term bi in a complex number written as $a + bi$
4. a number equal to $\sqrt{-1}$
5. solutions to functions and equations that have a negative value for the discriminant
6. a number of the form bi where b is a real number and is not equal to 0
7. the term a in a complex number written as $a + bi$

a. imaginary roots (imaginary zeros)

b. the number i

c. imaginary numbers

d. pure imaginary number

e. complex numbers

f. real part of a complex number

g. imaginary part of a complex number

Remember

The set of complex numbers is the set of all numbers written in the form $a + bi$, where a and b are real numbers. Imaginary numbers are complex numbers where b is not equal to 0 and real numbers are complex numbers where b is equal to 0.

Lesson 4 Assignment

Practice

1. Rewrite each radical using i.

   a. $\sqrt{-16}$
   b. $\sqrt{-27}$
   c. $\sqrt{-200}$
   d. $5 + \sqrt{-20}$

2. Classify each number according to its most specific number set.

   a. $\frac{-4}{\sqrt{9}}$
   b. $\frac{\sqrt{-4}}{9}$
   c. $9 - \sqrt{-4}$
   d. $-4 - \sqrt{9}$

3. Mr. Brown writes the expression $(3 + i)(7 - i)$ on the board and asks his students to rewrite it using the distributive property. The work of two students is shown below. Which student simplified the expression correctly? What mistake did the other student make?

   Student 1

   $$\begin{aligned}(3 + i)(7 - i) &= 21 - 3i + 7i - i^2 \\ &= 21 + 4i + 1 \\ &= 22 + 4i\end{aligned}$$

   Student 2

   $$\begin{aligned}(3 + i)(7 - i) &= 21 - 3i + 7i - i^2 \\ &= 21 + 4i - 1 \\ &= 20 + 4i\end{aligned}$$

4. Jayden claims that $\sqrt{-16} \cdot \sqrt{-4}$ is equal to 8. Mia claims that $\sqrt{-16} \cdot \sqrt{-4}$ is equal to -8.
   Who is correct? What mistake did the other student make? Support your answer with work.

Lesson 4 Assignment

5. Victoria identifies $\frac{6i}{4}$ as an imaginary number and a rational number. Is Victoria correct? Explain how you determined your answer.

6. Perform each operation.

a. $(8 + 5i) + (2 - i)$

b. $(4 + 7i) - (1 - 3i)$

c. $7 + 6i - 3 + 5i - 9i$

d. $(9 + 2i)(3 + 4i)$

e. $\left(\frac{3}{4} + i\right)\left(\frac{3}{4} - i\right)$

f. $(5 - i)(4 - 6i)$

Lesson 4 Assignment

7. Solve each quadratic equation

   a. $-2 = 3x^2 - 4x - 7$

   b. $12 = -2x^2 - 7x + 3$

Prepare

Solve each linear inequality and graph the solution set on a number line.

1. $-3(2x - 5) + 10 \geq 16$

2. $6(x - 4) > -2(x + 8)$

Lesson 4 Assignment

3. $2(3x + 7) < 2x - 6$ or $5(x - 1) > 10$

4. $-2 \leq 3(x - 3) \leq 5$

Name: ______________________

Quadratic Functions and Equations

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me,** 2 represents **I am building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Quadratic Functions and Equations* topic by:

TOPIC 1: *Quadratic Functions and Equations*	Beginning of Topic	Middle of Topic	End of Topic
explaining that there are three forms of quadratic equations: standard, vertex, and factored forms.	◯	◯	◯
rewriting a quadratic equation given in standard form into vertex form.	◯	◯	◯
analyzing a graph and stating the key attributes of the graph.	◯	◯	◯
identifying the key attributes that are easily identifiable in each of the three forms.	◯	◯	◯
solving quadratic equations using the properties of equality and square roots, factoring, completing the square, and the quadratic formula.	◯	◯	◯
recognizing a linear, exponential, or quadratic function by its equation or graph.	◯	◯	◯
explaining that complex solutions for a quadratic equation result when the discriminant is negative in the quadratic formula or when the graph of the equation does not intersect the *x*-axis.	◯	◯	◯

continued on the next page

TOPIC 1 SELF-REFLECTION *continued*

TOPIC 1: *Quadratic Functions and Equations*	Beginning of Topic	Middle of Topic	End of Topic
understanding that $i = \sqrt{-1}$.	○	○	○
identifying a complex number in the form $a + bi$.	○	○	○
using properties to add, subtract, and multiply complex numbers.	○	○	○
solving quadratic equations with real number coefficients and imaginary solutions.	○	○	○
factoring a quadratic expression using complex numbers.	○	○	○

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Exploring Quadratic Functions and Equations* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 1 SUMMARY

Quadratic Functions and Equations

LESSON 1 Forms of Quadratic Functions

Quadratic functions can be written in different forms.

Standard form: $f(x)) = ax^2 + bx + c$, where a does not equal 0.
Factored form: $f(x) = a(x - r_1)(x - r_2)$, where a does not equal 0.
Vertex form: $f(x) = a(x - h)^2 + k$, where a does not equal 0.

The graphs of quadratic functions can be described using key attributes: x-intercept(s), y-intercept, vertex, axis of symmetry, and concavity. The key attributes of a function can be determined using different methods depending on the form of the function.

Standard Form Methods to Determine Key Attributes	
Axis of Symmetry $x = \frac{-b}{2a}$	**x-intercept(s)** Substitute 0 for y, and then solve for x using the quadratic formula, factoring, or graphing technology.
Vertex Use $\frac{-b}{2a}$ to determine the x-coordinate of the vertex. Then substitute that value into the equation and solve for y.	**y-intercept** c-value

KEY TERMS

- standard form of a quadratic equation [forma estandarizada de una ecuación cuadrática]
- factored form of a quadratic equation [forma factorizada de una ecuación cuadrática]
- vertex form of a quadratic equation [forma de vértice de una ecuación cuadrática]
- concavity of a parabola [concavidad de una parábola]
- quadratic formula [fórmula cuadrática]
- the number i [el número i]
- imaginary zeros [ceros imaginarios]
- imaginary roots [raíces imaginarias]
- complex numbers [números complejos]
- real part of a complex number [parte real de un número complejo]
- imaginary part of a complex number [parte imaginaria de un número complejo]
- imaginary numbers [números imaginarios]
- pure imaginary number [número imaginario puro]
- Fundamental Theorem of Algebra [Teorema fundamental del álgebra]

Factored Form Methods to Determine Key Attributes	
Axis of Symmetry $x = \frac{r_1 + r_2}{2}$	***x*-intercept(s)** $(r_1, 0), (r_2, 0)$
Vertex Use $\frac{r_1 + r_2}{2}$ to determine the *x*-coordinate of the vertex. Then substitute that value into the equation and solve for *y*.	***y*-intercept** Substitute 0 for *x*, and then solve for *y*.

Vertex Form Methods to Determine Key Attributes	
Axis of Symmetry $x = h$	***x*-intercept(s)** Substitute 0 for *y*, and then solve for *x* using the quadratic formula, factoring, or graphing technology.
Vertex (h, k)	***y*-intercept** Substitute 0 for *x*, and then solve for *y*.

Concavity of a parabola describes whether a parabola opens up or opens down. A parabola is concave up if it opens upward; a parabola is concave down if it opens downward. When the leading coefficient *a* is negative, the graph of the quadratic function opens downward and has a maximum. When *a* is positive, the graph of the quadratic function opens upward and has a minimum.

You can identify the axis of symmetry and the vertex of any quadratic function written in standard form by completing the square.

Consider the equation $y = ax^2 + bx + c$.

Step 1: $$y - c = ax^2 + bx$$

Step 2: $$y - c = a\left(x^2 + \frac{b}{a}x\right)$$

Step 3: $$y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right)$$

Step 4: $$y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

Step 5: $$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

LESSON 2 Writing Quadratic Functions

You can write a unique quadratic function given a vertex and a point on the parabola.

For example, consider a parabola with vertex (5, 2) that passes through the point (4, 9).

Substitute the given values into the vertex form of the function and solve for a.

$$f(x) = a(x - h)^2 + k$$
$$9 = a(4 - 5)^2 + 2$$
$$9 = a(-1)^2 + 2$$
$$9 = 1a + 2$$
$$7 = a$$

Finally, substitute the a-value into the function.

$$f(x) = 7(x - 5)^2 + 2$$

You can write a unique quadratic function given the roots and a point on the parabola.

For example, consider a parabola with roots at (–2, 0) and (4, 0) that passes through the point (1, 6).

Substitute the given values into the vertex form of the function and solve for a.

$$f(x) = a(x - r_1)(x - r_2)$$
$$6 = a(1 + 2)(1 - 4)$$
$$6 = a(3)(-3)$$
$$6 = -9a$$
$$-\frac{2}{3} = a$$

Finally, substitute the a-value into the function.

$$f(x) = -\frac{2}{3}(x + 2)(x - 4)$$

You can determine a unique quadratic function algebraically given three reference points. Substitute the x- and y-values of each point into the standard form, $y = ax^2 + bx + c$ to create a system of three equations. Then, use elimination and substitution to solve for a, b, and c.

Consider the three points: A (2, 1), B (–1, –2), and C (3, –10).

To write an equation in standard form to represent a parabola that passes through three given points, begin by substituting the x- and y-values of each point into $y = ax^2 + bx + c$.

Point A: $1 = a(2)^2 + b(2) + c$
$1 = 4a + 2b + c$

Equation A: $1 = 4a + 2b + c$

Point B: $-2 = a(-1)^2 + b(-1) + c$
$-2 = a - b + c$

Equation B: $-2 = a - b + c$

Point C: $-10 = a(3)^2 + b(3) + c$
$-10 = 9a + 3b + c$

Equation C: $-10 = 9a + 3b + c$

Now, use linear combinations and substitution to solve for a, b, and c.

STEP 1: Subtract Equation B from A:

$$\begin{array}{r} 1 = 4a + 2b + c \\ -(-2 = a - b + c) \\ \hline 3 = 3a + 3b \end{array}$$

STEP 2: Subtract Equation B from C:

$$\begin{array}{r} -10 = 9a + 3b + c \\ -(-2 = a - b + c) \\ \hline -8 = 8a + 4b \end{array}$$

STEP 3: Solve the equation from Step 1 in terms of a.

$3 = 3a + 3b$

$3 - 3b = 3a$

$1 - b = a$

STEP 4: Substitute the value for a into the equation from Step 2.

$-8 = 8(1 - b) + 4b$

$-8 = 8 - 4b$

$16 = 4b$

$4 = b$

STEP 5: Substitute the value for b into the equation from Step 3.

$a = 1 - (4)$

$a = -3$

STEP 6: Substitute the values for a and b into Equation A.

$1 = 4a + 2b + c$

$1 = 4(-3) + 2(4) + c$

$1 = -12 + 8 + c$

$1 = -4 + c$

$5 = c$

STEP 7: Substitute the values for a, b, and c into the standard form of a quadratic.

$y = -3x^2 + 4x + 5$

Solving Quadratic Equations

You can use factoring and the zero product property to solve quadratics in the form $y = ax^2 + bx + c$.

For example, you can solve the quadratic equation $x^2 - 4x = -3$.

$$x^2 - 4x = -3$$

$$x^2 - 4x + 3 = -3 + 3$$

$$x^2 - 4x + 3 = 0$$

$$(x - 3)(x - 1) = 0$$

$$(x - 3) = 0 \quad \text{or} \quad (x - 1) = 0$$

$$x - 3 + 3 = 0 + 3 \quad \text{or} \quad x - 1 + 1 = 0 + 1$$

$$x = 3 \quad \text{or} \quad x = 1$$

For a quadratic function that has zeros but cannot be factored, there is another method for solving the quadratic equation. Recall completing the square is a process for writing a quadratic expression in vertex form, which then allows you to solve for the zeros.

For example, you can calculate the roots of the equation $x^2 - 4x + 2 = 0$.

Isolate $x^2 - 4x$.

$$x^2 - 4x + 2 - 2 = 0 - 2$$

$$x^2 - 4x = -2$$

Determine the constant term that would complete the square.

$$x^2 - 4x + ? = -2 + ?$$

Add this term to both sides of the equation.

$$x^2 - 4x + 4 = -2 + 4$$

$$x^2 - 4x + 4 = 2$$

Factor the left side of the equation.

$$(x - 2)^2 = 2$$

Determine the square root of each side of the equation.

$$\sqrt{(x - 2)^2} = \sqrt{2}$$

$$(x - 2) = \pm\sqrt{2}$$

Set the factor of the perfect square trinomial equal to each square root of the constant and solve for x.

$$x - 2 = \sqrt{2} \quad \text{or} \quad x - 2 = -\sqrt{2}$$

$$x = 2 + \sqrt{2} \quad \text{or} \quad x = 2 - \sqrt{2}$$

$$x \approx 3.41 \quad \text{or} \quad x \approx 0.59$$

The roots are approximately 3.41 and 0.59.

The **quadratic formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b and c represent real numbers and $a \neq 0$.

For example, given the function $f(x) = 2x^2 - 4x - 3$ you can identify the values of a, b and c.

$$a = 2;\ b = -4;\ c = -3$$

Then you use the quadratic formula to solve.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{16 + 24}}{4}$$

$$x = \frac{4 \pm \sqrt{40}}{4}$$

$$x \approx \frac{4 + 6.325}{4} \approx 2.581 \quad \text{or} \quad x \approx \frac{4 - 6.325}{4} \approx -0.581$$

The roots are $x \approx 2.581$ and $x \approx -0.581$.

LESSON 4 Imaginary and Complex Numbers

In order to calculate the square root of any real number, there must be some way to calculate the square root of a negative number. That is, there must be a number such that when it is squared, it is equal to a negative number. For this reason, mathematicians defined what is called **the number *i***. The number i is a number such that $i^2 = -1$.

For example, you can rewrite the expression $\sqrt{-25}$ using i.

Factor out -1.	$\sqrt{-25} = \sqrt{(-1)(25)}$
Rewrite the radical expression.	$= \sqrt{-1} \cdot \sqrt{25}$
Apply the square root on $\sqrt{25}$.	$= 5\sqrt{-1}$
Rewrite $\sqrt{-1}$ as i.	$= 5i$

So, $\sqrt{-25}$ can be rewritten as $5i$.

Functions and equations that have imaginary solutions have **imaginary roots** or **imaginary zeros**, which are the solutions.

The set of **complex numbers** is the set of all numbers written in the form $a + bi$, where a and b are real numbers. The term a is called the **real part of a complex number**, and the term bi is called the **imaginary part of a complex number**.

The set of **imaginary numbers** is a subset of the set of complex numbers. A **pure imaginary number** is a number of the form $a + bi$, where b is not equal to 0.

When operating with complex numbers involving i, combine like terms by treating i as a variable (even though it is a constant).

For example, consider the sum of $(2 + 3i) + (-5 + i)$.

$$\begin{aligned}(2 + 3i) + (-5 + i) &= (2 + (-5)) + (3i + \mathrm{i}) \\ &= -3 + 4i\end{aligned}$$

You can also multiply complex numbers using the distributive property.

For example, consider the product of $(2 + i)(2 - i)$.

$$\begin{aligned}(2 + i)(2 - i) &= 2(2) + 2(-i) + i(2) + i(-i) \\ &= 4 - 2i + 2i - i^2 \\ &= 4 - i^2 \\ &= 4 - (-1) = 5\end{aligned}$$

You can use the same methods you used to solve quadratic equations with real solutions to solve quadratic equations with imaginary solutions.

For example, consider the function $f(x) = x^2 - 2x + 2$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 - 8}}{2}$$

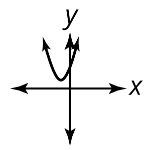
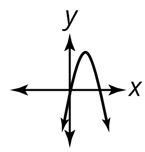
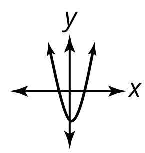
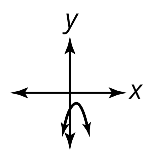
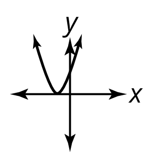
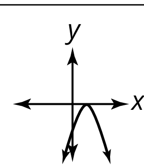
$$x = \frac{2 \pm \sqrt{-4}}{2}$$

$$x = \frac{2 \pm 2i}{2}$$

$$x = 1 \pm i$$

The **Fundamental Theorem of Algebra** states that any polynomial equation of degree n must have, exactly n complex roots or solutions.

Any root may be a multiple root. The table shows the number of real and imaginary roots for different quadratic equations.

Location of Vertex	Concavity	Sketch	Number of x-Intercepts	Number and Type of Roots
Above the x-axis	Up		0	2 imaginary roots
	Down		2	2 real roots
Below the x-axis	Up		2	2 real roots
	Down		0	2 imaginary roots
On the x-axis	Up		1	1 unique root
	Down		1	1 unique root

TOPIC 2

The Golden Gate Bridge, an iconic suspension bridge in San Francisco, features parabolic curves in its main cables, which distribute the tension and weight efficiently across the towers and anchorages. These parabolas are formed naturally by the uniform load of the suspended bridge deck, showcasing the elegance of engineering and mathematics in design.

Applications of Quadratics

1 Solving Quadratic Inequalities

OBJECTIVES

- Solve a quadratic inequality by calculating the roots of the quadratic equation which corresponds to the inequality and testing values within intervals determined by the roots.
- Connect the graphical representation of a quadratic function and the solution to a corresponding quadratic inequality represented on a number line and describe the features of each in written form.
- Use interval notation to record the solutions to quadratic inequalities.

KEY TERM

- interval notation

You have interpreted the solution sets to linear inequalities on a coordinate plane. You have also solved quadratic equations using a variety of methods.

How can you interpret the solution sets to quadratic inequalities on a coordinate plane using what you know about solving quadratic equations?

Getting Started

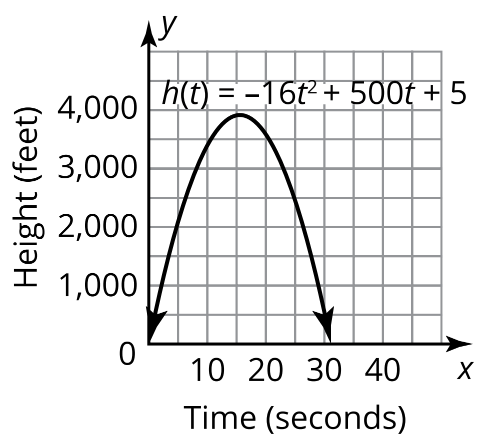
It Has Its Ups and Downs

People use fireworks at many Fourth of July celebrations. The Fourth of July marks Congress's passage of the Declaration of Independence in 1776 and has served as a federal holiday in the United States since 1941.

A firework is shot straight up into the air with an initial velocity of 500 feet per second from 5 feet off the ground. The graph of the function that represents this situation is shown.

Review the definitions of **select** and **estimate** in the Academic Glossary.

1. Use the graph to estimate when the firework will be at each given height off the ground.



a. 0 feet

b. 1000 feet

c. 2500 feet

d. 3900 feet

2. Describe any patterns you notice for the number of times the firework reaches a given height.

3. Draw a horizontal line on the graph to represent when the firework is 2000 feet off the ground.

   a. When is the firework higher than 2000 feet? Circle this portion of the graph.

   b. When is the firework below 2000 feet? Draw a box around this portion of the graph.

   c. Write a quadratic inequality that represents the times when the firework is below 2000 feet.

Remember . . .

A vertical motion model is a quadratic equation of the form $y = -16t^2 + v_0t + h_0$.

ACTIVITY
1.1 Solving Quadratic Inequalities

Just like with the other inequalities you have studied, the solution to a quadratic inequality is the set of values that satisfy the inequality.

WORKED EXAMPLE

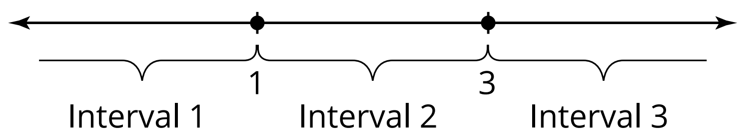
Let's determine the solution of the quadratic inequality $x^2 - 4x + 3 < 0$.

Write the corresponding quadratic equation. $\qquad x^2 - 4x + 3 = 0$

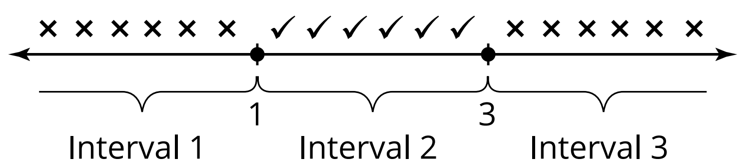
Calculate the roots of the quadratic equation using an appropriate method.

$$(x - 3)(x - 1) = 0$$
$$(x - 3) = 0 \quad \text{or} \quad (x - 1) = 0$$
$$x = 3 \quad \text{or} \quad x = 1$$

Plot the roots to divide the number line into three regions.



Choose a value from each interval to algebraically test in the original inequality.



Try $x = 0$	Try $x = 2$	Try $x = 4$
$0^2 - 4(0) + 3 < 0$	$2^2 - 4(2) + 3 < 0$	$4^2 - 4(4) + 3 < 0$
$3 < 0$ ×	$4 - 8 + 3 < 0$	$16 - 16 + 3 < 0$
	$-1 < 0$ ✓	$3 < 0$ ×

Identify the solution set as the interval(s) in which your test value(s) satisfies the inequality.

Interval 2 satisfies the original inequality, so the solution includes all numbers between 1 and 3.

Solution: $x \in (1, 3)$

The symbol $\in$ is read "is an element of," "is in," or "belongs to." The notation $x \in (1, 3)$ has the same meaning as $1 < x < 3$. This notation, called **interval notation**, represents a range of values on the number line.

1. Analyze the Worked Example.

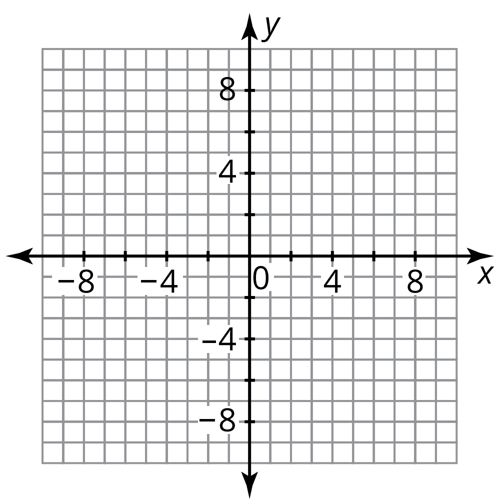
   a. How would the solution set change if the inequality was less than or equal to? Explain your reasoning.

   b. How would the solution set change if the inequality was greater than or equal to? Explain your reasoning.

The notation $x \in [1, 3]$ means the same as $1 \le x \le 3$.

Ask Yourself . . .
How does representing mathematics in multiple ways help to communicate reasoning?

2. Graph $y = x^2 - 4x + 3$ on the coordinate plane shown and label the roots of the equation and the vertex. Then describe how the graph supports that the solution set for the associated quadratic inequality $x^2 - 4x + 3 < 0$ is $1 < x < 3$.



Ethan correctly determined the roots of the quadratic inequality $2x^2 - 14x + 27 \geq 7$ to be $x = 5$ and $x = 2$. However, he incorrectly determined the solution set. His work is shown.

Ethan

$2x^2 - 14x + 27 = 7$
$2x^2 - 14x + 20 = 0$
$2(x^2 - 7x + 10) = 0$
$2(x - 5)(x - 2) = 0$
$x = 5$ or $x = 2$

$x = 1$	$x = 3$	$x = 6$
$2(1)^2 - 14(1) + 27 \geq 7$	$2(3)^2 - 14(3) + 27 \geq 7$	$2(6)^2 - 14(6) + 27 \geq 7$
$2 - 14 + 27 \geq 7$	$18 - 42 + 27 \geq 7$	$72 - 84 + 27 \geq 7$
$15 \geq 7$ ✓	$3 \geq 7$ ×	$15 \geq 7$ ✓

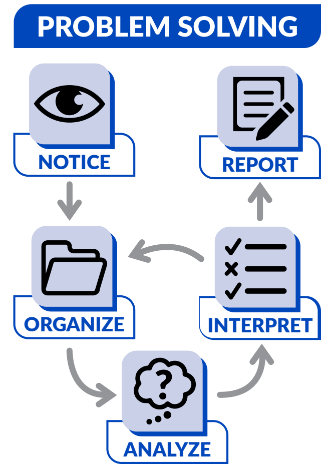
Solution: $x \in (-\infty, 1]$ or $x \in [6, \infty)$

Ask Yourself . . .
When testing values from each interval, could you use the factored form of the inequality rather than the original inequality?

3. Describe Ethan's error. Then, determine the correct solution set for the inequality.

ACTIVITY 1.2

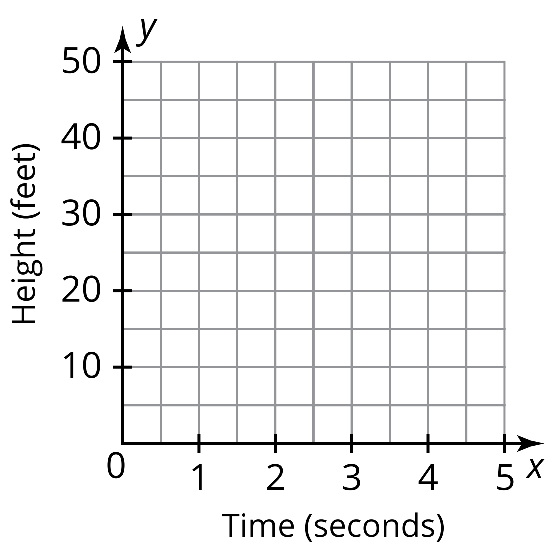
Modeling Quadratic Inequalities



A water balloon is launched from a machine upward from a height of 10 feet with an initial velocity of 46 feet per second.

1. Identify the variables and write a quadratic function to represent this situation.

2. Use technology to sketch the graph of the function.



Review the steps and questions from the problem-solving model.

3. Draw a horizontal line on the graph to represent when the balloon is 30 feet off the ground.

   a. Circle the portion of the graph that represents when the balloon is above 30 feet.

b. Write and solve an inequality to determine when the balloon is above 30 feet. Use the graph to explain your solution.

4. Determine when the balloon is at or below 43 feet. Interpret your solution in terms of the model you graphed.

Talk the Talk

Boom! Boom!

In the Getting Started activity, a firework was shot straight up into the air with an initial velocity of 500 feet per second from 5 feet off the ground. The function representing the situation was identified as $h(t) = -16t^2 + 500t + 5$. You determined the firework would be above 2000 feet between about 5 seconds and 27 seconds.

Suppose a second firework was shot straight up into the air with an initial velocity of 500 feet per second from the ground.

1. Predict whether the second firework will be above 2000 feet for more time, less time, or the same amount of time as the first firework.

2. Write a quadratic inequality to represent when the second firework will be above 2000 feet.

3. Determine when the second firework will be above 2000 feet.

4. Was your prediction made in Question 1 correct?

5. Use technology to compare the graph of the first firework to the graph of the second firework. What do you notice?

Lesson 1 Assignment

Write

Describe the difference between the solutions of a quadratic equation and the solutions of a quadratic inequality.

Remember

The solution set of a quadratic inequality is determined by first solving for the roots of the quadratic equation, then determining which interval(s) created by the roots satisfy the inequality.

Practice

1. A nutrition company has determined that the fixed cost associated with producing cases of its special health bars is \$1000. The variable cost is $\frac{3}{4}x + 25$ dollars per case that they produce. The selling price of the cases of health bars is $135 - \frac{1}{4}x$ per case that they sell.

The cost function is the variable cost times the number of products sold. The revenue function is the selling price times the number of products sold. The profit function = revenue function − cost function.

   a. Determine the cost function $C(x)$ for this product based on the number of cases, x, that they produce and sell. Simplify if necessary.

   b. Determine the revenue function $R(x)$ for this product based on the number of cases, x, that they produce and sell. Simplify if necessary.

   c. The profit that a company makes is the difference between the revenue and the cost. Determine the profit function $P(x)$ for this product.

d. Determine when the company will break even.

e. If they make and sell fewer than 10 cases of health bars, will they have a positive or negative profit? Explain your reasoning.

f. If they make and sell more than 100 cases of health bars, will they have a positive or negative profit? Explain your reasoning.

g. Determine how many units the company must produce and sell to make a profit of at least $1800.

Lesson 1 Assignment

2. Solve each inequality.

a. $2y^2 + 2y - 12 > 0$

b. $x^2 + 6x \leq 0$

c. $4b^2 + 14b + 16 < 10$

d. $a^2 \geq 4(2a - 3)$

e. $2t^2 > 9t + 18$

f. $k^2 + 3k + 2 < -3(k + 2)$

Prepare

Solve each system of equations.

1. $\begin{cases} y = 2x - 5 \\ y = x - 1 \end{cases}$

2. $\begin{cases} 3x + y = 2 \\ y = 5x - 6 \end{cases}$

3. $\begin{cases} 2x + y = 7 \\ x = 3y - 11 \end{cases}$

4. $\begin{cases} -3x + y = 7 \\ -x + y = 1 \end{cases}$

2

Systems of Linear and Quadratic Equations

OBJECTIVE

- Solve systems involving a linear equation and a quadratic equation, and verbally explain the relationship between the two functions.

You have solved systems of linear equations graphically by determining the point of intersection and algebraically using substitution.

How can you use these same methods to solve systems involving a linear and a quadratic equation?

Getting Started

Block That Kick!

A punter kicks a football. The height of the football, in meters, is modeled by the function $h(t) = -4.9t^2 + 20t + 0.75$, where t represents time, in seconds. A blocker can only attempt to knock down the football as it travels upward from the punter's foot. The height in meters of the approaching blocker's hands is modeled by the function $h(t) = -0.6t + 3$, where t represents the same time. Can the blocker knock down the football?

1. Describe the shape of the functions that model the football's height over time and the height of the blocker's hands over time.

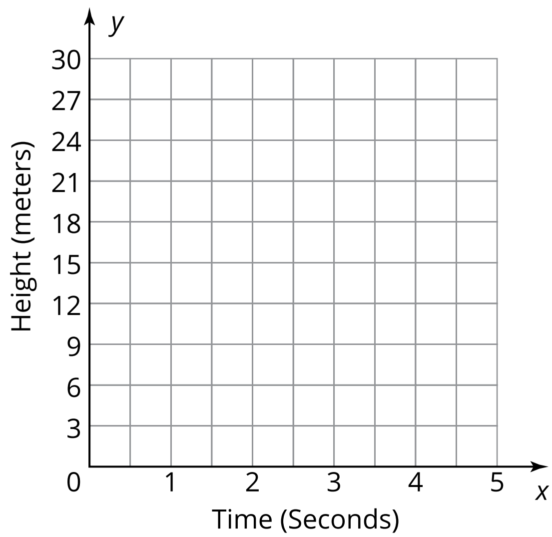
2. Sketch a graph of the situation. Do you think it is possible for the blocker to knock down the football? Explain your reasoning.

ACTIVITY 2.1

Modeling a System with a Linear and a Quadratic Equation

A system of equations can involve nonlinear equations, such as quadratic equations. The scenario described in the previous activity models the relationship between a quadratic and a linear equation.

1. Use technology to sketch the graph of the system described in the previous activity.



2. How many solutions does the system have? Explain your reasoning.

3. Does every solution make sense in the context of the problem situation? Explain your reasoning.

4. Use the graph to approximate at what point the blocker can block the football. Interpret your solution in the context of the problem.

ACTIVITY
2.2 Solving a System With a Linear and a Quadratic Equation

Methods for solving a system of nonlinear equations can be similar to methods for solving a system of linear equations.

1. Consider the system of a linear equation and a quadratic equation shown.

$$\begin{cases} y = 2x + 7 \\ y = x^2 + 4 \end{cases}$$

Ask yourself . . .
Since y is equal to two different expressions, can you set the expressions equal to each other?

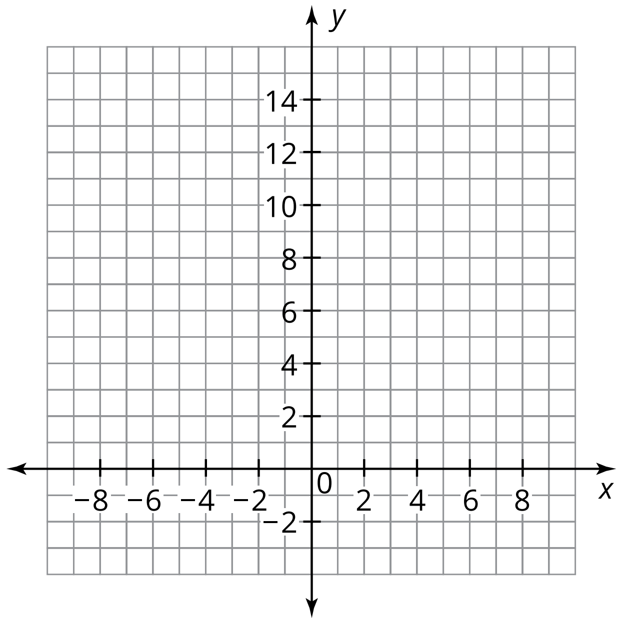
a. Write a new equation you can use to solve this system.

b. Solve the resulting equation for x.

c. Calculate the corresponding values for y.

d. Identify the solution(s) to the system of equations.

e. Graph each equation of the system and identify the points of intersection.

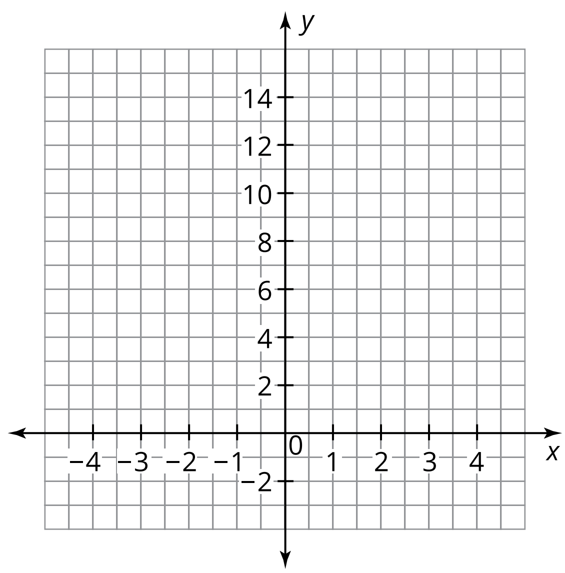


f. What do you notice about the solutions you calculated algebraically and the points of intersection you determined graphically?

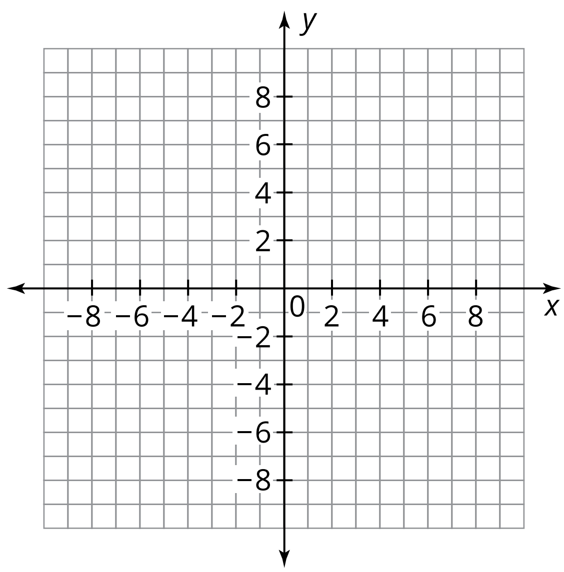
2. Think about the graphs of a linear equation and a quadratic equation. Describe the different ways in which the two graphs can intersect and provide a sketch of each case.

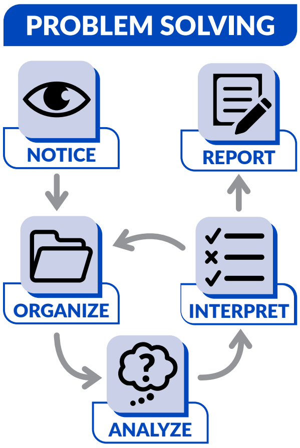
3. Solve each system of equations algebraically over the set of real numbers. Then verify the solution graphically.

a. $\begin{cases} y = -2x + 4 \\ y = 4x^2 + 2x + 5 \end{cases}$



b. $\begin{cases} y = -4x - 7 \\ y = 3x^2 + x - 3 \end{cases}$





ACTIVITY 2.3 Formulating Systems with a Linear and a Quadratic Equation

1. Noah and his sister are playing a guessing game. Noah tells his sister that he is thinking of two positive numbers. The first number minus the second number is 15. The square of the first number minus 20 times the second number is equal to 300.

   a. Write a system of one linear and one quadratic equation to represent the two numbers. Be sure to define your variables.

   b. Solve the system of equations.

   c. What are the two numbers that Noah is thinking of? Explain your reasoning.

2. A farmer is planning two types of fences for a rectangular piece of land. The cost of the first type of fence increases at a constant rate and can be determined as 4 times the length of the fence minus 8 dollars. The second type of fence, made with higher-quality material, has costs that depend on both the length and the area of the fence. Its cost can be calculated as the square of the length minus 5 times the length plus 6 dollars. The farmer wants to know at what lengths the costs of the two types of fences will be the same.

   a. Write a system of one linear and one quadratic equation to represent the situation. Be sure to define your variables.

   b. Solve the system of equations.

   c. What are the measures of the length of the fence that will make the cost equal? Explain your reasoning.

3. At an archery competition, a student shoots an arrow whose height changes with the square of the horizontal distance, increasing at first before falling. A safety line stretched across the range decreases slightly as it moves. The arrow's height follows the equation: square of the distance plus 2 times the distance, increased by 4. The safety line's height decreases at a constant rate, of 2 times the distance plus the height equals 10.

   a. Write a system of one linear and one quadratic equation to represent the situation. Be sure to define your variables.

   b. Solve the system of equations.

   c. What is the distance where the height of the arrow and the safety line are the same? Explain your reasoning.

Talk the Talk

System Solutions

1. A system of equations consisting of two linear equations has how many possible solutions?

2. A system of equations consisting of a linear equation and a quadratic equation has how many possible solutions?

3. Explain why a system of equations consisting of a linear equation and a quadratic equation cannot have an infinite number of solutions.

4. A teacher gives students a puzzle involving two numbers. The first hint she gives is that the first number is 2 more than twice the second number. The second hint is that the sum of the square of the first number and twice the square of the second number equals 44.

   a. Write a system of one linear and one quadratic equation to represent the two numbers. Be sure to define your variables.

   b. Solve the system of equations.

   c. What are the two numbers? Explain your reasoning.

Lesson 2 Assignment

Write

Describe how a graph can be used to determine the solutions to a system of nonlinear equations in your own words.

Remember

A system of equations consisting of a linear and a quadratic equation can have no solution, one solution, or two solutions.

Practice

1. Each month, a company that builds children's bikes, must keep track of their costs and revenue. Their costs consist of fixed costs that include rent, utilities, and workers' salaries, as well as the variable cost to make the bikes. The company's costs can be represented by the function $C(x) = 25x + 900$. The company's revenue for every bike sold can be represented by the function $R(x) = 100x - x^2$.

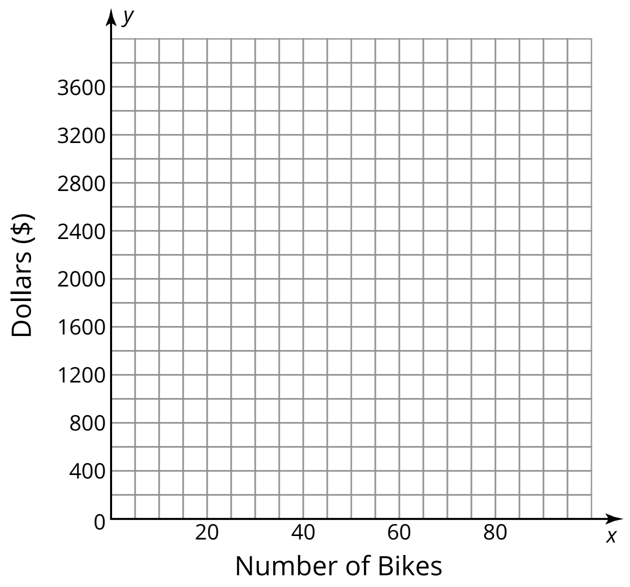
   a. Write a system of equations to represent the problem situation.

   b. Determine the break-even point(s) for the month.

   c. What is the solution to this system of equations? Explain what the solution means in terms of the problem and why it is reasonable.

Lesson 2 Assignment

d. Verify the solution by graphing both the cost and the revenue equations.

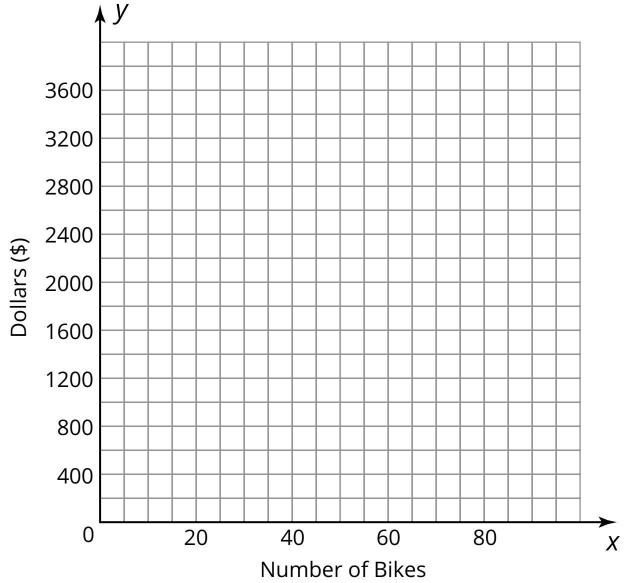


2. Due to the rising costs of running a business, the bike company anticipates fixed costs in the next year to be $1800 per month, whereas the cost to make each bike will stay at $25 per bike.

   a. Write a new system of equations to represent the updated situation.

   b. Determine the number of bikes the company will now need to make for one month to break even if the revenue from selling bikes remains the same. Is this solution reasonable?

c. Verify the solution by graphing both the revenue and the cost equations.



d. What does the company need to do to be able to break even for the month?

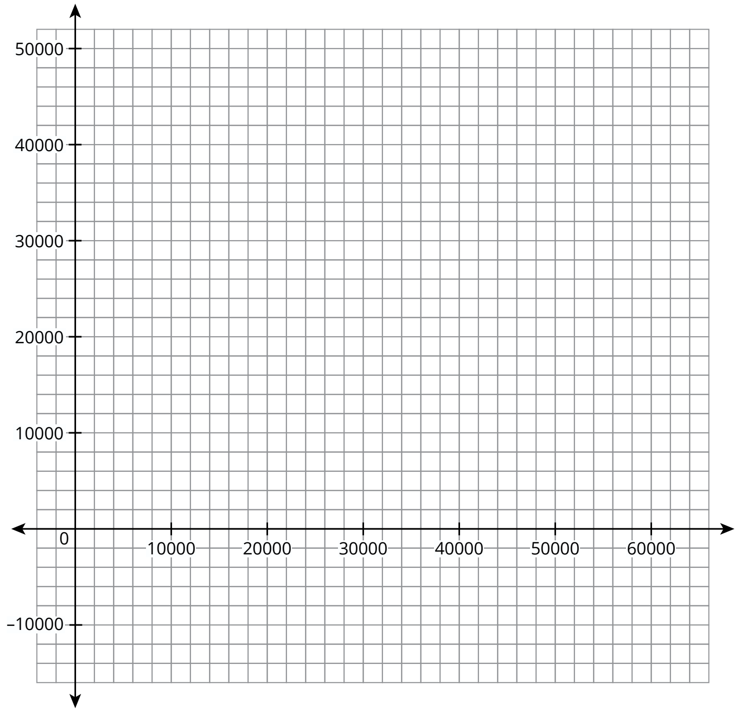
Lesson 2 Assignment

3. A hamburger store is analyzing their revenue and their expenses. The chief financial officer has determined that their revenue is modeled by $y = 2.44x - \frac{x^2}{20{,}000} - 5000$ and their expenses are modeled by $y = 5000 + 0.56x$, where x represents the number of hamburgers sold.

   a. Write a system of equations to represent the problem situation.

   b. What is the solution to this system of equations? Explain what the solution means in terms of the problem.

   c. Verify the solution by sketching a graph of both the revenue and expenses equations.



4. A hamburger diner is analyzing their revenue and their expenses. The chief financial officer has determined that their revenue is modeled by $y = 2.44x - \frac{x^2}{20{,}000} - 5000$ and their expenses are modeled by $y = 5000 + 0.56x$, where x represents the number of hamburgers sold.

   a. Use substitution to write a new equation that can be used to solve the system.

   b. Use the quadratic formula to solve the resulting equation for x. Round your answer to the nearest whole number and explain why it makes sense to do so.

c. Calculate the corresponding value(s) for y. Determine the solution(s) to the system of equations.

d. Interpret the results.

Lesson 2 Assignment

Prepare

1. Determine a linear regression function that best models the data.

x	y
1	32
2	35
3	34
4	35
5	39
6	38
7	40
8	42
9	41

3 Modeling with Linear and Quadratic Functions

OBJECTIVES

- Use a linear or quadratic function to model data and write an explanation of how it represents the data in real-world scenarios.
- Use first and second differences to determine whether a table of values represents a linear or quadratic function.
- Make predictions using regression functions.

KEY TERMS

- regression function
- coefficient of determination

You know how to model data with regression functions.

How can you determine whether a linear or quadratic regression function best models a set of data?

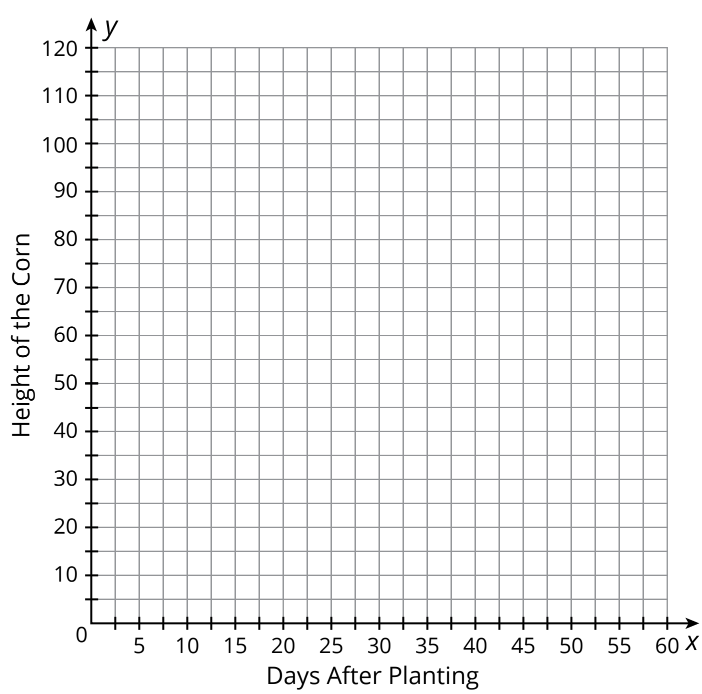
Getting Started

Stalk the Line

In 2023, Texas farmers planted approximately 2.5 million acres of corn, with about 2.1 million acres harvested for grain, according to the United States Department of Agriculture. When corn is planted, it has several stages of growth. The table shows the height of one corn stalk during the vegetative growth phase.

Days After Planting	Height of the Corn (inches)
25	18
30	30
35	45
40	60
45	78
50	100

1. Create a scatterplot of the data. Then, use a piece of uncooked spaghetti to create a trend line for the data.



2. Write an equation for your trendline.

3. Compare your trendline to your classmate's trendlines. What do you notice? Who's trendline do you think fits the data best? Explain your reasoning.

4. Use graphing technology to determine the linear regression function for the data. How does your trendline compare to the linear regression function?

Remember . . .
A linear regression function is also known as the line of best fit.

ACTIVITY 3.1

Comparing Linear and Quadratic Data

Remember, you can determine whether a table represents linear or quadratic data by analyzing the first and second differences.

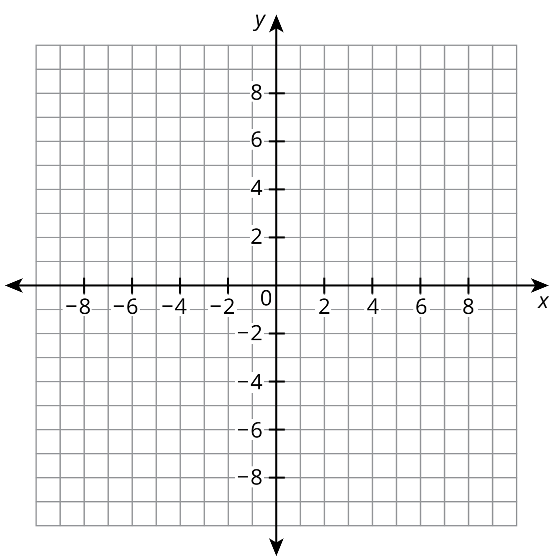
1. Complete each table to determine the first and second differences. Then, graph each function.

a. $f(x) = 2x$

x	y
−3	−6
−2	−4
−1	−2
0	0
1	2
2	4
3	6

First Differences

Second Differences

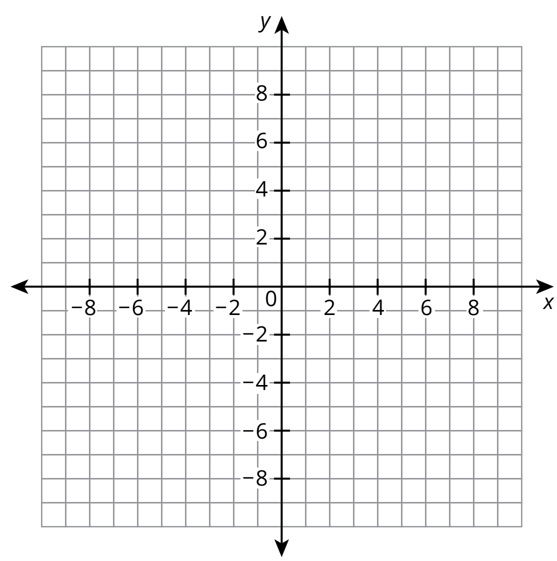


b. $g(x) = 2x^2$

x	y
−3	18
−2	8
−1	2
0	0
1	2
2	8
3	18

First Differences

Second Differences



2. What do you notice about the first and second differences in the tables and the function family each graph belongs to?

3. Use the first and second differences to determine whether each table represents a linear or quadratic function.

a.

x	y
−2	−6
−1	−9
0	−12
1	−15
2	−18

First Differences

Second Differences

b.

x	y
−2	12
−1	3
0	0
1	3
2	12

First Differences

Second Differences

c.

x	y
−1	1
0	0
1	3
2	10
3	21

First Differences

Second Differences

d.

x	y
−3	3
−2	4
−1	5
0	6
1	7

First Differences

Second Differences

Ask Yourself . . .
Notice all the tables have an interval of 1. How would your process for determining the first and second differences change if the interval was not 1?

You can also use graphing technology to help you determine whether a data set is best represented by a linear or a quadratic function. Recall that a **regression function** is a function that best models the relationship between two variables in a scatterplot. You can use the regression function to make predictions from the data. Any function can model a scatterplot, but there is generally one function that fits the data best.

You may also recall that the **coefficient of determination** (r_2) describes the strength of the relationship between the data and the regression function. The value ranges from 0 to 1, with a value of 1 indicating a perfect fit between the regression function and the data.

A group of students went door-to-door collecting donations for a local charity The table displays the number of donations received for various donation amounts.

Donation Amount (dollars)	Number of Donations
2	400
3	360
4	320
5	280
6	240
7	200
8	160
9	120
10	80
11	40
12	0

4. Use graphing technology to determine whether a linear or quadratic regression function best fits the data. Write the function. Then, explain your reasoning.

5. Use your regression function to predict the number of donations the group collected for each donation amount. Explain whether the number of donations makes sense in terms of the problem situation.

   a. $1 donations

   b. $20 donations

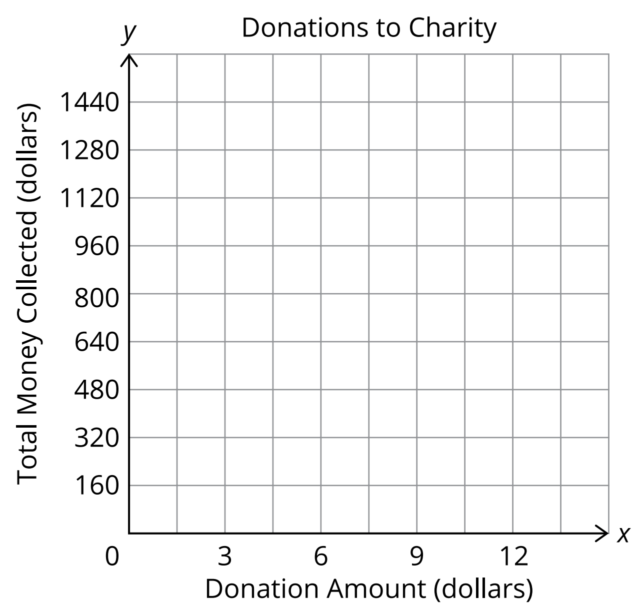
6. Based on the data and your regression function, what would happen to the number of donations when the donation amount increases to $13? How would this affect the charity?

7. Complete the table by calculating the total money collected for each donation amount.

Donation Amount (dollars)	Number of Donations	Total Money Collected (dollars)
2	400	
3	360	
4	320	
5	280	
6	240	
7	200	
8	160	
9	120	
10	80	
11	40	
12	0	

Ask Yourself . . .
How else can you represent this information?

8. Create a scatterplot of the donation amount and the total money collected for each donation amount.



9. Choose a regression function that best models the data. Sketch the graph of the regression function on the coordinate plane. Explain why the regression function best models the data.

Use your regression function to answer each question.

10. Determine the y-intercept and interpret its meaning in terms of this problem situation.

11. Determine the x-intercepts and interpret the meaning of each in terms of this problem situation.

12. Predict the total amount collected for each donation amount given. Describe whether or not the solution makes sense in terms of this problem situation.

    a. $1.00

b. $15.00

13. Predict the donation amount for each total amount collected. Describe whether or not the solution makes sense in terms of this problem situation.

    a. $300

    b. $1000

14. Next year, you will no longer go door-to-door collecting donations. It will be easier to sell booster certificates for a fixed amount to display in windows. Use the regression function to determine how much you should charge for each booster certificate in order to maximize the amount of money collected.

    $x = \frac{-b}{2a}$

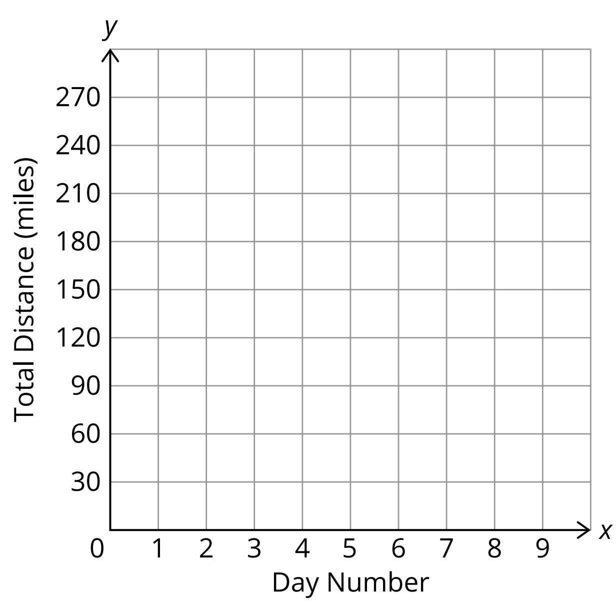
    $x = \frac{-480}{2(-40)} = 6$

ACTIVITY
3.2 Choosing the Best Model

The table shows the odometer record for the first 10 days of a 20-day bike trip.

Day Number	Total Distance (miles)
1	12
2	27
3	49
4	72
5	86
6	103
7	129
8	158
9	184
10	212

1. Create a scatterplot of the data. Based on the graph, predict which regression function will best fit the data: linear or quadratic?



2. Calculate a linear and quadratic regression function using graphing technology. What do you notice? Does this change your prediction? Explain your reasoning.

Review the definition of **explain your reasoning** in the Academic Glossary.

3. Use the linear and the quadratic regression functions to predict the number of total miles biked after 20 days. Which prediction seems more reasonable?

4. Do you think the correlation coefficient always indicates the best fit of the data? Explain your reasoning.

ACTIVITY
3.3 Modeling with Systems of Equations

A photographer specializes in taking senior portraits. She records the amount of revenue she earns for each senior portrait package that she sells.

Number of Portrait Packages Sold	Revenue (dollars)
5	575
10	1100
15	1575
20	2000
25	2375
30	2700
35	2975
40	3200

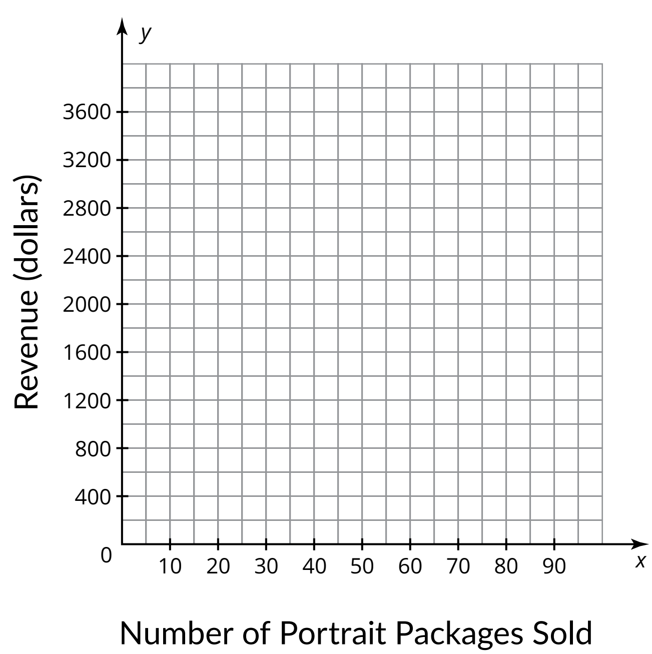
1. Which family of functions best represents the problem situation? Explain your reasoning.

2. Use graphing technology to write a regression function for the situation. Be sure to define your variables.

3. Each month, the photographer must keep track of her costs and revenue. Her costs consist of a fixed amount of $1400, which includes rent, utilities, and workers' salaries, as well as $30 per package to print the portraits. Write a linear equation to model the photographer's costs.

4. Write a system of one linear and one quadratic equation to represent this problem situation.

5. Graph the system of equations.



6. Solve the system of equations.

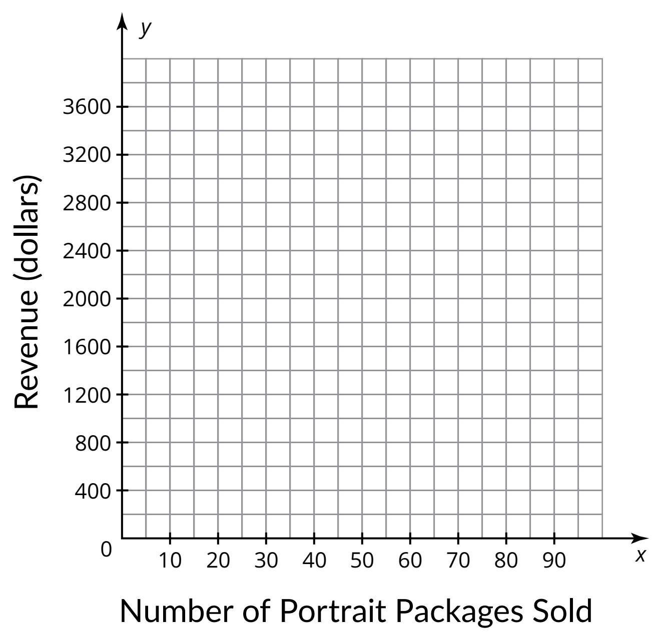
7. Explain the solution(s) in terms of this problem situation.

8. Determine the amount of portrait packages that the photographer needs to sell in order to make a profit. Is this solution reasonable in terms of the problem situation?

Due to the rising costs of running a business, the photographer anticipates fixed costs in the next year to be $2400 per month, whereas the costs for portrait packages will increase to $35 per portrait package.

9. Write a new system of equations to reflect the changes in the cost.

10. Graph the system.



11. Determine the solution(s) to the system of equations.

12. Explain the solution(s) in terms of the problem situation.
Are the solutions reasonable?

13. What critical decision will the photographer have to make based on the data, the system of equations, and the graph that represent the problem situation?

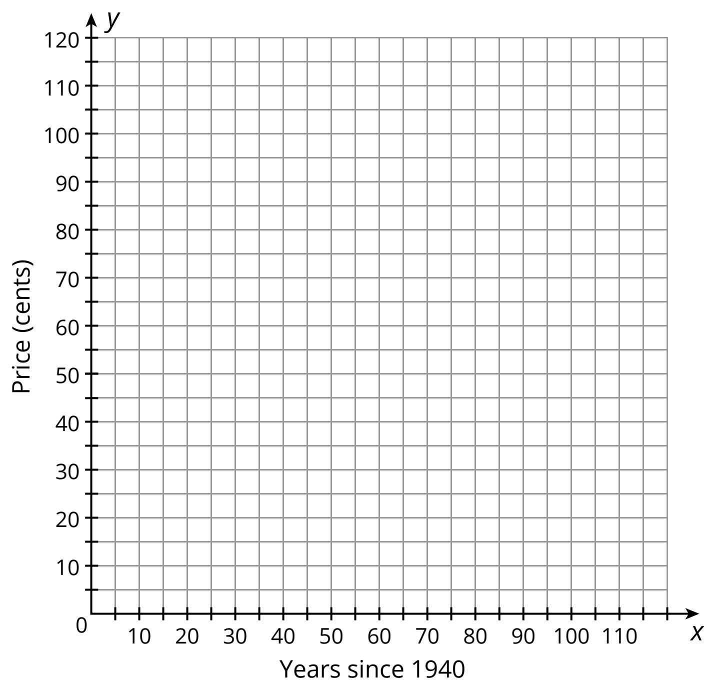
Talk the Talk

Postage Prices

The table describes the price of one first-class postage stamp in each specified year.

Year	Price (cents)
1940	3
1950	3
1960	4
1970	6
1980	15
1990	25
2000	33
2010	44
2020	55
2024	73

1. Create a scatterplot of the data on the coordinate plane.



Ask Yourself . . .
How can you use regression functions in everyday life?

2. Use graphing technology to determine a linear regression function.

3. Use a graphing calculator to determine a quadratic regression function.

4. Which function best fits the data? Explain your reasoning.

5. Use the best-fit regression function to answer each question.
Predict the cost of stamps in the year 2030.

6. When will stamps cost 90 cents?

7. Consider the data in the table and the regression function. How might advancements in technology have affected the cost of stamps?

Lesson 3 Assignment

Write

Explain how can you determine whether a set of data is best represented by a linear or quadratic regression function.

Remember

You can use linear and quadratic regression functions to model real-world situations.

Practice

1. Use the first and second differences to determine whether each table represents a linear or quadratic function.

a.

x	y
−2	20
−1	13
0	8
1	5
2	4

First Differences

Second Differences

b.

x	y
−2	7
−1	6.5
0	6
1	5.5
2	5

First Differences

Second Differences

c.

x	y
−2	−22
−1	−16
0	−10
1	−4
2	2

First Differences

____

____

____

____

Second Differences

____

____

____

d.

x	y
−2	−36
−1	−22
0	−14
1	−12
2	−16

First Differences

____

____

____

____

Second Differences

____

____

____

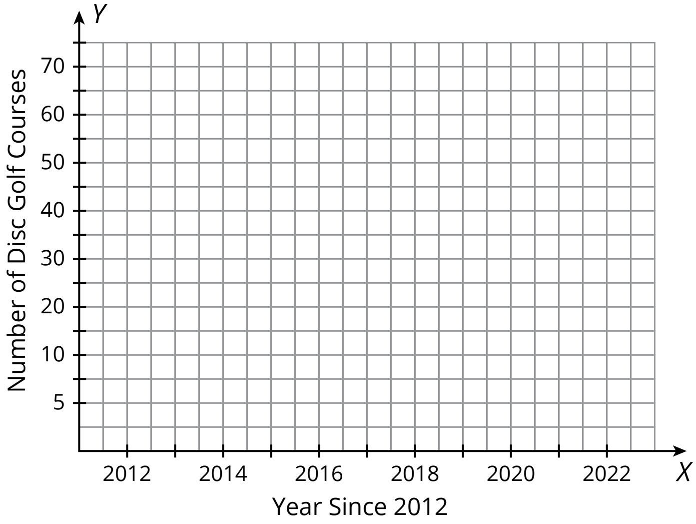
Lesson 3 Assignment

2. The table shows the number of disc golf courses in a certain area from 2012 to 2022.

Year	Number of Disc Golf Courses
2012	17
2013	20
2014	23
2015	27
2016	31
2017	36
2018	41
2019	47
2020	53
2021	60
2022	67

a. Predict whether a linear or quadratic regression function will best fit the data. Explain your reasoning.

b. Create a scatterplot of the data.



c. Does your scatterplot change or support your answer to part (a)? Explain your reasoning.

d. Use graphing technology to determine linear and quadratic regression functions for the data. Do the regression functions change or support your answer to part (a)? Explain your reasoning.

Lesson 3 Assignment

e. Use the regression function that better fits the data to predict how many disc golf courses will be in the area for 2033. Does your answer make sense in terms of the problem situation? Explain your reasoning.

3. A personal fitness trainer offers specialized one-on-one training sessions. She keeps track of the revenue she earns in a table.

Number of Training Sessions Sold	Revenue (dollars)
5	725
10	1400
15	2025
20	2600
25	3125
30	3600
35	4025
40	4400
45	4725
50	5000

Lesson 3 Assignment

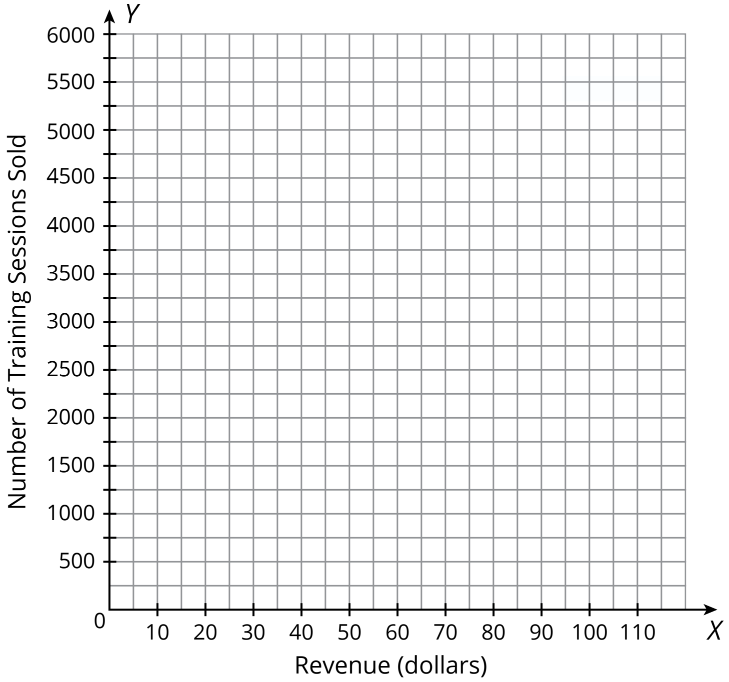
1. Which family function best represents the problem situation? Explain your reasoning.

2. Use graphing technology to write a regression function for the situation. Be sure to define your variables.

3. Each month the trainer keeps track of her costs and revenue. She has a fixed cost of $900 as well as $50 per training session sold. Write a linear equation to model the fitness trainer's costs.

4. Write a system of one linear and one quadratic equation to represent the problem situation.

5. Graph the system of equations.



6. Solve the system of equations.

Lesson 3 Assignment

7. Explain the solution(s) in terms of this problem situation.

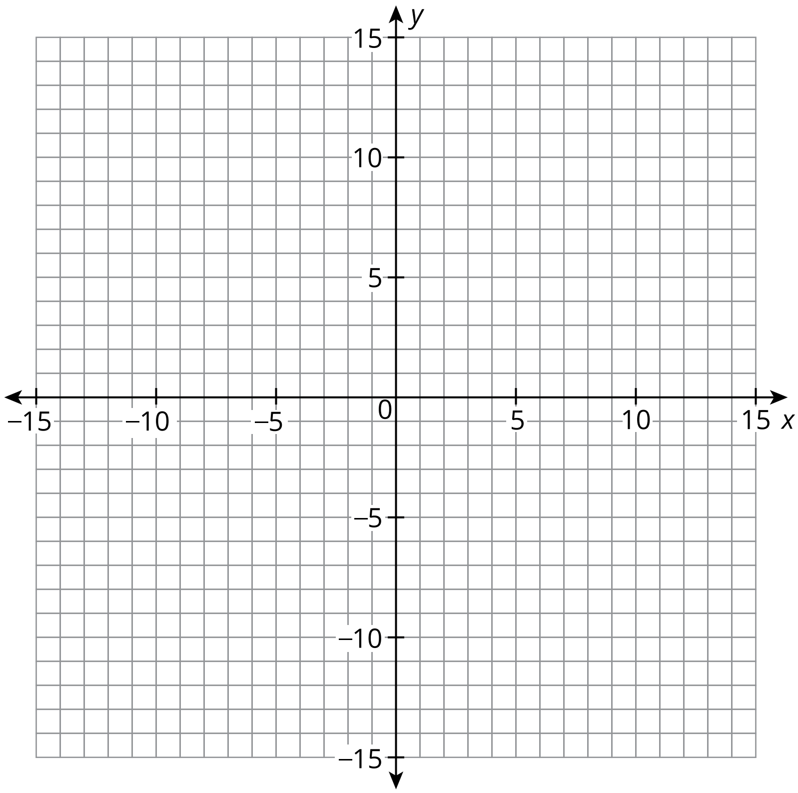
8. Determine the amount of training sessions that the trainer needs to sell in order to make a profit. Is this solution reasonable in terms of the problem situation?

Lesson 3 Assignment

Prepare

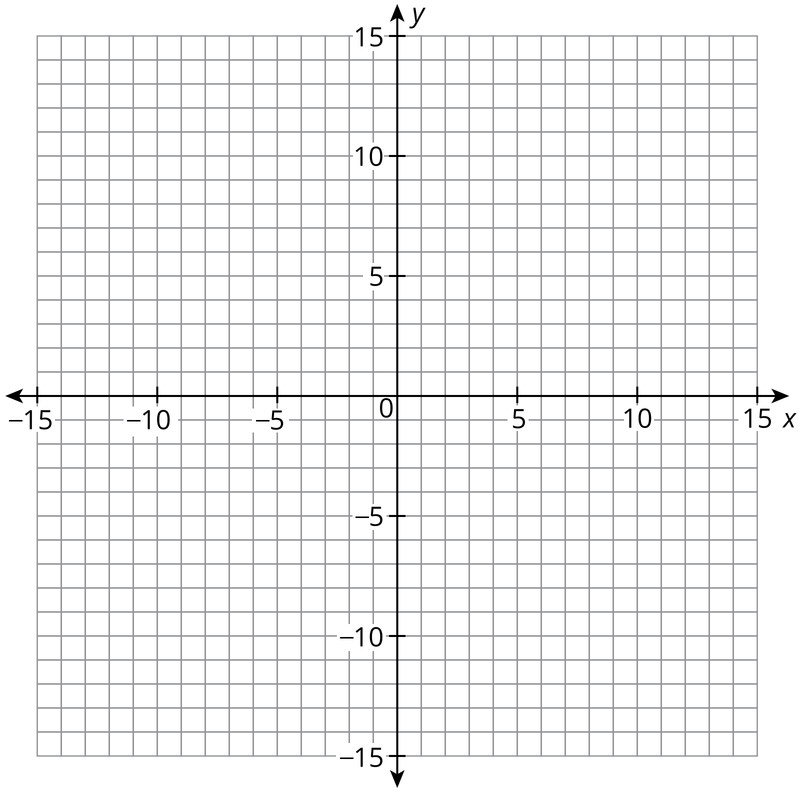
Use graphing technology to graph each equation and its inverse. Determine the vertex and axis of symmetry of each graph.

1. $y = \frac{1}{8}x^2$



Lesson 3 Assignment

2. $y = -\frac{1}{8}x^2$



4 Equation of a Parabola

OBJECTIVES

- Write equations for parabolas in geometric form given certain attributes.
- Solve and verbally describe the solutions to problems in real-world contexts using attributes of parabolas.

KEY TERMS

- parabola
- focus
- directrix
- vertex of a parabola
- geometric form of a parabola

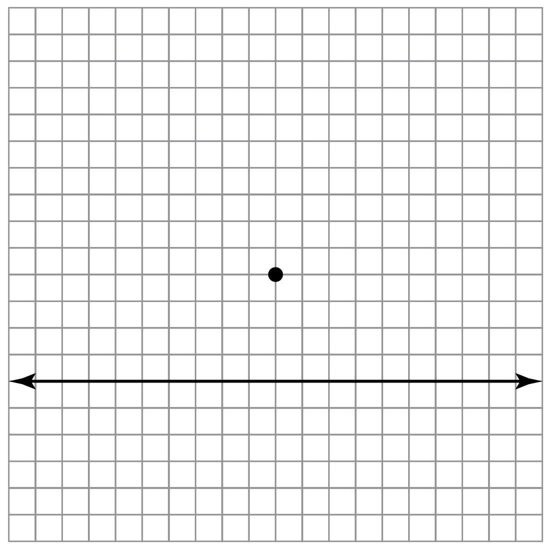
You have studied parabolas as quadratic functions.

What are the attributes of a parabola determined by the set of points equidistant from both a given point and a given line?

Getting Started

Hocus Pocus . . . Here's the Focus!

Consider the grid with the plotted point and graphed line. Can you create a set of points that are the same distance from both the point and the line?



1. Construct a circle with a radius of 1 unit using the point as the circle's center.

   a. What is the relationship between the points on the circle and the plotted point?

   b. Are any of the points on the circle the same distance from the plotted point and the graphed line? Explain your reasoning.

2. Construct a circle with a radius of 2 units using the point as the circle's center.

   a. Are any of the points on the circle the same distance from the plotted point and the graphed line? Explain your reasoning.

   b. Plot a point on the circle that is the same distance from the circle's center and the line.

Remember . . .
Concentric circles are two or more circles that share a common center but have different radii.

3. Continue to construct a total of eight concentric circles using the point as the center with the radius of each successive circle increasing by 1 unit.

   a. How can you determine which points on each new circle are the same distance from the circle's center and the line?

   b. Plot the points on each circle that are the same distance from the circle's center and the line.

Think About . . .
Will any of the points be below the graphed line?

4. Connect the points you plotted with a smooth curve.

   a. What shape did you draw?

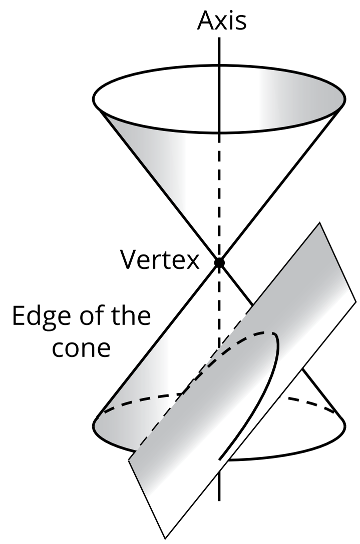
   b. What do all the points on the curve have in common?

ACTIVITY 4.1 Exploring Equations of Parabolas

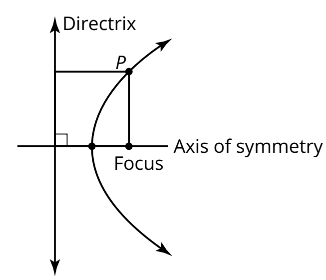
You previously studied parabolas as quadratic functions and recognized the equation of a parabola as $y = x^2$. You analyzed equations and graphed parabolas based on the position of the vertex and additional points determined by using x-values on either side of the axis of symmetry.

When a plane intersects one nappe of the double-napped cone parallel to the edge of the cone, the curve that results is a parabola.

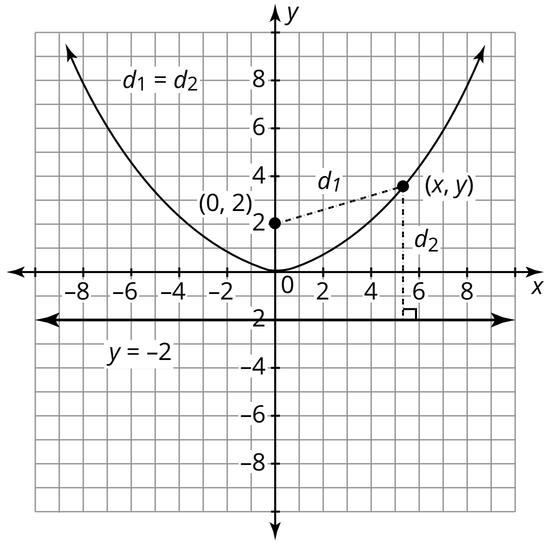
Remember . . .
The axis of symmetry of a parabola is a line that passes through the parabola and divides it into two symmetrical parts that are mirror images of each other.



In this lesson, you will explore a parabola as a *set of points* to determine more information about the parabola. A **parabola** is the set of all points in a plane that are equidistant from both a *focus* and a *directrix*. The **focus** is a point which lies inside the parabola on the axis of symmetry. The **directrix** is a line that is perpendicular to the axis of symmetry and lies outside the parabola and does not intersect the parabola. The **vertex of a parabola** is the point on the axis of symmetry which is exactly midway between the focus and the directrix. It is also the point where the parabola changes direction.



The parabola shown is defined such that all points on the parabola are equidistant from the point (0, 2) and the line $y = -2$. One point on the parabola is labeled as (x, y).



1. Determine the equation of the parabola by completing the steps.

   a. Let d_1 represent the distance from (x, y) to (0, 2). Write an equation using the distance formula to represent d_1. Simplify the equation.

   b. Let d_2 represent the distance from (x, y) to the line $y = -2$. Write an equation using the distance formula to represent d_2. Simplify the equation.

c. What do you know about the relationship between d_1 and d_2?

d. Write an equation for the parabola using Question 1, parts (a) through (c). Simplify the equation so that one side of the equation is x^2.

e. What is another name for the line y = −2 and the point (0, 2)?

You have studied parabolas in terms of quadratic functions. The graph opened up or down, had a minimum or maximum *y*-value, and had an x^2 term in the equation.

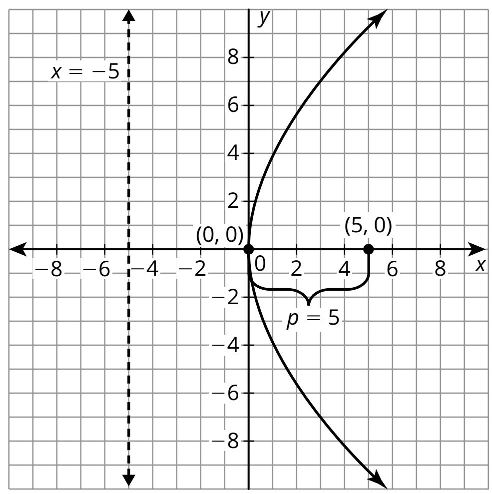
However, this is not true for all parabolas. Parabolas, when studied in *geometric form*, can also open horizontally. These parabolas have a y^2-term in the equation and an axis of symmetry that is parallel to the *x*-axis.

The **geometric form of a parabola** centered at the origin is an equation of the form $x^2 = 4py$, where p represents the difference between the *y*-coordinates of the focus and the vertex, or $y^2 = 4px$, where p represents the difference between the *x*-coordinates of the focus and the vertex.

In Question 1, you used the distance formula and the coordinate (x, y) to derive the geometric form of a parabola. You can use the geometric form to write the equation for a parabola if you know its vertex and its focus or directrix.

WORKED EXAMPLE

Consider the graph of the parabola. You can write an equation for this parabola using the direction of opening, vertex, and focus.



The orientation is horizontal, so the geometric form of the parabola is $y^2 = 4px$.

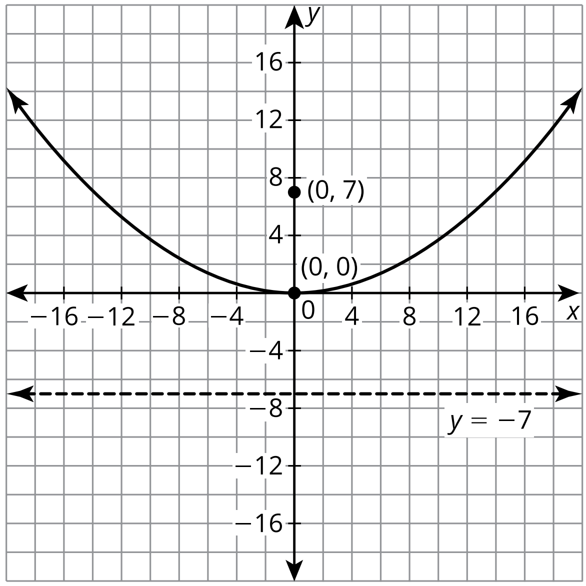
The vertex of the parabola is (0, 0) and the focus is (5, 0). The focus is 5 units from the vertex along the axis of symmetry. Therefore, $p = 5$.

Substitute p into the geometric form equation.

$$y^2 = 4px$$
$$y^2 = 4(5)x$$
$$y^2 = 20x$$

The geometric form of the parabola is $y^2 = 20x$.

2. Consider the graph of the parabola.

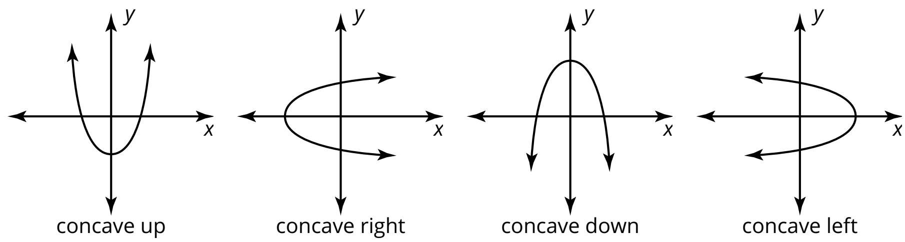


a. Identify the vertex of the parabola and its orientation.

b. Determine the value of *p*. Explain your reasoning.

c. Write the equation of the parabola in geometric form.

Recall, the concavity of a parabola describes the orientation of the curvature of the parabola. A parabola can be concave up, concave down, concave right, or concave left, as shown.



3. Analyze the parabola in the Worked Example.

   a. Describe its concavity and symmetry.

   b. What is the relationship between the parabola's equation and its orientation?

4. Analyze the parabola in Question 2.

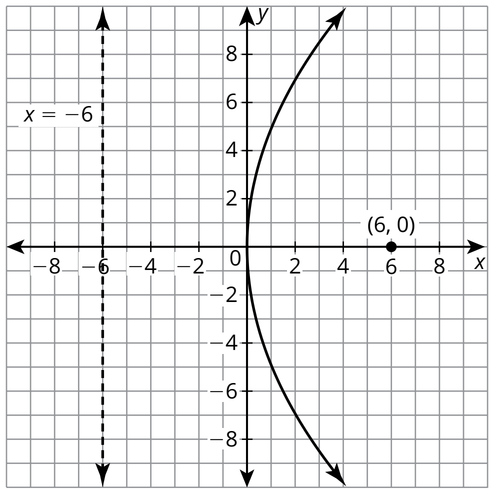
   a. Describe its concavity and symmetry.

   b. What is the relationship between the parabola's equation and its orientation?

5. For each graph of a parabola with vertex at the origin, write its equation in geometric form. Then, identify the symmetry and concavity of each parabola.

Ask Yourself . . .
Is the value of p positive or negative?

a.

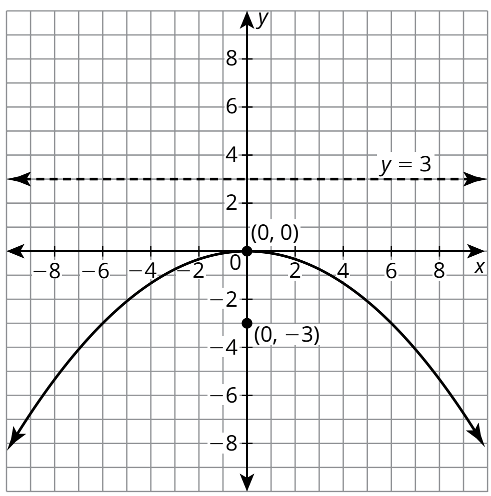


Equation:

Symmetry:

Concavity:

b.

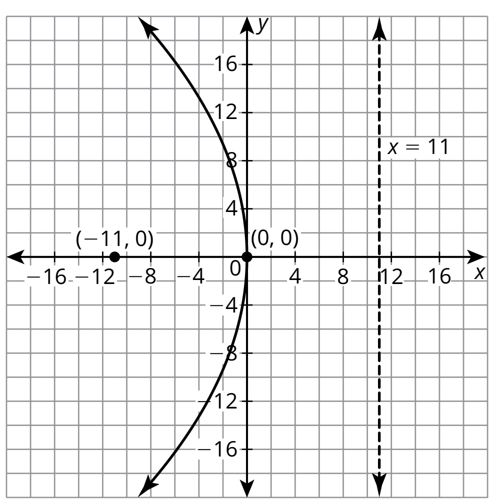


Equation:

Symmetry:

Concavity:

c.



Equation:

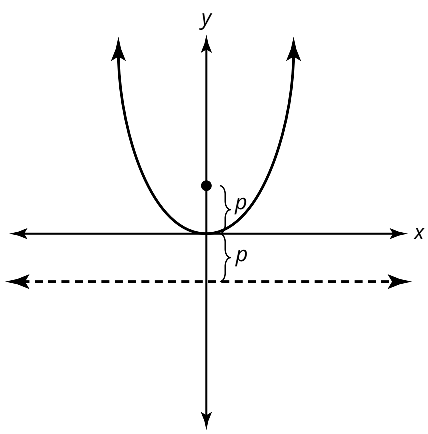
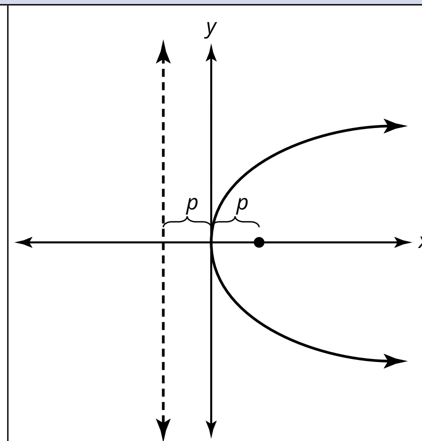
Symmetry:

Concavity:

> **Ask Yourself . . .**
> What patterns do you notice?

6. How does the value of p affect the concavity of the parabola?

7. Complete the table to generalize the key attributes of a parabola with vertex at the origin.

Parabola (Vertex at the Origin)		
Graph		
Equation of Parabola		
Orientation of Parabola		
Axis of Symmetry		
Coordinates of Vertex		
Concavity		
Coordinates of Focus		
Equation of Directrix		

8. Using the table from Question 7, write the geometric form of the equation of the parabola with its vertex at the origin, based on the given attributes. Let (x, y) represent a point on the parabola.

   a. focus at (0, 4)

   b. directrix at $y = 3$

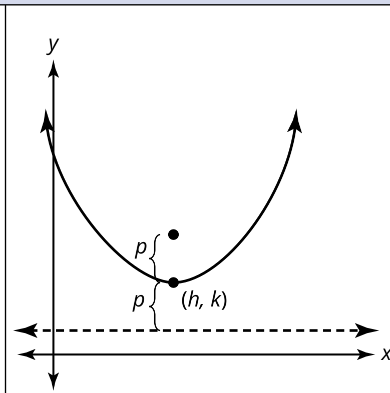
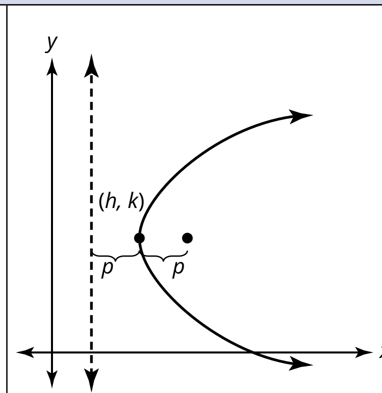
   c. horizontal axis of symmetry and a p-value of 5

   d. concave left and focus at (–1, 0)

ACTIVITY
4.2

Writing the Equation of a Parabola

1. Complete the table to analyze the key attributes of a parabola with vertex (h, k).

Parabola (Vertex at (h, k))		
Graph		
Orientation of Parabola		
Axis of Symmetry		
Coordinates of Vertex		
Coordinates of Focus		
Equation of Directrix		
Concavity		

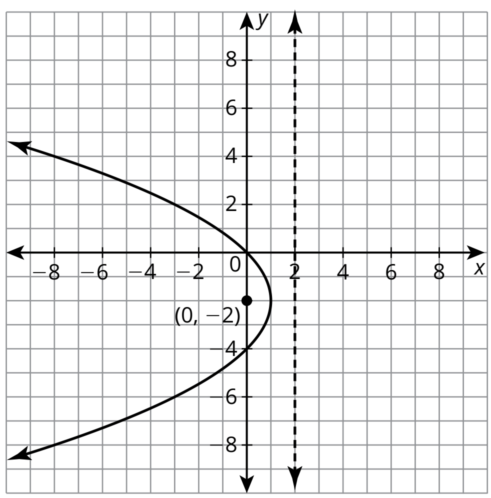
2. Compare the table with the table you created in the previous activity. How did each attribute change when the vertex was translated from the origin to (h, k)?

3. Recall that the geometric form of a parabola is an equation of the form $x^2 = 4py$ or $y^2 = 4px$ when the parabola is centered at the origin. Write the equation for the geometric form of a parabola centered at the vertex (h, k).

The process for writing the equation for the geometric form of a parabola translated (h, k) units is similar to the process for writing an equation that is centered at the origin.

The vertex and focus both lie on the axis of symmetry. You can determine the value of p by calculating the number of units between the vertex and the focus along the axis of symmetry.

4. Consider the graph of the translated parabola.



a. Identify the orientation and the axis of symmetry. Draw and label the axis of symmetry on the graph.

b. Determine the vertex of the parabola. Draw and label it on the graph.

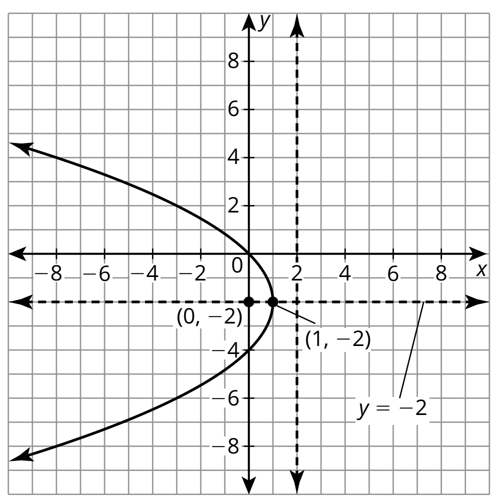
c. State the concavity of the parabola. How will this affect the equation?

d. Use the axis of symmetry to calculate the value of p. Explain your reasoning.

Once you determine the vertex, axis of symmetry, and p-value, you can write the equation for the geometric form of a translated parabola.

WORKED EXAMPLE

You can use the graph to write the equation of a translated parabola.



Substitute the values of the vertex and p into the equation for the geometric form of a translated parabola.

$$(y - k)^2 = 4p(x - h)$$

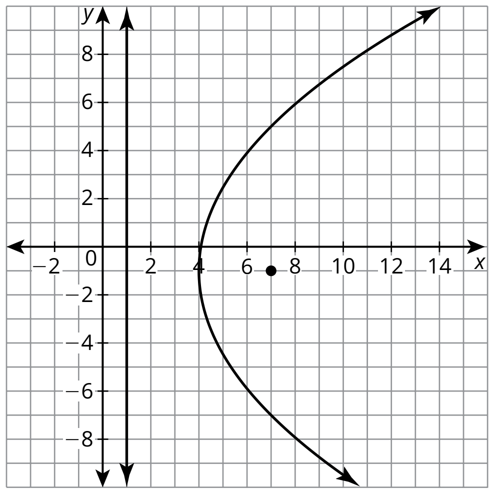
$$(y - (-2))^2 = 4(-1)(x - 1)$$

$$(y + 2)^2 = -4(x - 1)$$

The equation for the geometric form of the translated parabola is $(y + 2)^2 = -4(x - 1)$.

5. Write the equation for the geometric form of each parabola. Label the vertex and the axis of symmetry.

a.



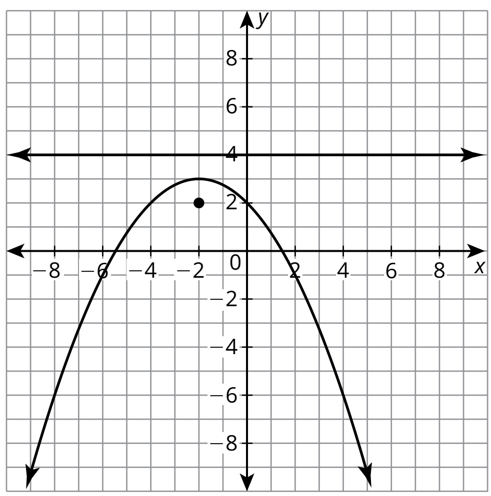
Vertex:

Axis of Symmetry:

p-value:

Equation:

b.



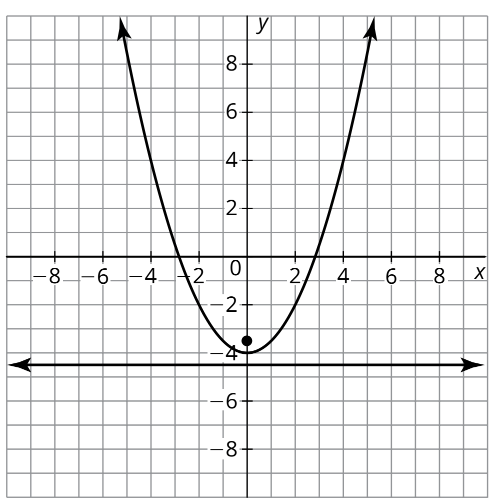
Vertex:

Axis of Symmetry:

p-value:

Equation:

c.



Vertex:

Axis of Symmetry:

p-value: $p =$

Equation:

6. Consider a parabola that has a vertex at (3, 5) and a focus at (3, 1).

   a. Determine the axis of symmetry, orientation, and concavity of the parabola. Justify your reasoning.

   b. Calculate the value of *p*. Justify your reasoning.

   c. Write the equation for the geometric form of the parabola.

   d. Calculate the equation for the directrix. Explain your reasoning.

7. Consider a parabola that has a vertex at (4, 1) and a directrix at $x = 2$.

   a. Determine the axis of symmetry, orientation, and concavity of the parabola. Justify your reasoning.

   b. Calculate the value of p. Justify your reasoning.

   c. Write the equation for the geometric form of the parabola.

   d. Calculate the coordinates for the focus. Explain your reasoning.

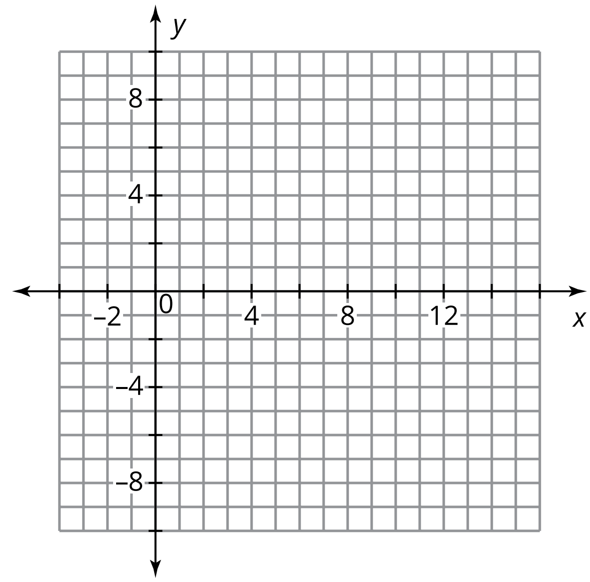
8. Determine the equation of a parabola in geometric form that has a focus at (3, 5), a directrix at $y = 1$, and is concave up.

9. Determine the equation of a parabola that has an axis of symmetry $y = 0$, vertex at (0, 0), and passes through the point (2, 4).

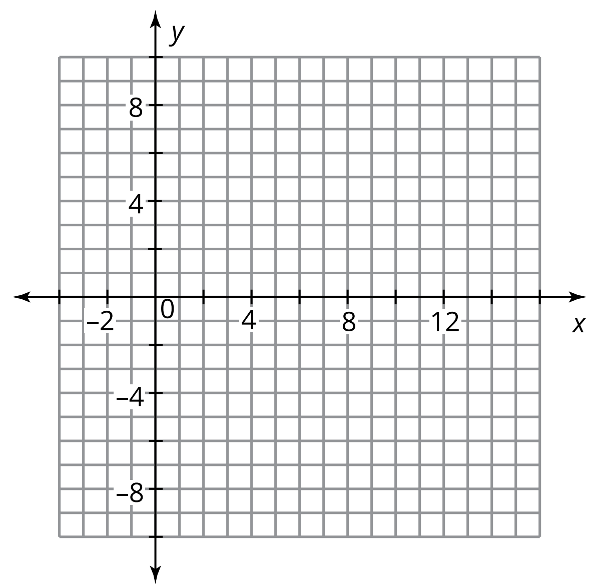
10. A parabola passes through the point (2, 4) and its directrix is $x = -1$. Determine whether the axis of symmetry is vertical or horizontal. Explain your reasoning.

11. Write an equation in geometric form for each parabola. Then, graph and label the parabola.

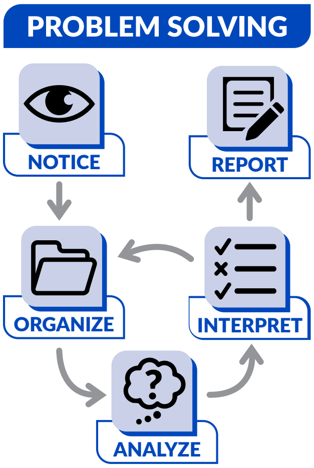
a. A parabola with a vertex at (3, 2) and a focus at (3, 4).



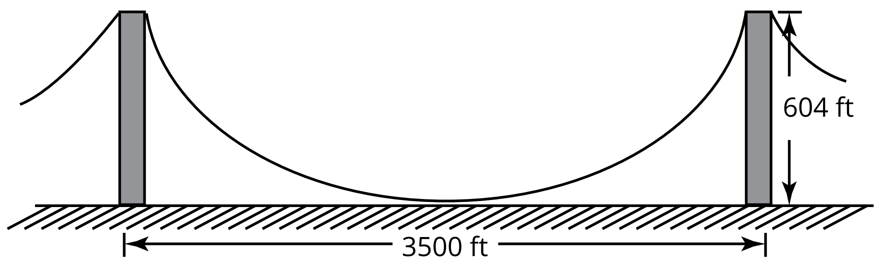
b. A parabola with a vertex at (4, 1) and a directrix at $x = 2$.



ACTIVITY 4.3 Applications of Parabolas



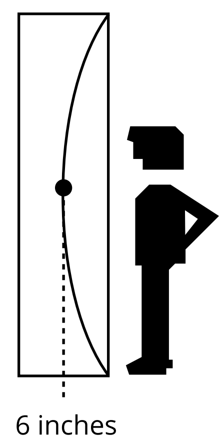
1. The George Washington Bridge opened in 1932, connecting Fort Lee, New Jersey to Manhattan, New York. It is a suspension bridge named after President George Washington. The main cables of a suspension bridge are parabolic. The parabolic shape allows the cables to bear the weight of the bridge evenly. The distance between the towers of the George Washington Bridge is 3500 feet and the height of each tower is about 604 feet. Write an equation for the parabola that represents the cable between the two towers.



2. A cross-section of NASA's Deep Space Station 56 (DSS-56) dish is a parabola. The satellite dish is 34 meters wide at its opening and 6.25 meters deep. The receiver of the satellite dish should be placed at the focus of the parabola. How far should the receiver be placed from the vertex of the satellite dish?

President Eisenhower signed the National Aeronautics and Space Act of 1958 into law on July 29, 1958, creating NASA.

3. Lighthouses use parabolic reflectors to focus and direct beams of light across the open water to alert and guide boats. These reflectors ensure that light travels in parallel rays, making them crucial for guiding ships safely along the coast. The focal length of a parabolic reflector is the distance from the vertex to the focus, where the light is concentrated before being projected outward. Consider the Port Isabel Lighthouse, which uses a parabolic reflector that is 72 inches tall, with its vertex positioned 6 inches from both the top and bottom edges. What is the focal length of this reflector?



Talk the Talk

Time to Focus

1. Founded by the National Park Service in 1935 to commemorate Thomas Jefferson's vision of a transcontinental United States, the Gateway Arch National Park (formerly known as the "Jefferson National Expansion Memorial") stretches from the Old Courthouse to the steps overlooking the Mississippi River. In between, the Gateway Arch rises high, a bold monument to the pioneering spirit. The Gateway Arch, located in St. Louis Missouri, is considered by some to be the tallest human-made monument in the Western Hemisphere. The shape of the arch can be modeled by a parabola. The arch is approximately 630 feet tall and spans 630 feet across at its base. Write an equation in geometric form that represents the shape of the arch.

2. Write the equation of the parabola using the given attributes.

a. focus at (2, 4) and vertex at (2, 2).

b. directrix at $y = -4$ and vertex at (0,0).

c. vertex at $(1, -2)$, p-value of -3, and opens down.

d. axis of symmetry is $y = -1$, vertex at $(5, -1)$, and x-coordinate of focus is 1.

e. opens to the right, vertex at (3,4), and distance between vertex and focus is 2 units.

f. focus at $(-5,0)$ and directrix at $x = 5$.

Lesson 4 Assignment

Write

1. parabola
2. focus
3. directrix
4. geometric form of a parabola
5. vertex of a parabola
6. concavity

a. $x^2 = 4py$ or $y^2 = 4px$

b. describes the orientation of the curvature of the parabola

c. a set of points in a plane that are equidistant from a fixed point and a fixed line

d. the maximum or minimum point of a parabola

e. the fixed point from which all points of a parabola are equidistant

f. the fixed line from which all points of a parabola are equidistant

Remember

A *parabola* is the set of all points in a plane that are equidistant from a fixed point, the *focus*, and a fixed line, the *directrix*. A parabola has an axis of symmetry, a vertex, and concavity. A parabola can be concave up, concave down, concave right, or concave left.

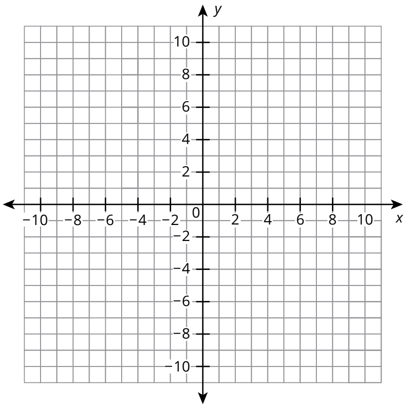
The geometric form of a parabola centered at the origin is an equation of the form $x^2 = 4py$ or $y^2 = 4px$. The geometric forms of parabolas with vertex (h, k) are $(x - h)^2 = 4p(y - k)$ and $(y - k)^2 = 4p(x - h)$.

Lesson 4 Assignment

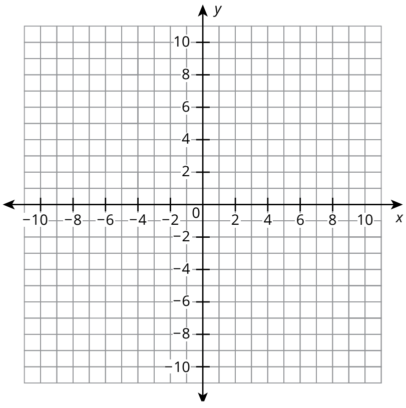
Practice

1. Write an equation in geometric form for each parabola. Then, graph and label the parabola.

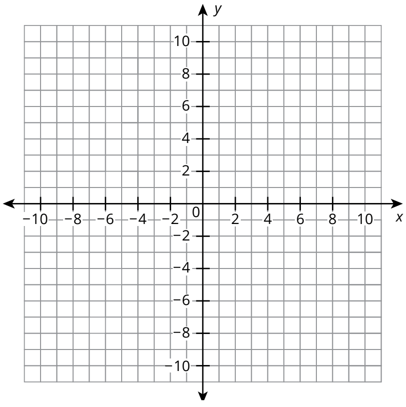
   a. A parabola with a vertex at (–2, 4) and a focus at (–2, 1).



   b. A parabola with vertex at (4, 2) and a directrix and $x = 8$.



   c. A parabola opens to the right, has an axis of symmetry at $y = 3$, and the focus is at (5, 3), two units from the vertex.



Lesson 4 Assignment

2. Determine the equation of each parabola with the given focus and directrix. Let (x, y) represent a point on the parabola.

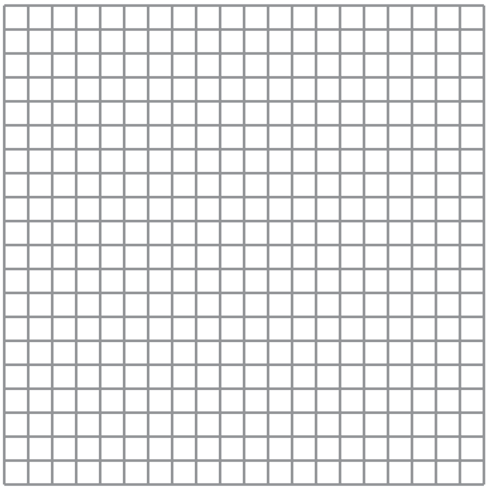
   a. focus: (0, 3); directrix: $y = -3$

   b. focus: (−4, 0); directrix: $x = 4$

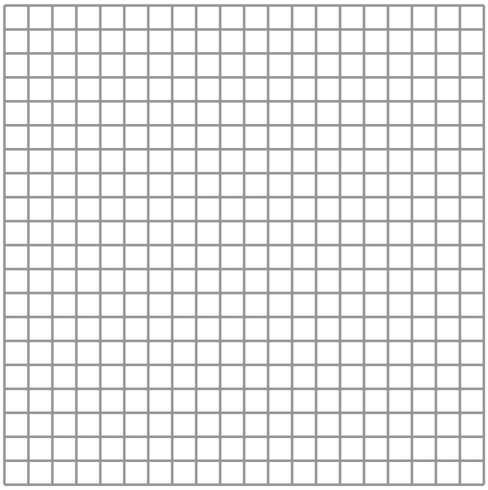
   c. focus: (7, 0); directrix: $x = -7$

3. Write an equation in geometric form for each parabola. Then, graph and label the parabola.

   a. A parabola with a vertex of (8, 6) and a focus of (6, 6).



   b. A parabola with a vertex of (1, 0) and a directrix of $y = -3$.



Lesson 4 Assignment

4. Write an equation in geometric form for each parabola.

   a. a parabola with a vertex at $(1, -2)$, opens to the right and passes through the point $(2, 1)$.

   b. a parabola with an axis of symmetry at $x = -2$, a vertex 10 units above the x-axis, and passes through the point $(-4, 6)$.

TOPIC 2 SELF-REFLECTION

Name: ______________________

Applications of Quadratics

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me,** 2 represents **I am building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Applications of Quadratics* topic by:

TOPIC 2: *Applications of Quadratics*	Beginning of Topic	Middle of Topic	End of Topic
identifying quantities represented in real-world situations modeled by quadratic equations.	◯	◯	◯
writing and solving a quadratic equation that best models a real-world situation.	◯	◯	◯
graphing a quadratic inequality on a coordinate plane with appropriate labels and scales.	◯	◯	◯
interpreting the solution set of a quadratic inequality.	◯	◯	◯
graphing and solving a system comprised of one linear equation and one quadratic equation.	◯	◯	◯
solving linear and quadratic systems using graphing and substitution.	◯	◯	◯
using technology to determine the function of best fit for a data set.	◯	◯	◯

continued on the next page

TOPIC 2 SELF-REFLECTION *continued*

TOPIC 2: *Applications of Quadratics*	Beginning of Topic	Middle of Topic	End of Topic
making predictions using a linear or quadratic regression function that represents a data set.			
writing an equation in geometric form of a parabola with given attributes.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Applications of Quadratics* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 2 SUMMARY

Applications of Quadratics Summary

LESSON 1 Solving Quadratic Inequalities

Just like with the other inequalities you have studied, the solution set to a quadratic inequality is the set of values that satisfy the inequality.

For example, consider the inequality $x^2 - 4x + 3 < 0$.

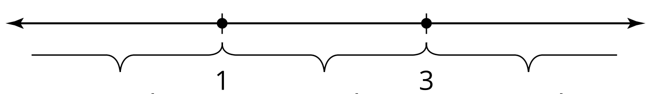
Write the corresponding quadratic equation.

$$x^2 - 4x + 3 = 0$$

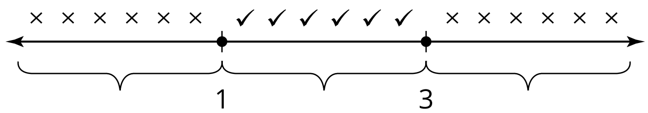
Calculate the roots of the quadratic equation using an appropriate method.

$$(x - 3)(x - 1) = 0$$
$$(x - 3) = 0 \text{ or } (x - 1) = 0$$
$$x = 3 \text{ or } \quad x = 1$$

Plot the roots to divide the number line into three regions.



Choose a value from each interval to test in the original inequality. Identify the solution set as the interval(s) in which your test value satisfies the inequality.



Interval 1	Interval 2	Interval 3
Try $x = 0$	Try $x = 2$	Try $x = 4$
$0^2 - 4(0) + 3 < 0$	$2^2 - 4(2) + 3 < 0$	$4^2 - 4(4) + 3 < 0$
$3 < 0$ ×	$4 - 8 + 3 < 0$	$16 - 16 + 3 < 0$
	$-1 < 0$ ✓	$3 < 0$ ×

Interval 2 satisfies the original inequality, so the solution set includes all numbers between 1 and 3. You can write the solution set using **interval notation** or inequalities.

Solution: $x \in (1, 3)$, or $1 < x < 3$.

KEY TERMS

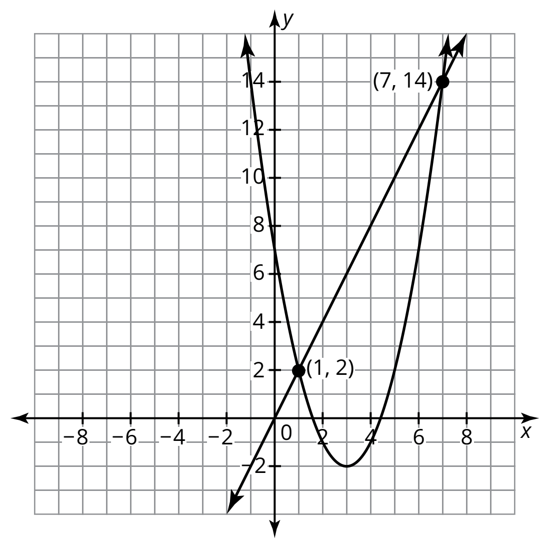
- interval notation [notación de intervalo]
- regression function [función de regresión]
- coefficient of determination [coeficiente de determinación]
- parabola [parábola]
- focus [foco]
- directrix [directriz]
- vertex of a parabola [vértice de una parábola]
- geometric form of a parabola [forma geométrica de una parábola]

LESSON 2 Solving Systems of Linear and Quadratic Equations

The graph of a linear equation and the graph of a quadratic equation can intersect at two points, at one point, or not at all.

The method to determine the solution or solutions of a system involving a quadratic equation and a linear equation is similar to solving a system of two linear equations. First, substitute one equation into the other. Then, solve the resulting equation for *x* and calculate the corresponding values for *y*. These values represent the point(s) of intersection. Finally, graph each equation of the system to verify the points of intersection.

$$\begin{cases} y = x^2 - 6x + 7 \\ y = 2x \end{cases}$$



$2x = x^2 - 6x + 7$

$0 = x^2 - 8x + 7$

$0 = (x - 7)(x - 1)$

$x - 7 = 0$	$x - 1 = 0$
$x = 7$	$x = 1$
$y = 2(7)$	$y = 2(1)$
$y = 14$	$y = 2$

The system has two solutions: (7, 14) and (1, 2).

LESSON 3

Modeling with Linear and Quadratic Functions

You can use first and second differences to determine whether a table of values represents a linear or quadratic function. Linear functions have constant first differences and second differences of 0. Quadratic functions have changing first differences and constant second differences.

Linear function: $f(x) = -x + 2$

x	$f(x)$
0	2
1	1
2	0
3	−1
4	−2

First Differences: −1, −1, −1, −1

Second Differences: 0, 0, 0

Quadratic function: $f(x) = 2x^2 - 3x$

x	$f(x)$
0	0
1	−1
2	2
3	9
4	20

First Differences: −1, 3, 7, 11

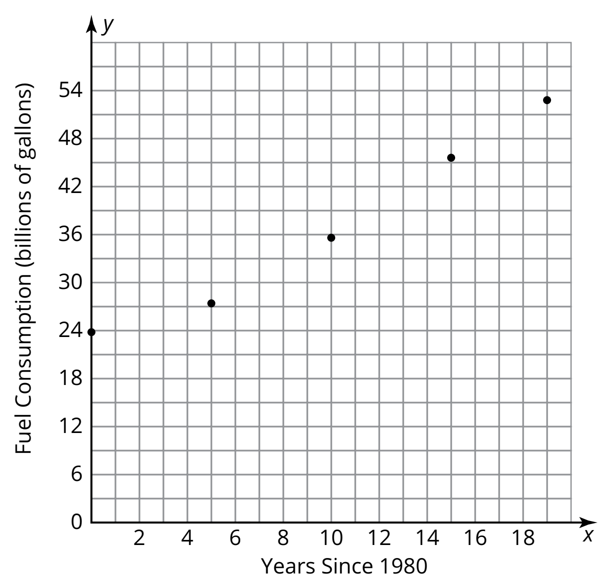
Second Differences: 4, 4, 4

You can also use graphing technology to help you determine whether a data set is best represented by a linear or a quadratic function. Recall that a **regression function** is a function that best models the relationship between two variables in a scatterplot. You can use the regression function to make predictions from the data. Any function can model a scatterplot, but there is generally one function that fits the data best. The **coefficient of determination** (r^2) describes the strength of the relationship between the data and the regression function. The value ranges from 0 to 1, with a value of 1 indicating a perfect fit between the regression function and the data.

Quadratic regression functions can be used to model real-world situations and make predictions.

For example, as vans, trucks, and SUVs have increased in popularity, the fuel consumption of these types of vehicles has also increased.

Years Since 1980	Fuel Consumption (billions of gallons)
0	23.8
5	27.4
10	35.6
15	45.6
19	52.8



The quadratic regression function that best fits the data is $f(x) = 0.0407x^2 + 0.809x + 23.3$. The r^2 value for the quadratic regression fit is 0.996. You can use regression functions to make predictions for the data.

For example, you can predict the fuel consumption in the year 2020 by substituting $x = 40$ into the regression function.

$f(40) = 0.0407(40)^2 + 0.809(40) + 23.3$

$f(40) \approx 121$

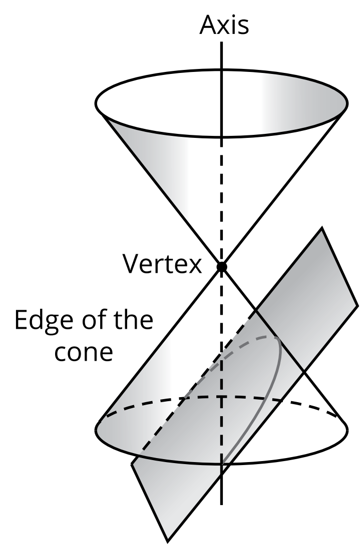
In 2020, fuel consumption will be about 121 billion gallons.

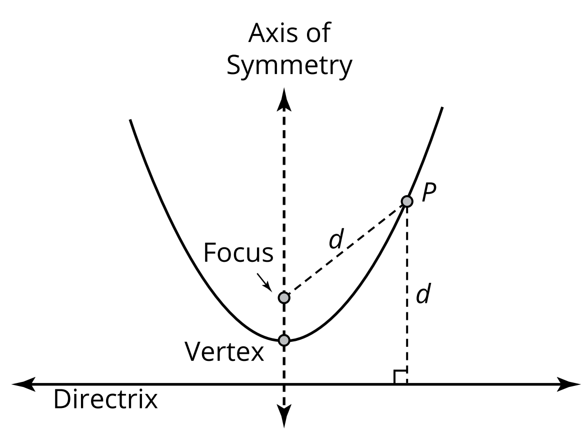
LESSON 4 Equation of a Parabola

You previously studied parabolas as quadratic functions. You analyzed equations and graphed parabolas based on the position of the vertex and additional points determined using *x*-values on either side of the axis of symmetry.

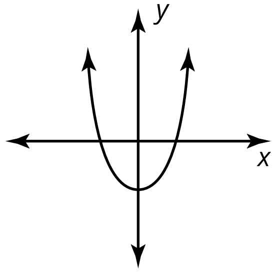
A parabola can also be described geometrically.

A **parabola** is the set of all points in a plane that are equidistant from a focus and a directrix. The **focus** is a point which lies inside the parabola on the axis of symmetry. The **directrix** is a line that is perpendicular to the axis of symmetry and lies outside the parabola and does not intersect the parabola. Thus, a parabola is equidistant from the focus and the directrix. The **vertex of a parabola** is the point on the axis of symmetry which is exactly midway between the focus and the directrix. It is also the point where the parabola changes direction.

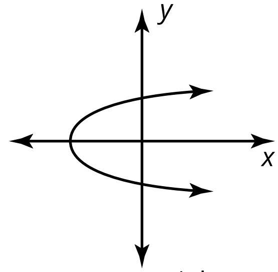




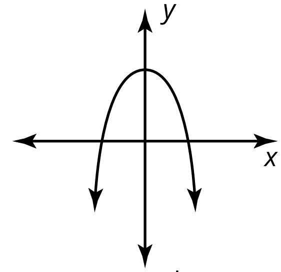
Recall the *concavity* of a parabola describes the orientation of the curvature of the parabola. A parabola can be concave up, concave down, concave right, or concave left, as shown.



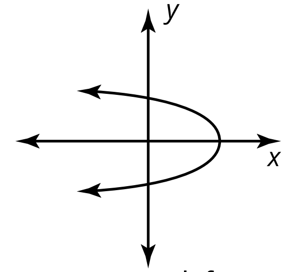
concave up



concave right

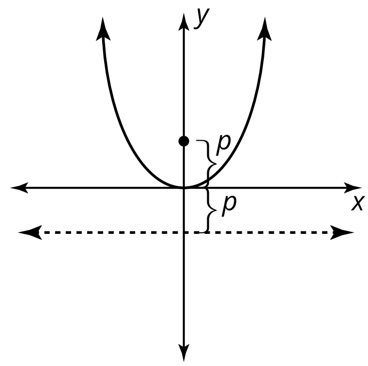
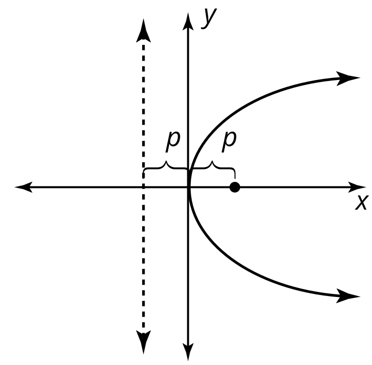


concave down

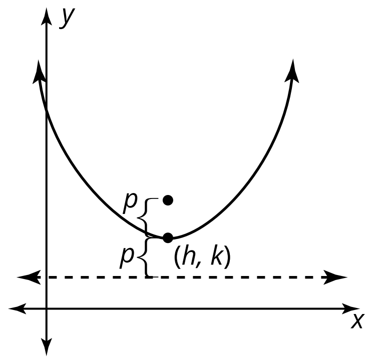
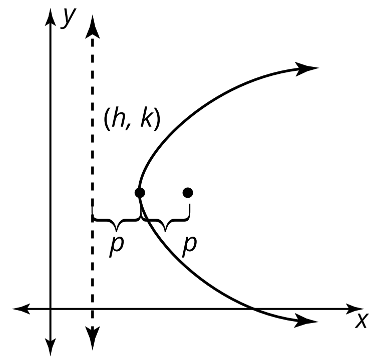


concave left

The **geometric form of a parabola** with a vertex at the origin is an equation of the form $x^2 = 4py$ or $y^2 = 4px$, where p represents the distance from the vertex to the focus.

Parabola (Vertex at the Origin)		
Graph		
Equation of Parabola	$x^2 = 4py$	$y^2 = 4px$
Orientation of Parabola	vertical	horizontal
Axis of Symmetry	y-axis	x-axis
Coordinates of Vertex	(0, 0)	(0, 0)
Coordinates of Focus	$(0, p)$	$(p, 0)$
Equation of Directrix	$y = p$	$x = p$
Concavity	up or down	right or left

The geometric forms of parabolas with a vertex at (h, k) are $(x - h)^2 = 4p(y - k)$ and $(y - k)^2 = 4p(x - h)$.

Parabola (Vertex at (h, k))		
Graph		
Equation of Parabola	$(x - h)^2 = 4p(y - k)$	$(y - k)^2 = 4p(x - h)$
Orientation of Parabola	vertical	horizontal
Axis of Symmetry	$x = h$	$y = k$
Coordinates of Vertex	(h, k)	(h, k)
Coordinates of Focus	$(h, k + p)$	$(h + p, k)$
Equation of Directrix	$y = k - p$	$x = h - p$
Concavity	up or down	right or left

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