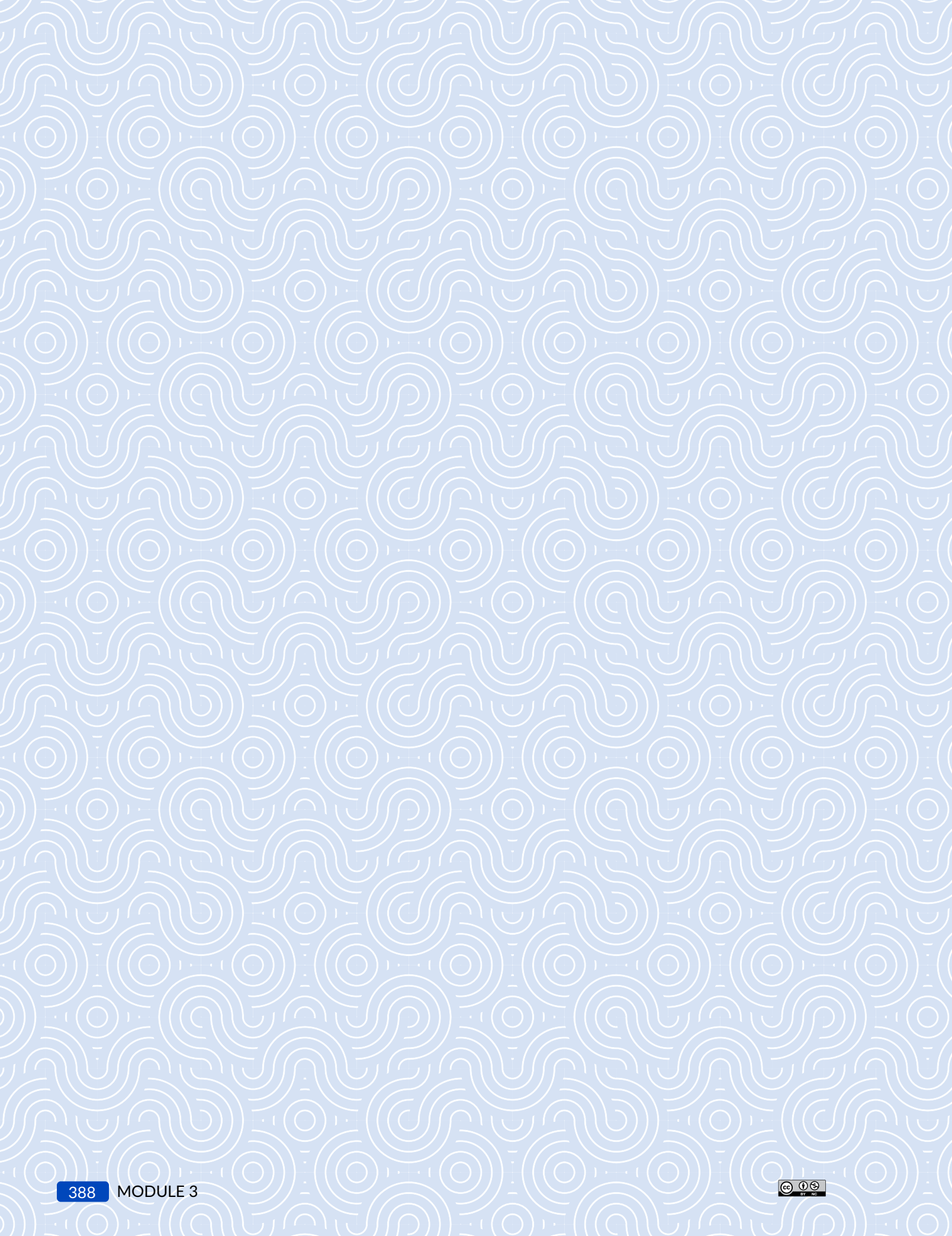


Analyzing Structure

TOPIC 1	Composing and Decomposing Functions	389
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TOPIC 3	Relating Factors and Zeros	557

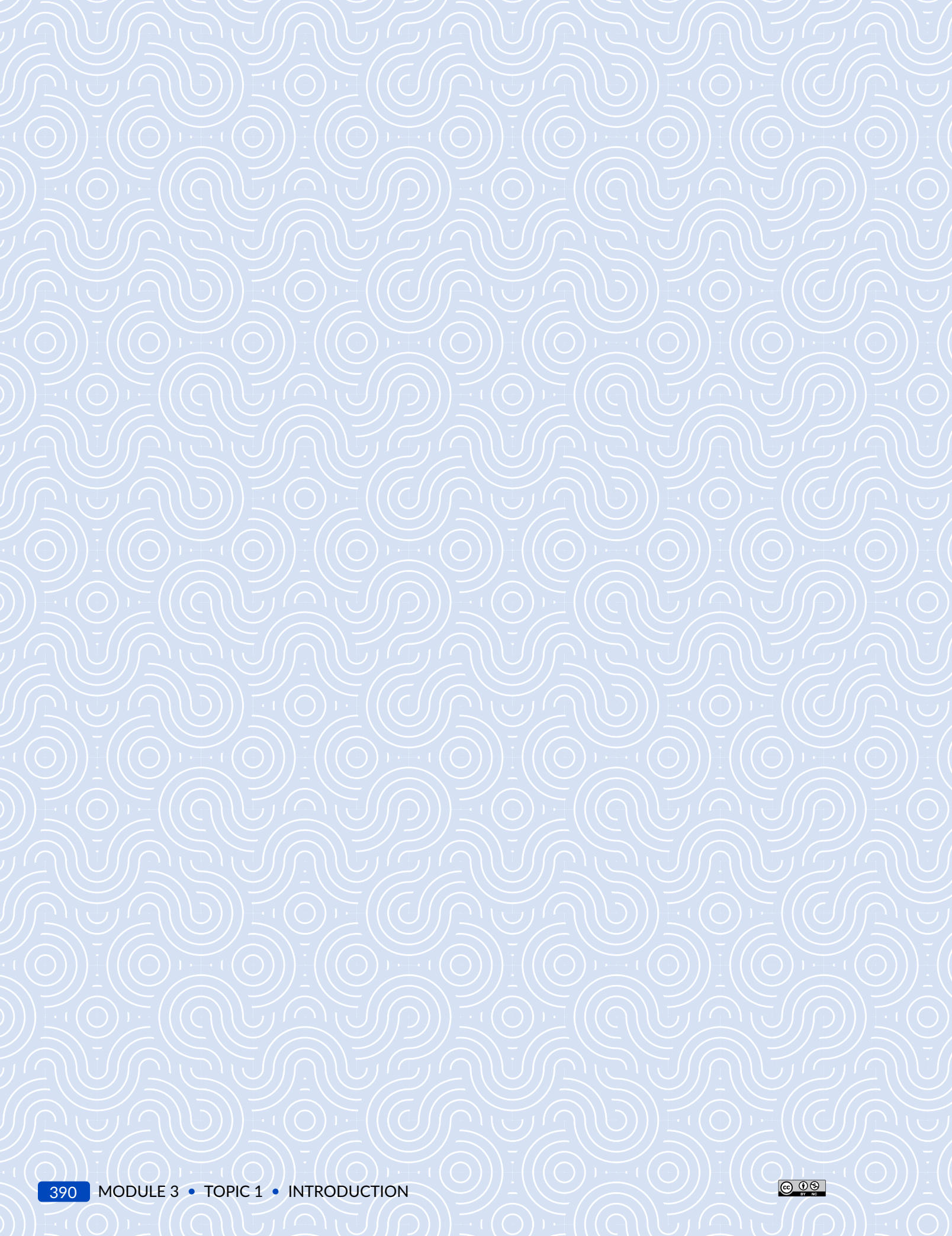




Tubes for producing musical sounds may be cylindrical, conical, or a combination, and are often called to make an instrument easier to carry. The musician opens and closes valves to change the tube length and diameter, thereby changing the pitch of the note. How might you calculate the volume of air moving through a trumpet, clarinet, or trombone?

Composing and Decomposing Functions

LESSON 1	Modeling with Functions	391
LESSON 2	Transforming Function Shapes	407
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LESSON 4	Decomposing Cubic Functions	447



1

Modeling with Functions

OBJECTIVES

- Use multiple representations of functions to model and solve problems.
 - Use multiple representations of functions to analyze problems with students engaged in reading, writing, and speaking during their analysis.
-

You have calculated the area of various geometric figures.

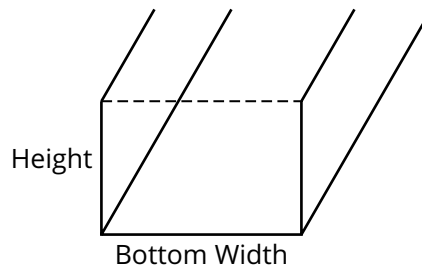
How can you use what you know about area to build an algebraic function?

Getting Started

Brain Drain

A nearby town hired a civil engineer to rebuild its storm drainage system. The drains in this town are open at the top to allow water to flow directly into them. While designing the drains, the engineer must keep in mind the height and the width of the drain. She needs to consider the height because the water cannot rise above the drain or it will flood the town and cause major destruction. However, the drain must also be wide enough that it will not get clogged by debris.

Rectangular sheets of metal will be used to build the drains. These sheets are bent up on both sides to represent the height of the drain. An end view of the drain is shown.



1. Use a sheet of paper to model a drain.
 - a. Compare your model of a drain to your classmates' models. Identify similarities and differences between your models.

- b. How does folding the sides of the drain affect the bottom width of the drain?
- c. Describe the drain that you think best fits the needs of the town. Explain your reasoning.

PROBLEM SOLVING



ACTIVITY

1.1

Modeling a Problem Situation

1. The sheets of metal being used to create the drain are 8.5 feet wide. The engineer wants to identify possible heights and bottom width measurements she could use to construct the drains.
 - a. Determine the bottom width for each given height and choose other height values to complete the table. Then, use the table to determine the width function, $w(h)$.

Ask Yourself . . .

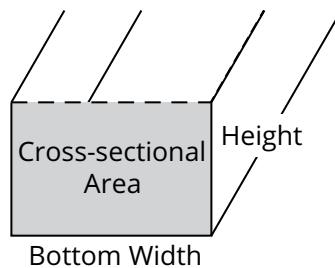
Which height values make sense for this situation?

Height of the Drain (feet)	Bottom Width of the Drain (feet)
0	
1.5	
3	
5	

- b. What function family do you think best represents the bottom width of the drain when given the height? Explain your reasoning.
- c. What is the reasonable domain and range of the width and height function $w(h)$?

2. The engineer needs to identify the measurements that allow the most water to flow through the drain. What does the engineer need to calculate? What does she need to consider?

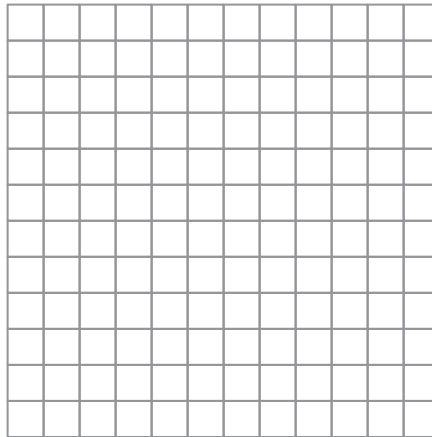
To determine the drain dimensions that allow the most water to flow through, the engineer must calculate the cross-sectional area. The cross-sectional area of a drain is shown.



.....
A *cross-section* is a plane figure created by the intersection of the solid and a plane.
.....

3. Describe how to determine the cross-sectional area of any drain.
4. Which function family best represents the cross-sectional area of a drain with a height of h feet? Explain your reasoning.

5. Predict and describe the drain with the maximum cross-sectional area.
6. Define a function $A(h)$ for the cross-sectional area of the drain with a height of h feet.
7. Use technology to graph the function $A(h)$. Then sketch the graph.



8. Analyze and interpret your graph.

a. What is the maximum cross-sectional area for the drain pipe?
How does the maximum cross-sectional area compare to your prediction from Question 5?

b. Identify and label the intercepts of $A(h)$ on your graph. Then describe the meaning of each in terms of this problem situation.

c. What is the reasonable domain and range of your area function, $A(h)$? How does it compare with the domain and range of your width function $w(h)$ and your height h from Question 1(c)?

.....
Think About...
Is there a way to determine the maximum cross-sectional area using the x-intercepts?
.....

9. Draw and label the drain with the greatest cross-sectional area.

10. Suppose the cross-sectional area is 6.5 square feet. Determine the measure of its height.
11. In this problem, you built a new function $A(h) = h(8.5 - 2h)$ using two existing functions.
- Identify the first factor in this function and its function family. Then describe what it represents in terms of this problem situation.
 - Identify the second factor in this function and its function family. Then describe what it represents in terms of this problem situation.
 - Identify the type of function created by the product of these two factors. Why does this happen?

Talk the Talk

Determine the Best Design

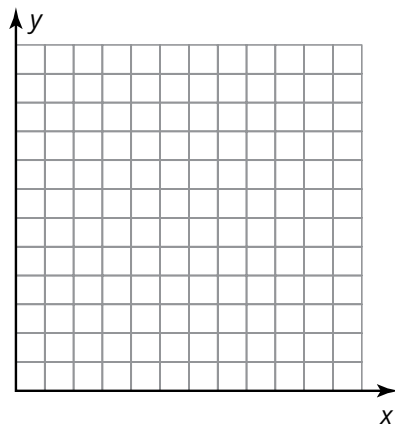
A civil engineering company is hired to design a new drainage system for your town. To construct one of the storm drains, a sheet of metal that is 15.25 feet wide is folded on both sides.

1. Which function family best represents the cross-sectional area? Explain your reasoning.



2. Define a function $A(h)$ for the cross-sectional area of the drain with a height of h feet.

3. Describe the drain that has the maximum cross-sectional area. Include a graph and at least 1 one representation in your description. Show all work and explain your reasoning.



Lesson 1 Assignment

Write

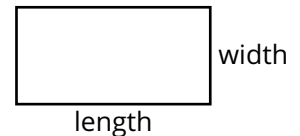
Explain how different representations of a function are useful to model a situation.

Remember

Tables, graphs, and equations can be used to model situations. A function created by the product of two linear factors is a quadratic function.

Practice

1. Mr. Park wants to fence in a rectangular field for his horse using the 600 feet of fence he has stored in his barn. He wants to maximize the area of the field in order to give his horse the most pasture possible. Help Mr. Park design his field to achieve the maximum area.



- a. Complete the table to show the length of the field for each given width.

Width (feet)	0	50	100	150	200	250	300
Length (feet)							

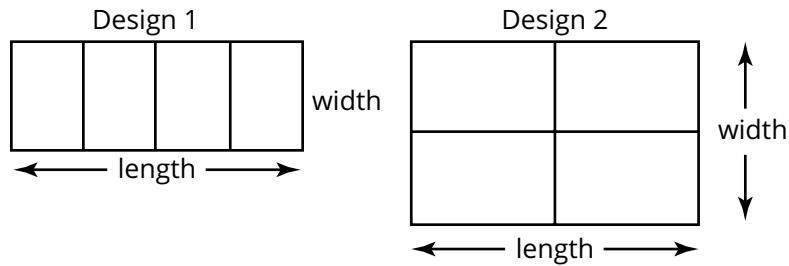
- b. Define the function $\ell(w)$ to represent the length of the field as a function of the width. What function family do you think best represents this? Explain your reasoning.

Lesson 1 Assignment

- c. Define the function $A(w)$ to represent the area of the field as a function of the width. What function family do you think best represents this? Explain your reasoning.
- d. Determine the maximum area of the field as well as the length and width that will result in the maximum area. Explain your reasoning.
- e. Suppose the area of the field is 18,404 square feet. Determine the measure of its length.

Lesson 1 Assignment

2. Ms. Martinez wants to fence in a rectangular area of her field using the 1200 feet of fence she has. She wants the area to have four congruent sections. She is trying to decide which of the two designs shown will give her animals the maximum fenced area.



Determine the design and the dimensions of the design that will give Ms. Martinez the maximum fenced area. Show your work and explain your reasoning.

Lesson 1 Assignment

Prepare

Use the distributive property to rewrite each expression.

1. $x(x^2 - 5)$

2. $(x - 1)(x + 4)$

3. $x(x + 1)\left(\frac{1}{2}x + 1\right)$

4. $(2x + 1)(x + 2)(x + 2)$

2

Transforming Function Shapes

OBJECTIVES

- Translate and dilate functions by non-constant values to create higher-degree functions.
- Describe the relationship between the linear factors of a function and its zeros both orally and in writing.
- Describe how the linear factors of a function combine to produce intervals of increase and decrease in the product function.
- Explain the behavior of function graphs at zeros represented by degree-1 factors and zeros represented by degree-2 factors.
- Sketch graphs of degree-3 functions given their linear factors.

.....

You have transformed functions by constant values to create new functions of the same degree.

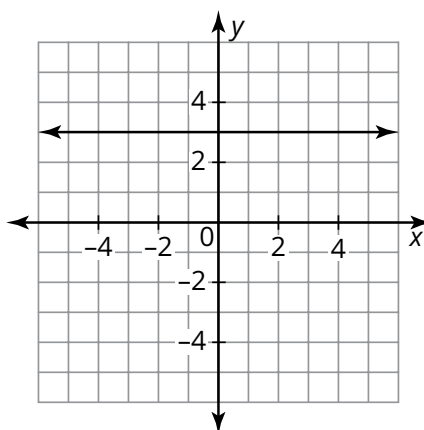
How can you use transformations to create higher-degree functions?

A Functional Transformation

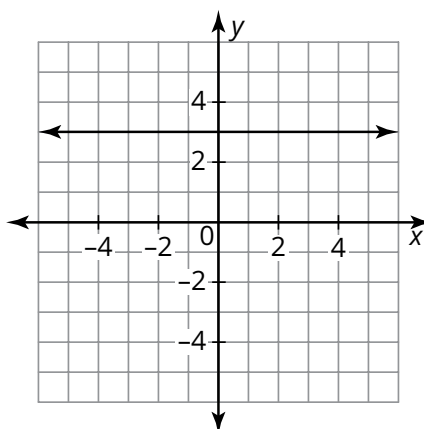
Consider the constant function $f(x) = 3$. The graph of this function is a horizontal line crossing the y -axis at $(0, 3)$.

Recall that you can translate this function vertically by adding a constant: $f(x) = 3 + d$. Every point on the line is translated vertically by the constant. But think about a transformation in which the d -value is not a constant. Suppose the function $f(x) = 3$ is translated by x to form $f(x) = 3 + x$. This would mean that the y -value of every point on the line $y = 3$ is translated by its corresponding x -value.

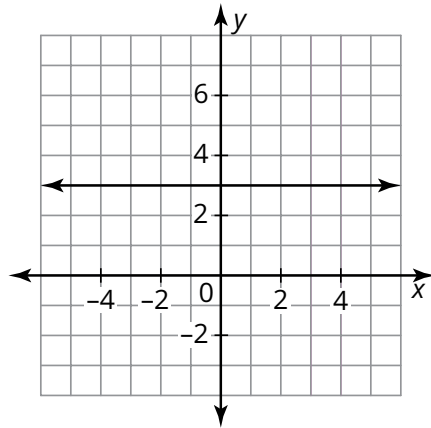
1. Consider the point of the original function at $x = 1$. This point has coordinates $(1, 3)$. Plot the location of the point after a vertical translation of x .



2. Consider the point of the original function at $x = -1$. This point has coordinates $(-1, 3)$. Plot the location of the point after a vertical translation of x .



3. Plot other points after applying a vertical translation of x . What function is formed by translating all the points on the line $f(x) = 3$ by x ?



4. What point does not change its location from the original function to the new function? Explain why.
5. Identify the zero of the new function. Explain how the zero is created.

Creating Functions of Higher Degree

Recall that you can dilate a function vertically by multiplying the function by a constant. For example, if the function is $f(x) = x$, then $f(x) = a \cdot x$ represents a vertical dilation of the original function.

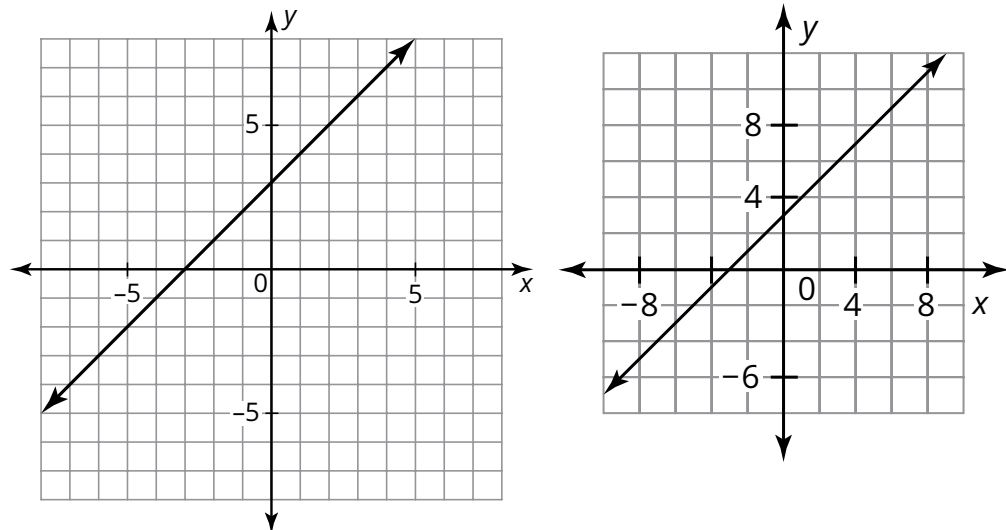
What happens when you transform a function by an a -value that is not a constant? Suppose the function $f(x) = x + 3$ is dilated by x , or $x \cdot f(x)$. This would mean that the y -value of every point on the line $y = x + 3$ is dilated according to its corresponding x -value.

Think About...

How can you use a table to help organize your calculations?

1. The graph shows the function $f(x) = x + 3$.

Choose several points on $f(x)$ and dilate each by multiplying by the x -value of that particular point. Then sketch the graph. Two points have already been dilated.



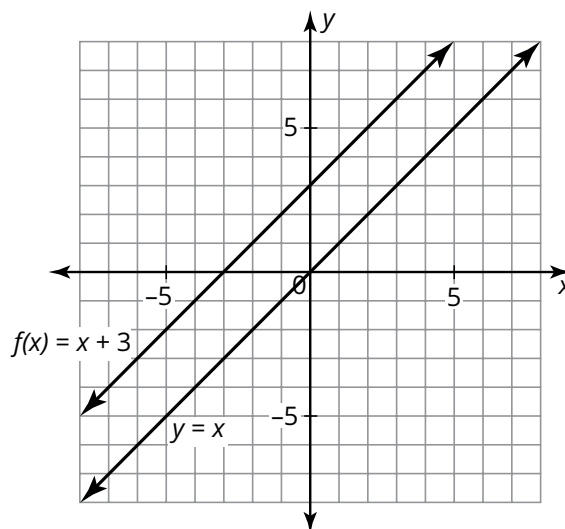
2. Write the equation to represent the new function $g(x) = x \cdot f(x)$. How did the dilation change the degree of the new function?

3. Identify the zeros of $g(x)$. How are these related to the two linear factors, $x + 3$ and x ?

Remember ...

The x -intercepts of a graph correspond to the zeros of the function, as they represent the points where the function's output is zero.

The graph shows the product of the linear factors used to create the new function $g(x) = x \cdot f(x)$. As you analyze the graph from left to right, consider that the new function $g(x)$ changes from decreasing to increasing at the point $(-1.5, -2.25)$.

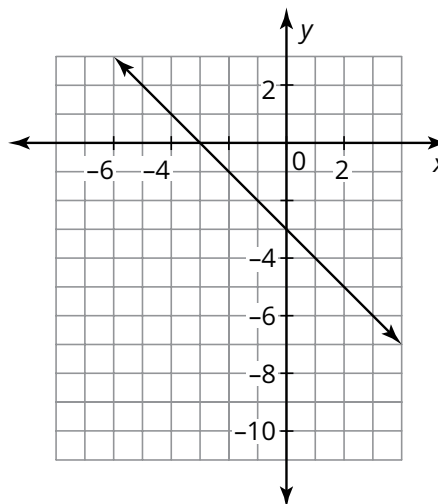
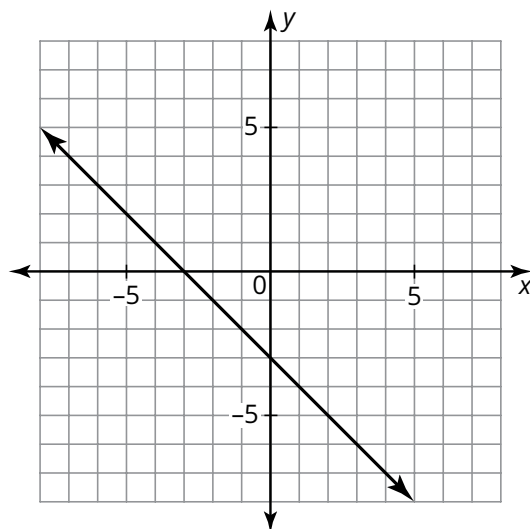


4. Analyze two intervals.
- Consider the interval between $x = -8$ and $x = -3$. Explain why the product of the linear factors results in $g(x)$ being positive and why the function is decreasing.

- b. Consider the interval between $x = -3$ and $x = 0$. Explain why the product of the linear factors results in $g(x)$ being negative and why the function changes from decreasing to increasing.

Suppose the function $h(x) = -(x + 3)$ is dilated by x , or $x \cdot h(x)$.

5. The graph shows the function $h(x) = -(x + 3)$. Choose several points on $h(x)$ and dilate each by multiplying by the x -value of that particular point. Then sketch the graph.

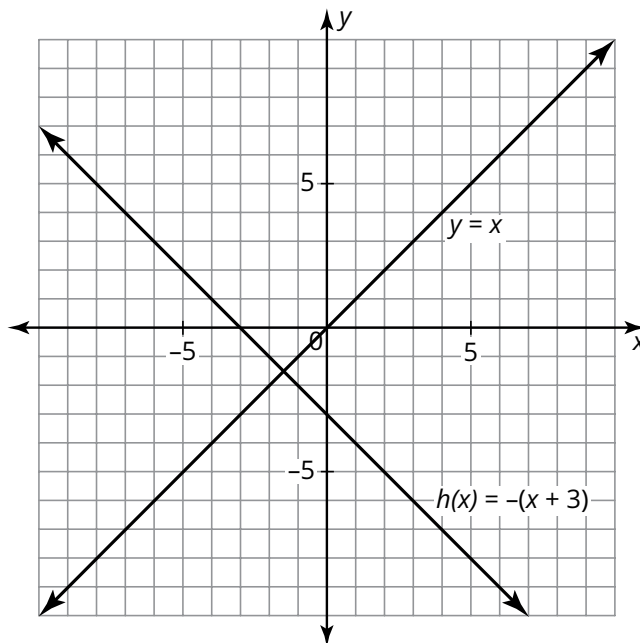


6. Write the equation to represent the new function $j(x) = x \cdot h(x)$. What function family does the new function belong to?

7. Identify the zeros of $j(x)$. How are these related to the two linear factors, $-(x + 3)$ and x ?

The graph shows the product of the linear factors used to create the new function $j(x) = x \cdot h(x)$. As you analyze the graph from left to right, consider that the new function $j(x)$ changes from decreasing to increasing at the point $(-1.5, 2.25)$.

8. Analyze the two intervals.



- a. Consider the interval between $x = -10$ and $x = -3$. Explain why the product of the linear factors results in $j(x)$ being negative and why the function is increasing.

- b. Consider the interval between $x = -3$ and $x = 0$. Explain why the product of the linear factors results in $j(x)$ being positive and why the function changes from increasing to decreasing.

9. What do you know about the linear function factors when the resulting quadratic function:

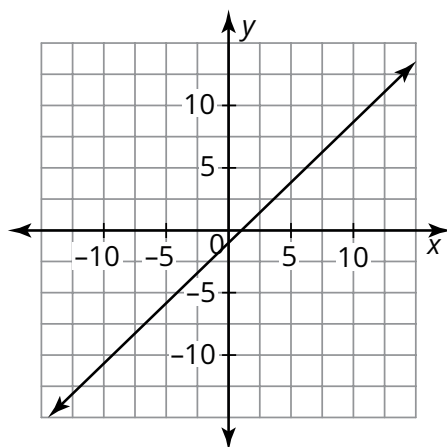
- a. Has a positive a -value?

- b. Has a negative a -value?

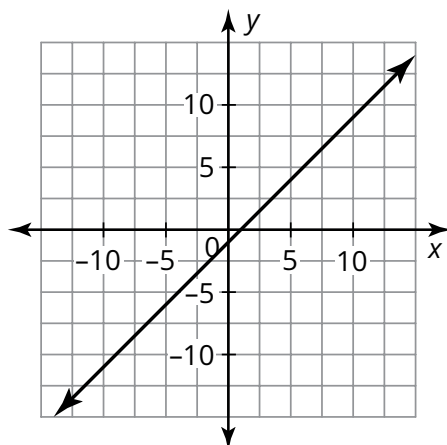
ACTIVITY
2.2**Creating New Zeros of Functions**

Let's create a function of degree 3. Start with the linear function $f(x) = x - 1$, as shown on the coordinate plane.

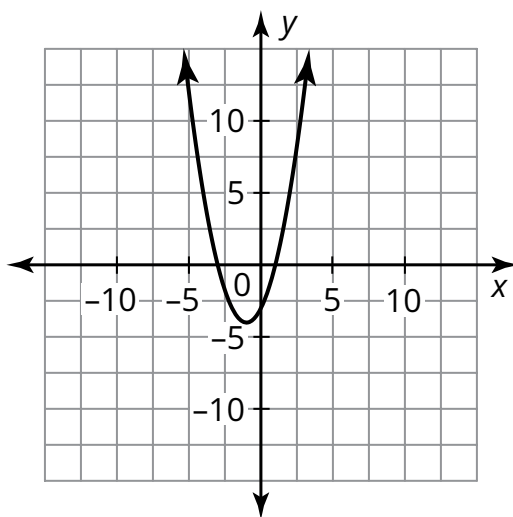
1. Dilate the function $f(x)$ by a factor of $x + 3$, or $(x + 3) \cdot f(x)$. Sketch the graph of the new function, $m(x)$. Write the equation for $m(x)$ and identify the zeros of the new function.



2. Dilate $m(x)$ by a factor of $x + 2$, or $(x + 2) \cdot m(x)$. Sketch the graph of the new function, $p(x)$. Write the equation for $p(x)$ and identify the zeros of the new function.



3. Analyze the changes between each original function and each new function.
- What do you notice about the dilations of each function and the zeros of each new function?
 - How do the zeros appear to be related to the increasing and decreasing behavior of the function? Explain why you think this behavior occurs.
4. The graph shows the function $m(x) = (x - 1)(x + 3)$.
- Dilate $m(x)$ by another factor of $x + 3$. Sketch the graph of the new function, $r(x) = (x + 3) \cdot m(x)$.



- b. Compare the degree-3 function $r(x)$ with the degree-3 function $p(x)$ you created in Question 2. How are the two functions similar and different?

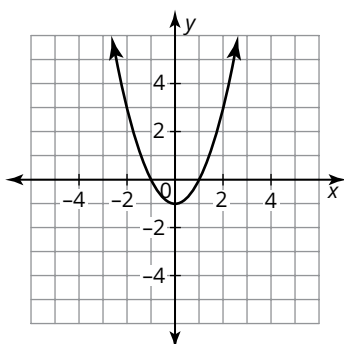
ACTIVITY
2.3

Practice with Functional Transformations

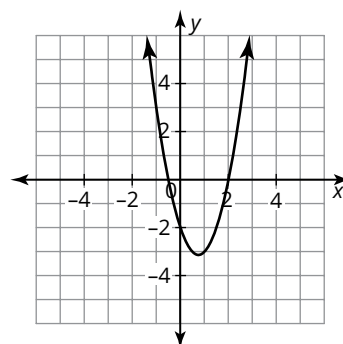
Let's continue to explore the dilation of quadratic functions.

1. Dilate each given function to create a new function of a higher degree. Sketch the graph and then identify the zeros of each new function.

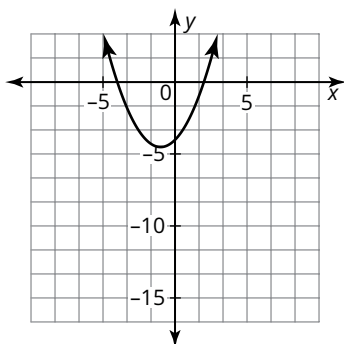
a. $f(x) = (x - 1)(x + 1)$
Sketch $x \cdot f(x)$.



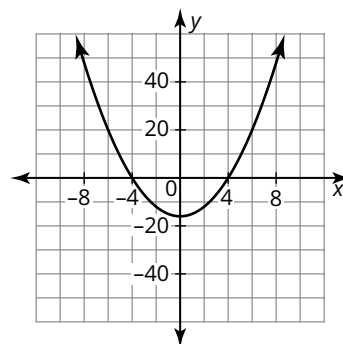
b. $g(x) = (2x + 1)(x - 2)$
Sketch $(x - 2) \cdot g(x)$.



c. $h(x) = \left(\frac{1}{2}x - 1\right)(x + 4)$
Sketch $(x + 4) \cdot h(x)$.



d. $j(x) = (x - 4)(x + 4)$
Sketch $(x - 2) \cdot j(x)$.



2. How do the factors of each equation in Question 1 describe the locations of the zeros of the function?

When the function $g(x)$ is dilated by $x - 2$, the new function contains a square factor, $(x - 2)^2$. Similarly, when the function $h(x)$ is dilated by $x + 4$, the new function contains a square factor, $(x + 4)^2$.

3. Describe how the graphs of the functions behave at zeros described by square factors. How is this different from the way the graphs behave at zeros described by linear factors?

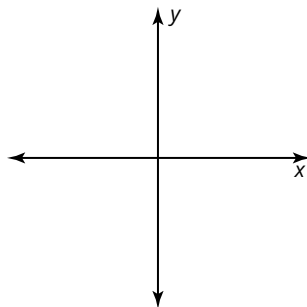


Talk the Talk

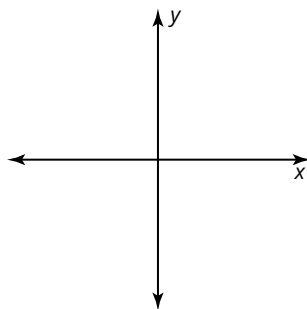
Test and Interpret

1. Predict the shape of each function. Sketch it on the coordinate plane and label the zeros. Use graphing technology to test each prediction.

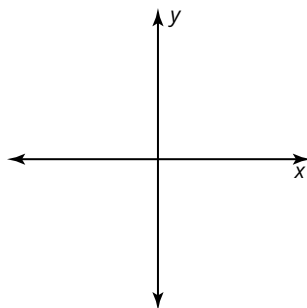
a. $f(x) = x(x - 10)(x + 4)$



b. $g(x) = (x + 3)(x - 3)(x + 3)$



c. $h(x) = (x^2)(x)$



2. For each function in question 1, rewrite the function in standard form. Explain how standard form can justify the location of where each sketch crosses the y -axis.

Lesson 2 Assignment

Write

Describe how you can use dilations to create a cubic function from a linear factor.

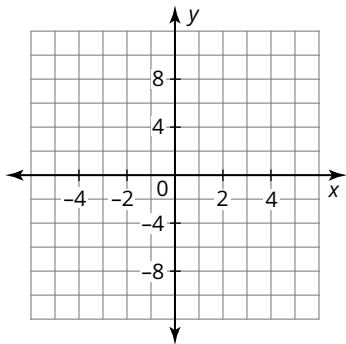
Remember

Functions can be translated and dilated by non-constant values, which apply a different transformation to each point of the function.

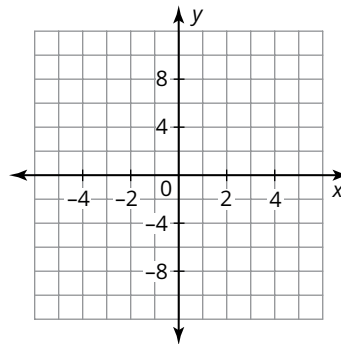
Practice

Dilate each function to create a new function of higher degree. Sketch the graph and identify the zeros of each new function.

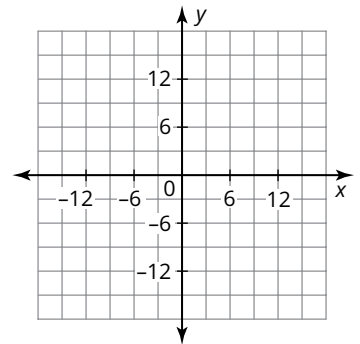
1. $f(x) = (x + 2)(x - 1)$
Dilate the function by x .



2. $g(x) = (x + 1)(2x - 1)$
Dilate the function by $x - 1$.

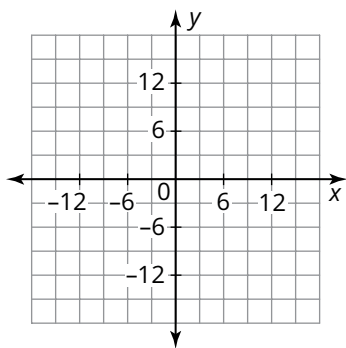


3. $h(x) = (x + 3)(x - 3)$
Dilate the function by $(x + \frac{1}{2})$.

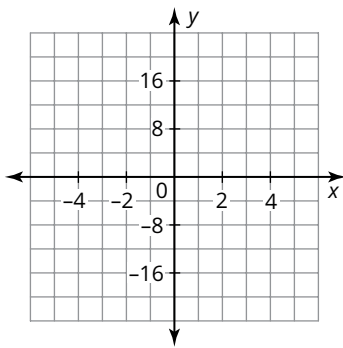


Lesson 2 Assignment

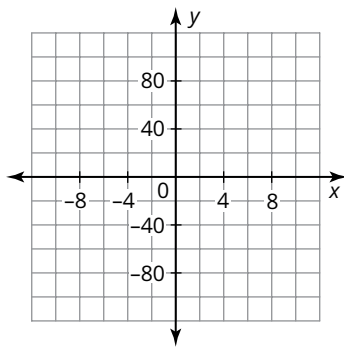
4. $m(x) = \left(\frac{1}{3}x - 2\right)(x + 1)$
Dilate the function by x .



5. $f(x) = (3x - 1)(3x + 1)$
Dilate the function by $x - 2$.



6. $g(x) = (x + 4)(x - 5)$
Dilate the function by $x + 3$.



Lesson 2 Assignment

Prepare

Use the distributive property to rewrite each expression.

1. $a(2a - 1)(5 + a)$

2. $(9 - x)(x + 3)$

3. $b^2(10 - b) + b^2$

4. $(w - 2)(w + 3)(w + 1)$

3

Exploring Cubic Functions

OBJECTIVES

- Represent cubic functions using words, tables, equations, and graphs.
- Interpret the key attributes of the graphs of cubic functions.
- Analyze cubic functions in terms of their mathematical context and write explanations of their use in real-world problem contexts.
- Connect the attributes and behaviors of a cubic function to its factors.
- Compare cubic functions with linear and quadratic functions.
- Build cubic functions from linear and quadratic functions.

KEY TERMS

- cubic function
- relative maximum
- relative minimum

.....

You have calculated the volume of various geometric figures. How can you use what you know about volume to build an algebraic function?

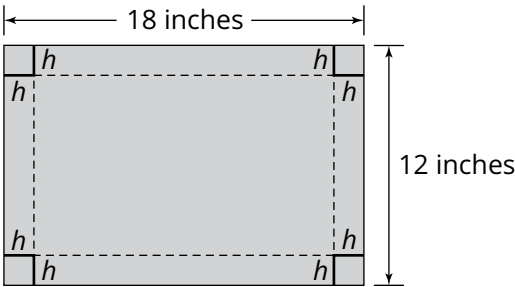
Getting Started

Our Business Is Growing

.....
It may help to create
a model of the planter
by cutting squares
out of the corners
of a sheet of paper
and folding.
.....

A company produces planter boxes. To make the boxes, a square is cut from each corner of a rectangular copper sheet. The sides are bent to form a rectangular prism without a top. Cutting different-sized squares from the corners results in differently sized planter boxes. The company takes sales orders from customers who request a sized planter box.

Each rectangular copper sheet is 12 inches by 18 inches. In the diagram, the solid lines indicate where the square corners are cut, and the dotted lines represent where the sides are bent for each planter box.



1. Complete the table given each planter box is made from a 12-inch by 18-inch copper sheet. Include an expression for each planter box’s height, width, length, and volume for a square corner side of length h .

Square Corner Side Length (inches)	Height (inches)	Width (inches)	Length (inches)	Volume (cubic inches)
0				
1				
2				
3				
4				
5				
6				
7				
h				

2. Analyze the relationship between the height, length, and width of each planter box. What are the dimensions of the largest sized square corner that can be cut to make a planter box? Explain your reasoning.

Ask Yourself . . .

What patterns do you notice in the table?

3. Write a function $V(h)$ to represent the volume of the planter box in terms of the corner side length h .

ACTIVITY
3.1

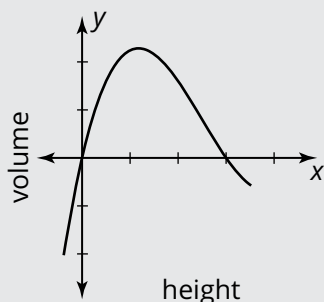
Building a Cubic Function from a Situation

Let's consider the graph of the *cubic function* you created.



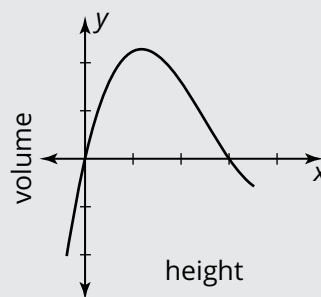
1. Mason, Sebastian, and Kayla each used graphing technology to analyze the volume function, $V(h)$, and to sketch the graph. They disagree about the shape of the graph.

Mason



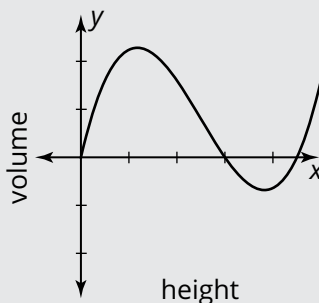
The graph increases and then decreases. It is a parabola.

Sebastian



The graph lacks a line of symmetry, so it can't be a parabola.

Kayla



I noticed the graph curves back up so it can't be a parabola.

Evaluate each student's sketch and rationale to determine who is correct. For the student(s) who is/are not correct, explain why the rationale is not correct.

.....
A **cubic function** is a function that can be written in the general form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.
.....

2. Represent the cubic function using graphing technology with the settings $[-10, 15] \times [-400, 400]$.

- a. Describe the given key attributes of the graph.

x-intercept(s):

y-intercept:

intervals of increase:

interval of decrease:

- b. What is the maximum volume of a planter box?
State the dimensions of this planter box. Explain your reasoning.

- c. Identify the domain of the function $V(h)$ using inequalities.
Is the domain the same or different in terms of the context of this problem? Explain your reasoning.

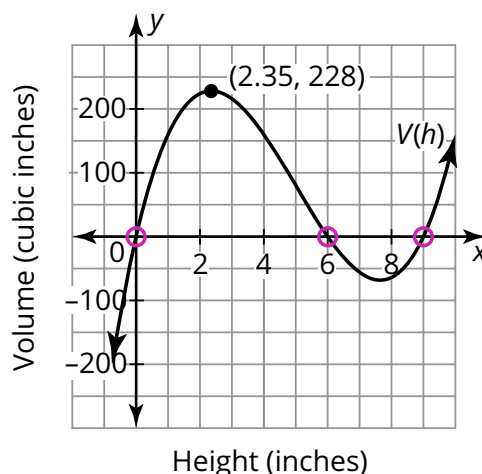
- d. Identify the range of the function $V(h)$ using inequalities.
Is the range the same or different in terms of the context of this problem? Explain your reasoning.

- e. What do the x-intercepts represent in this problem situation? Do these values make sense in terms of this problem situation? Explain your reasoning.

.....
Think About ...

In this problem, you are determining the maximum value graphically. How will your solution strategy change when using the table or equation?
.....

A graph may reveal different key attributes within a given domain. The function $V(h) = h(12 - 2h)(18 - 2h)$ has x -intercepts at $x = 0$, $x = 6$, and $x = 9$.



As the input values for height increase, the output values for volume approach infinity. Therefore, the function doesn't have a maximum; however, the point (2.35, 228) is a *relative maximum* within the domain interval of (0, 6). A **relative maximum** is the highest point in a particular section of a graph.

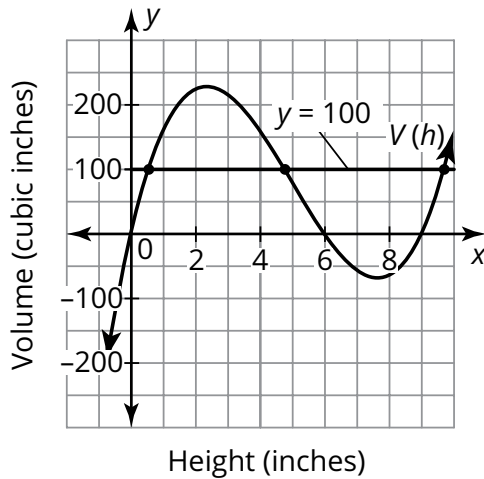
Similarly, as the values for height decrease, the output values approach negative infinity. Therefore, a *relative minimum* occurs at (7.65, 268.16). A **relative minimum** is the lowest point in a particular section of a graph.

The function $V(h)$ represents all of the possible volumes for a given height h . A horizontal line is a powerful tool for working backwards to determine the possible values for the height when the volume is known.

Suppose a customer ordered a particular planter box with a volume of 100 cubic inches, but did not specify the height of the planter box.

WORKED EXAMPLE

You can determine the possible heights from the graph of $V(h)$. Suppose the given volume of a planter box is 100 cubic inches.



- Draw a horizontal line at $y = 100$.
- Identify each point where $V(h)$ intersects with $y = 100$, or where $V(h) = 100$.

The first point of intersection is represented using function notation as $V(0.54) = 100$.

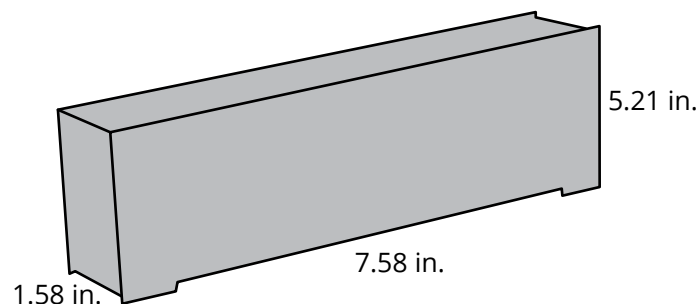
.....
Why doesn't a cubic function have an absolute minimum or absolute maximum value?
.....

3. Consider the Worked Example.

- Use graphing technology to determine where $V(h) = 100$. Then write the intersection points in function notation. What do the intersection points mean in terms of this problem situation?
- How many differently sized planter boxes can the company make to fill this order? Explain your reasoning.

4. A neighborhood beautifying committee would like to purchase a variety of planter boxes with volumes of 175 cubic inches to add to business windowsill storefronts. Determine the planter box dimensions that the company can create for the committee. Show all work and explain your reasoning.
5. An intern claims that he can no longer complete an order because he spilled a cup of coffee on the sales ticket. Help him complete the order by determining the missing dimensions from the information that is still visible. Explain how you determined possible unknown dimensions of each planter box.

Sales Ticket	
Base Area:	12 square inches
Height:	
Length:	
Width:	
Volume:	



A customer sent this email:

To Whom It May Concern,

I would like to purchase several planter boxes, all with a height of 5 inches. Can you make one that holds 100 cubic inches of dirt?

Please contact me at your earliest convenience.

Thank you,
Ms. Perez

.....
Think About . . .

How is the volume function built in this problem?

6. Write a response to this customer, showing all calculations.

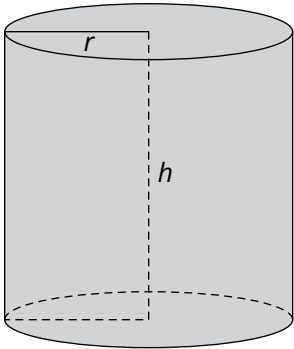


ACTIVITY
3.2

Building a Cubic Function from a Quadratic and Linear Function

The company also makes cylindrical planters for city sidewalks and storefronts. The cylindrical planters come in a variety of sizes, but all have a height that is twice the radius.

1. Why do you think the company might want to manufacture different sizes of a product, but maintain a constant ratio of height to radius?



Remember ...

A constant ratio makes the cylindrical planters similar.

Volume of a cylinder:
 $V = (\text{base area})(\text{height})$
 Area of a circle: $A = \pi r^2$

2. Consider differently sized cylindrical planters.

- a. Complete the table.

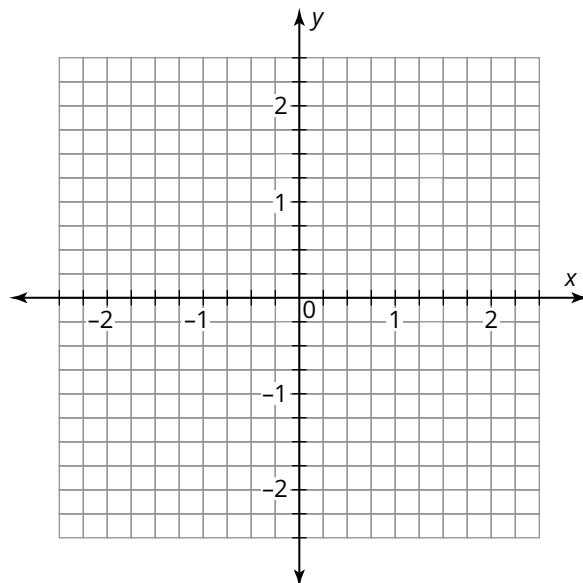
Radius	Height (inches)	Base Area (square inches)	Volume (cubic inches)
0			
1			
2			
3			
4			
			2000
x			

- b. Describe how you determined the volume when you are given the radius.

- c. Describe your method to determine the base area and the height when you are given the volume.
- d. Analyze your table of values. For every unit increase in the radius, describe the rate of change in the height, area of the base, and volume of each planter.

The base area function $A(x) = \pi x^2$ and the height function $h(x) = 2x$ are multiplied to build the volume function $V(x) = (\pi x^2)(2x)$.

3. Use technology to sketch and label the functions $A(x)$, $h(x)$, and $V(x)$ on the coordinate plane shown.



4. Determine the domain and range of $V(x)$ based on the problem situation using interval notation.

.....

A cubic function is a function that can be written in the general form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

.....

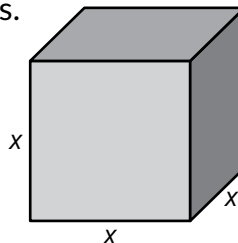
The two cubic functions you have sketched in this lesson represent the two general shapes of a cubic function.

5. How can you describe these two general shapes?

ACTIVITY
3.3

The Cubic Parent Function

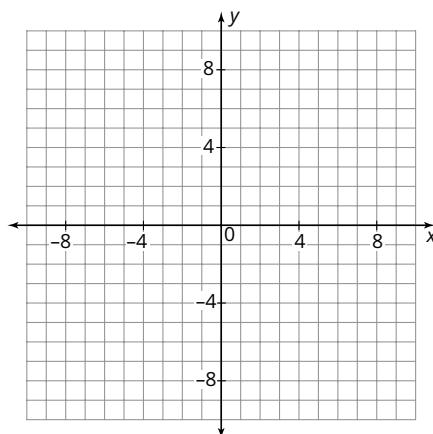
The company also makes cube-shaped containers. Consider a cube, which has equal length, width, and height, x .



1. Write the equation for the volume of the cube, $V(x)$, given the side length x .

The equation you wrote in Question 1 represents the parent function for the cubic family of functions.

2. Graph the parent cubic function.



3. Use the graph to identify the key attributes of the cubic parent function.

domain using inequalities:

range using inequalities:

x -intercept(s):

y -intercept(s):

relative maximum or relative minimum:

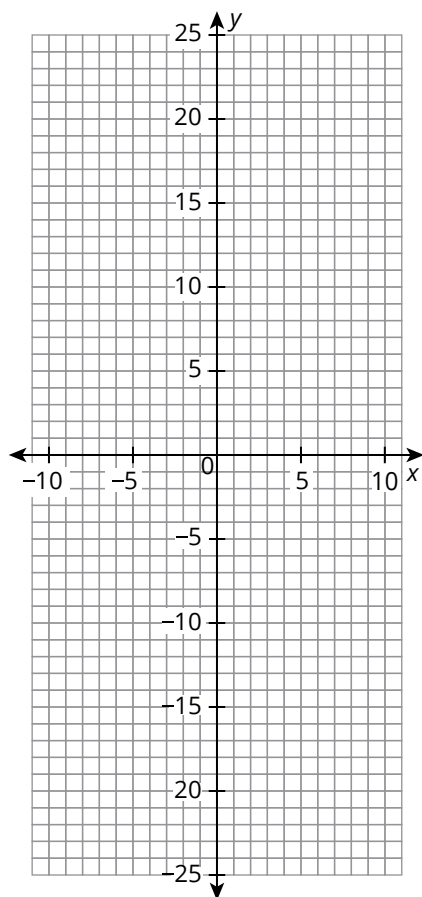


Talk the Talk

Cubic Connections

Use technology to sketch the functions on the coordinate plane. Then, identify the key attributes of their graph. Write the domain and range using inequalities.

1. $f(x) = (x - 2)(x^2 + 4x + 5)$



domain:

range:

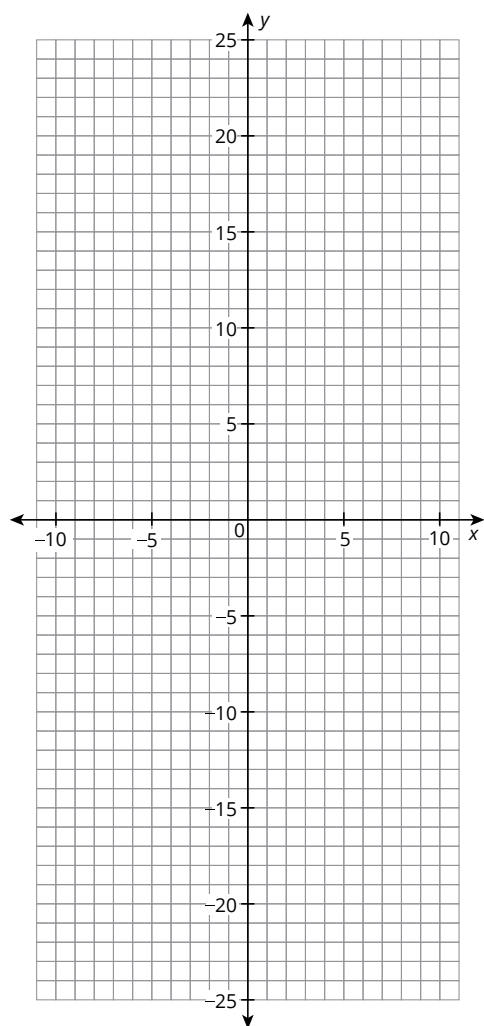
x-intercept(s):

y-intercept:

relative maximum:

relative minimum:

2. $g(x) = x(x + 3)(x - 4)$



domain:

range:

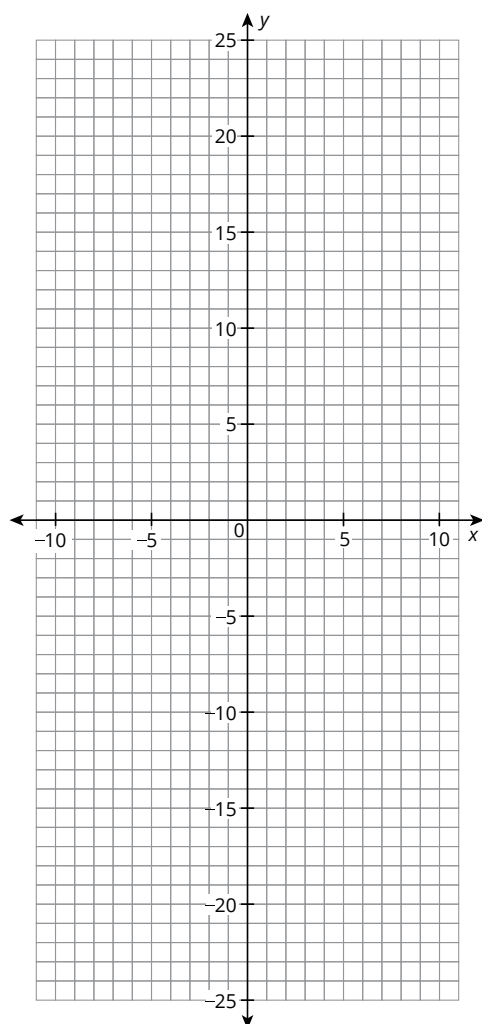
x-intercept(s):

y-intercept:

relative maximum:

relative minimum:

3. $h(x) = -x^3 + 2x^2 + 5x - 6$



domain:

range:

x-intercept(s):

y-intercept:

relative maximum:

relative minimum:

Lesson 3 Assignment

Write

Provide an example of each key term.

1. relative minimum
2. relative maximum
3. cubic function

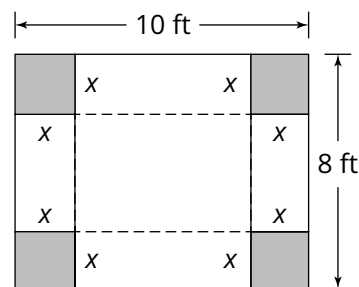
Remember

A cubic function is a polynomial function of degree 3 that can be written in the form

$f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. The graph has 2 general shapes.

Practice

1. Adriana is an engineer at a manufacturing plant. Her boss asks her to use rectangular metal sheets to build storage bins with the greatest possible volume. Each rectangular sheet is 8 feet by 10 feet. Adriana's sketch shows the squares to be removed from the corners of each sheet. The dashed lines indicate where the metal sheets will be folded before they are welded to form the prism-shaped storage bins without tops.



- a. Write a function $V(x)$ to represent the volume of a bin in terms of the side length, x , of the removed squares. Explain your reasoning.
- b. Graph the function $V(x)$ using technology. Determine the domain and range of the function using inequalities. Determine the domain and range of the function as they relate to this problem situation. Explain your reasoning.

Lesson 3 Assignment

- c. Determine the maximum volume of a bin. What are the dimensions of a bin with the maximum volume?

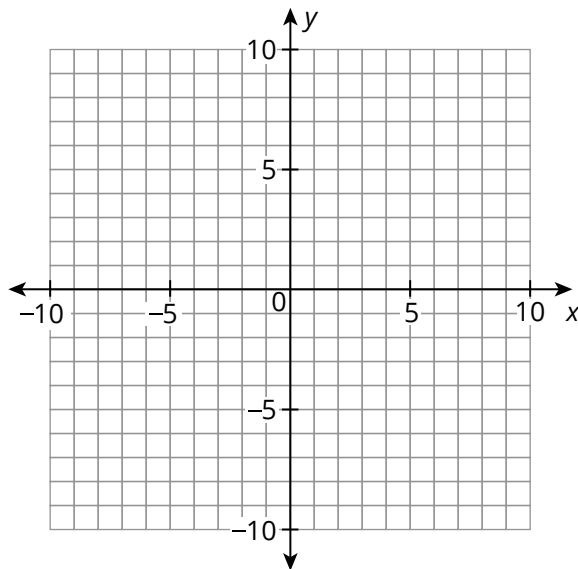
- d. Determine any relative maximums or relative minimums of $V(x)$. Then, determine the intervals over which the function is increasing and decreasing.

- e. Determine the x - and y -intercepts of the graph of $V(x)$. What do they represent in this problem situation?

- f. Adriana's boss asks her to make several bins with volumes of exactly 40 cubic feet. Determine the bin dimensions that will work.

Lesson 3 Assignment

2. Use technology to sketch the function $f(x) = x^3 - 2x + 4$ on the coordinate plane. Then, describe the given key attributes of its graph. Write the domain and range as inequalities.



domain:

range:

x-intercept(s)

y-intercept :

relative maximum:

relative minimum:

Lesson 3 Assignment

Prepare

Determine the possible values for x in each equation.

1. $2x - 5 = 0$

2. $(x + 4)(x - 4) = 0$

3. $x(x + 3) = 0$

4. $(x - 5)(x + 1)(x - 2) = 0$

4

Decomposing Cubic Functions

OBJECTIVES

- Connect the characteristics and behaviors of quadratic and cubic functions to their factors and describe them verbally.
- Build cubic functions from linear and quadratic functions graphically.
- Determine the zeros of functions and the multiplicity of the functions' zeros.
- Compare cubic functions with linear and quadratic functions.
- Show how linear, quadratic, and cubic functions demonstrate the Fundamental Theorem of Algebra.

KEY TERM

- multiplicity

.....

You have built functions to represent two- and three-dimensional shapes.

What does the structure of the equation reveal about the graph of the function?

A Difference in Degree

You know that another name for a linear function that is not a constant function is a polynomial of degree 1 because the greatest exponent in a linear function equation is 1. A quadratic function is a polynomial of degree 2, and a cubic function is a polynomial of degree 3.

1. Determine the degree of each function.

a. $f(x) = x$

b. $g(x) = x - 1$

c. $h(x) = x(x - 1)$

d. $j(x) = (x - 1)(x - 1)$

e. $k(x) = x(x - 1)(x - 1)$

f. $m(x) = (x)(x)(x - 1)$

Recall that the zeros of a function are the points on the graph of the function where the function's value is zero. For example, given the function $f(x) = x$, the value of the function is 0 when $x = 0$. Likewise, given $g(x) = x - 1$, the value of the function is 0 when $x = 1$.

Multiplicity is how many times a particular number is a zero for a given function. The zero in each of the functions f and g , for example, has a multiplicity of 1.

2. For the functions given in parts (c) through (f), identify the number of zeros and the multiplicity of each zero.

c.

d.

e.

f.

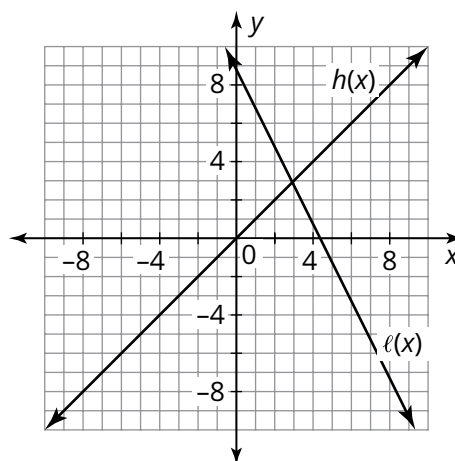
3. What do you notice about the zeros of a function, counted with multiplicity, and the degree of the function?

Decomposing a Quadratic Function

Consider the quadratic function you created to represent the cross-sectional area of the rain gutter in a previous lesson. This function was written as a product of two linear functions: one representing the height, $h(x) = x$, and one representing the length, $\ell(x) = -2x + 8.5$. Let's consider how to sketch the graph of the quadratic function without technology using the two linear functions.

The graph shows the two linear functions. Dashed lines are drawn through each x -intercept to divide the graph into three regions.

- Region A: From negative infinity to the leftmost zero.
- Region B: The region between the two zeros.
- Region C: From the rightmost zero to positive infinity.



1. Label the zeros of each function with open points on the coordinate plane and on the number line below the coordinate plane.

Ask Yourself . . .

How many zeros, counted with multiplicity, should the product function have?

2. Consider the three regions. For each region:

- a. Determine whether the output values of the product of the functions, $h(x) \cdot \ell(x)$ is positive or negative. Label the number line accordingly.

Ask Yourself . . .

Is the product positive, negative, or zero? Does the point representing the product fall above, below, or on the x -axis?

- b. Choose an x -value in the region. Evaluate the functions $h(x)$, $\ell(x)$, and $h(x) \cdot \ell(x)$ for that x -value. Plot each point on the coordinate plane.

3. Describe the product $h(x) \cdot \ell(x)$ at the intersection point of the two linear functions. Plot the point that represents the product of the two functions at this point on the coordinate plane.

4. Describe the product $h(x) \cdot \ell(x)$ at the zeros of the two linear functions. Plot the point that represents the products of the two functions at these points on the coordinate plane.

5. Use the information from Questions 2 and 3 to sketch the function $h(x) \cdot \ell(x)$ on the coordinate plane.

.....

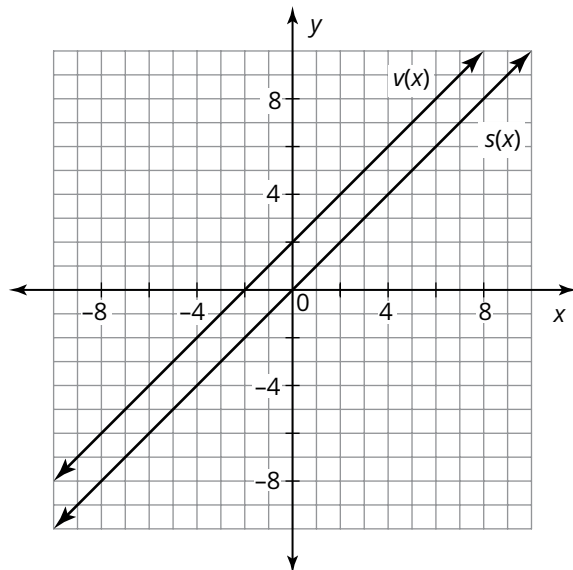
Remember ...

The zero product property states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

.....

6. How does the zero product property relate to the x-intercepts of the three functions?

7. Analyze the graphs of $s(x)$ and $v(x)$.



.....
Think About...
Predict the function family of your sketch before you get started!
.....

- a. Sketch the graph of $p(x)$ if $p(x) = s(x) \cdot v(x)$.
- b. Identify the x -intercepts of $p(x)$. Explain the relationship between the x -intercepts of $p(x)$ and the x -intercepts of $s(x)$ and $v(x)$.
- c. Identify the vertex of $p(x)$. What is the relationship between the vertex of $p(x)$ and the functions $s(x)$ and $v(x)$?

Decomposing Cubic Functions

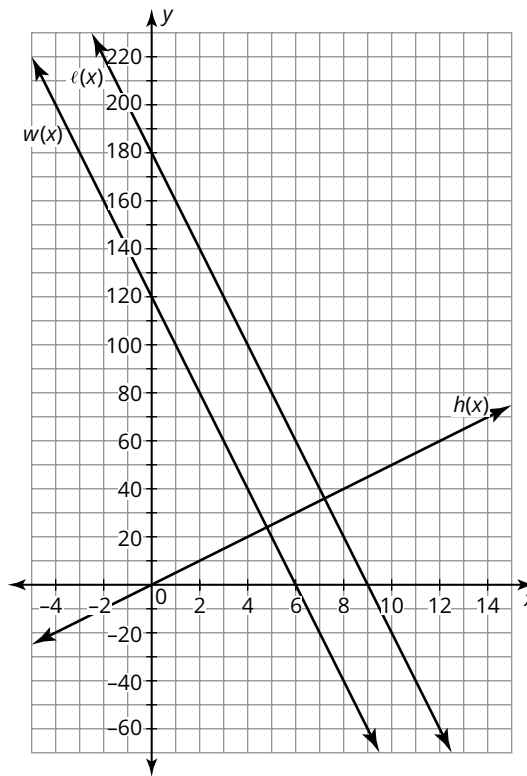
Think About...

How do the zeros help you construct a sketch of the function defined by the factors of the polynomial?

Let's now use three linear functions to sketch their cubic function product.

Recall that the volume function $V(x) = x(18 - 2x)(12 - 2x)$ for the planter boxes in a previous lesson was built by multiplying three linear functions representing length, width, and height.

You can sketch the graph of this cubic function without technology using the three linear functions. The linear functions that represent the length, width, and height of the planter boxes are shown on the graph.



1. Follow the instructions to compose the graph of $V(x)$ using $w(x)$, $l(x)$, and $h(x)$.
 - a. Draw dashed vertical lines through the x -intercepts to divide the coordinate plane into regions.

b. Determine whether the output values for the product are positive or negative in each region. Label each region. Explain how you know where $V(x)$ lies, above or below the x -axis.

c. How do you know where $V(x)$ crosses the x -axis?

d. Choose an x -value in each region. Evaluate each function and then determine the product. Plot each point on the coordinate plane.

e. Sketch the graph of $V(x)$.

.....

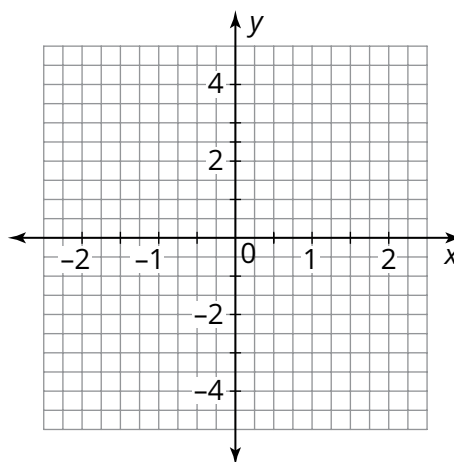
Think About ...

Which x -values do you want to choose in each region to help you understand the shape of the graph? Is one value enough?

.....

Recall that the volume function for the cylindrical planter in a previous lesson was built from a quadratic function, $A(x) = \pi x^2$, representing the area of the base and a linear function, $h(x) = 2x$, representing the height.

2. Sketch the graph of the cubic volume function that is the product of the quadratic and linear functions. Show all your work and explain your reasoning.



.....
Remember...

The Fundamental Theorem of Algebra states that a degree n polynomial has, counted with multiplicity, exactly n zeros.

.....

3. Explain how the two cubic functions presented in this activity demonstrate the Fundamental Theorem of Algebra.

ACTIVITY 4.3

Multiplying to Create Polynomials

You can multiply to determine whether a product written in factored form is equivalent to a quadratic or cubic function written in general form.

Consider, for example, the cubic function $f(x) = 3x^3 + 16x^2 + 12x - 16$. Is the function equivalent to $g(x) = (x + 2)(3x - 2)(x + 4)$?

WORKED EXAMPLE

You can determine the product of the linear factors $(x + 2)(3x - 2)(x + 4)$ using multiplication tables.

Step 1:

Choose 2 of the binomials, multiply, and then combine like terms.

·	x	2
$3x$	$3x^2$	$6x$
-2	$-2x$	-4

Step 2:

Multiply the product from step 1 with the remaining binomial. Then combine like terms.

·	x	4
$3x^2$	$3x^3$	$12x^2$
$4x$	$4x^2$	$16x$
-4	$-4x$	-16

$$(x + 2)(3x - 2)(x + 4) = 3x^3 + 16x^2 + 12x - 16$$

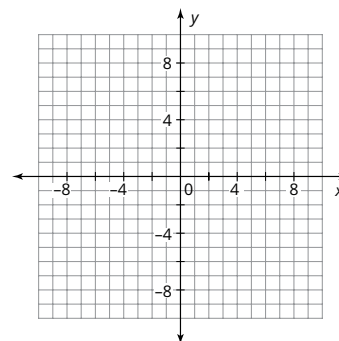
1. Analyze the Worked Example for the multiplication of three binomials.
 - a. How can you use graphing technology to verify that the expression in factored form is equivalent to the product written in general form?
 - b. How does the cubic function demonstrate the Fundamental Theorem of Algebra? Explain your reasoning.

Ask Yourself . . .

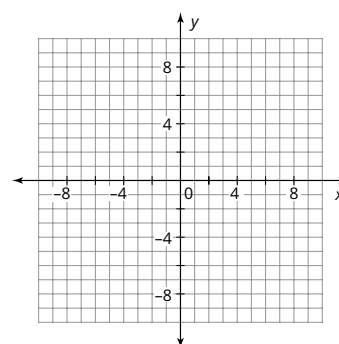
How do the factors of the given expression relate to the x-intercepts of the graph?

2. Determine each product algebraically. Show all your work and then use technology to verify your product is correct. Finally, sketch the graph and explain how the function demonstrates the Fundamental Theorem of Algebra.

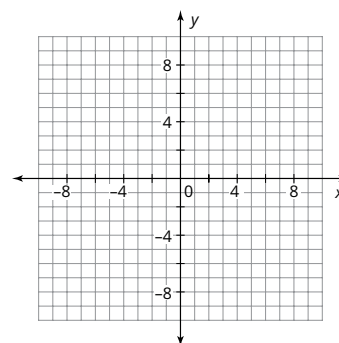
a. $(x + 2)(-3x + 2)(2x + 1)$



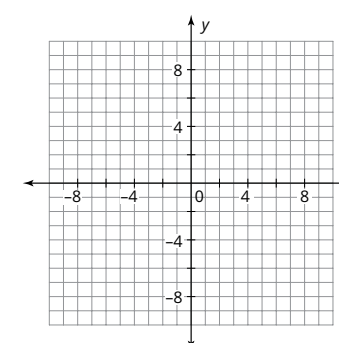
b. $(x + 2)(5x + 1)$



c. $(2x - 1)(x - 1)$



d. $(x + 2)^3$



3. Avery determined the product of three linear factors.

Avery



The function $f(x) = (x + 2)^3$ is equivalent to $f(x) = x^3 + 8$.

- a. Create a multiplication table to show why Avery is incorrect.
Explain your reasoning.
- b. How many x -intercepts does the function $f(x) = (x + 2)^3$ have?
How many zeros are counted with multiplicity? Explain
your reasoning.



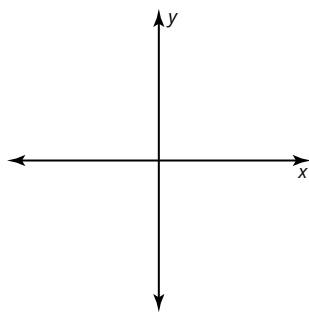
Talk the Talk

A Difference in Kind

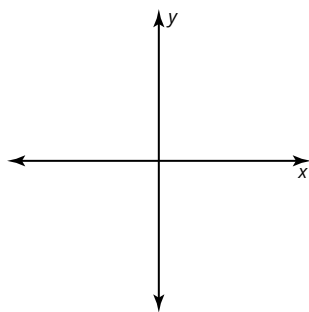
In this lesson, you sketched polynomial functions using factors and rewrote them in general form.

1. Use the factors to sketch each cubic function and label the zeros. Then, rewrite the function in general form.

a. $f(x) = (x + 3)(-x + 5)(2 - 2x)$



b. $g(x) = (x - 2)^3$



2. How are cubic functions similar to linear functions? How are they different?

.....

Think About ...

Consider the functions as graphs, tables, equations, and situations. What is the degree of the function? How many zeros, counted with multiplicity, does it have?

.....

3. How are cubic functions similar to quadratic functions? How are they different?

Lesson 4 Assignment

Write

In your own words, explain what the term *multiplicity* means.

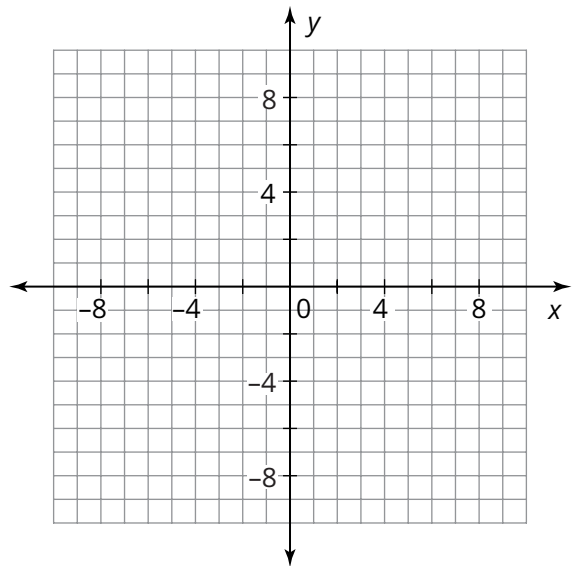
Remember

The zero product property states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero. The Fundamental Theorem of Algebra states that a degree- n polynomial has, counted with multiplicity, exactly n zeros.

Practice

1. Consider the functions $k(x) = x - 1$, $m(x) = x + 2$, $n(x) = x - 3$, and $f(x) = k(x) \cdot m(x) \cdot n(x)$.

- a. Graph $k(x)$, $m(x)$, and $n(x)$.
- b. Determine the degree of the function $f(x)$. Explain your reasoning.

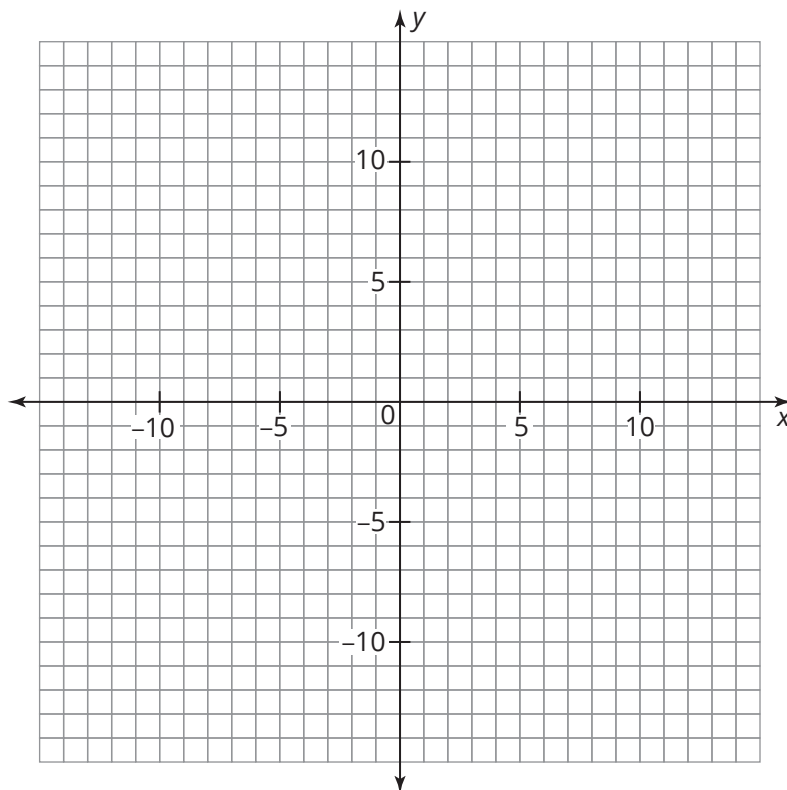


Lesson 4 Assignment

- c. Determine the zeros of $f(x)$. Explain your reasoning.
- d. Determine the intervals over which the value of $f(x)$ is positive.
Determine the intervals over which the value of $f(x)$ is negative.
Explain your reasoning.
- e. Sketch $f(x)$.

Lesson 4 Assignment

2. Consider the functions $p(x) = 3 + 2x$, $q(x) = -x + 1$ and $r(x) = x + 4$.
- Determine its product algebraically. Use technology to verify your product is correct.
 - Sketch the graph and explain how the function demonstrates the Fundamental Theorem of Algebra.



Lesson 4 Assignment

Prepare

Evaluate each expression for $x = 2$ and $x = -3$.

1. x^4

2. $-x^2 + 1$

3. $(-x)^2 - 1$

4. x^8

Composing and Decomposing Functions

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

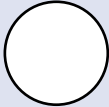
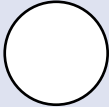
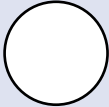
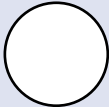
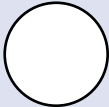
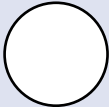
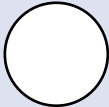
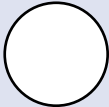
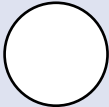
Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Composing and Decomposing Functions* topic by:

TOPIC 1: <i>Composing and Decomposing Functions</i>	Beginning of Topic	Middle of Topic	End of Topic
interpreting complex expressions by examining their variables.			
using a graph to identify zeros of polynomial functions.			
recognizing the linear and quadratic factors that compose polynomials.			
sketching a graph of a polynomial using factors and zeros.			
approximating solutions to a polynomial function graphically by inspecting where the graph intersects a constant function.			
identifying transformations of a function on a graph.			

continued on the next page

TOPIC 1 SELF-REFLECTION
continued

TOPIC 1: <i>Composing and Decomposing Functions</i>	Beginning of Topic	Middle of Topic	End of Topic
describing the effects of transformations by variable functions on the parent functions.			
using volume formulas for cylinders and prisms to build polynomial functions.			
adding, subtracting, and multiplying polynomials.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

- Describe a new strategy you learned in the *Composing and Decomposing Functions* topic.

- What mathematical understandings from the topic do you feel you are making the most progress with?

- Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 1 SUMMARY

Composing and Decomposing Functions Summary

KEY TERMS

- cubic function [función cúbica]
- relative maximum [máximo relativo]
- relative minimum [mínimo relativo]
- multiplicity [multiplicidad]

LESSON

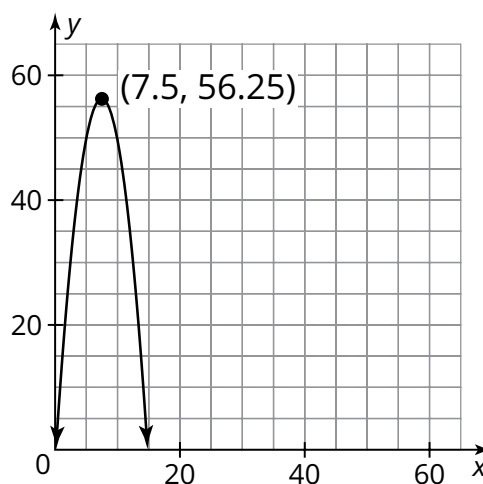
1

Modeling with Functions

A physical model or drawing is often helpful when describing and analyzing a real-world problem. You can use the model, a graph, or a table of values to help you determine which function family best represents a problem situation.

A quadratic function is used to model area. You can determine the maximum area by identifying and interpreting the coordinates of the vertex of the graph.

For example, consider a situation in which Jasmine wants to build a rectangular pen for her rabbit from a length of foldable mesh wire. She has a total length of 30 feet of mesh wire and wants to determine the maximum area for the pen.



The function $A(w) = w(15 - w)$, or $A(w) = -w^2 + 15w$, models the area of the pen where w represents the width of the pen.

The graph of the function has a vertex at $(7.5, 56.25)$. This means that the maximum area the pen can be is 56.25 square feet when the width of the pen is 7.5 feet.

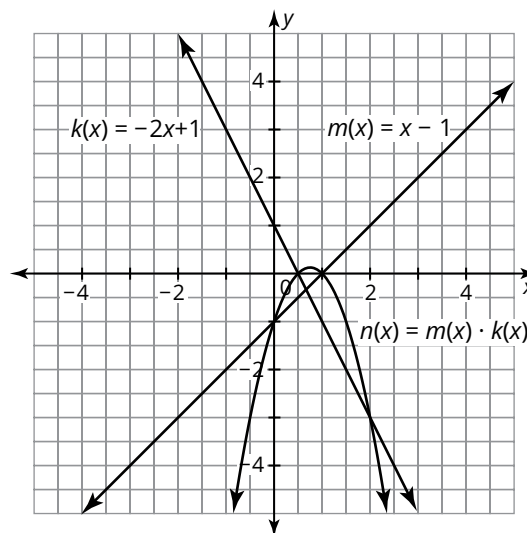
You can use the graph or the quadratic formula to solve for unknown values x values in a quadratic equation.

Recall that you can dilate a function vertically by multiplying the function by a constant, the a -value. If the function is $f(x) = x$, then $f(x) = ax$ represents a vertical dilation of the original function. When a linear function is dilated vertically by multiplying the function by a linear function that is not a constant function, the resulting function is a second-degree function.

For example, consider the graphs of $k(x) = -2x + 1$, $m(x) = x - 1$, and $n(x) = m(x) \cdot k(x)$. In this case, the a -value in $n(x)$ representing the vertical dilation of $k(x)$ is $m(x)$, or $(x - 1)$.

Since $n(x)$ is a vertical dilation of $k(x)$ by a linear function, $m(x)$, the resulting function is a second-degree function. Written as an equation, $n(x) = (x - 1)(-2x + 1)$, or $n(x) = -2x^2 + 3x - 1$.

The function $n(x)$ shares zeros with both $k(x)$ and $m(x)$.

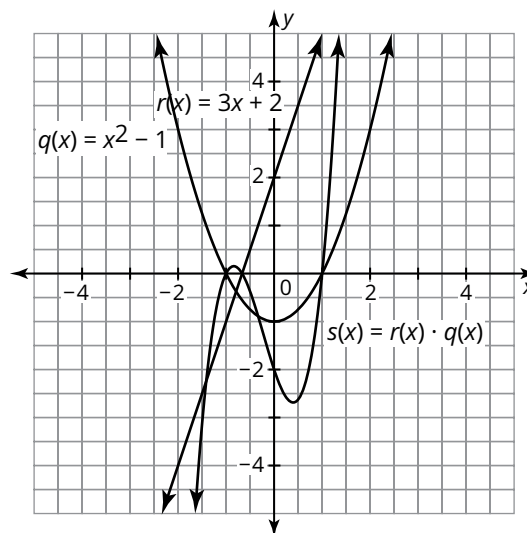


When a quadratic function is dilated vertically by multiplying the function by a linear function that is not a constant function, the resulting function is a third-degree function.

For example, consider the graphs of $q(x) = x^2 - 1$, $r(x) = 3x + 2$, and $s(x) = r(x) \cdot q(x)$. In this case, the a -value in $s(x)$ representing the vertical dilation of $q(x)$ is $r(x)$, or $(3x + 2)$.

Since $s(x)$ is a vertical dilation of $q(x)$ by a linear function, $r(x)$, the resulting function is a third-degree function. Written as an equation, $s(x) = (3x + 2)(x^2 - 1)$, or $s(x) = 3x^3 + 2x^2 - 3x - 2$.

The function $s(x)$ shares zeros with both $q(x)$ and $r(x)$.



When functions are written in factored form, the zeros are observable, and the intervals of increase and decrease are determined by the sign of the a -value.

The shape of the graph of functions can be predicted using observable characteristics of the equation when the equation is written in factored form. Somewhere between any two zeros of a function, the graph changes directions from increasing to decreasing or vice versa. The graph must change directions to intersect with the x-axis.

LESSON

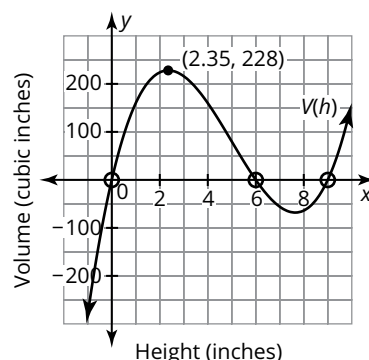
3

Exploring Cubic Functions

The formula for volume, $V = \ell \cdot w \cdot h$, can represent a cubic function when length and width are given in terms of height. A **cubic function** is a function that can be written in the general form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. A cubic function is a polynomial function of degree 3.

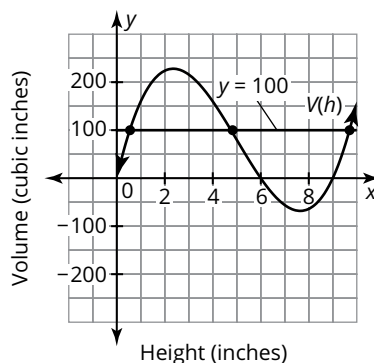
As the input values for height increase, the output values for volume approach infinity. Therefore, the function doesn't have a maximum. But, it can have a **relative maximum**, or a highest point in a particular section of a graph. Similarly, as the values for height decrease, the output values approach negative infinity. Therefore, a **relative minimum** can occur at the lowest point in that particular section of a graph.

For example, the function $V(h) = h(12 - 2h)(18 - 2h)$ models the volume of a planter box with height, h . The graph of the function is shown. The function has a relative maximum at $(2.35, 228)$ and a relative minimum at $(7.65, -68.16)$. The relative maximum can be interpreted in the context of the situation. A height of 2.35 inches will produce a prism with a maximum volume of 228 cubic inches.



A horizontal line is a powerful tool for working backwards to determine the possible values for height when the volume is known.

For example, consider a situation in which someone wants a planter box with a volume of 100 cubic inches. You can determine the possible heights for the planter from the graph of $V(h)$.



Draw a horizontal line at $y = 100$ and identify each point where $V(h)$ intersects with $y = 100$, or where $V(h) = 100$.

The points of intersection are $(0, 0.54)$, $(0, 4.76)$, and $(0, 9.7)$. The planter box can have a height of 0.54 inch, 4.76 inches, or 9.7 inches and have a volume of 100 cubic inches.

LESSON

4

Decomposing Cubic Functions

Recall that the zeros of a function are the points on the graph of the function where the function's value is zero. **Multiplicity** is how many times a particular number is a zero for a given function.

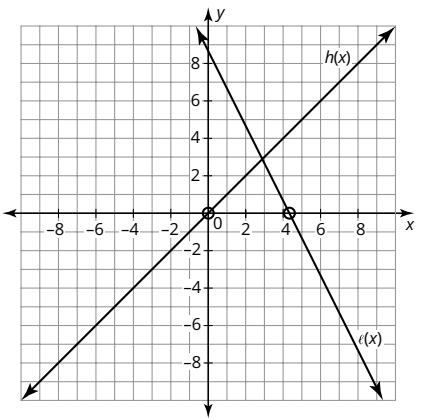
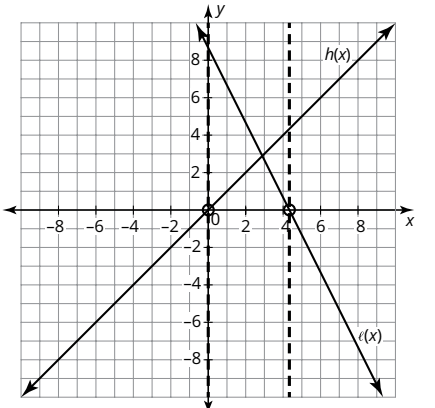
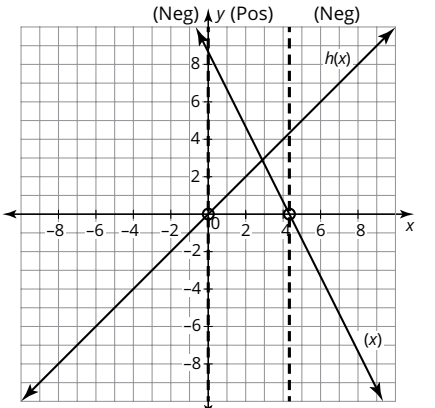
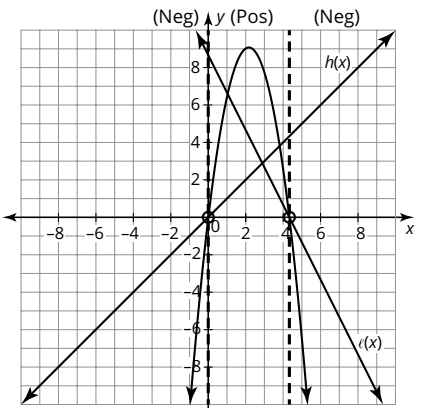
For example, for $f(x) = x - 5$, the value of the function f is 0 when $x = 5$. This zero has a multiplicity of 1. For $g(x) = (x - 3)(x - 3)$, the value of the function g is 0 when $x = 3$. This zero has a multiplicity of 2.

The zero product property can be used to identify when a factor is equal to 0. At the x -intercepts, the values of linear functions are zero, so the product of one linear function and another at a zero point will be zero.

Recall that the Fundamental Theorem of Algebra states that a degree- n polynomial has, counted with multiplicity, exactly n zeros. Cubic functions have exactly 3 zeros, with multiplicity.

You can use two linear functions to sketch their quadratic function product.

For example, consider the functions $\ell(x) = -2x + 8.5$ and $h(x) = x$. Use each linear function to sketch the graph of $A(x) = \ell(x) \cdot h(x)$.

Description	Graphical Display
Graph each factor as an individual function. The x-intercepts for each function are circled.	
Draw dashed vertical lines through the x-intercepts. The coordinate plane is now divided into 3 sections: $(-\infty, 0)$, $(0, 4.25)$, and $(4.25, \infty)$.	
Determine whether the output values for each function in the interval are positive or negative. - Values above the x-axis are positive. - Values below the x-axis are negative. - Determine the location of the quadratic function by calculating whether the product of the factors is positive or negative over each interval.	
- Use the x-intercepts and the sign of the output value over each interval to sketch the graph. - The new function will cross the x-axis with each of the x-intercepts becoming the factors. - The graph will increase or decrease depending on whether the output is positive or negative as it moves from one interval to the next.	

When multiplying three linear factors, the result is always a cubic function. First, choose two of the factors to multiply and combine like terms. Then, multiply that product with the remaining factor and combine like terms. The original expression and the new expression can be graphed to verify that they are equivalent.

For example, consider the cubic function, $V(x) = x(12 - 2x)(18 - 2x)$. Multiply the factors to rewrite it as a polynomial with degree 3.

$$\begin{aligned}x(12 - 3x)(18 - 2x) &= (12x - 3x^2)(18 - 2x) \\&= 216x - 24x^2 - 54x^2 + 6x^3 \\&= 6x^3 - 78x^2 + 216x\end{aligned}$$

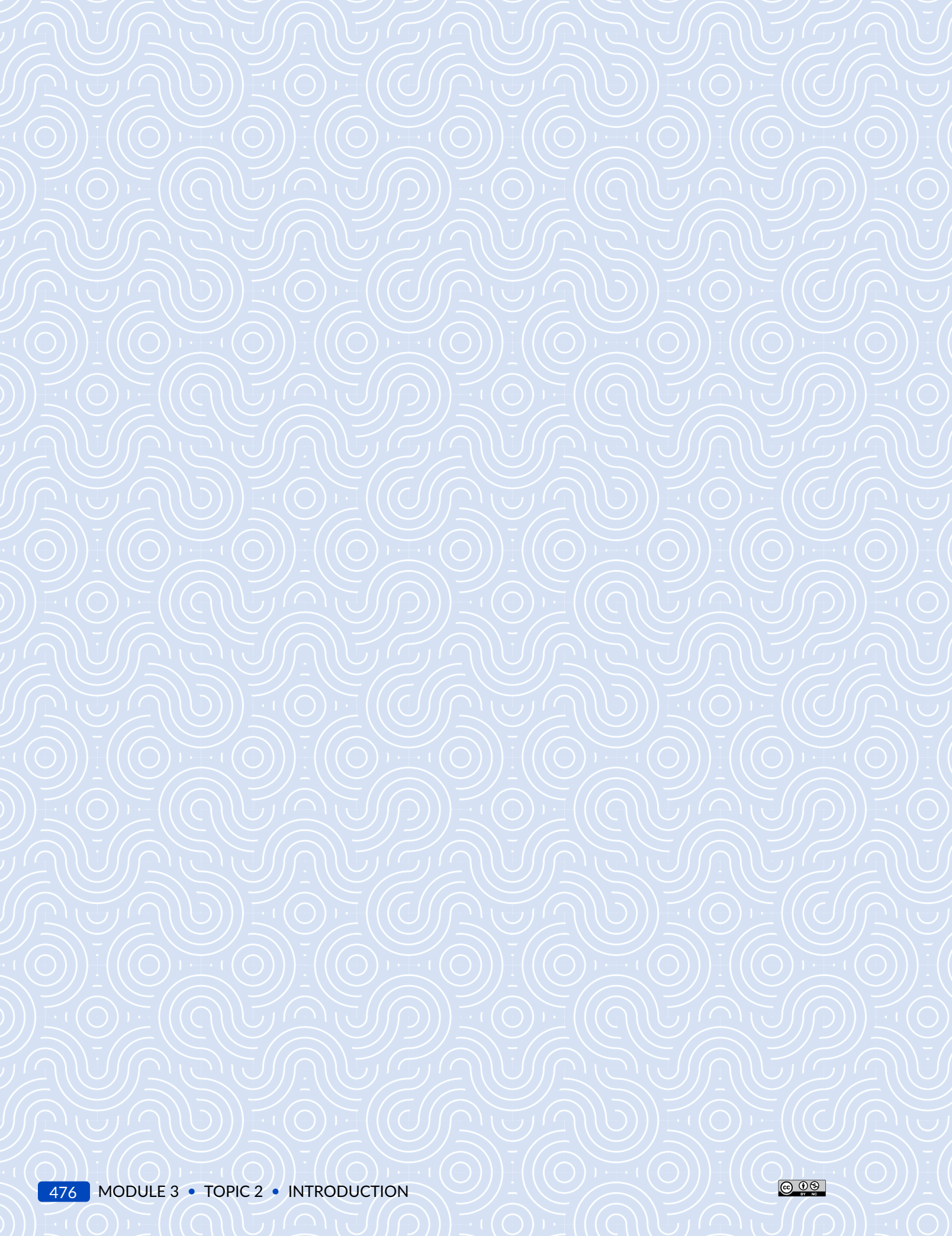
The product of a quadratic function and a linear function is also a cubic function.



Port Houston's Bayport Container Terminal is a key hub for global trade. As one of the busiest container ports in the U.S., Bayport efficiently handles thousands of 20- and 40-foot containers, connecting businesses to international markets.

Attributes of Cubic Functions

LESSON 1	Symmetric Functions	477
LESSON 2	Transformations of Cubic Functions	491
LESSON 3	Key Attributes of Cubic Functions	527



1

Symmetric Functions

OBJECTIVES

- Use graphs and transformations to determine the symmetry of a function.
- Determine whether a function is symmetric about a line or symmetric about a point based on a function equation or graph, and explain reasoning through writing.

KEY TERMS

- power function
- symmetric about a line
- symmetric about a point

.....

You have analyzed and built quadratic and cubic functions from functions of a lesser degree.

What are the characteristics of the graphs of power functions?

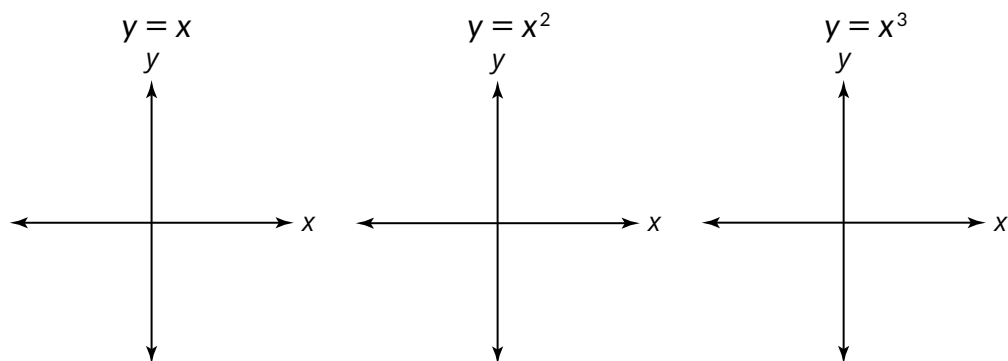
Getting Started

.....
When $a = 1$, the power function is also the parent function for the family of function.
.....

Power Functions

You have studied linear functions, quadratic functions, and cubic functions. A common type of polynomial function, which you have also studied, is a *power function*. A **power function** is a function of the form $P(x) = ax_n$, where n is a non-negative integer.

1. Sketch a graph of each power function.

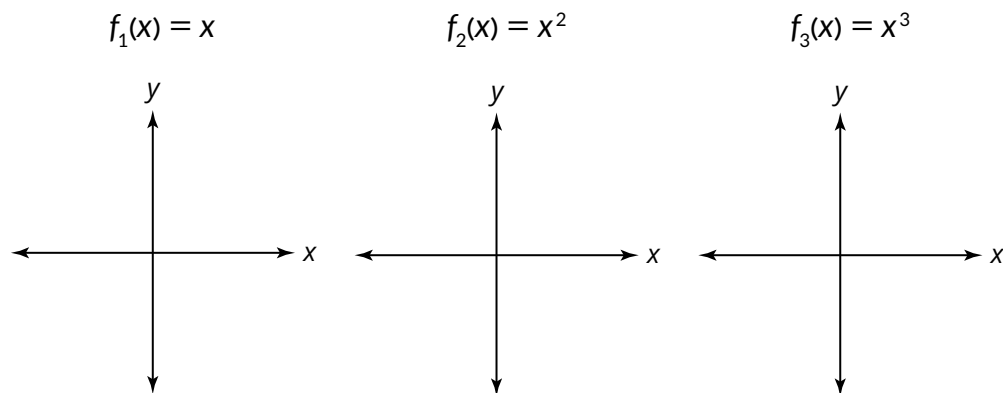


2. For each graph:
Identify the key attributes of each function, including any maximum or minimum values if applicable.

Exploring Symmetry

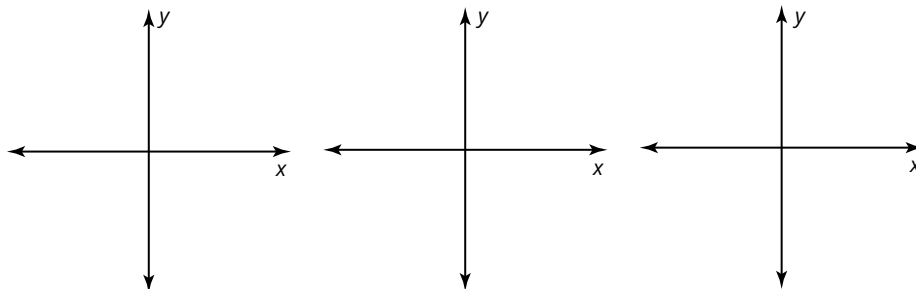
In this activity, you will make connections between reflections and the symmetry of the graph of the function.

1. Consider the sequence of graphs shown.



- a. What happens when the a -value is negative? Write each function in terms of x , and then sketch it.

$$-f_1(x) = \underline{\hspace{2cm}} \qquad -f_2(x) = \underline{\hspace{2cm}} \qquad -f_3(x) = \underline{\hspace{2cm}}$$



Remember ...

Changes to the a -value in the transformation form of a function can cause a vertical dilation of the function and/or a reflection across the x -axis.

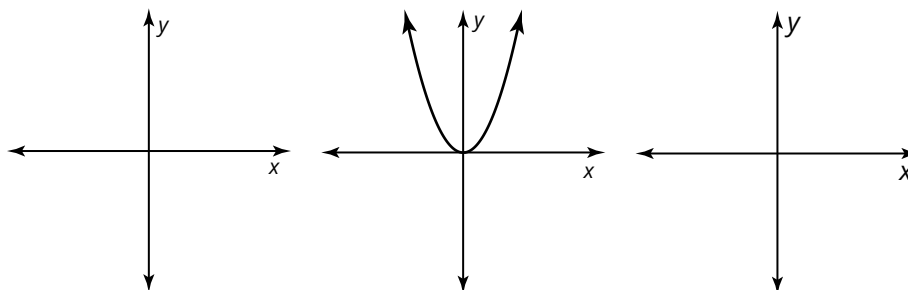
Changes to the b -value of a function can cause a horizontal dilation of the function and/or a reflection across the y -axis.

- b. What happens when the b -value is negative? Write each function in terms of x , and then sketch it.

$$f_1(-x) = \underline{\hspace{2cm}}$$

$$f_2(-x) = \underline{\hspace{2cm}}$$

$$f_3(-x) = \underline{\hspace{2cm}}$$

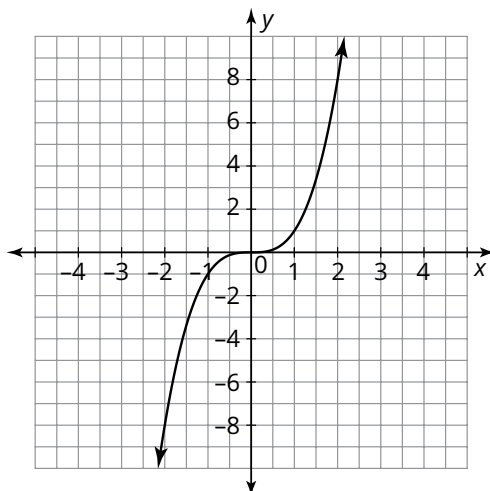


If a graph is **symmetric about a line**, the line divides the graph into two identical parts. Special attention is given to the line of symmetry when it is the y -axis.

2. Determine which graph(s) from Question 1 are symmetric about a line. What is the line of symmetry?



3. Analyze the graph shown.



Parker says that the graph has no line of symmetry because if she reflected the graph across the x - or y -axis, it would not be a mirror image.

Carlos says that the graph has no line of symmetry because if he looks at the x -values 1 and -1 , the y -values are not the same, so there can't be symmetry about the y -axis. Also, if he looks at the y -values 8 and -8 , the x -values are not the same, so there can't be symmetry about the x -axis.

Isaiah said that there is some type of symmetry. He looks at the point $(2, 8)$ and notices that the point $(-2, -8)$ is also on the graph. Likewise, he looks at the point $(1, 1)$ and notices that the point $(-1, -1)$ is also on the graph. He concluded that it must have a reflection across the x - and y -axis at the same time.

Who's correct? Explain your reasoning.

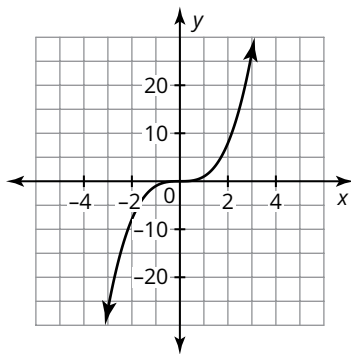
.....
Review the definition of **explain your reasoning** in the Academic Glossary.
.....

The graph of the parent function $f(x) = x^3$ function is *symmetric about a point*, in particular the origin. A function is **symmetric about a point** if each point on the graph has a point the same distance from the central point, but in the opposite direction. When the point of symmetry is the origin, the graph is reflected across the x -axis and the y -axis. If you replace both (x, y) with $(-x, -y)$, the function remains the same.

WORKED EXAMPLE

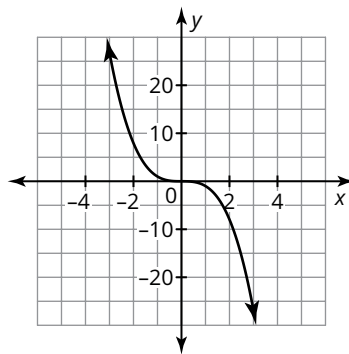
Consider the function $f(x) = x^3$. You can think of the point of symmetry about the origin as a double reflection.

$$f_1(x) = x^3$$



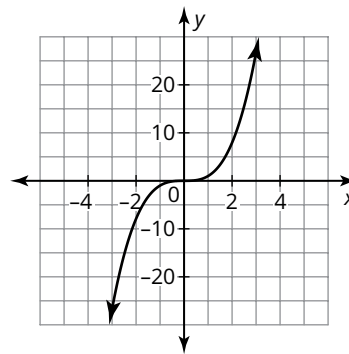
The function $f_1(x)$ is shown.

$$f_2(x) = f_1(-x) \\ = (-x)^3$$



The function $f_1(x)$ is reflected across the y -axis to produce f_2 .

$$f_3(x) = -f_2(x) \\ = -((-x)^3) \\ = x^3$$



The function $f_2(x)$ is reflected across the x -axis to produce f_3 .

If a graph is symmetric about the y -axis, then $f(x) = f(-x)$.

If a graph is symmetric about the origin, then $f(x) = -f(-x)$.

- Use what you know about a - and b -value transformations to describe symmetry about the y -axis and symmetry about a point in your own If a graph is symmetric about the origin, then.

5. Explain why Nahimana is correct. Use function notation to write her conclusion.

Nahimana

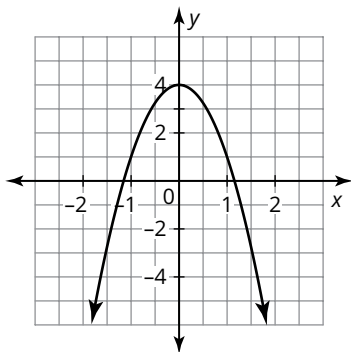


The reflection of $f(x) = x^3$ across the y -axis produces the same graph as its reflection across the x -axis.

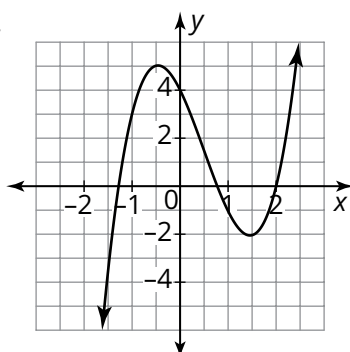
You determine algebraically whether a function is symmetric about the y -axis by evaluating the function at $f(-x)$. If $f(-x) = f(x)$, then the function is symmetric about the y -axis. You can also evaluate the function at $-f(x)$. If $-f(-x) = f(x)$, or $-f(x) = f(-x)$, then the function has symmetry about the origin.

6. State whether the graph of each function shown is symmetric about a line or a point.

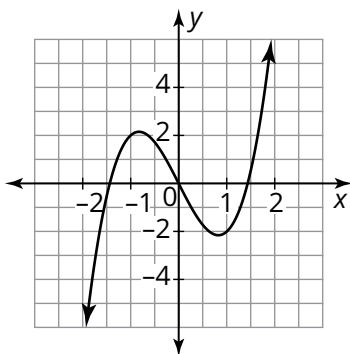
a.



b.



c.



Ask Yourself . . .

What observations can you make?



Talk the Talk

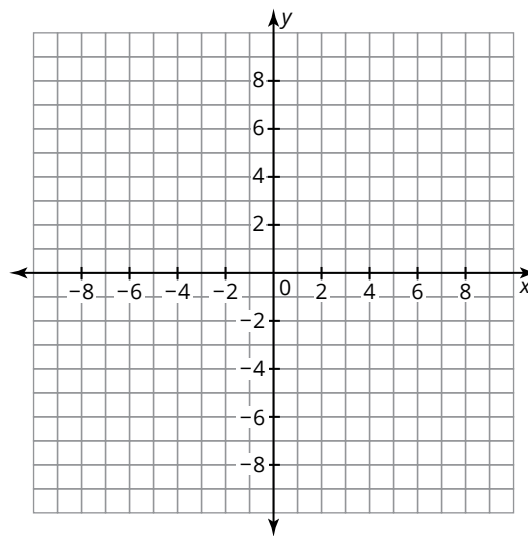
Symmetry in Motion

1. Graph and label each function.

$$f(x) = x^3$$

$$g(x) = (x - 3)^3$$

$$h(x) = 3x^3$$



2. Describe the key attributes of each function. Be sure to include whether the graph is symmetric about a point, symmetric about a line, or neither.

Lesson 1 Assignment

Write

Choose the term from the box that best completes each statement.

symmetric about a point power function symmetric about a line

1. A function is _____ if the line divides the graph into two identical parts.
2. A function is _____ if each point on the graph has a point the same distance from the central point but in the opposite direction.
3. A(n) _____ is a function of the form $P(x) = ax^n$, where n is a non-negative integer.

Remember

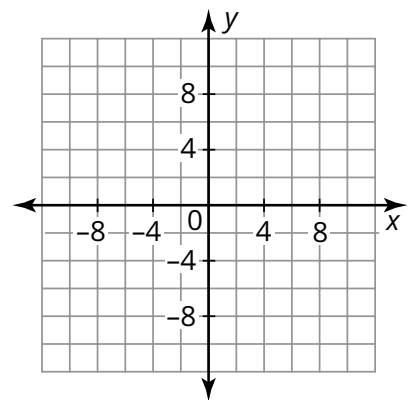
A power function is a function of the form $P(x) = ax^n$, where n is a non-negative integer.

If a graph is symmetric about the y -axis, then $f(x) = f(-x)$. If a graph is symmetric about the origin, then $f(x) = -f(-x)$.

Practice

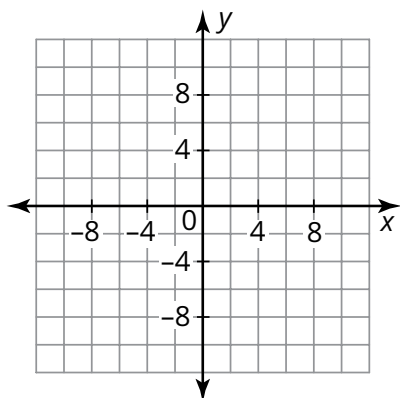
1. Graph each function. Determine whether the function is symmetric about a line, a point, or neither.

a. $f(x) = x^3 + x$

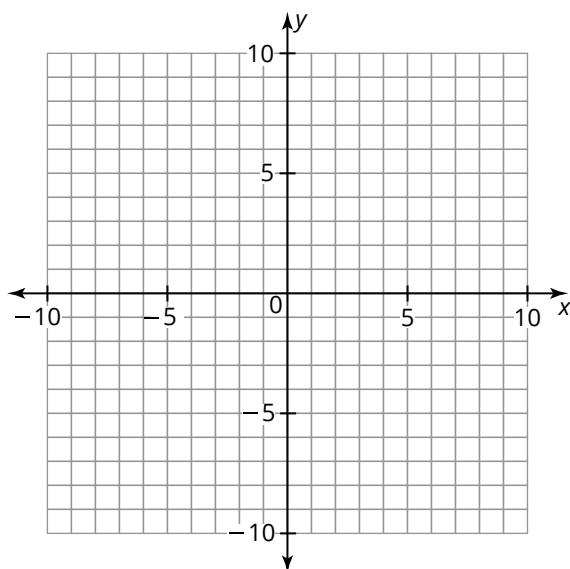


Lesson 1 Assignment

b. $f(x) = x^3 + x^2 - 6x$

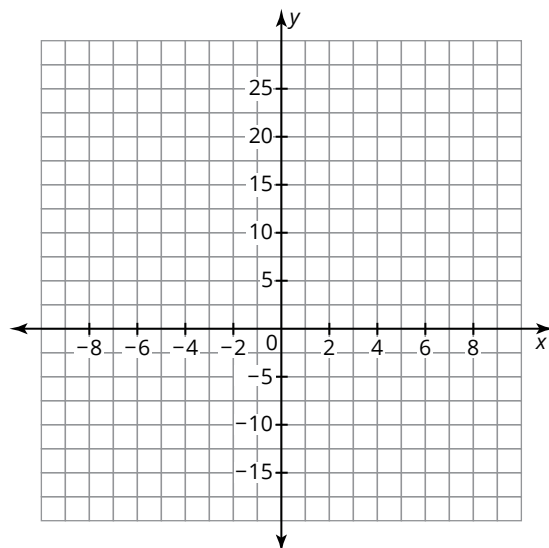


c. $f(x) = x^3 - 4x + 3$



Lesson 1 Assignment

d. $f(x) = 2x^2 + 9$



Lesson 1 Assignment

Prepare

1. Describe the graph of $y = x^2$.

2. Describe the graph of $y = x^3$.

2

Transformations of Cubic Functions

OBJECTIVES

- Dilate and translate cubic functions.
- Use power functions to build cubic functions.
- Explore the possible graphs of cubic functions and use precise mathematical language to describe their key features.

KEY TERM

- polynomial function

You have transformed linear, quadratic, and absolute value functions.

How do these transformations compare to transforming a cubic function?

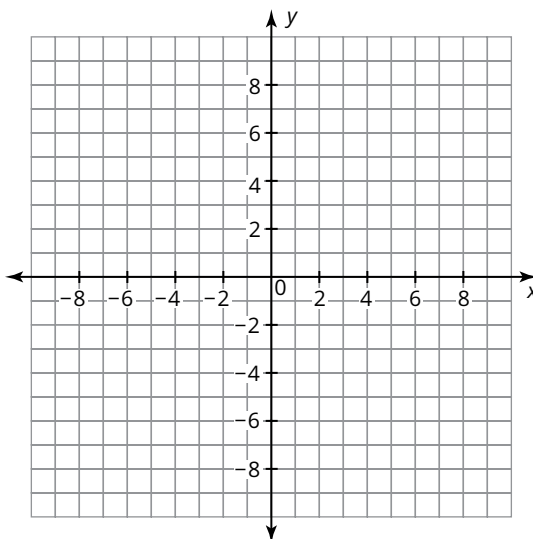
Getting Started

Applying Some Foundation

Let's consider a set of key points and the property of symmetry to graph the parent cubic function, $f(x) = x^3$.

1. Complete the table for the given key points. Then, graph the points on the coordinate plane shown.

x	$f(x) = x^3$
0	
1	
2	



.....
The graph of the parent cubic function is symmetric about the origin. So, $f(x) = -f(-x)$.
.....

2. Use the property of symmetry to determine two points other than the key points. Then, use these points to graph the parent cubic function on the coordinate plane shown.

3. Identify the key attributes of the cubic parent function.

- a. Domain using interval notation:

b. Range using interval notation:

c. x -intercept(s):

d. y -intercept(s):

e. Maximum or minimum:

f. Symmetry:

ACTIVITY 2.1

Sorting Transformations of Polynomial Functions

Linear, quadratic, and cubic functions are all examples of *polynomial functions*. A **polynomial function** is a function that can be written in the form

$$\blacksquare x^n + \blacksquare x^{n-1} + \dots \blacksquare x^2 + \blacksquare x + \blacksquare$$

In a polynomial function, the coefficients, represented by each $\blacksquare$, are complex numbers and the exponents of the variables are nonnegative integers.

You know that a polynomial function of degree 3 is a cubic function.

Transformations performed on any polynomial function $f(x)$ to form a new function $g(x)$ can be described by the transformational function:

$$g(x) = af(b(x - c)) + d$$

Recall that the constants a and d affect the outside of the function (the output values). For example, if $a = 2$, then you can multiply each y -coordinate of $f(x)$ by 2 to determine the y -coordinates of $g(x)$. The constants b and c affect the inside of the function (the input values). For example, if $b = 2$, then you can multiply each x -coordinate of $f(x)$ by $\frac{1}{2}$ to determine the x -coordinates of $g(x)$.

Function Form	Equation Information	Description of Transformation of Graph
$y = af(x)$	$ a > 1$	vertical stretch of the graph by a factor of a units
	$0 < a < 1$	vertical compression of the graph by a factor of a units
	$a < 0$	reflection across the x -axis
$y = f(bx)$	$ b > 1$	compressed horizontally by a factor of $\frac{1}{ b }$
	$0 < b < 1$	stretched horizontally by a factor of $\frac{1}{ b }$
	$b < 0$	reflection across the y -axis
$y = f(x - c)$	$c > 0$	horizontal shift right c units
	$c < 0$	horizontal shift left c units
$y = f(x) + d$	$d > 0$	vertical shift up d units
	$d < 0$	vertical shift down d units

1. Cut out the Function Transformation Cards located at the end of the lesson. Match the function representation with its corresponding coordinate pair. For each function representation or coordinate pair that does not have a match, provide a match by writing the appropriate equation or coordinate pair on a blank card.
2. Explain why Lucas's reasoning is not correct. Rewrite the transformation to identify the correct b - and c -values.

Lucas



I can tell from the expression that the transformation given by $f(2x + 5)$ has a b -value of 2 and a c -value of 5.

3. Complete the table to show the coordinates of $g(x) = af(b(x - c)) + d$ after each type of transformation performed on $f(x)$.

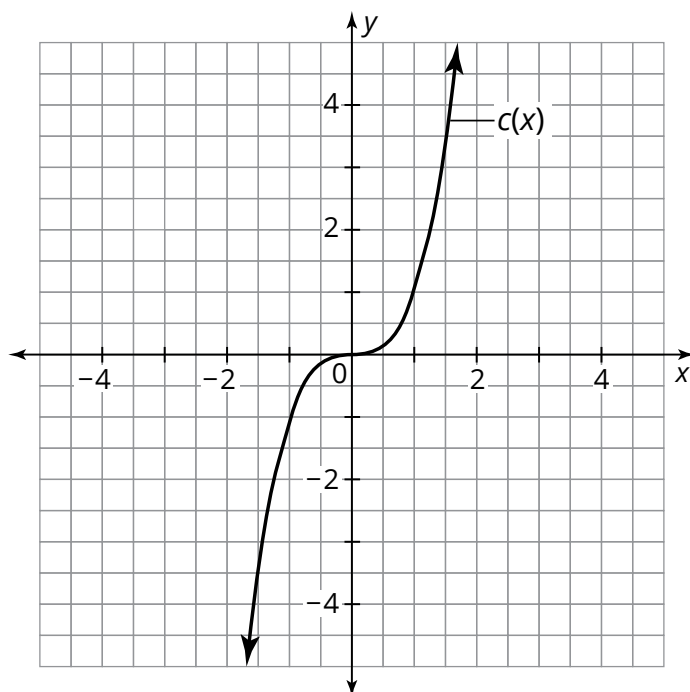
Type of Transformation Performed on $f(x)$	Coordinates of $f(x) \rightarrow$ Coordinates of $g(x)$
Vertical Stretch or Compression Dilation by a Factor of a	$(x, y) \rightarrow (\text{—————}, \text{—————})$
Horizontal Stretch or Compression Dilation by a Factor of b	$(x, y) \rightarrow (\text{—————}, \text{—————})$
Horizontal Translation of c units	$(x, y) \rightarrow (\text{—————}, \text{—————})$
Vertical Translation of d units	$(x, y) \rightarrow (\text{—————}, \text{—————})$
All four transformations: $a, b, c,$ and d	$(x, y) \rightarrow (\text{—————}, \text{—————})$

ACTIVITY 2.2

Building and Sketching Cubic Functions

Let's use what you know about function transformations to graph transformations of the cubic parent function, $c(x) = x^3$.

- The graph of the parent cubic function $c(x) = x^3$ is shown.



- Suppose that $g(x) = 2c(x)$. Use reference points and properties of symmetry to complete the table of values for $g(x)$. Then, graph and label $g(x)$ on the same coordinate plane as $c(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $g(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

- Suppose that $h(x) = \frac{1}{2}c(x)$. Use reference points and properties of symmetry to complete the table of values for $h(x)$. Then, graph and label $h(x)$ on the same coordinate plane as $c(x)$ and $g(x)$.

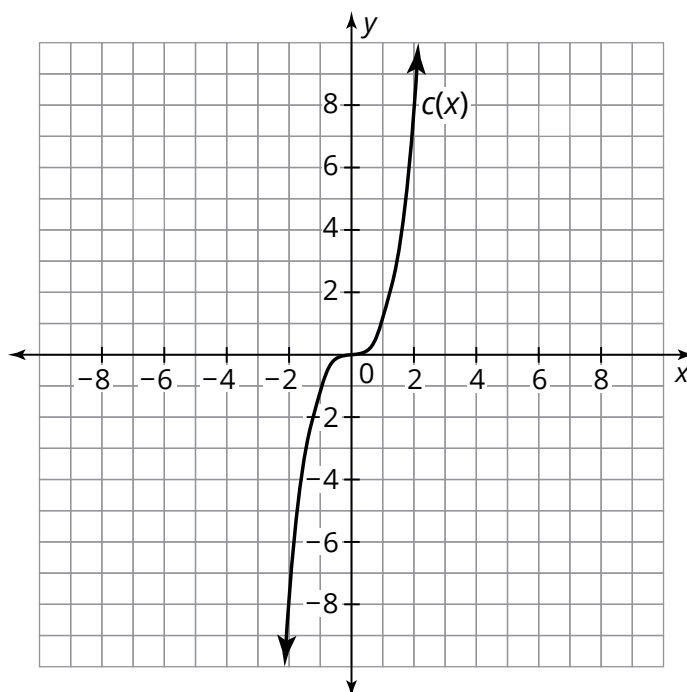
Reference Points on $c(x)$	→	Corresponding Points on $h(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

- c. Suppose that $j(x) = -2c(x)$. Use reference points and properties of symmetry to complete the table of values for $j(x)$. Then, graph and label $j(x)$ on the same coordinate plane as $c(x)$, $g(x)$ and $h(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $j(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

- d. Describe the symmetry of $g(x)$, $h(x)$, and $j(x)$. How does the symmetry of $g(x)$, $h(x)$, and $j(x)$ compare to the symmetry of $c(x)$?

2. The graph of the parent cubic function $c(x) = x^3$ is shown.



- a. Suppose that $u(x) = c(2x)$. Use reference points and properties of symmetry to complete the table of values for $u(x)$. Then, graph and label $u(x)$ on the same coordinate plane as $c(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $u(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

- b. Suppose that $v(x) = c\left(\frac{1}{2}x\right)$. Use reference points and properties of symmetry to complete the table of values for $v(x)$. Then, graph and label $v(x)$ on the same coordinate plane as $c(x)$ and $u(x)$.

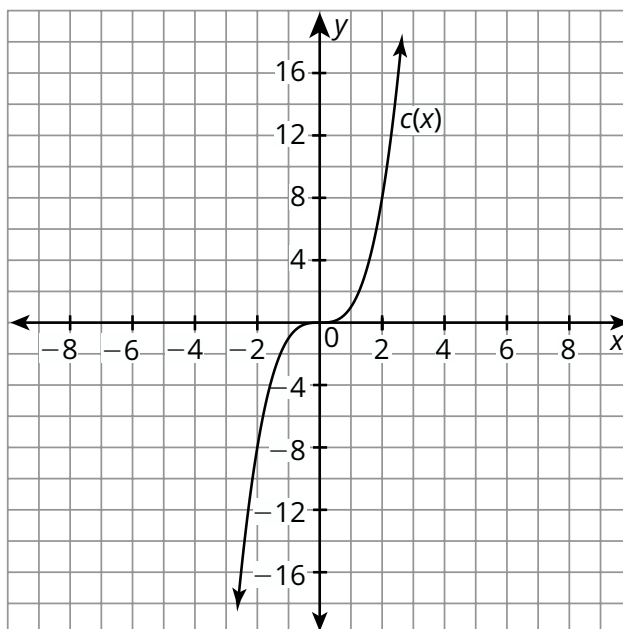
Reference Points on $c(x)$	→	Corresponding Points on $v(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

- c. Suppose that $w(x) = c(-2x)$. Use reference points and properties of symmetry to complete the table of values for $w(x)$. Then, graph and label $w(x)$ on the same coordinate plane as $c(x)$, $u(x)$, and $v(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $w(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

- d. Describe the symmetry of $u(x)$, $v(x)$, and $w(x)$. How does the symmetry of $u(x)$, $v(x)$, and $w(x)$ compare to the symmetry of $c(x)$?

3. The graph of the parent cubic function $c(x) = x^3$ is shown.



- a. Suppose that $m(x) = c(x - 5)$. Use reference points and properties of symmetry to complete the table of values for $m(x)$. Then, graph and label $m(x)$ on the same coordinate plane as $c(x)$.

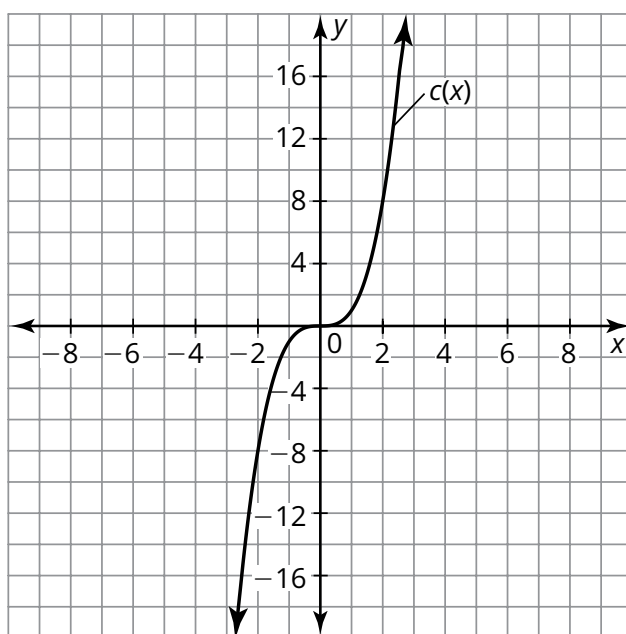
Reference Points on $c(x)$	→	Corresponding Points on $m(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

- b. Suppose that $n(x) = c(x + 5)$. Use reference points and properties of symmetry to complete the table of values for $n(x)$. Then, graph and label $n(x)$ on the same coordinate plane as $c(x)$ and $m(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $n(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

- c. Describe the symmetry of $m(x)$ and $n(x)$. How does the symmetry of $m(x)$ and $n(x)$ compare to the symmetry of $c(x)$?

4. The graph of the parent cubic function $c(x) = x^3$ is shown.



- a. Suppose that $j(x) = c(x) + 5$. Use reference points and properties of symmetry to complete the table of values for $j(x)$. Then, graph and label $j(x)$ on the same coordinate plane as $c(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $j(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

- b. Suppose that $k(x) = c(x) - 5$. Use reference points and properties of symmetry to complete the table of values for $k(x)$. Then, graph and label $k(x)$ on the same coordinate plane as $c(x)$ and $j(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $k(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

- c. Describe the symmetry of $j(x)$ and $k(x)$. How does the symmetry of $j(x)$ and $k(x)$ compare to the symmetry of $c(x)$?

5. Complete the table to summarize the effects that transformations have on the parent cubic function $c(x) = x^3$. The first row has been completed for you.

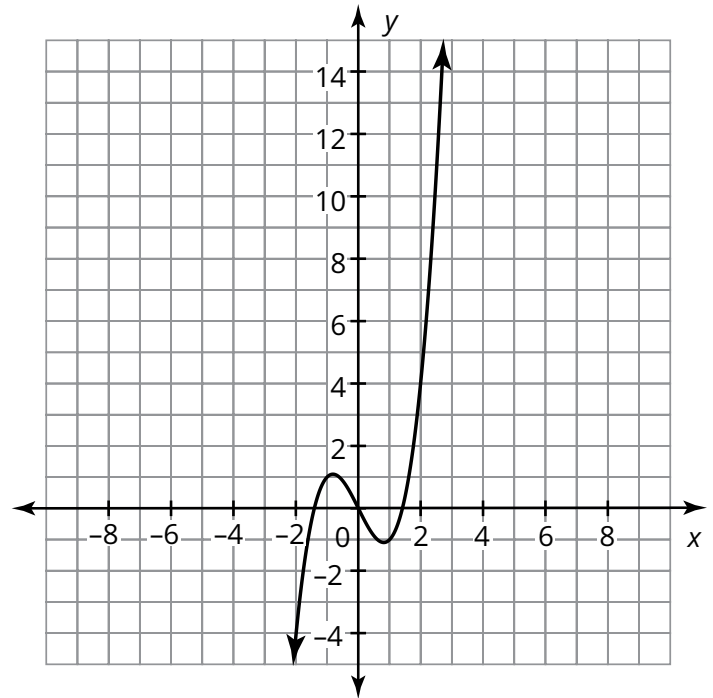
Effects of Rigid Motions on the Basic Cubic Function $c(x) = x^3$		
Rigid Motion	New Transformed Function $p(x)$ in Terms of $c(x)$	Description of Symmetry of $p(x)$
Vertical Stretch Dilation	$p(x) = ac(x)$, $ a > 1$	Symmetric about the point $(0, 0)$
Vertical Compression Dilation		
Horizontal Stretch Dilation		
Horizontal Compression Dilation		
Reflection across x-axis		
Reflection across y-axis		
Vertical Translation		
Horizontal Translation		

Let's use what you know about function transformations to graph polynomials.

6. The graph of the function $f(x) = x^3 - 2x$ is shown.

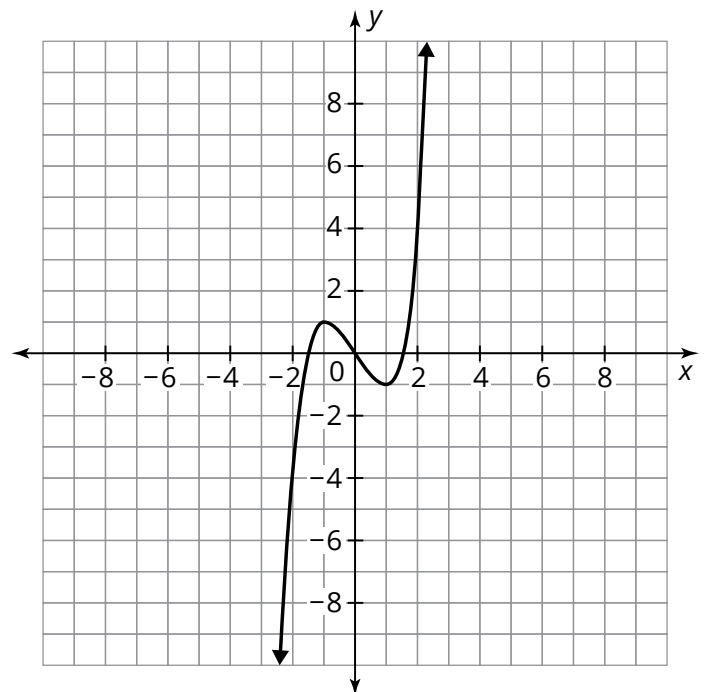
- a. Suppose that $g(x) = 2f(x)$. Use the given reference points to complete the table of values for $g(x)$. Then, use properties of symmetry to graph and label $g(x)$ on the same coordinate plane as $f(x)$.

Reference Point on $f(x)$	Corresponding Point on $g(x)$
(0, 0)	
(1, -1)	
(2, 4)	



- b. Suppose that $h(x) = -2f(x)$. Use the given reference points to complete the table of values for $h(x)$. Then, use properties of symmetry to graph and label $h(x)$ on the same coordinate plane as $f(x)$.

Reference Point on $f(x)$	Corresponding Point on $h(x)$
(0, 0)	
(1, -1)	
(2, 4)	

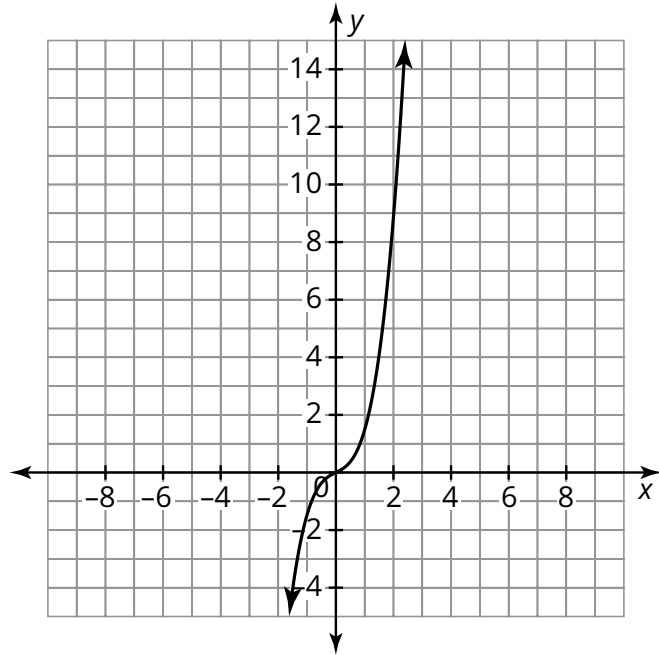


7. The graph of the function $d(x) = x^3 + \frac{1}{2}x$ is shown.

a. Suppose that $j(x) = d(2x)$.

Use the given reference points to complete the table of values for $j(x)$. Then, use properties of symmetry to graph and label $j(x)$ on the same coordinate plane as $d(x)$.

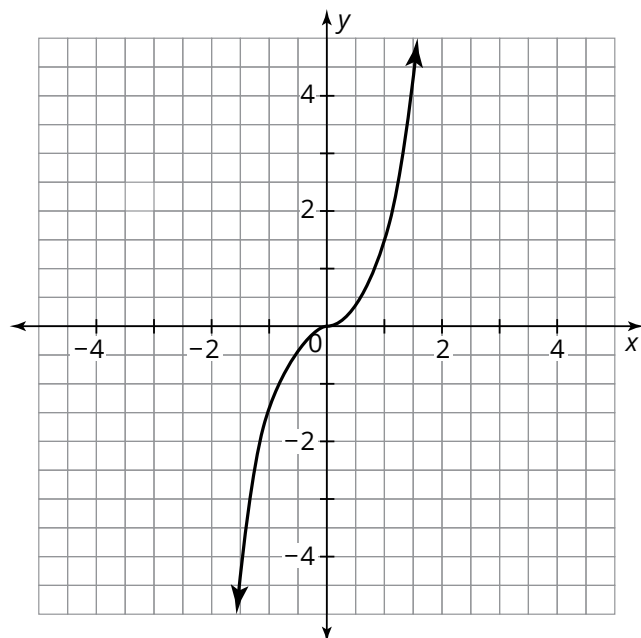
Reference Points on $d(x)$	Corresponding Points on $j(x)$
$(0, 0)$	
$(1, \frac{3}{2})$	
$(2, 9)$	



b. Suppose that $k(x) = d(-2x)$.

Use reference points to complete the table of values for $k(x)$. Then, use properties of symmetry to graph and label $k(x)$ on the same coordinate plane as $d(x)$.

Reference Points on $d(x)$	Corresponding Points on $k(x)$
$(0, 0)$	
$(1, \frac{3}{2})$	
$(2, 9)$	



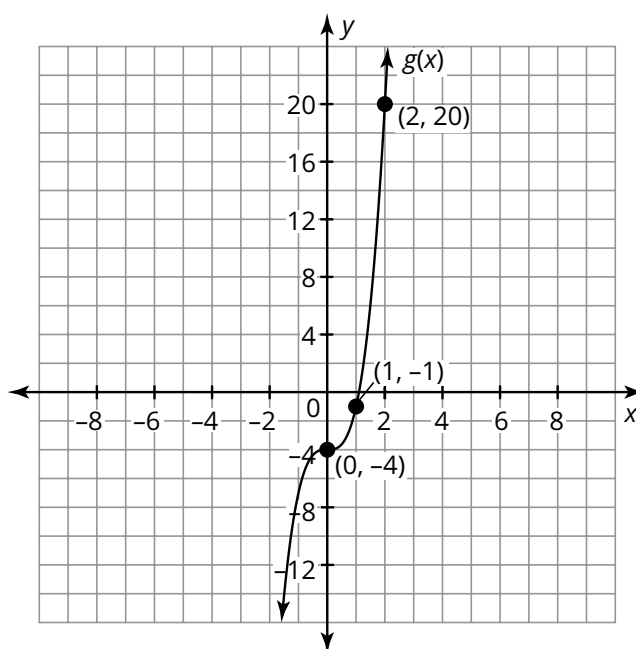
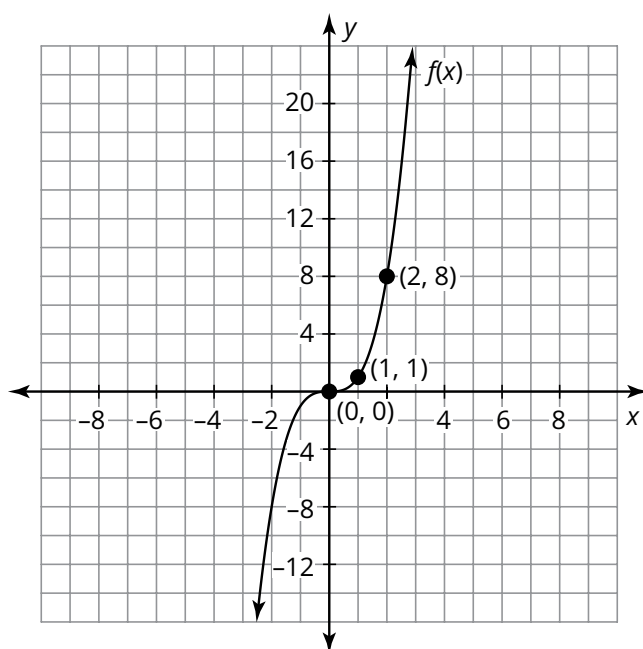
ACTIVITY 2.3

Determining Multiple Transformations of Polynomial Functions

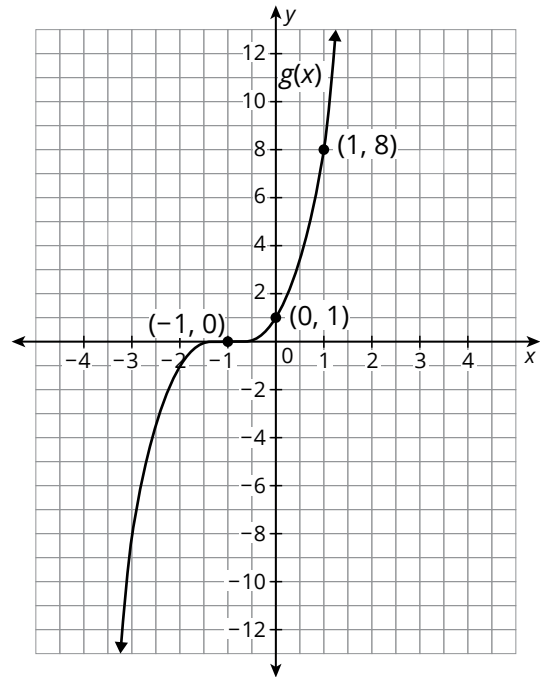
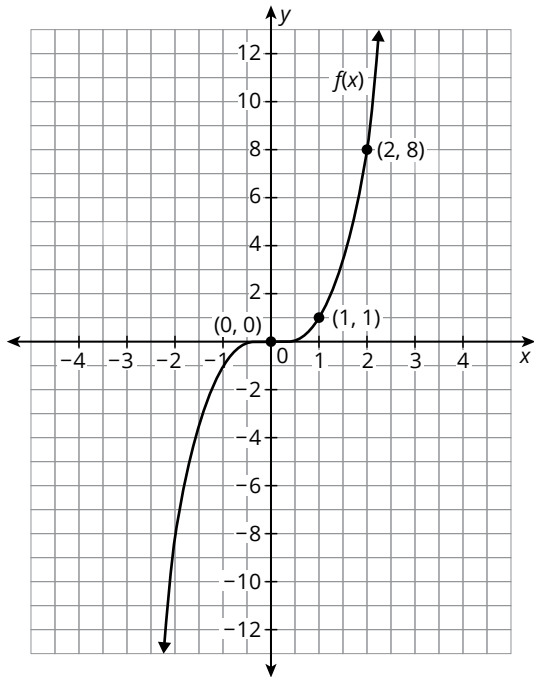
You have graphed transformations of functions. In this activity, you will interpret the transformation given in the graph.

1. Analyze the graphs of $f(x)$ and $g(x)$. For each reference point on $f(x)$, the corresponding points after the transformations are shown on $g(x)$. Describe the transformations performed on $f(x)$ to create $g(x)$. Then, write an equation for $g(x)$ in terms of $f(x)$.

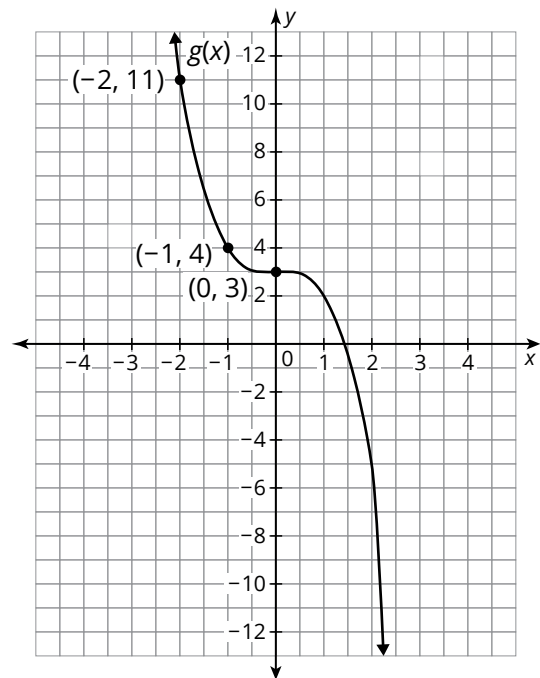
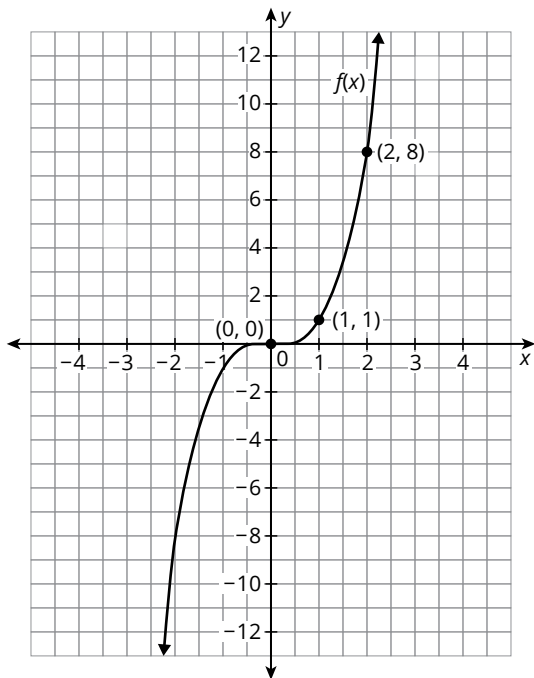
a. $g(x) =$ _____



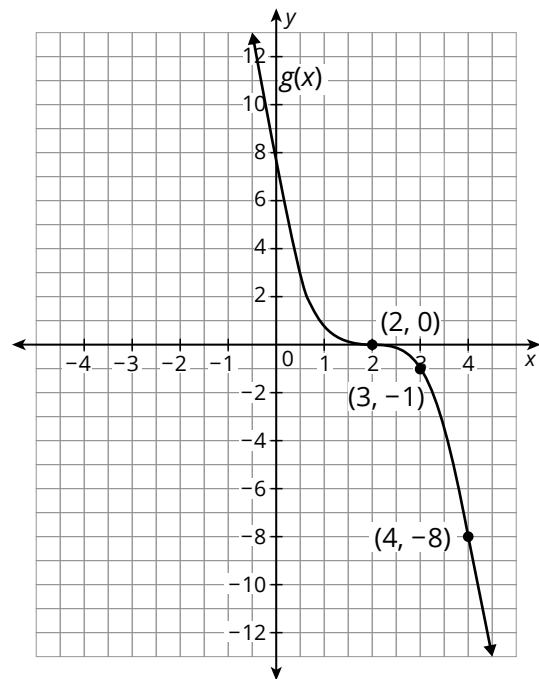
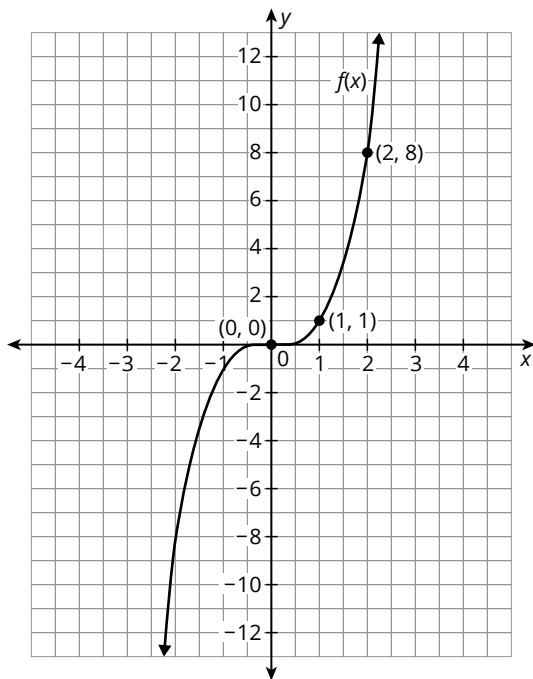
b. $g(x) =$ _____



c. $g(x) =$ _____



d. $g(x) =$ _____



2. The equation for a cubic function $c(x)$ is given. The equation for the transformed function $t(x)$ in terms of $c(x)$ is also given. Describe the transformation(s) performed on $c(x)$ that produced $t(x)$. Then, write an equation for $t(x)$ in terms of x .

a. $c(x) = x^3$
 $t(x) = 0.5c(-x)$

b. $c(x) = x^2$
 $t(x) = 2c(x + 3)$

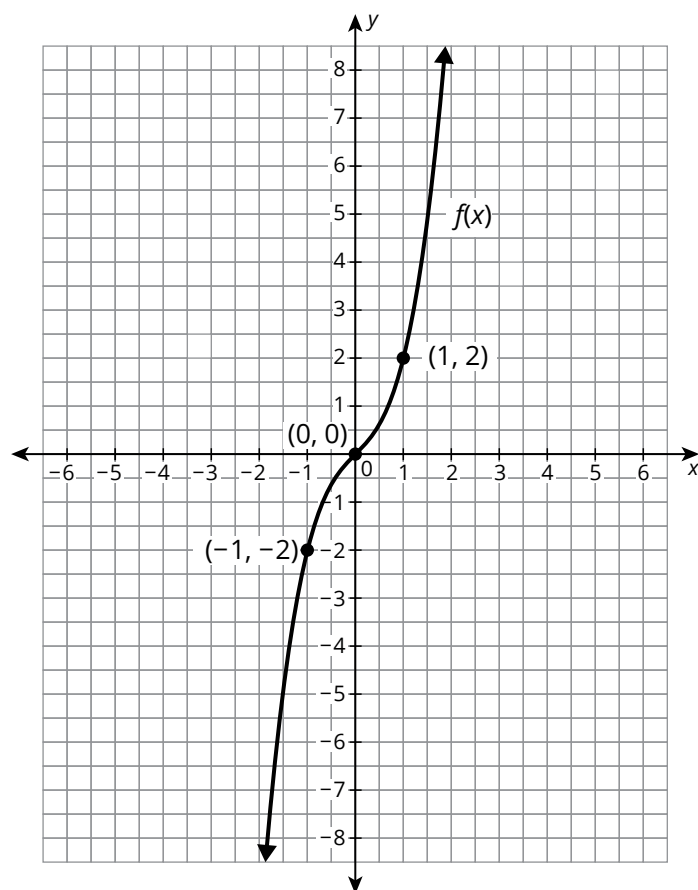
c. $c(x) = x^3 + 2$
 $t(x) = -c(x - 2) + 4$

d. $c(x) = x^3 - 1$
 $t(x) = c(3x) - 4$

Talk the Talk

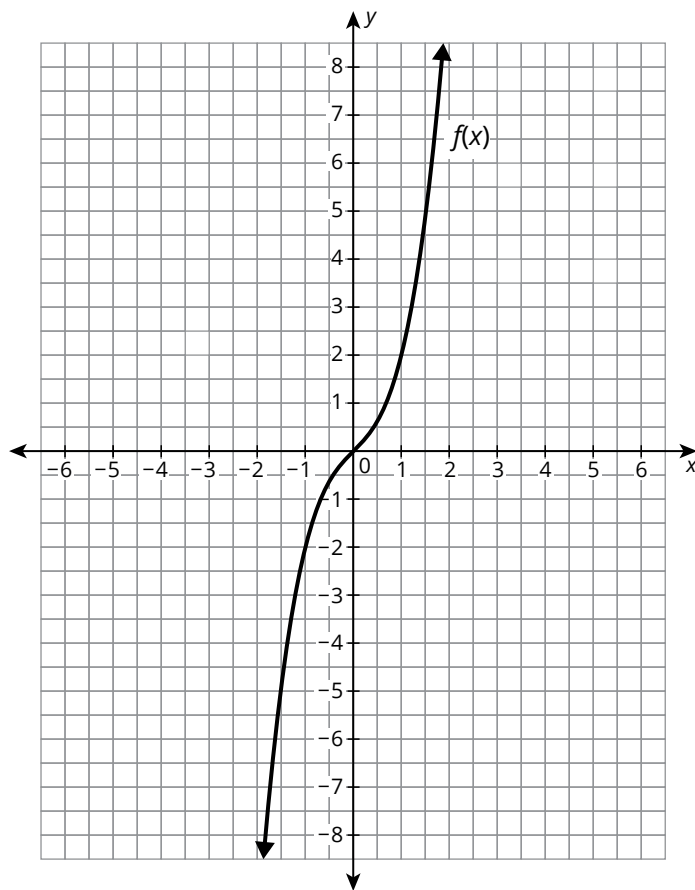
Just a Few Highlights

1. The graph of the function $f(x) = x^3 + x$ is shown. Use the given reference points to complete the table of values for the transformed function. Use the properties of symmetry to graph and label the transformed function on the same coordinate plane as $f(x)$. Then, describe the transformations performed on $f(x)$ to create the transformed function.



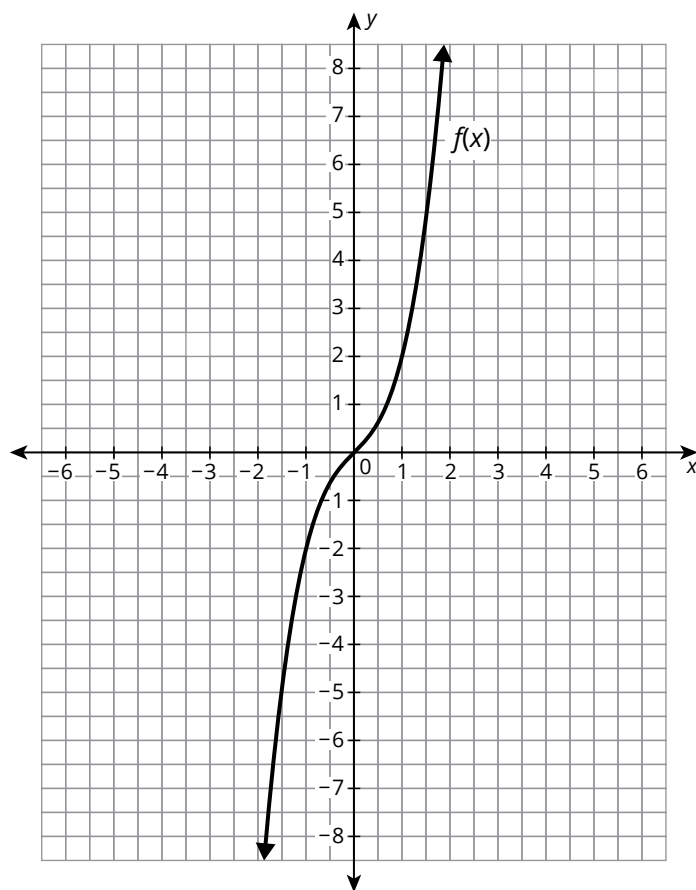
a. $g(x) = 3f(x)$

Reference Points on $f(x)$	Corresponding Points on $g(x)$
(0, 0)	
(1, 2)	
(-1, -2)	



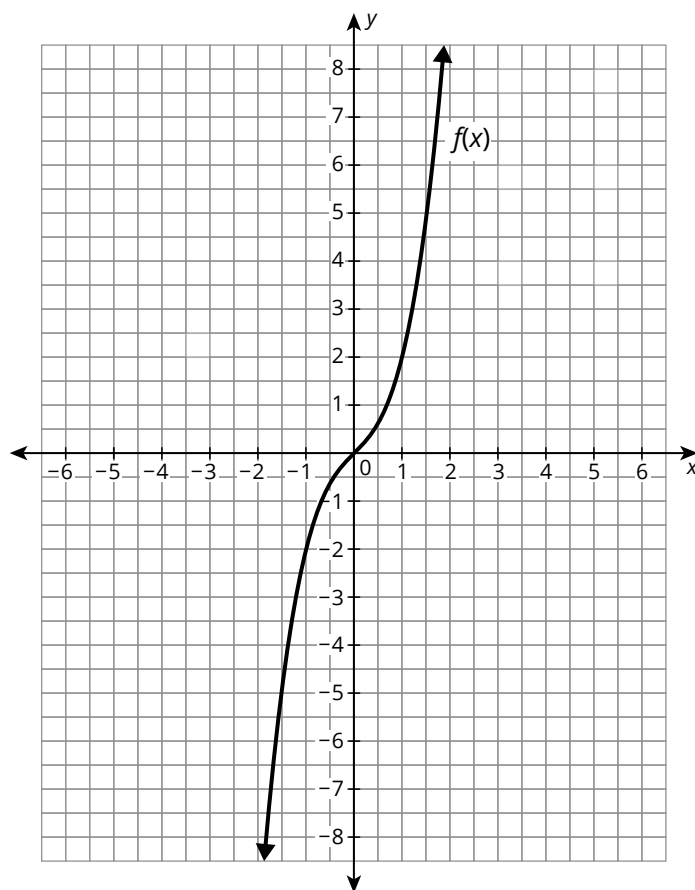
b. $h(x) = -\frac{1}{3}f(x)$

Reference Points on $f(x)$	Corresponding Points on $h(x)$
(0, 0)	
(1, 2)	
(-1, -2)	

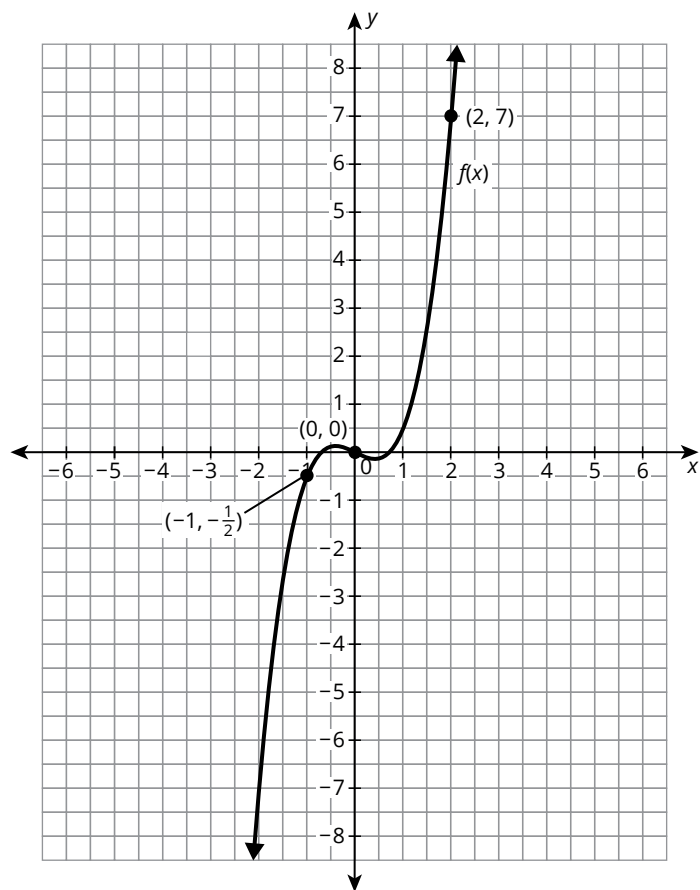


c. $k(x) = f(x) + 3$

Reference Points on $f(x)$	Corresponding Points on $k(x)$
(0, 0)	
(1, 2)	
(-1, -2)	

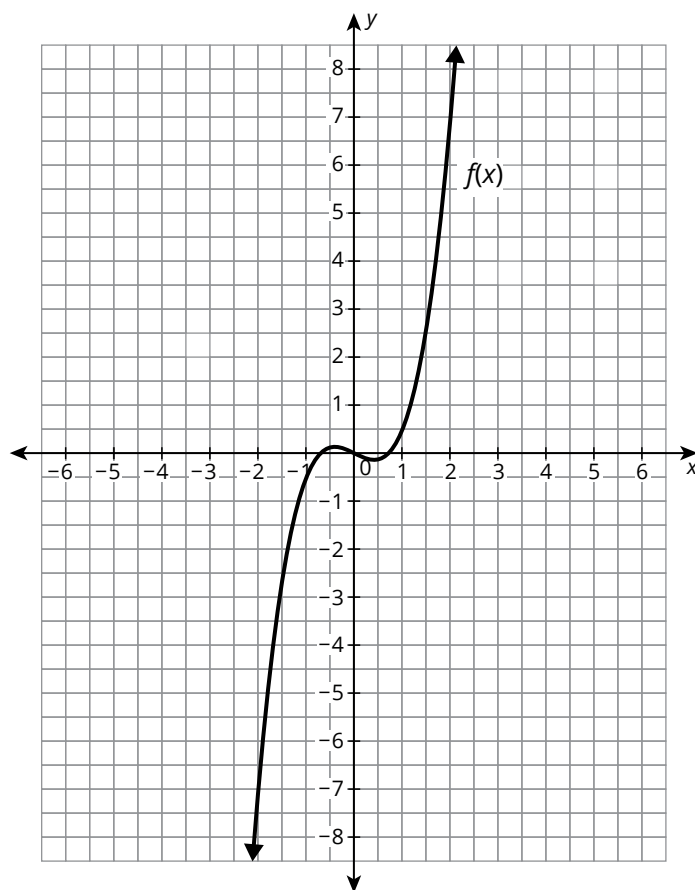


2. The graph of the function $f(x) = x^3 - \frac{1}{2}x$ is shown. Complete the table of values for the transformation of $f(x)$. Use the properties of symmetry to graph and label the transformed function on the same coordinate plane as $f(x)$. Then, describe the transformations performed on $f(x)$ to create the transformed function.



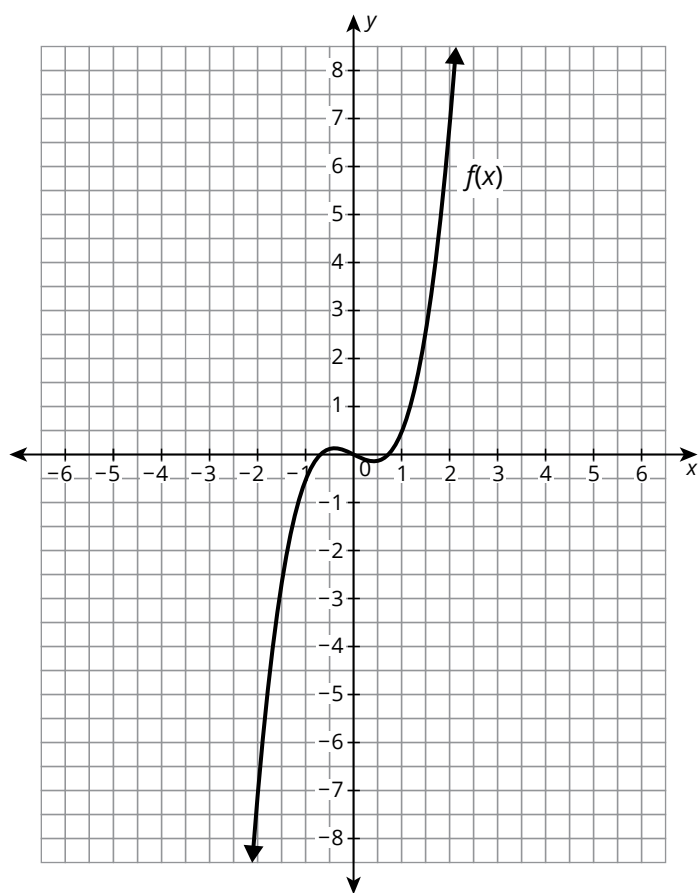
a. $j(x) = 2f(x + 1)$

Reference Points on $f(x)$	Corresponding Points on $j(x)$
$(0, 0)$	
$(2, 7)$	
$(-1, -\frac{1}{2})$	



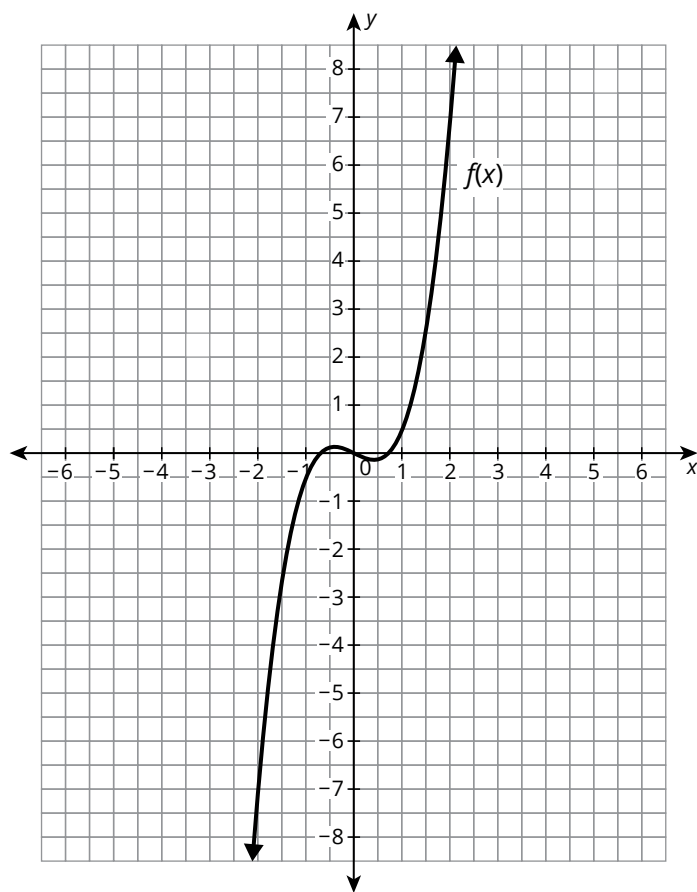
b. $m(x) = f(-2x) - 1$

Reference Points on $d(x)$	Corresponding Points on $m(x)$
$(0, 0)$	
$(2, 7)$	
$(-1, -\frac{1}{2})$	



c. $n(x) = f(x - 2) + 3$

Reference Points on $d(x)$	Corresponding Points on $j(x)$
$(0, 0)$	
$(2, 7)$	
$(-1, -\frac{1}{2})$	



Function Transformation Cards



$f\left(\frac{1}{2}x\right)$	$f(x - 5)$	$f(2x + 5)$	$f(x) + 5$	$f(-x)$
$f(x) - 5$	$-f(x)$	$\frac{1}{2}f(x)$	$2f(x)$	$f(2x)$
$\left(x, \frac{1}{2}y\right)$	$(-x, y)$	$(x, 2y)$	$(x + 5, y)$	$\left(\frac{1}{2}x, y\right)$
$(x, y + 5)$	$(x, -y)$	$(2x, y)$	$(-x, -y)$	$(x - 5, y)$
$(x, y - 5)$				

Why is this page blank?

So you can cut out the Quadratic Equations Cutouts on the other side

Lesson 2 Assignment

Write

Explain how transforming a cubic function is similar to transforming a linear, quadratic, or absolute value function.

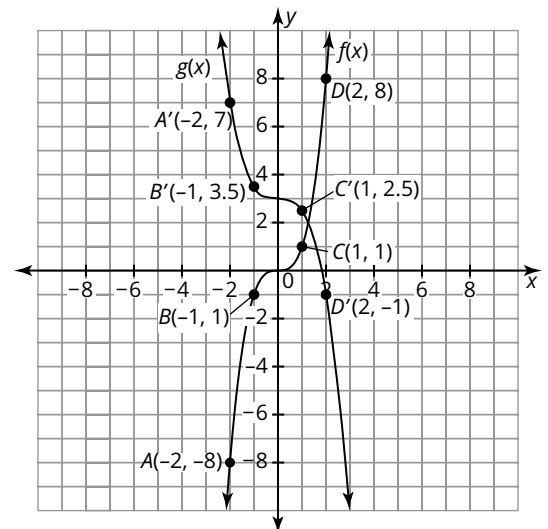
Remember

The function $g(x) = af(b(x - c)) + d$ is the transformation function where the constants a and d affect the output values of the function and the constants b and c affect the input values of the function.

Practice

1. Analyze the graphs of the functions $f(x)$ and $g(x)$.

a. Write the equation for $f(x)$.



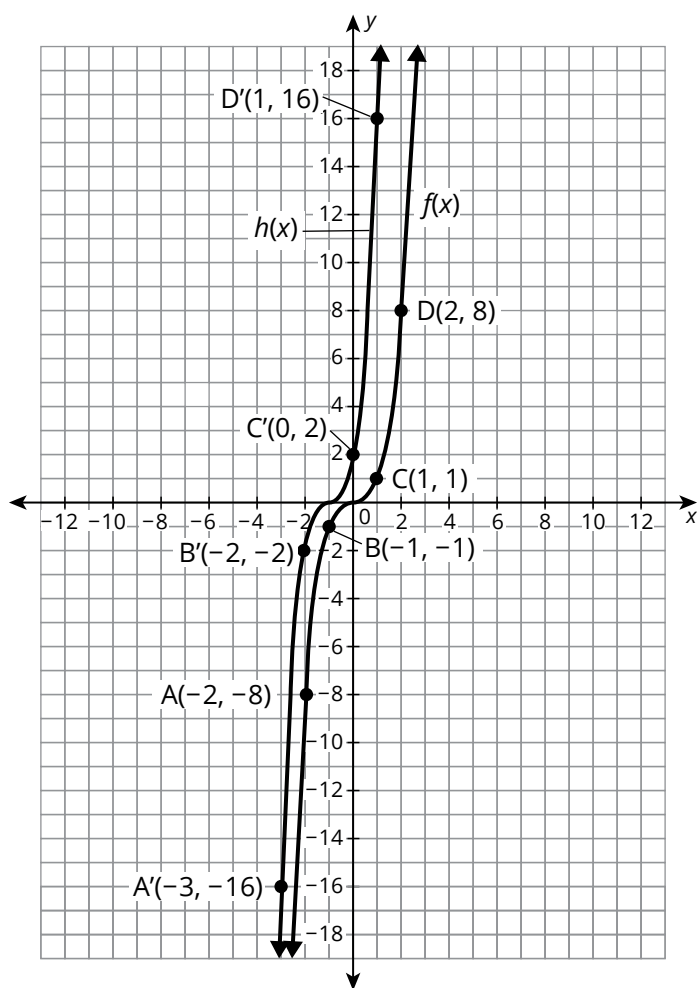
b. The function $g(x)$ is a transformation of the function $f(x)$. Describe the transformations performed on $f(x)$ that result in the function $g(x)$. Explain your reasoning.

Lesson 2 Assignment

c. Write the equation for $g(x)$.

d. Describe the symmetry of $g(x)$.

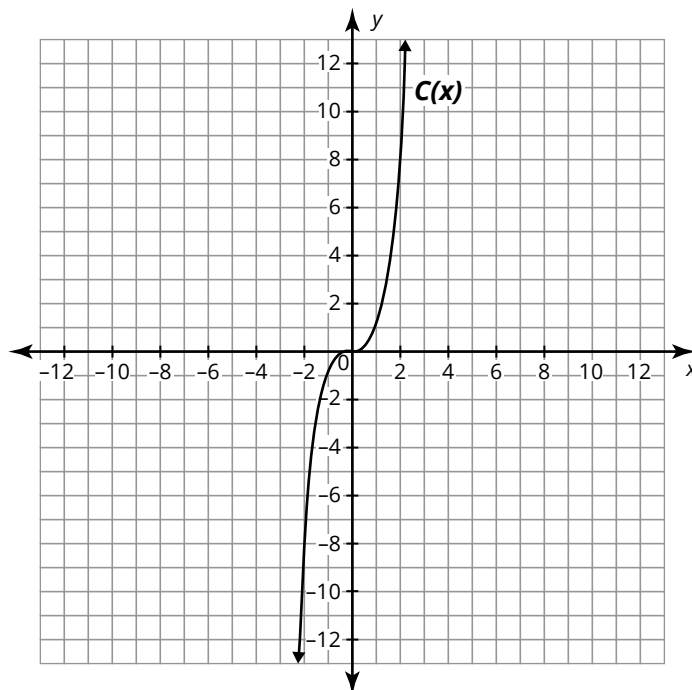
2. Use the graph of the function $f(x)$ from Question 1. Analyze the graphs of the functions $f(x)$ and $h(x)$.



Lesson 2 Assignment

- a. The function $h(x)$ is a transformation of the function $f(x)$. Describe the transformation performed on $f(x)$ that result in the function $h(x)$. Explain your reasoning.
- b. Write the equation for $h(x)$.
- c. Describe the symmetry of $h(x)$.

3. The graph of the cubic function $c(x) = x^3$ is shown.

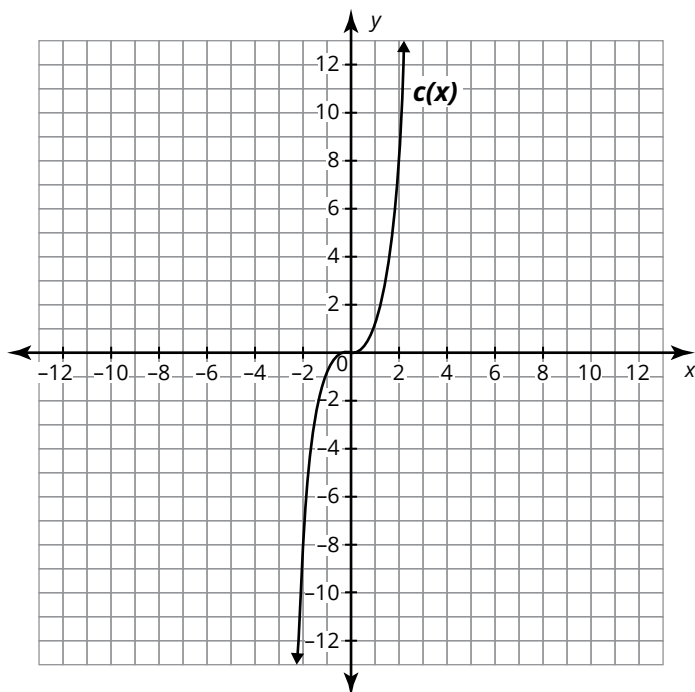


Lesson 2 Assignment

- a. The function $h(x) = \frac{1}{4}(x - 6)^3$ is a transformation of $c(x)$. Complete the table.

Reference Points on $c(x)$	→	Corresponding Points on $h(x)$
(x, y)	→	
$(-2, -8)$	→	
$(-1, -1)$	→	
$(0, 0)$	→	
$(1, 1)$	→	
$(2, 8)$	→	

- b. Graph the function $h(x) = \frac{1}{4}(x - 6)^3$ on the same coordinate plane as $c(x)$.



Lesson 2 Assignment

- c. Determine the transformation(s) performed on $c(x)$ that result in the function $h(x)$.

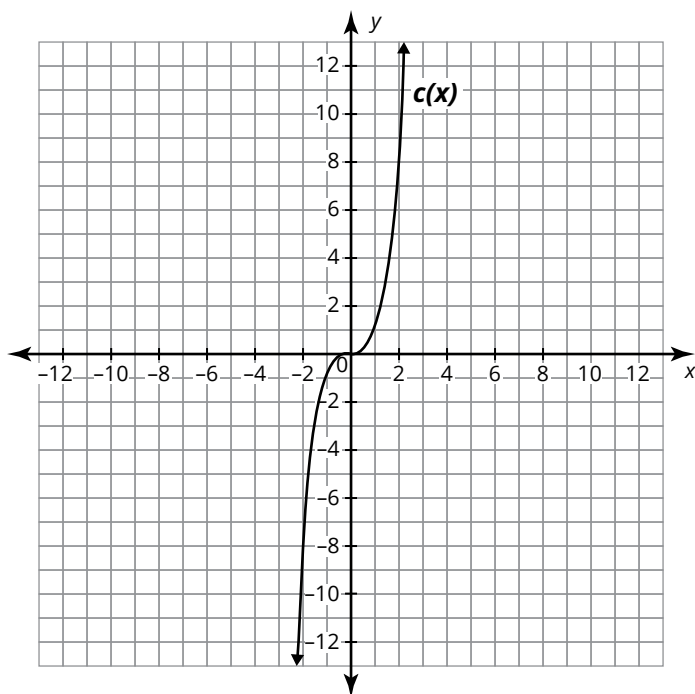
4. The function $g(x) = \left(-\frac{1}{2}x\right)^3 - 4$ is a transformation of $c(x)$.

- a. Complete the table.

Reference Points on $c(x)$	→	Corresponding Points on $g(x)$
(x, y)	→	
$(-2, -8)$	→	
$(-1, -1)$	→	
$(0, 0)$	→	
$(1, 1)$	→	
$(2, 8)$	→	

Lesson 2 Assignment

- b. Graph the function $g(x) = \left(-\frac{1}{2}x\right)^3 - 4$ on the same coordinate plane as $c(x)$.



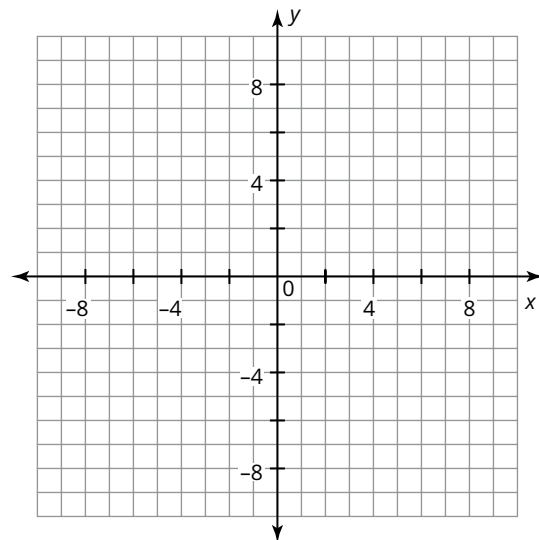
- c. Determine the transformation(s) performed on $c(x)$ that result in the function $g(x)$.

Lesson 2 Assignment

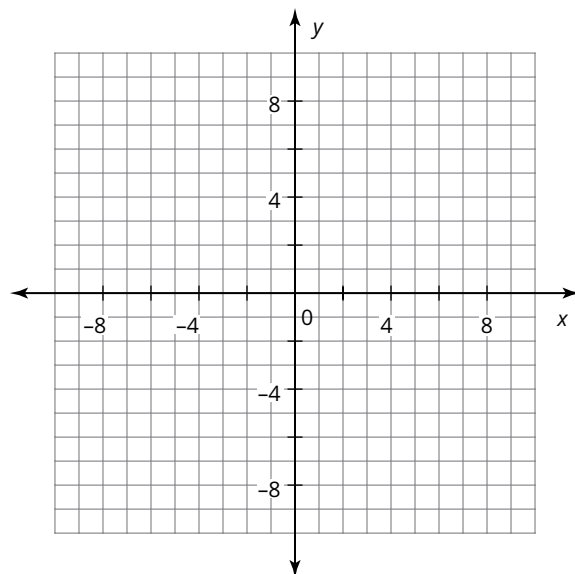
Prepare

1. Sketch graphs of cubic functions with the given number of zeros.
Describe the domain, range, and zeros of each function.

a. 1 zero

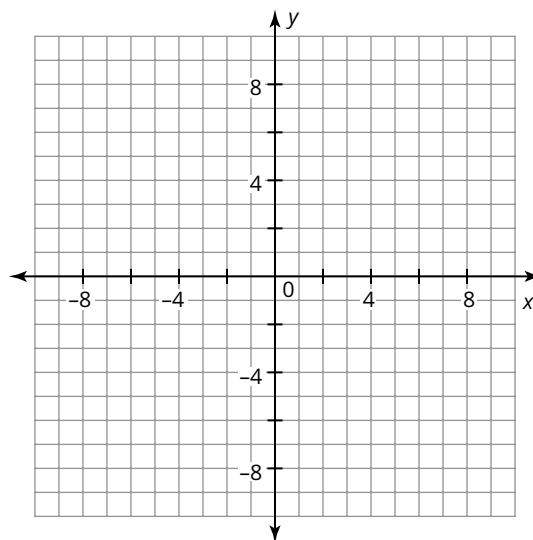


b. 2 zeros



Lesson 2 Assignment

c. 3 zeros



3

Key Attributes of Cubic Functions

OBJECTIVES

- Interpret key attributes of a cubic function in the context of a problem situation.
- Generalize orally and in written form about the key attributes of polynomial functions.
- Sketch the graph of a cubic function given certain key attributes.

KEY TERMS

- absolute maximum
- absolute minimum
- extrema

.....

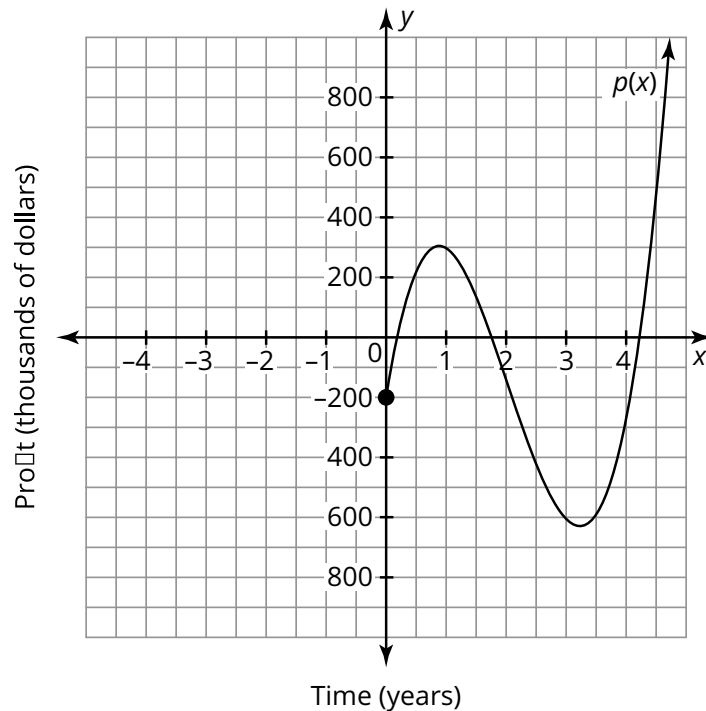
You have investigated quadratic functions, power functions, and cubic functions. What are some of the key attributes of cubic functions?

Getting Started

Play is Our Work

The cubic function $c(x)$ models the profits of a video game company, from its original business plan, through its first five years of business.

Video Game Company's Profits Over Years 0–5



.....
Think About . . .

What is the maximum
number of solutions
for a given profit?

.....

1. Use the graph to estimate when the company achieved each profit.
Then explain how you determined your estimate.

a. \$800,000

b. \$200,000

c. Greater than \$200,000

d. The company is losing money

e. The company is making a profit

2. Mei makes the statement about the company's profits. Explain the error in Mei's reasoning.

Mei



The video game company is making a profit after 3.5 years because the graph is increasing.

3. Do you think a cubic function is an appropriate model for this scenario? Explain your reasoning.

Ask Yourself . . .

How can you use (insert the math students are learning, for example ratios) in everyday life?

Maxima and Extrema of Polynomial Functions

Consider the situation and cubic function modeled in the Getting Started.

1. Use the graph to estimate the x-intercepts of the function. What do the x-intercepts mean in the context of this problem situation?
2. Write the domain and range of the problem situation using interval notation.
3. Describe the interval(s) when the company's profit is:
 - a. Increasing.
 - b. Decreasing.

You have learned about minimums, maximums, zeros, and the general shape of a graph. Now, you will combine all that information to generalize the key attributes for a polynomial of any degree.

Recall that a relative maximum is the highest point in a particular section of a function's graph, and a relative minimum is the lowest point in a particular section of the graph. Similarly, the **absolute maximum** is the highest point in the entire graph, and the **absolute minimum** is the lowest point in the entire graph. The set of relative maximums, relative minimums, absolute maximums, and absolute minimums may also be referred to as **extrema**. The extrema are also called *extreme points* and *extremum*.

4. Consider the graph that represents the video game company's profit in the Getting Started.
Identify all relative maximums and relative minimums.

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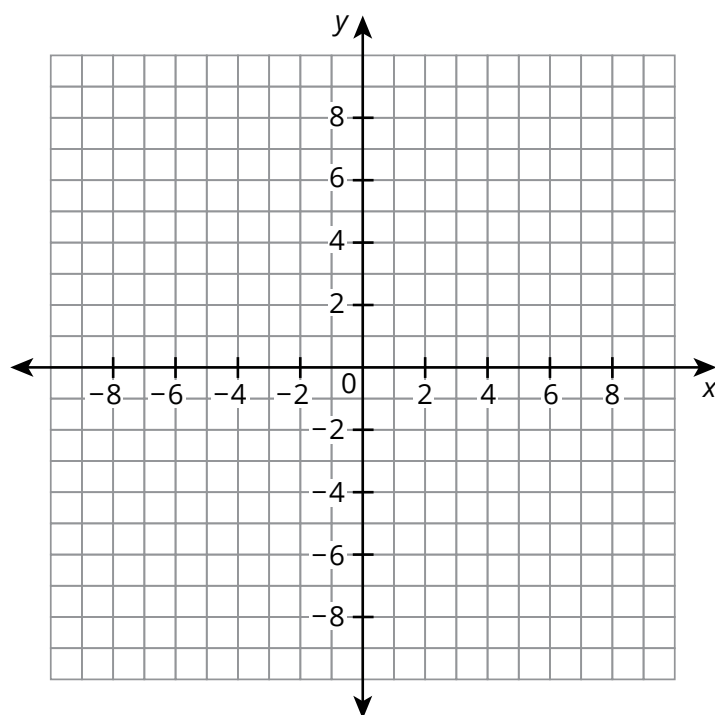
An absolute maximum can also be considered a relative maximum. Usually only the more significant label of absolute maximum is used, with the term relative maximum implied. The same reasoning can be applied to an absolute minimum and a relative minimum.

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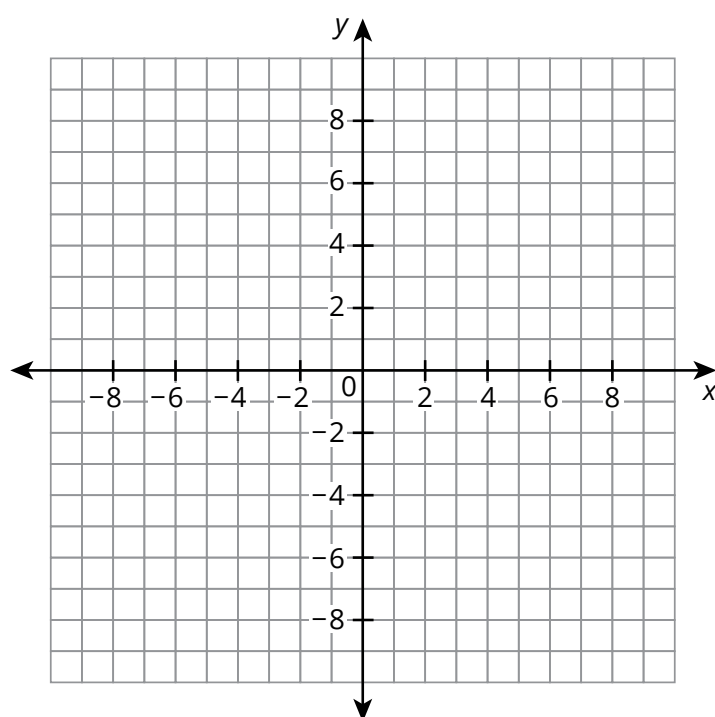
5. Identify any absolute maximums and absolute minimums.

6. Identify the extrema of the graph of $f(x)$, if any.

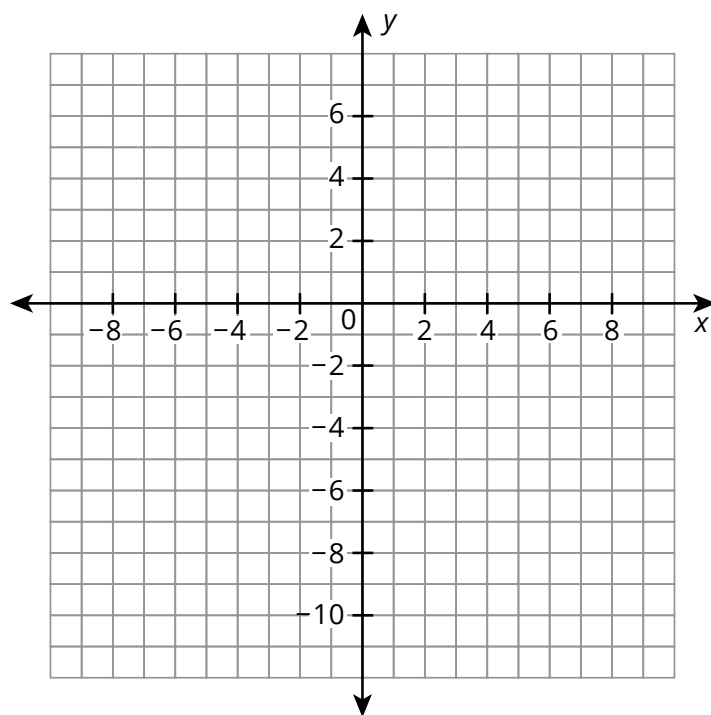
a.



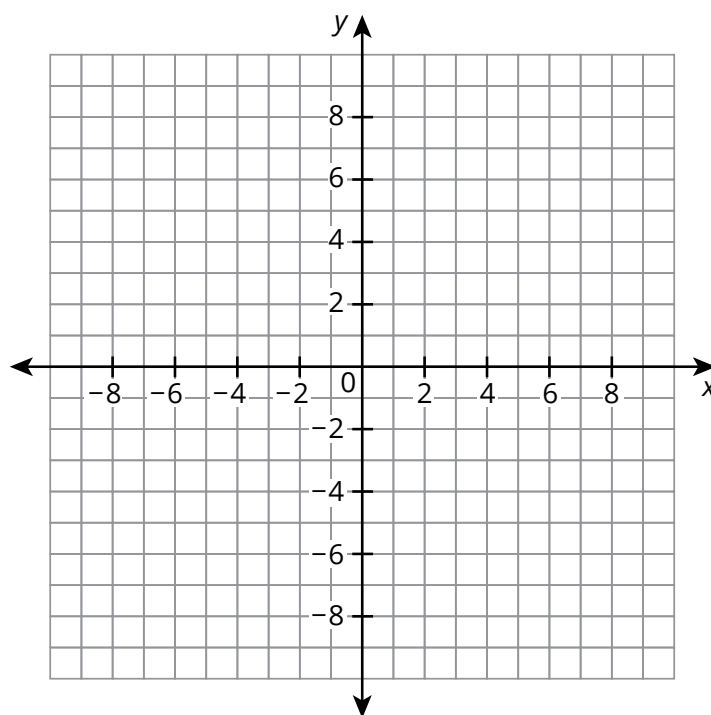
b.



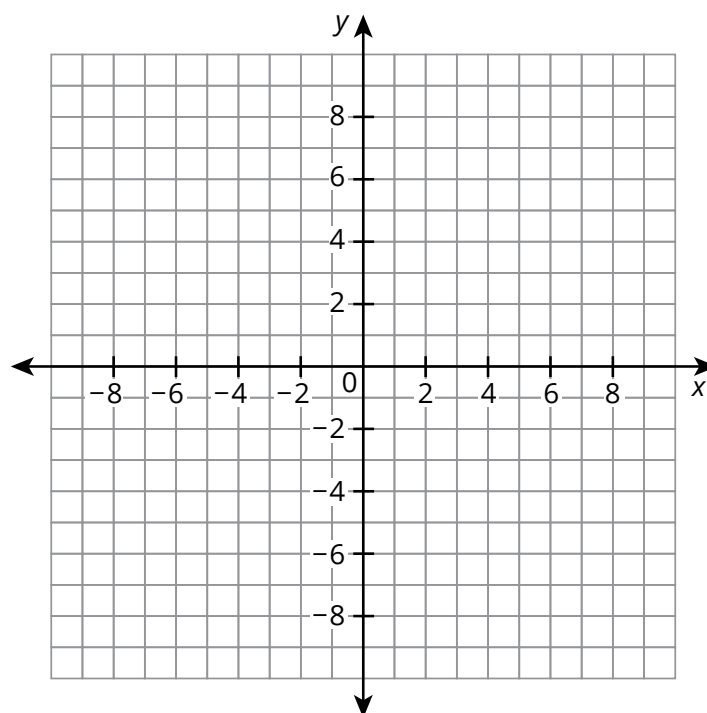
c.



d.



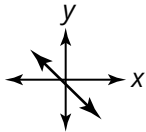
e. cubic (ends open circle)

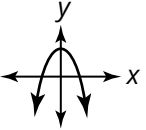
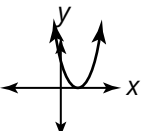
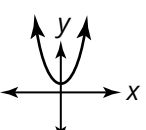


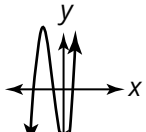
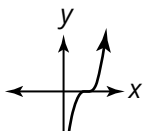
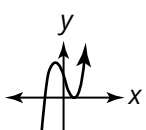
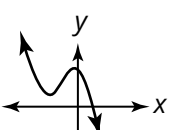
7. Will a cubic function always have an absolute maximum or an absolute minimum? Explain your reasoning.

Zeros of Polynomial Functions

The zeros of a polynomial function can be real, imaginary, or a combination of both. The real zeros of the function may all be distinct, or can have varying degrees of multiplicity. The tables summarize all possible combinations of zeros for a linear, quadratic, and a cubic function.

Linear Function	
Zeros	Graph
1 real zero	

Quadratic Function	
Zeros	Graph
2 real distinct	
1 real (multiplicity 2)	
2 imaginary	

Cubic Function	
Zeros	Graph
3 real distinct	
1 real (multiplicity 3)	
1 real (multiplicity 1) 1 real (multiplicity 2)	
1 real 2 imaginary	

1. Analyze the tables.
 - a. Describe any similarities or differences you notice.
 - b. Describe the number of graphs of each function you could sketch given the combination of zeros. Explain your reasoning.
2. How does the combination of zeros help you decide on the shape of the graph?

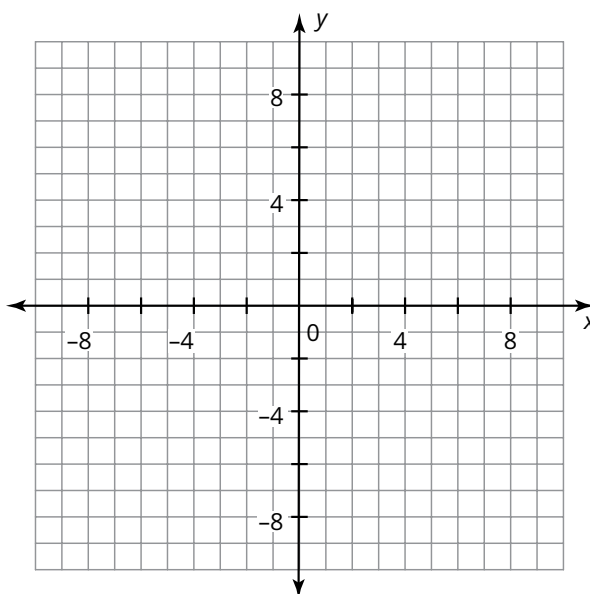
Sketching Cubic Functions

You have composed and decomposed various cubic functions. In this activity, you will sketch cubic given key attributes.

1. Use the coordinate plane to sketch each polynomial cubic.

a. Key Attributes:

- has an a -value less than 0
- relative maximum at $x = 3$



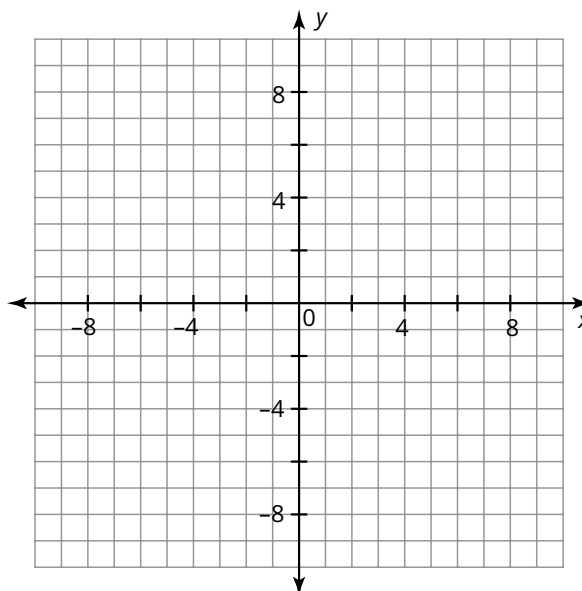
PROBLEM SOLVING

**Ask Yourself . . .**

Did you make a plan to solve the problem?

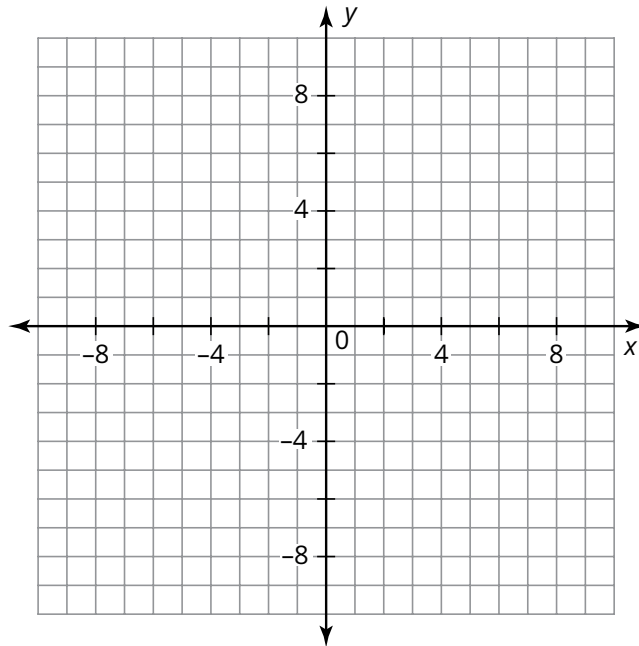
b. Key Attributes:

- always increasing
- y -intercept at 5
- x -intercept at -1.7



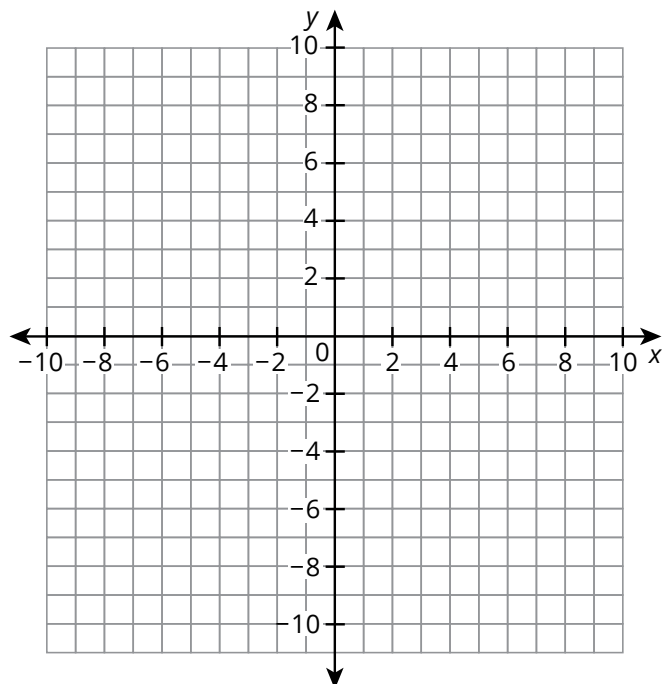
c. Key Attributes:

- three x -intercepts
- relative maximum at $y = 3$
- an a -value greater than zero



d. Key Attributes:

- two x -intercepts
- negative a -value



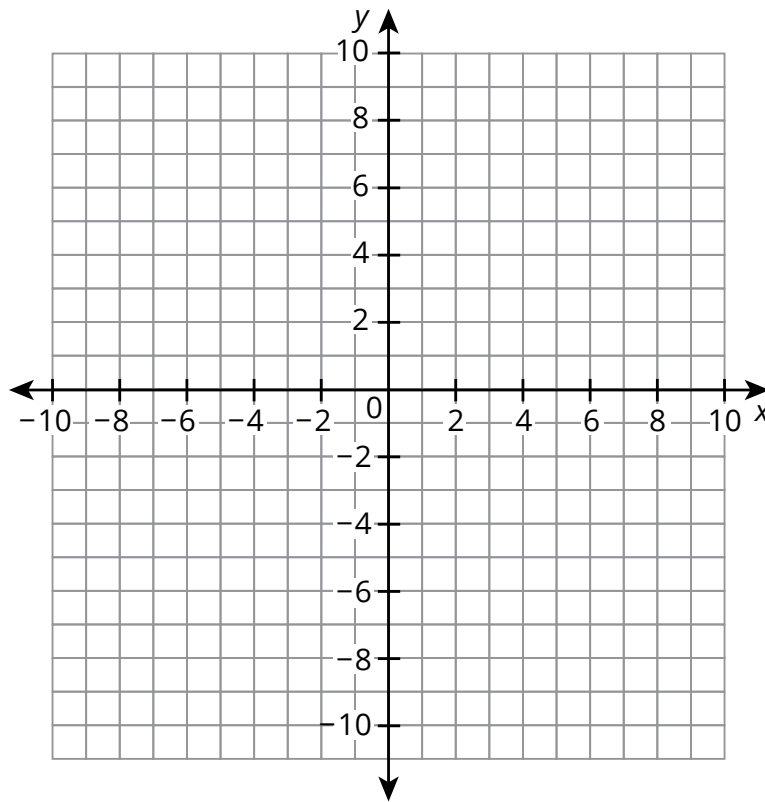
Ask Yourself ...

How does representing mathematics in multiple ways help to communicate reasoning?

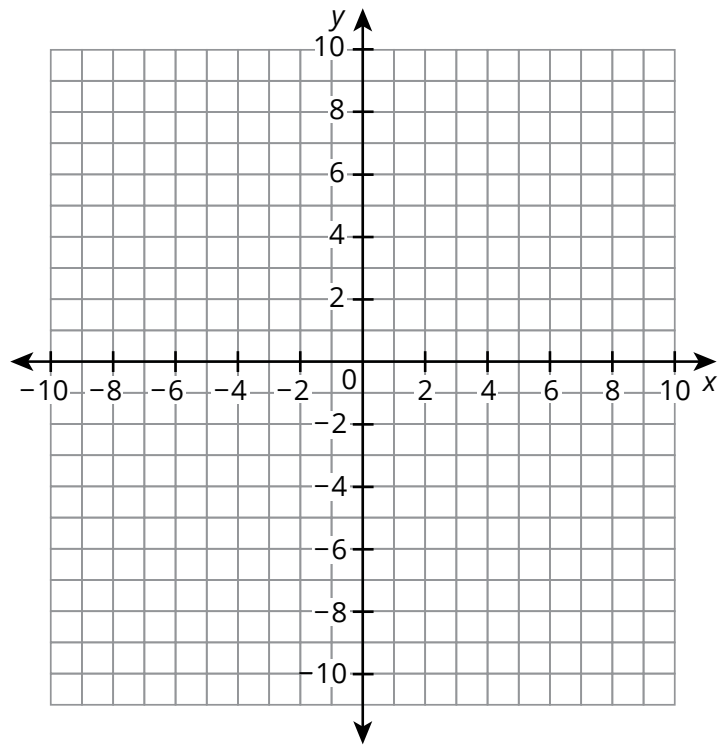
2. Use the coordinate plane to sketch each cubic function. Then, identify their key attributes:

- domain
- range
- x-intercepts
- y-intercept
- symmetry
- relative maximum (if it exists)
- relative minimum (if it exists)

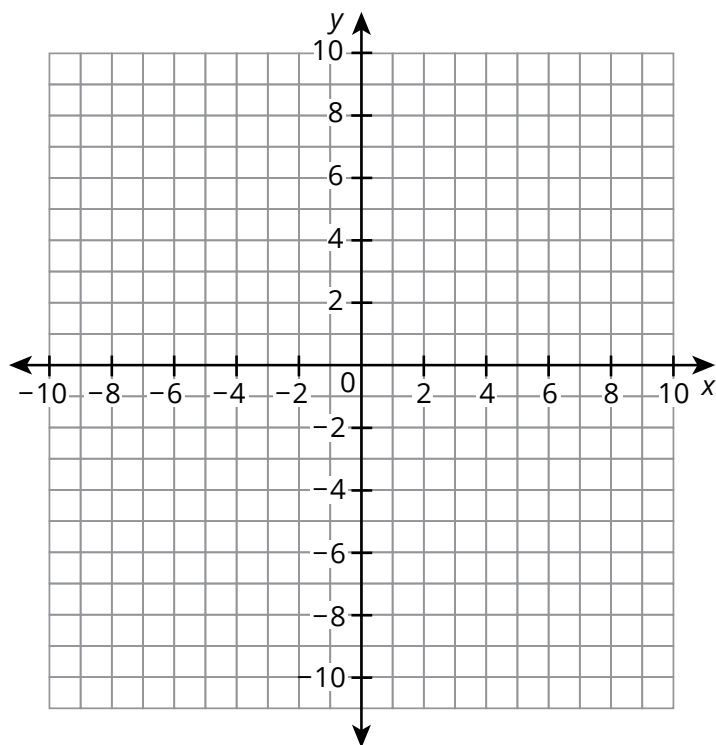
a. $f(x) = 2(x - 1)^3 + 3$



b. $g(x) = -\frac{1}{4}x(x - 4)(x + 5)$



c. $h(x) = (x + 2)^3$ for $-4 < x \leq 0$





Talk the Talk

Shapeshifting

Consider all the different shapes of graphs of polynomial functions you saw in this lesson.

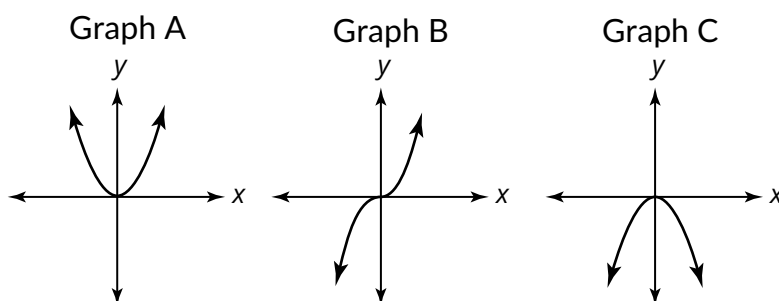
1. Describe what each attribute of the graph tells you about the polynomial function.

- a. x-intercept(s)

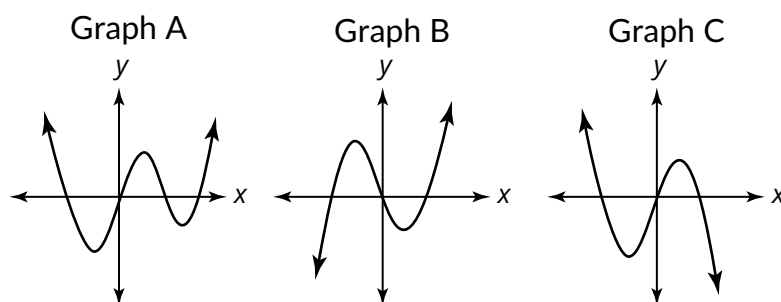
- b. extrema

2. Choose the possible graph(s) for each given polynomial function $f(x)$.

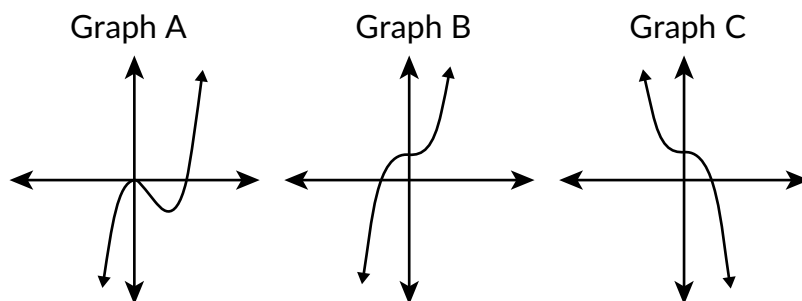
- a. Which graph(s) could be the graph of $f(x) = 2x^3$?



b. Which graph(s) could be the graph of $f(x) = -x^3 - x^2 + 6x$?



c. Which graph(s) could be the graph of $f(x) = x^3 + 1$?

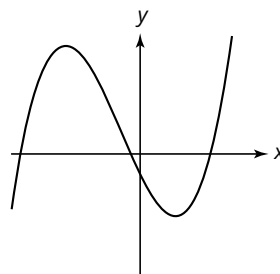


3. Analyze each graph. Circle the function(s) which could model the graph. Describe your reasoning to either eliminate or choose each function.

$$f_1(x) = 0.5(x + 7)(x + 1)(x - 5) - 3$$

$$f_2(x) = -2(x + 7)(x + 1)(x - 5) - 3$$

$$f_3(x) = 2(x + 7)(x + 1)$$



Lesson 3 Assignment

Write

Define each term in your own words.

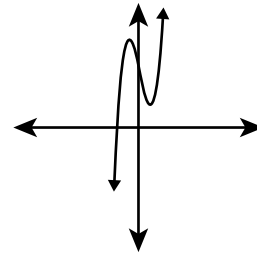
1. absolute maximum
2. absolute minimum
3. extrema

Remember

- The maximum number of x -intercepts is the same as the degree of the function.
- The combination of real and imaginary roots of a polynomial function are equal to the degree of the polynomial and can be used to determine the shape of its graph.

Practice

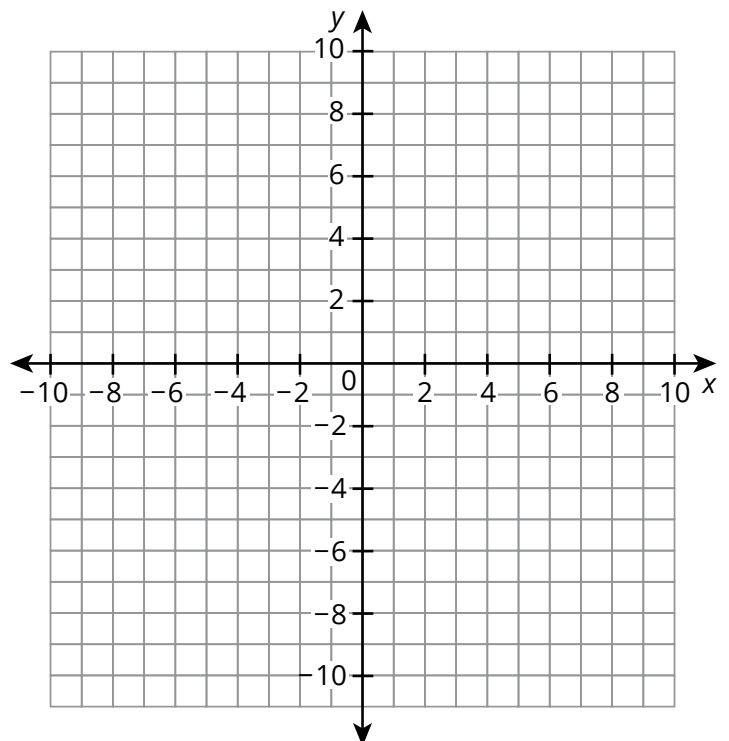
1. Describe the combination of real and imaginary roots for the graphed cubic function. Include the multiplicity of each real root.



2. Sketch a graph of a cubic function with the attributes given.

a. Attributes:

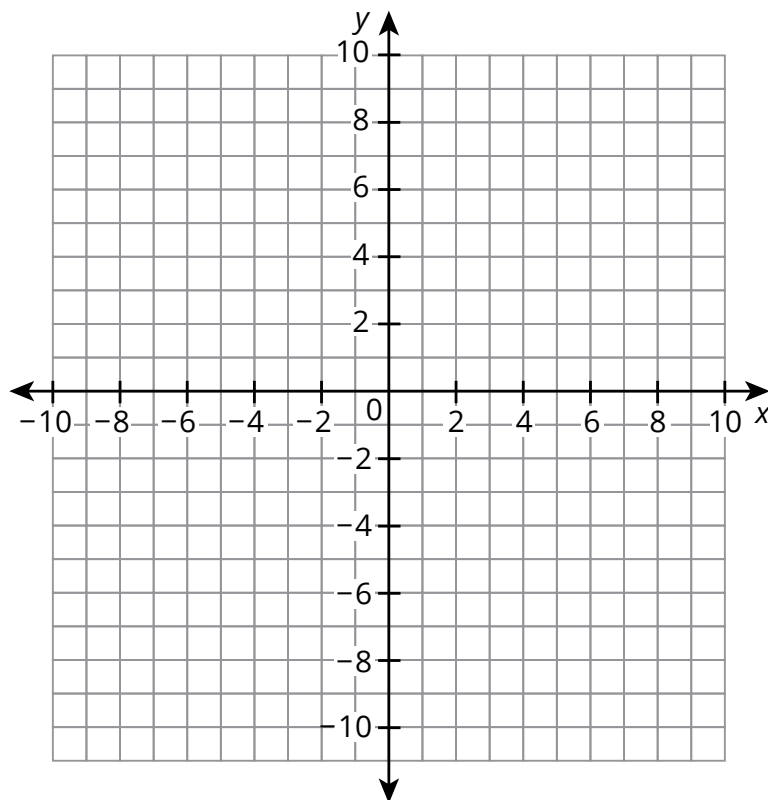
- 3 real roots
- relative minimum at $x = 1$



Lesson 3 Assignment

b. Attributes:

- always decreasing
- x-intercept at $x = -2$
- y-intercept at -3



3. Determine the key attributes of the cubic function $h(x) = \frac{2}{5}(x + 3)(x - 1)(x - 5)$.

Lesson 3 Assignment

Prepare

Solve each equation for x .

1. $2x^2 - 4 = 8$

2. $(x - 1)^3 - 5 = 0$

3. $3(x - 6)^4 + 11 = 15$

4. $x^3 - 27 = 0$

Attributes of Cubic Functions

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

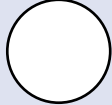
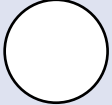
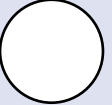
Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Attributes of Cubic Functions* topic by:

TOPIC 2: <i>Attributes of Cubic Functions</i>	Beginning of Topic	Middle of Topic	End of Topic
recognizing when a function is symmetric about a line or a point.			
identifying graphical transformations of cubic functions.			
describing the effects of transformations on the equations of cubic functions.			
identifying the multiplicity of zeros of a cubic function by the factored form of the equation or from a graph.			
explaining how the multiplicity of zeros indicates the behavior of the graph of a cubic function.			
identifying key attributes of cubic functions from a graph including domain, range, extrema, symmetry, and intercepts.			
creating graphs of cubic functions that model key attributes.			
modeling real-world situations using cubic functions.			

continued on the next page

TOPIC 2 SELF-REFLECTION
continued

TOPIC 2: <i>Attributes of Cubic Functions</i>	Beginning of Topic	Middle of Topic	End of Topic
using a graph to approximate solutions to a cubic function.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

- Describe a new strategy you learned in the *Attributes of Cubic Functions* topic.

- What mathematical understandings from the topic do you feel you are making the most progress with?

- Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 2 SUMMARY

Attributes of Cubic Functions Summary

LESSON

1

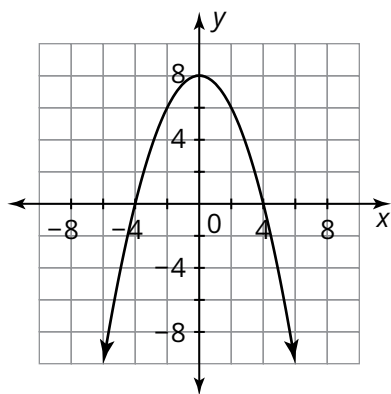
Symmetric Functions

A **power function** is a function of the form $P(x) = ax^n$, where n is a non-negative integer. The graph of a power function raised to an odd degree increases from left to right (or right to left if $a < 0$), flattening near the origin, as the absolute value of the power increases. The graph of a power function raised to an even degree is a concave up (or down if $a < 0$) parabola, flattening near the origin as the absolute value of the power increases.

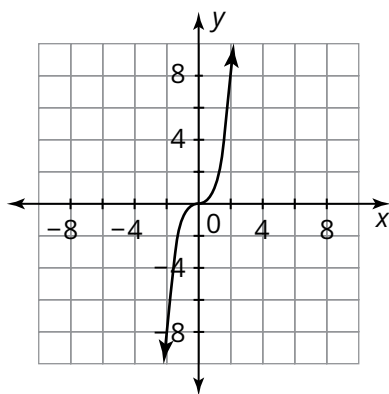
If a graph is **symmetric about a line**, the line divides the graph into two identical parts. The graph of the cubic power function is symmetric about a point, in particular the origin. A function is **symmetric about a point** if each point on the graph has a point the same distance from the central point but in the opposite direction. When the point of symmetry is the origin, the graph is reflected across the x -axis and the y -axis. If you replace (x, y) with $(-x, -y)$, the function remains the same. You can think of the point of symmetry about the origin as a double reflection.

If a graph is symmetric about the y -axis, then $f(x) = f(-x)$. If a function is symmetric about the origin, then $f(x) = -f(-x)$.

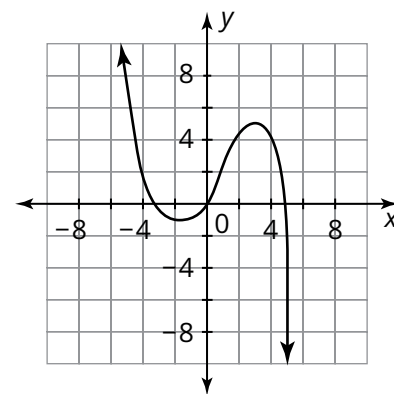
For example, consider the functions shown.



The function is symmetric about the y -axis.



The function is symmetric about the origin.



The function is not symmetric.

KEY TERMS

- power function
- symmetric about a line [simétrico respecto a una línea]
- symmetric about a point [simétrico respecto a un punto]
- polynomial function [función polinomial]
- absolute maximum [máximo absoluto]
- absolute minimum [mínimo absoluto]
- extrema [extremos]

Linear, quadratic, and cubic functions are all examples of polynomial functions. A **polynomial function** is a function that can be written in the form

$$\blacksquare x^n + \blacksquare x^{n-1} + \dots \blacksquare x^2 + \blacksquare x + \blacksquare.$$

In a polynomial function, the coefficients, represented by each $\blacksquare$, are complex numbers and the exponents are non-negative integers.

Transformations performed on any polynomial function $f(x)$ to form a new function $g(x)$ can be described by the transformation function:

$$g(x) = af(b(x - c)) + d$$

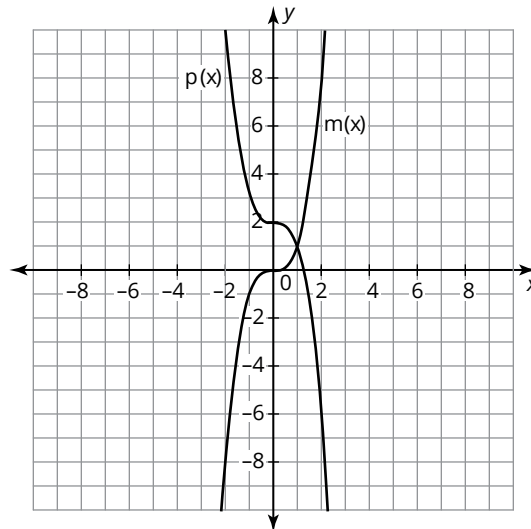
Function Form	Equation Information	Description of Transformation of Graph
$y = af(x)$	$ a > 1$	vertical stretch of the graph by a factor of a units
	$0 < a < 1$	vertical compression of the graph by a factor of a units
	$a < 0$	reflection across the x-axis
$y = f(bx)$	$ b > 1$	compressed horizontally by a factor of $\frac{1}{ b }$
	$0 < b < 1$	stretched horizontally by a factor of $\frac{1}{ b }$
	$b < 0$	reflection across the y-axis
$y = f(x - c)$	$c > 0$	horizontal shift right c units
	$c < 0$	horizontal shift left c units
$y = f(x) + d$	$d > 0$	vertical shift up d units
	$d < 0$	vertical shift down d units

For example, consider the function $m(x) = x^3$.

$$p(x) = -m(x) + 2$$

Reference Point on $m(x)$	$\rightarrow$	Corresponding Point on $p(x)$
(0, 0)	$\rightarrow$	(0, 2)
(1, 1)	$\rightarrow$	(1, 1)
(2, 8)	$\rightarrow$	(2, -6)

There is a horizontal line of reflection at $y = 1$. The function $p(x)$ is symmetric about the point $(0, 2)$.



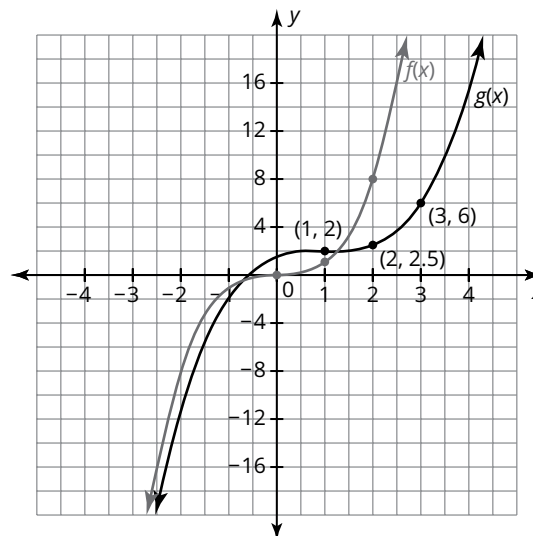
To determine multiple transformations of cubic functions, observe how corresponding points have changed on a graph and use the data to write the new function. Or, describe the transformations based on the equation of the transformed function in terms of the original function.

For example, consider the functions $f(x) = x^3$ and $g(x) = 0.5f(x - 1) + 2$.

$$a = 0.5, c = 1, d = 2$$

$$(x, y) \rightarrow (x + 1, 0.5y + 2)$$

The graph of the function $f(x)$ has been vertically compressed by a factor of 0.5, and translated 1 unit to the right and 2 units up to produce the graph of $g(x)$.



LESSON

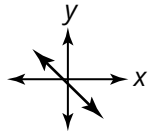
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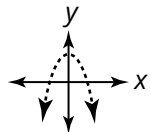
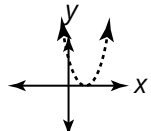
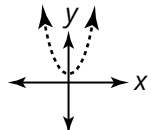
Key Attributes of Cubic Functions

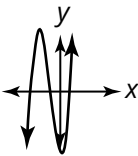
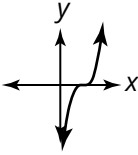
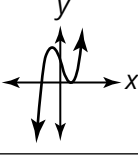
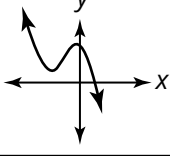
The **absolute maximum** is the highest point in the entire graph, and the **absolute minimum** is the lowest point in the entire graph. The set of relative maximums, relative minimums, absolute maximums and absolute minimums, may also be referred to as **extrema**. The extrema are also called *extreme points* and *extremum*.

Cubic Functions	
Possible number of x -intercept(s)	3, 2, or 1
Possible number of y -intercept(s)	1
Number of possible relative extrema	2 or none
Number of possible absolute extrema	None unless a partial function

The zeros of a polynomial function can either be real, imaginary, or a combination of both. The real zeros of the function may all be distinct, or can have varying degrees of multiplicity. The tables summarize all possible combinations of zeros for a linear, quadratic, and a cubic function.

Linear Function	
Zeros	Graph
1 real	

Quadratic Function	
Zeros	Graph
2 real distinct	
1 real (multiplicity 2)	
2 imaginary	

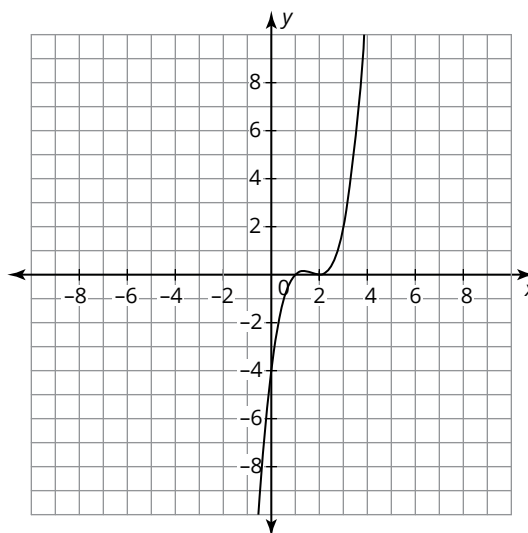
Cubic Function	
Zeros	Graph
3 real distinct	
1 real (multiplicity 3)	
1 real (multiplicity 1) 1 real (multiplicity 2)	
1 real 2 imaginary	

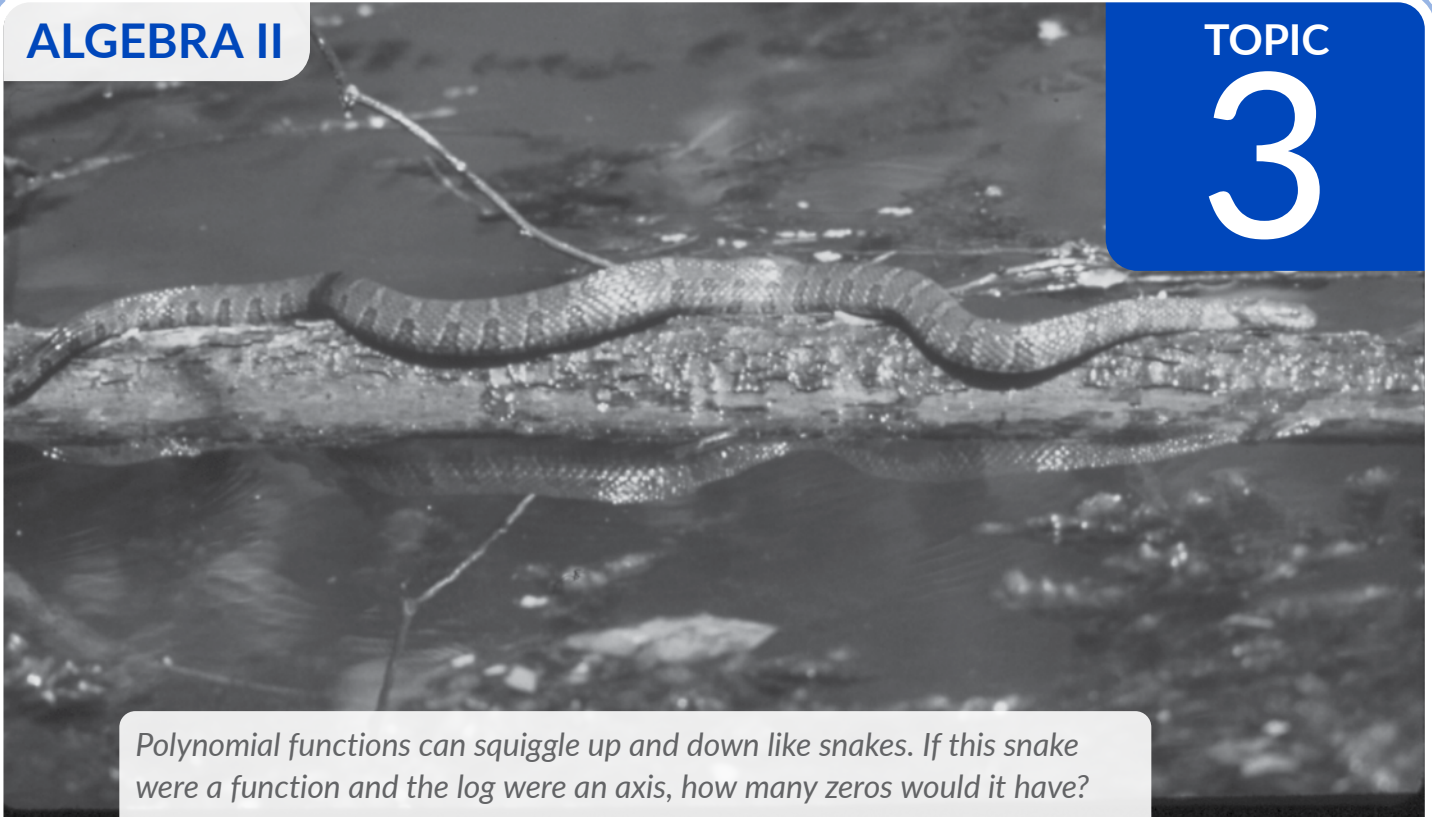
Key attributes of cubic functions, such as the number and kind of extrema, the a -value, degree of the function, intercepts, etc., can be used to sketch the graph of the function.

For example, consider a cubic function with the following key attributes and a possible sketch of its graph.

- negative a -value
- y -intercept of -4
- x -intercepts of 1 and 2

The function modeled by the graph could have the equation $f(x) = (x - 1)(x - 2)^2$.

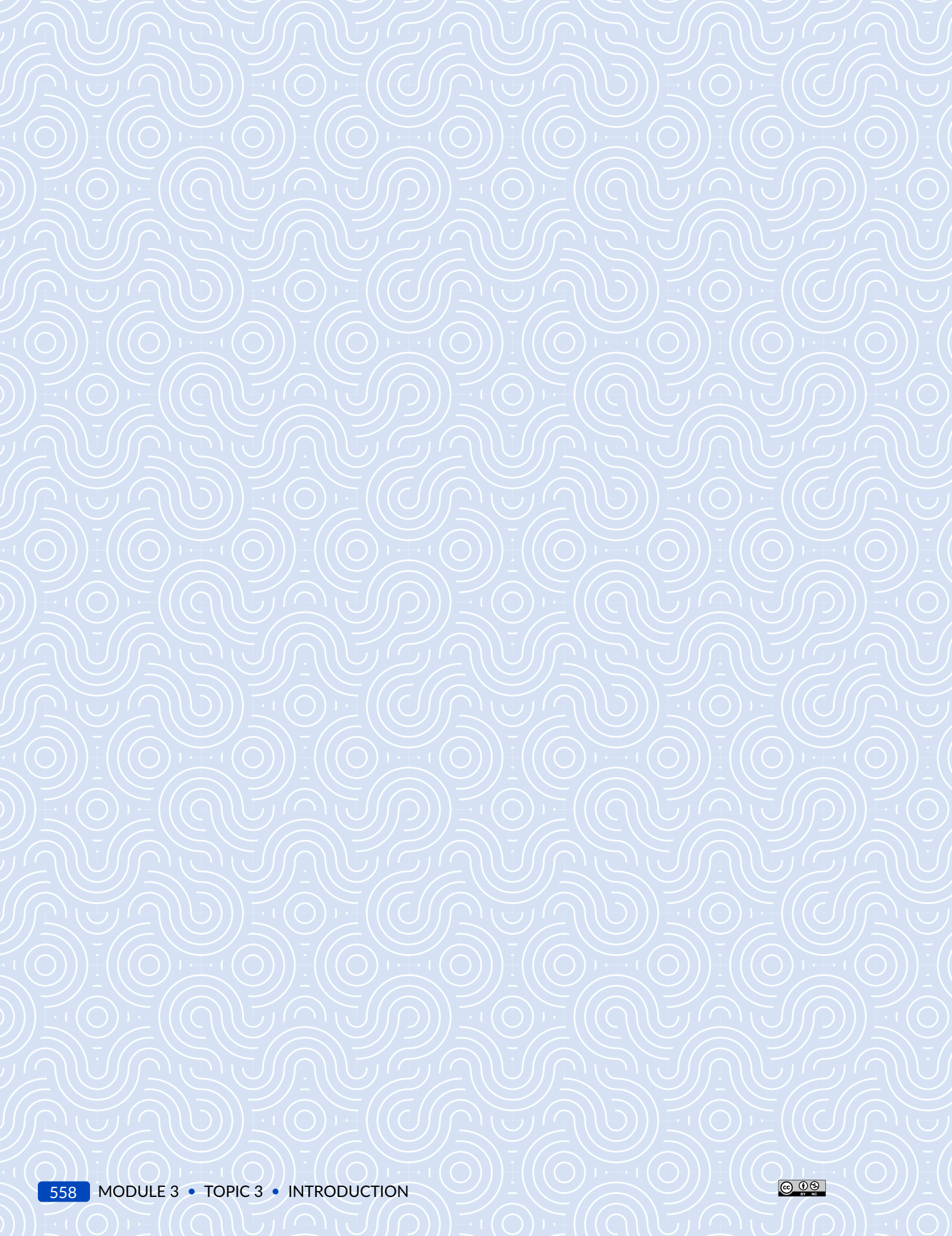




Polynomial functions can squiggle up and down like snakes. If this snake were a function and the log were an axis, how many zeros would it have?

Relating Factors and Zeros

LESSON 1	Adding, Subtracting, and Multiplying Polynomials	559
LESSON 2	Factoring Polynomials to Identify Zeros	573
LESSON 3	Polynomial Division	597
LESSON 4	Rational Root Theorem	625



1

Adding, Subtracting, and Multiplying Polynomials

OBJECTIVES

- Add and subtract polynomials and restate the sum or difference aloud.
- Use a multiplication table to multiply polynomials.
- Use the distributive property to multiply polynomials.

.....

You have performed operations with complex numbers and degree one and degree two polynomials.

How can you use that knowledge to add, subtract, and multiply higher-order polynomials?

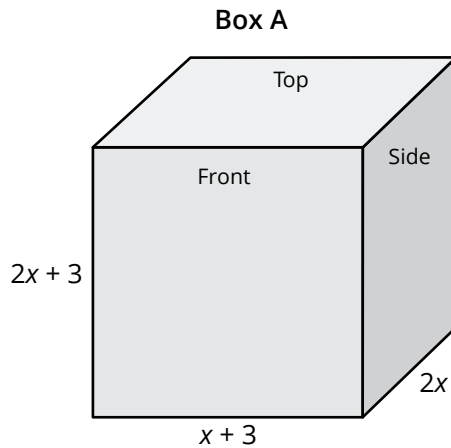
Getting Started

PROBLEM SOLVING

Package Delivery



Camilla owns a business that ships packages overseas. The size of the boxes varies, but all sides are rectangular and are made to order. All dimensions are in inches.



Box B

Dimensions:

$$\text{Height} = 4x$$

$$\text{Width} = x + 2$$

$$\text{Length} = x + 5$$

Ask Yourself ...

Did you make a plan to solve the problem?

1. Which box can hold more? How much larger is its capacity than the capacity of the other box? Show your work.

Adding and Subtracting Polynomials

In Algebra I, you added degree 1 and degree 2 polynomials. In this activity, you will practice adding and subtracting polynomials of higher degrees.

WORKED EXAMPLE

You can add and subtract polynomials by grouping terms that have the same variable and exponent.

To determine the sum $(3x^4 - 2x^2 + 5x + 1) + (2x^4 + x^3 + 3x - 5)$:

Step 1: Use the distributive property to rewrite the expressions without grouping symbols. $3x^4 - 2x^2 + 5x + 1 + 2x^4 + x^3 + 3x - 5$

Step 2: Use the commutative property of Addition to rearrange the terms. $3x^4 + 2x^4 + x^3 - 2x^2 + 5x + 3x + 1 - 5$

Step 3: Use the associative property of addition to group like terms. $(3x^4 + 2x^4) + x^3 - 2x^2 + (5x + 3x) + (1 - 5)$

Step 4: Simplify the like terms. $5x^4 + x^3 + 2x^2 + 8x - 4$

To determine the difference $(3x^4 - 2x^2 + 5x + 1) - (2x^4 + x^3 + 3x - 5)$:

Step 1: Use the distributive property to rewrite the expressions without grouping symbols. $3x^4 + 2x^2 + 5x + 1 - 2x^4 - x^3 - 3x + 5$

Step 2: Rewrite the terms using addition. $3x^4 + (-2x^2) + 5x + 1 + (-2x^4) + (-x^3) + (-3x) + 5$

Step 3: Use the commutative property of addition to rearrange the terms. $3x^4 + (-2x^4) + (-x^3) + (-2x^2) + 5x + (-3x) + 1 + 5$

Step 4: Use the associative property of addition to group like terms. $(3x^4 + (-2x^4)) + (-x^3) + (-2x^2) + (5x + (-3x)) + (1 + 5)$

Step 5: Simplify the grouped terms. $x^4 - x^3 - 2x^2 + 2x + 6$

1. Analyze each student's work. Determine the error and make the necessary corrections.

Isaiah



$$3x^3 + (2 + 5x^3) = 8x^6 + 2$$

Carlos



$$\begin{aligned} 2x^4 - (4x^4 + 6x + 5) \\ 2x^4 - 4x^4 + 6x + 5 \\ -2x^4 + 6x + 5 \end{aligned}$$

Sarah



$$\begin{aligned} (4x^4 + 2x^2 - 5) + (3x^4 + 7) \\ (4x^4 + 3x^4) + 2x^2 - (5 + 7) \\ 7x^4 + 2x^2 - 12 \end{aligned}$$

2. Determine the sum or difference of each. Show your work.

a. $(3x^3 - 7x) + (5x^2 + 4x) - (6x^2 - 2)$

b. $(7x^2y - 9y^2z + 2x^2z) - (10x^2y + 4y^2z - 3x^2z)$

c. $(8x^4 - 2x^3 + 7x + 4) - (x^4 + 9x + 3)$

d. $(-2x^3 + 3x^2 - 1) + (2x^3 + 5x^2 + 1)$

e. $(x^4 - 2x^2) - (3x + 6) + (2x^4 + 8x + 1)$

f. $(6x^4 + 3x^3 - 4x) - (6x^4 + x^3 - 4x + 7)$

g. $(4x^3 + 5x - 2) + (7x^2 - 8x + 9)$

h. $(9x^4 - 5) - (8x^4 - 2x^3 + x)$

.....
Since the sum of
the exponents for
each item in part
(b) is 3, each term
has a degree of 3.
.....

Multiplying Polynomials

In a previous lesson, you used multiplication tables to multiply three linear factors. You can also use a multiplication table to multiply higher-order polynomials.

WORKED EXAMPLE

Determine the product $(4x^3 - 2x^2 + 3x - 1)(2x^4 - x^2 + 4x)$.

Step 1: Use a multiplication table to determine the product of the factors.

.	$4x^3$	$-2x^2$	$3x$	-1
$2x^4$	$8x^7$	$-4x^6$	$6x^5$	$-2x^4$
$-x^2$	$-4x^5$	$2x^4$	$-3x^3$	x^2
$4x$	$16x^4$	$-8x^3$	$12x^2$	$-4x$

Step 2: Combine like terms.

$$(4x^3 - 2x^2 + 3x - 1)(2x^4 - x^2 + 4x)$$

$$= 8x^7 - 4x^6 + 2x^5 + 16x^4 - 11x^3 + 13x^2 - 4x$$

1. Use a multiplication table to determine each product.

a. Determine the product $(5x^4 + 2x^3 - x + 4)(3x^3 - 6x^2 + 8x)$.

b. Determine the product $(7x^3 - x^2 + 9x - 1)(5x^4 + 3x^3 - 9)$.

2. Could you use a multiplication table to multiply more than two polynomials? Explain your reasoning.

You can also use the distributive property to determine the product of polynomials.

WORKED EXAMPLE

Use the distributive property to determine the product $(3x^5 - 6x^3 - 4x)(x^3 + 2x - 1)$.

First use the distributive property to multiply each term of $x^3 + 2x - 1$ by the polynomial $3x^5 - 6x^3 - 4x$.

$$3x^5(x^3 + 2x - 1) - 6x^3(x^3 + 2x - 1) - 4x(x^3 + 2x - 1)$$

Then, distribute.

$$(3x^8 + 6x^6 - 3x^5) + (-6x^6 - 12x^4 + 6x^3) + (-4x^4 - 8x^2 + 4x)$$

Finally, combine like terms and write the solution in standard form.

$$3x^8 - 3x^5 - 16x^4 + 6x^3 - 8x^2 + 4x$$

3. Use the distributive property to determine each production.

a. $(7x^4 + 3x^2 - 5x)(2x^5 - 4x^3 + 8x)$

b. $(3x^7 - 4x^5 - 3x^4)(2x^3 + 5x^2 - 6x)$



Talk the Talk

Let the Operations Begin!

1. Determine the sum or the difference for each expression.

a. $(3x^4 - 5x^3 + 12x - 7) + (x^4 + 4x^3 - 8x^2 - 9x - 1)$

b. $(2x^3 + 9x^2 - 4x - 6) - (5x^4 + 7x^3 - 8x + 2)$

2. Use a multiplication table or the distributive property to determine the product of $(5x^5 + 9x^4 - 4x + 7)(2x^6 - 3x^3 - 8x^2)$.

Lesson 1 Assignment

Write

In your own words, explain how to use the distributive property to multiply two polynomials.

Remember

To add or subtract polynomials, use the distributive property to rewrite the expressions without grouping symbols. Then, combine like terms and rewrite the sum or difference in standard form.

You can use a multiplication table or the distributive property to multiply polynomials.

Practice

1. Determine the sum or difference of each expression.

a. $(3x^4 - 7x^3 + 5x^2 - 9) + (6x^4 - 3x^2 + 2x + 3)$

b. $(6x^3 - x) + (2x^2 + 5x + 2) - (4x^3 + 7x^2)$

c. $(4x^2y + 2xy^2 + x^2z - 7xz^2) + (4x^2y + 3xy^2 - 8xz^2)$

Lesson 1 Assignment

d. $(7x^4 - x^2 + 3) - (5x^2 - 9x + 6) + (2x^4 + 8)$

e. $(2x^2y - xy^2 - 7xz^2) - (3x^2y - 5xy^2 + 6xz^2)$

f. $(x^3 + 8x^2 - 5x + 1) - (x^4 - 4x^3 - 3x + 5)$

2. Use a multiplication table to determine each product. Show your work.

a. $(2x^3 - x^2 + 6x)(3x^4 - 9x^2 + 5x - 6)$

Lesson 1 Assignment

b. $(x^4 + 5x^3 + 7x + 4)(4x^4 - 8x^2 + 3x)$

3. Use the distributive property to determine each product. Show your work.

a. $(-x^5 + 2x^3 + 7x + 4)(4x^4 - 8x^2 + 3x)$

b. $(3x^6 - 9x^4 + 2x^3)(9x^5 - 8x^4 - 7x^2)$

Lesson 1 Assignment

Prepare

1. $2x^2 - 4 = 8$

2. $(x - 1)^3 - 5 = 0$

3. $3(x - 6)^4 + 11 = 15$

4. $x^3 - 27 = 0$

2

Factoring Polynomials to Identify Zeros

OBJECTIVES

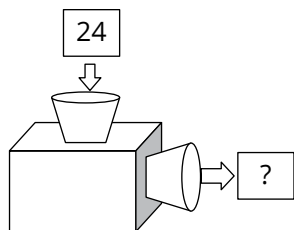
- Factor higher-order polynomials.
- Distinguish between factoring polynomial equations over the set of real numbers and over the set of complex numbers.
- Identify zeros of polynomials when suitable factorizations are available.
- Use the zeros of a polynomial to sketch a graph of the function and provide written and verbal explanations of how you sketched the graph of the function.

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You have determined factors of degree-2 equations.
How can you factor higher-degree polynomial functions?

Getting Started

Factor Tree Factory



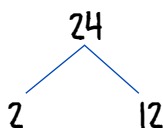
At the Factor Tree Factory, a factor machine takes any whole number as input and outputs one of its factor pairs.

1. Suppose the number 24 is entered into the machine.
 - a. What factor pairs might you see as the output?
 - b. How do you know whether two numbers are a factor pair of 24?
 - c. Can 5 be an output value? Explain your reasoning.

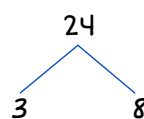


2. Jasmine and Luna each begin a factor tree for 24 using different outputs from the factor machine.

Jasmine



Luna



Jasmine says both factor trees will show the same prime factorization when completed. Luna says because they each started with a different factor pair, the prime factorizations will be different. Who's correct? Complete each factor tree to justify your answer.

3. Consider the expression $2 \cdot 2 \cdot 2 \cdot 3$.
- How does $2 \cdot 2 \cdot 2 \cdot 3$ relate to the factors determined by Jasmine and Luna?
 - How does it relate to 24?
4. If you know a factor of a given whole number, how can you determine another factor?
5. What is the remainder when you divide a whole number by any of its factors? Explain your reasoning.
6. Create a factor tree to show the prime factorization of each number.
- 66
 - 210

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Remember ...

A greatest common factor can be a variable, a constant, or both.

Throughout your previous mathematics courses, you have applied the idea that a whole number can be decomposed into a product of its factors to solve a variety of problems. In this lesson, you will explore different methods of decomposing a polynomial into a product of its factors. Once you have factored a polynomial, you can use the factors to identify the zeros and then use the zeros to sketch a graph.

To begin factoring any polynomial, always look for a greatest common factor (GCF). You can factor out the greatest common factor of the polynomial and then factor what remains.



1. Lucas and Nahimana each attempt to factor $3x^3 + 12x^2 - 36x$ by factoring out the greatest common factor.

Lucas

$$3x^3 + 12x^2 - 36x$$

$$3x(x^2 + 4x - 12)$$

Nahimana

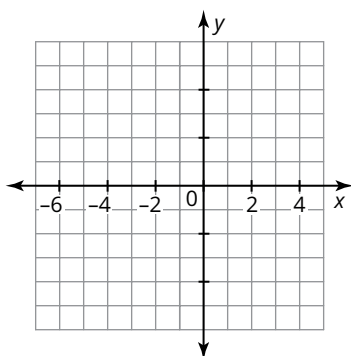
$$3x^3 + 12x^2 - 36x$$

$$3(x^3 + 4x^2 - 12x)$$

- a. Analyze each student's work. Determine which student is correct and explain the inaccuracy in the other student's work.

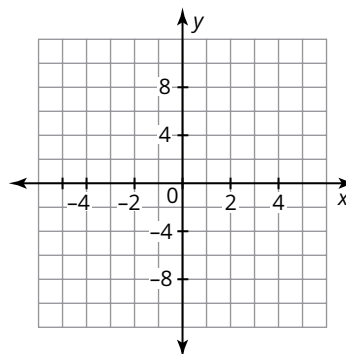
- b. Completely factor the expression that Lucas and Nahimana started to factor.

- c. Use the factors to identify the zeros of $f(x) = 3x^3 + 12x^2 - 36x$. Then, sketch the graph of the polynomial.



2. Factor each polynomial function and identify the zeros. Then, use the factors to sketch a graph of the function defined by the polynomial.

a. $f(x) = 3x^3 + 3x^2 - 6x$

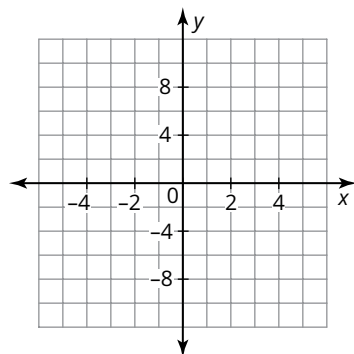


Remember ...

Look for a greatest common factor first. Identify the zeros and other key points before graphing.

b. $f(x) = 2x^4 + 6x^2$

c. $f(x) = 3x^2 - 3x - 6$



3. Analyze the factored form and the corresponding graphs in Questions 1 and 2. What do the graphs in Question 1 and Question 2, parts (a) and (b) have in common that the graph of Question 2, part (c) does not? Explain your reasoning.
4. Write a statement about the graphs of all polynomials that have a monomial GCF that contains a variable.



5. Sebastian and Kayla each attempt to factor $f(x) = x^3 - 2x^2 + 2x$. Analyze their work.

Sebastian

First, I removed the GCF, x .

The expression $x^2 - 2x + 2$ cannot be factored, so

$$f(x) = x(x^2 - 2x + 2).$$

Kayla

$$f(x) = x(x^2 - 2x + 2)$$

$$x^2 - 2x + 2 = 0$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{-4}}{2}$$

$$x = \frac{2 \pm 2i}{2}$$

$$x = 1 \pm i$$

The function in factored form is

$$f(x) = (x)[x - (1 + i)][x - (1 - i)].$$

- a. If you consider the set of real numbers, who's correct? If you consider the set of complex numbers, who's correct? Explain your reasoning.
- b. Use the distributive property to rewrite Kayla's function to verify that the function in factored form is equivalent to the original function in standard form.
- c. Identify the zeros of the function $f(x)$.

Remember ...

The set of complex numbers is the set of numbers that includes both real and imaginary numbers.

6. Analyze each expression.

$x^2 + 4$	$x^2 - 4$	$x^2 + 2x + 5$	$x^2 + 4x - 5$
$-x^2 + x + 12$	$x^2 + 4x - 1$	$-x^2 + 6x - 25$	

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Some functions can be factored over the set of real numbers. However, all functions can be factored over the set of complex numbers.

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- a. Sort each expression based on whether it can be factored over the set of real numbers or over the set of imaginary numbers.

Complex Factors	
Real Factors	Imaginary Factors

- b. Factor each expression over the set of real or complex numbers.

ACTIVITY
2.2

Using Structure to Factor Polynomials

There are multiple strategies you can use to factor polynomials. You can use the structure of a polynomial to help you choose the best approach to factoring it.

Certain polynomials in quadratic form may have common factors in some of the terms, but not in all terms. In this case, it may be helpful to write the terms as a product of two terms. You can then substitute the common term with a variable, z , and factor as you would any polynomial in quadratic form. This method of factoring is called *chunking*.

WORKED EXAMPLE

You can use chunking to factor $9x^2 + 21x + 10$.

Notice that the first and second terms both contain the common factor $3x$.

$9x^2 + 21x + 10 = (3x)^2 + 7(3x) + 10$ Rewrite terms as a product of common factors.

$$= z^2 + 7z + 10$$

Let $z = 3x$.

$$= (z + 5)(z + 2)$$

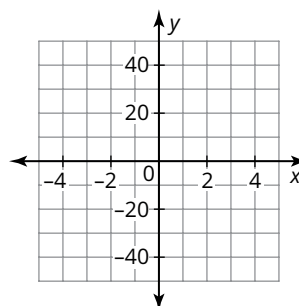
Factor the quadratic.

$$= (3x + 5)(3x + 2)$$

Substitute $3x$ for z .

The factored form of $9x^2 + 21x + 10$ is $(3x + 5)(3x + 2)$.

1. Use chunking to factor and identify the zeros of $f(x) = 25x^2 + 20x - 21$. Then, sketch the polynomial.



Recall, a special form of a polynomial is a perfect square trinomial. A perfect square trinomial has first and last terms that are perfect squares and a middle term that is equivalent to 2 times the product of the first and last term's square root. You used this form to factor quadratic expressions. For example, the quadratic expression $4x^2 - 12x + 9$ is a perfect square trinomial since 4 and 9 are both perfect squares and $12 = \sqrt{4} \cdot \sqrt{9}$. The factored form of the expression is $(2x - 3)^2$. You can also use this strategy to factor degree four polynomial expressions that are perfect square trinomials.

Factoring a perfect square trinomial can occur in two forms.

$$a^2 - 2ab + b^2 = (a - b)^2$$

$$a^2 + 2ab + b^2 = (a + b)^2$$

2. Determine which of the polynomial expression(s) is a perfect square trinomial and write it as the square of a sum or difference. If it is not a perfect square trinomial, explain why not.

a. $64x^2 - 32x + 4$



b. $x^4 + 14x^2 - 49$

c. $9x^4 + 6x^2 + 1$

d. $16x^4 - 40x^2 + 100$

In polynomials of four terms, you may notice that although not all terms share a common factor, pairs of terms might share a common factor. In this situation, you can factor by grouping.

3. Carlos factors the polynomial expression $x^3 + 3x^2 - x - 3$.

Carlos



$$x^3 + 3x^2 - x - 3$$

$$x^2(x + 3) - 1(x + 3)$$

$$(x + 3)(x^2 - 1)$$

$$(x + 3)(x + 1)(x - 1)$$

- a. Explain the steps Carlos took to factor the polynomial expression.

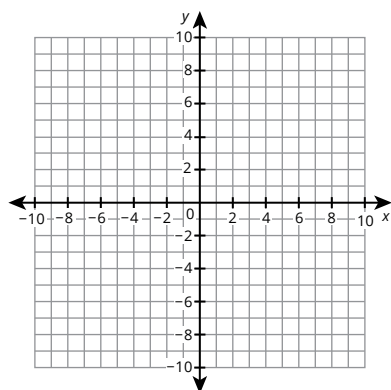
$$x^3 + 3x^2 - x - 3$$

$$x^2(x + 3) - (1x + 3) \quad \text{Step 1:}$$

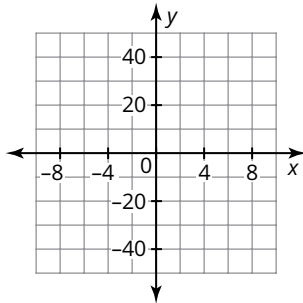
$$(x + 3)(x^2 - 1) \quad \text{Step 2:}$$

$$(x + 3)(x + 1)(x - 1) \quad \text{Step 3:}$$

- b. Use the factors to identify the zeros of $f(x) = x^3 + 3x^2 - x - 3$ and then sketch the graph.



4. Use factor by grouping to factor and identify the zeros of $f(x) = x^3 + 7x^2 - 4x - 28$. Then, sketch the polynomial.



5. Camilla and Isaiah both factor the polynomial expression $x^3 + 2x^2 + 4x + 8$. Analyze the set of factors in each student's work. Describe the set of numbers over which each student factored.

Camilla



$$x^3 + 2x^2 + 4x + 8$$

$$x^2(x + 2) + 4(x + 2)$$

$$(x^2 + 4)(x + 2)$$

Isaiah



$$x^3 + 2x^2 + 4x + 8$$

$$x^2(x + 2) + 4(x + 2)$$

$$(x^2 + 4)(x + 2)$$

$$(x + 2i)(x - 2i)(x + 2)$$

Review the definition of **describe** in the Academic Glossary.

According to the Fundamental Theorem of Algebra, any polynomial function of degree n must have exactly n complex factors:
 $f(x) = (x - r_1)(x - r_2) \dots (x - r_n)$ where $r \in \{\text{complex numbers}\}$.

6. Use factor by grouping to factor each polynomial over the set of real numbers.

a. $x^4 - 5x^3 + 6x^2 - 30x$

b. $x^3 + 3x^2 + 2x + 6$

c. $2x^3 - 8x^2 + 3x - 12$

d. $3x^4 + 6x^3 + 9x^2 + 18x$

7. Identify which expressions in Question 6 can be further factored over the set of complex numbers. Then, factor those expressions completely.

Some degree-4 polynomials, written as a trinomial $ax^4 + bx^2 + c$, have the same structure as quadratics. When this is the case, the polynomial may be factored using the same methods you would use to factor a quadratic. This is called *factoring using quadratic form*.

WORKED EXAMPLE

Factor $x^4 - 29x^2 + 100$ using quadratic form.

$$x^4 - 29x^2 + 100$$

Determine whether you can factor the given trinomial into 2 factors.

$$(x^2 - 4)(x^2 - 25)$$

Determine whether you can continue to factor each binomial.

$$(x - 2)(x + 2)(x - 5)(x + 5)$$

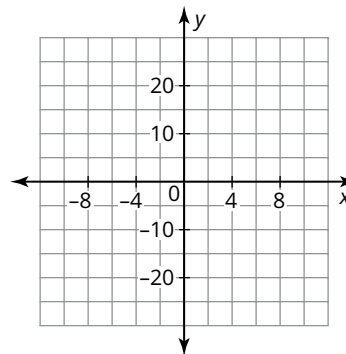
Remember ...

You can use the difference of two squares to factor a binomial of the form $a^2 - a^2$.

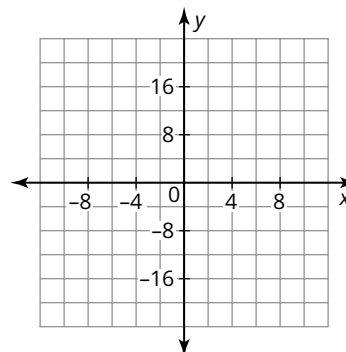
The binomial $a^2 - a^2 = (a + b)(a - b)$.

8. Factor each polynomial over the set of complex numbers, then sketch a graph of the function.

a. $f(x) = x^4 - 4x^3 - x^2 + 4x$



b. $f(x) = x^4 - 10x^2 + 9$



Ask Yourself ...

What tools or strategies can you use to solve this problem?

Factoring Polynomials

You have learned many different methods of factoring polynomial expressions. Depending on the polynomial, some methods of factoring are more efficient than others.

1. Factor each polynomial expression over the set of real numbers.
Explain why you chose the method you used to factor.

a. $x^3 - x^2 - 4x + 4$

b. $x^4 - 2x^3 + 4x - 8$

c. $3x^3 + 9x^2 + 12x + 36$

d. $6x^4 + 9x^3 - 2x - 3$

e. $4x^4 - 12x^2 + 9$

f. $x^4 + 6x^2 + 8$

g. $12x^3 - 18x^2 + 6x$

h. $2x^4 - 7x^2 + 3$

2. Which polynomial expressions from Question 1 can be further factored over the set of complex numbers? Factor these polynomials completely over the set of complex numbers.



Talk the Talk

Fracture It to Factor It

Some methods of factoring you used in this lesson include:

- Factoring Out the Greatest Common Factor
- Factoring by Grouping
- Perfect Square Trinomials

Depending on the polynomial, some methods of factoring are more efficient than others.

1. Complete the table on the next page by matching each polynomial with the method of factoring you would use from the bulleted list given. Explain why you chose the factoring method for each polynomial. Finally, write the polynomial in factored form over the set of real numbers.

Polynomial	Method of Factoring	Reason	Factored Form
$x^4 - 12x^2 + 36$			
$2x^4 + 10x^3 + 12x^2$			
$x^3 + 2x^2 + 7x + 14$			
$2x^3 + 9x^2 - 2x - 9$			
$x^4 + 5x^3 - 4x^2 - 20x$			
$6x^4 + 9x^3 - 8x^2 - 12x$			

Lesson 2 Assignment

Write

Describe the similarity between factoring a quadratic expression and factoring a degree-four expression using the perfect square trinomial method.

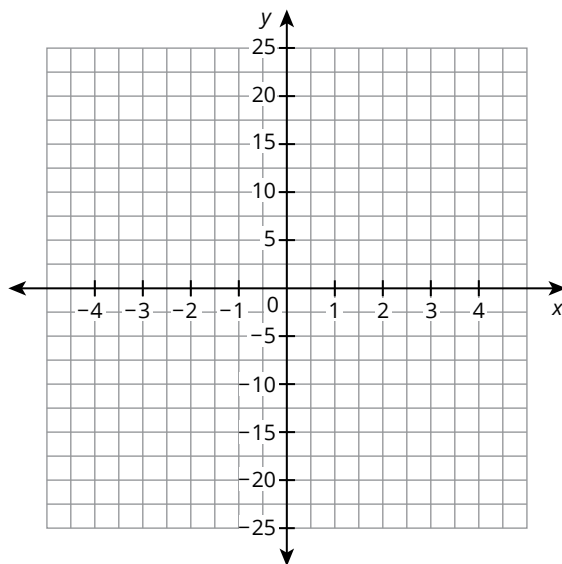
Remember

You can factor out the GCF of a polynomial and then factor what remains. Analyzing the structure of a polynomial can help you decide the most efficient method for factoring. Once you have factored a polynomial, you can use the factors to identify the zeros and then use the zeros to sketch a graph.

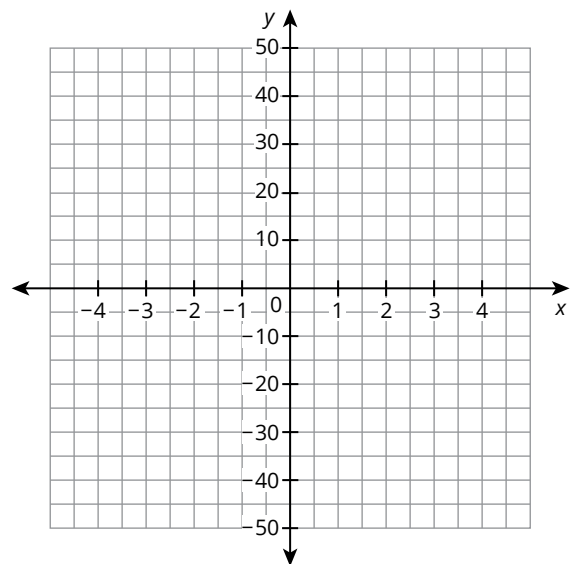
Practice

- Factor each polynomial over the set of real numbers to determine its linear factors. Use the factors to sketch the graph of each polynomial.

a. $f(x) = 25x^2 - 10x - 24$

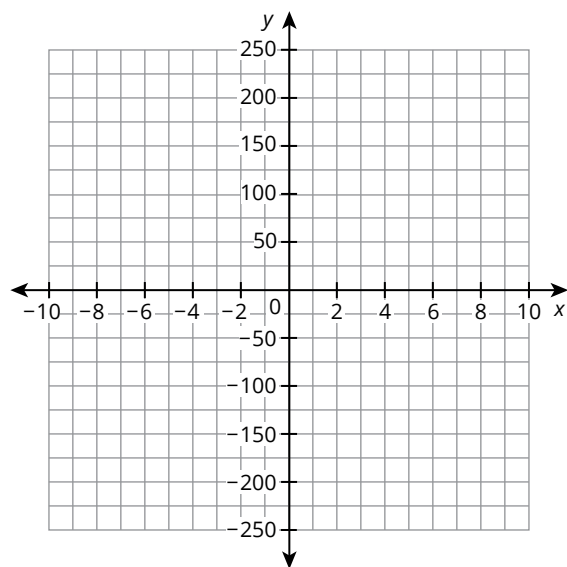


b. $f(x) = x^3 - 4x^2 - 9x + 36$

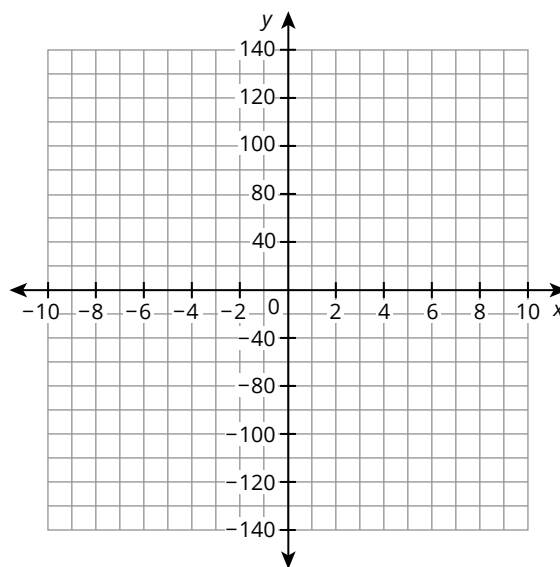


Lesson 2 Assignment

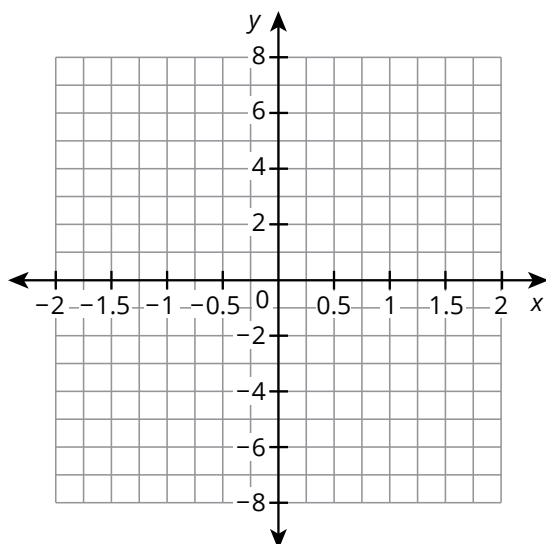
c. $f(x) = x^4 - 25x^2 + 144$



e. $f(x) = 2x^4 - 34x^2 + 32$



d. $f(x) = 6x^3 + 5x^2 - 6x - 5$



Lesson 2 Assignment

2. Factor each expression over the set of real numbers.

a. $x^3 + 4x^2 - x - 4$

b. $x^3 + 5x^2 - 2x - 10$

c. $3x^4 + 14x^3 + 3x^2 + 14x$

d. $2x^4 + 10x^3 + 4x^2 + 20x$

3. Factor each expression over the set of complex numbers.

a. $x^3 - x^2 + x - 1$

b. $5x^3 - 7x^2 + 20x - 28$

c. $x^4 + 13x^2 + 36$

d. $2x^4 + 7x^2 + 6$

Prepare

Determine each quotient.

1. $9x \div 3x$

2. $4x^2 \div 2x$

3. $x^3 \div x$

3

Polynomial Division

OBJECTIVES

- Describe similarities between polynomials and integers.
- Determine factors of a polynomial using one or more roots of the polynomial.
- Compare polynomial long division to integer long division.
- Determine factors through polynomial long division.
- Use the remainder theorem to evaluate polynomial equations and functions.

KEY TERMS

- factor theorem
- polynomial long division
- remainder theorem
- synthetic division

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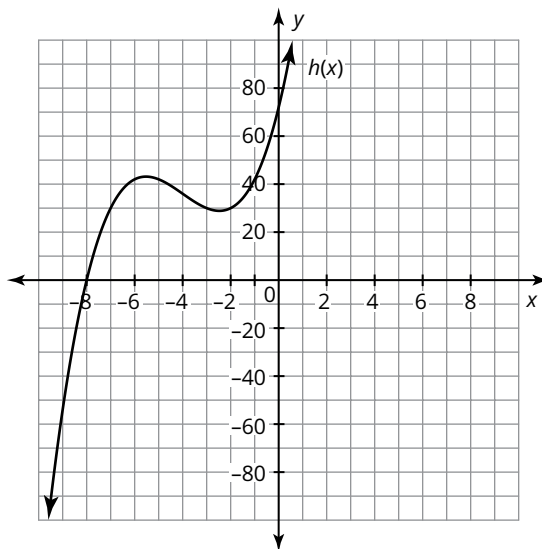
You know how to divide integers using the long division algorithm.
How can you use a similar algorithm to divide polynomials?

Getting Started

The $x - r$ Factor

You have analyzed the graphs of polynomials to determine the type and location of the zeros. How can you determine the factors of a polynomial given its algebraic representation and one of its factors?

1. Analyze the graph of the function $h(x) = x^3 + 12x^2 + 41x + 72$.



- a. Describe the key attributes of $h(x)$.
- b. Describe the number and types of zeros of $h(x)$.

The **factor theorem** states that a polynomial function $p(x)$ has $(x - r)$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$.

You can use the factor theorem to show that a linear expression is a factor of a polynomial.

WORKED EXAMPLE

Consider the graph of the polynomial function $h(x) = x^3 + 12x^2 + 41x + 72$ in Question 1.

The graph appears to have a zero at $(-8, 0)$, so a possible linear factor of the polynomial is $(x + 8)$.

Determine the value of the polynomial at $x = -8$, or $h(-8)$.

$$\begin{aligned} h(-8) &= (-8)^3 + 12(-8)^2 + 41(-8) + 72 \\ &= -512 + 768 + (-328) + 72 \\ &= 0 \end{aligned}$$

So, $(x + 8)$ is a linear factor of the polynomial function.

2. Consider that $d(x) = (x + 8)$, and $d(x) \cdot q(x) = h(x)$.

a. What do you know about the function $q(x)$?

b. Can you write the algebraic representation for $q(x)$?
Explain your reasoning.

To solve $0 = x^3 + 12x^2 + 41x + 72$, you need to factor the polynomial and use the zero product property to determine its zeros.

Remember ...

Recall that $a \div b$ is $\frac{a}{b}$, where $b \neq 0$.

The Fundamental Theorem of Algebra states that every polynomial equation of degree n must have n roots. This means that every polynomial can be written as the product of n factors of the form $(ax + b)$. For example, $2x^2 - 3x - 9 = (2x + 3)(x - 3)$. You knew this polynomial would have two factors since the polynomial had a degree of $n = 2$.

If 2 is a factor of 24, then 24 can be divided by 2 without a remainder. In the same way, the factors of a polynomial divide into that polynomial without a remainder.

Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree. The process is similar to integer long division.

WORKED EXAMPLE

Integer Long Division	Polynomial Long Division
$3660 \div 12$ or $\begin{array}{r} 3660 \\ 12 \overline{) 3660} \\ \underline{36} \\ 0 \\ \underline{0} \\ 0 \end{array}$	$(x^3 + 12x^2 + 41x + 72) \div (x + 8)$ $\begin{array}{r} \textcircled{A} x^2 + \textcircled{D} 4x + \textcircled{G} 9 \\ x + 8 \overline{) x^3 + 12x^2 + 41x + 72} \\ \underline{\textcircled{B} (x^3 + 8x^2)} \\ \textcircled{E} 4x^2 + 41x \\ \underline{-(4x^2 + 32x)} \\ \textcircled{F} 9x + 72 \\ \underline{\textcircled{H} (9x + 72)} \\ \text{Remainder } 0 \end{array}$ or $\frac{x^3 + 12x^2 + 41x + 72}{x + 8}$ A. Divide $\frac{x^3}{x} = x^2$. B. Multiply $x^2(x + 8)$, and then subtract. C. Bring down $41x$. D. Divide $\frac{4x^2}{x} = 4x$. E. Multiply $4x(x + 8)$, and then subtract. F. Bring down $+72$. G. Divide $\frac{9x}{x} = 9$. H. Multiply $9(x + 8)$, and then subtract.

1. Analyze the Worked Example that shows integer long division and polynomial long division.

- a. Rewrite each expression as a product of its factors.

$$3660 = \underline{\hspace{2cm}} \cdot \underline{\hspace{2cm}}$$

$$x^3 + 12x^2 + 41x + 72 = \underline{\hspace{2cm}} \cdot \underline{\hspace{2cm}}$$

- b. Is $h(x)$ completely factored? Explain your reasoning.

- c. Rewrite the function as a product of its linear factors.

- d. Determine the zeros of the function.

2. The expression $(x - 7)$ is a factor of $x^3 - 10x^2 + 11x + 70$.
Solve $0 = x^3 - 10x^2 + 11x + 70$ over the set of complex numbers.

You can use the same process to determine the quotient of a polynomial and a non-linear divisor.

WORKED EXAMPLE

Polynomial Long Division by Degree Two

$$(2x^4 - 11x^2 - 63) \div (2x^2 + 7) \quad \text{or} \quad \frac{2x^4 - 11x^2 - 63}{2x^2 + 7}$$

$$\begin{array}{r}
 \textcircled{\text{A}} \quad x^2 \qquad \qquad \textcircled{\text{D}} \quad -9 \\
 2x^2 + 7 \overline{) 2x^4 + 0x^3 - 11x^2 + 0x - 63} \\
 \textcircled{\text{B}} \quad -(2x^4 \qquad + 7x^2) \qquad \downarrow \qquad \downarrow \textcircled{\text{C}} \\
 \hline
 -18x^2 + 0x - 63 \\
 \textcircled{\text{E}} \quad -(-18x^2 \qquad - 63) \\
 \hline
 \text{Remainder } 0
 \end{array}$$

A. Divide $\frac{2x^4}{2x^2} = x^2$.

B. Multiply $x^2(2x^2 + 7)$, and then subtract.

C. Bring $0x - 63$.

D. Divide $\frac{-18x^2}{2x^2} = -9$.

E. Multiply $-9(2x^2 + 7)$, and then subtract.

3. Determine each quotient.

a. $(2x^3 - x^2 - 11x - 12) \div (x - 3)$

b. $(8x^3 + 22x^2 + 21x + 9) \div (4x^2 + 5x + 3)$

c. $(12x^4 - 8x^3 + 6x - 4) \div (3x - 2)$

d. $(-2x^4 + 5x^3 - 3x^2 - 2x + 8) \div (x^2 - x - 2)$

4. In what ways are the integer and polynomial long division algorithms similar?

Factoring Special Binomials

Recall that you can use the difference of squares to factor a binomial of the form $a^2 - b^2$. The binomial $a^2 - b^2 = (a + b)(a - b)$.

1. Use the difference of squares to factor each binomial over the set of real numbers.

a. $x^2 - 64$

b. $x^4 - 16$

c. $x^8 - 1$

d. $x^4 - y^4$

Now, let's consider expressions composed of perfect cubes, such as $f(x) = x^3 - 8$.

2. Consider Mason's and Sebastian's work.

Mason

I can use the Properties of Equality to determine the factors.

$$x^3 - 8 = 0$$

$$x^3 = 8$$

$$x = 2$$

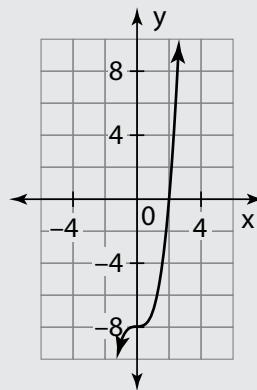
$$f(x) = (x - 2)(x - 2)(x - 2)$$

or

$$f(x) = (x - 2)^3$$



Sebastian



I looked at the graph of $f(x) = x^3$ and could see that $x = 2$ is one of the zeros, but the other two zeros are imaginary.



a. Describe Mason's error.

b. Use long division to factor $f(x) = x^3 - 8$ over the set of real numbers.

You can rewrite the expression $x^3 - 27$ as $(x)^3 - (3)^3$, and $x^3 + 27$ as $(x)^3 + (3)^3$. When you factor sums and differences of cubes, there is a special factoring formula you can use, which is similar to the difference of squares for quadratics.

To determine the formula for the difference of cubes, generalize the difference of cubes as $a^3 - b^3$.

WORKED EXAMPLE

To determine the factor formula for the difference of cubes, factor out $(a - b)$ by considering $(a^3 - b^3) \div (a - b)$.

$$\begin{array}{r}
 a^2 + ab + b^2 \\
 a - b \overline{) a^3 - 0a^2b + 0ab^2 - b^3} \\
 \underline{-(a^3 - a^2b)} \\
 a^2b + 0ab^2 \\
 \underline{-(a^2b - ab^2)} \\
 ab^2 - b^3 \\
 \underline{-(ab^2 - b^3)} \\
 0
 \end{array}$$

Therefore, the difference of cubes can be rewritten in factored form as:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

3. Use properties of equality to determine one zero. Then, factor each polynomial function over the set of real numbers.

a. $f(x) = x^3 - 27$

b. $g(x) = x^3 + 27$

.....
When performing long division, make sure that the dividend is in descending order. If any powers are not included, use a zero to help with spacing.
.....

4. Determine the formula for the sum of cubes by dividing $a^3 + b^3$ by $(a + b)$.

5. Use the sum or difference of cubes to factor each binomial over the set of real numbers.

a. $x^3 + 125$

b. $8x^3 - 1$

c. $x^6 - 8$

d. $x^9 + y^9$

The Remainder Theorem

You learned that the process of dividing polynomials is similar to the process of dividing integers. Sometimes when you divide two integers, there is a remainder, and sometimes there is not a remainder. What does each case mean? In this activity, you will investigate what the remainder means in terms of polynomial division.

1. Determine the quotient for each. Show all of your work.

a. $(x^3 + 2x^2 - 5x + 16) \div (x - 4)$

b. $(4x^4 + 5x^2 - 7x + 9) \div (2x - 3)$

c. $(6x^4 - 11x^3 - 2x^2 - x + 2) \div (3x^2 + 2x + 1)$

Ask Yourself . . .

Is the dividend in descending order?
Are there any powers of x not included?

d. $(2x^3 + 13x^2 + 19x + 30) \div (2x^2 + 3x + 4)$

2. Consider Question 1 parts (a) through (d) to answer each.

a. When there is a remainder, is the divisor a factor of the dividend?
Explain your reasoning.

b. Describe the remainder when you divide a polynomial by one of its factors.

Remember from your experiences with division that:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

or

$$\text{dividend} = (\text{divisor}) (\text{quotient}) + \text{remainder}.$$

It follows that any polynomial, $p(x)$, can be written in the form:

$$\frac{p(x)}{\text{linear expression}} = \text{quotient} + \frac{\text{remainder}}{\text{linear expression}}$$

or

$$p(x) = (\text{linear expression}) (\text{quotient}) + \text{remainder}.$$

Generally, the linear expression is written in the form $(x - r)$, the quotient is represented by $q(x)$, and the remainder is represented by R .

$$p(x) = (x - r) q(x) + R$$

3. Consider each dividend in Questions 1a, 1b, and 1c as a function, $p(x)$.

a. In part (a), evaluate $p(x)$ for $x = 4$.

b. In part (b), evaluate $p(x)$ for $x = \frac{3}{2}$.

c. In part (c), evaluate $p(x)$ for $x = -\frac{2}{3}$.

d. How does the remainder relate to the divisor in each problem?

.....
Remember...

The factor theorem states that a polynomial has a linear polynomial as a factor if and only if the remainder is zero. Therefore, if $R = 0$, then $f(r) = 0$, and $(x - r)$ is a factor of $f(x)$.
.....

4. What conclusion can you make about any polynomial evaluated at r ?

The **remainder theorem** states that when any polynomial function, $f(x)$, is divided by a linear expression of the form $(x - r)$, the remainder $R = f(r)$, or the value of the function when $x = r$.

5. Given $p(x) = x^3 + 6x^2 + 5x - 12$ and $\frac{p(x)}{(x - 2)} = x^2 + 8x + 21 \text{ R } 30$, Parker says that $p(-2) = 30$ and Avery says that $p(2) = 30$. Without performing any calculations, who is correct? Explain your reasoning.



6. The function $f(x) = 4x^2 + 2x + 9$ generates the same remainder when divided by $(x - r)$ and $(x - 2r)$, where r is not equal to 0. Calculate the value(s) of r .

7. Determine the unknown in each.

a. $\frac{x}{7} = 18$ R 2. Determine x .

b. $\frac{p(x)}{x+3} = 3x^2 + 14x + 15$ R 3. Determine the function $p(x)$.

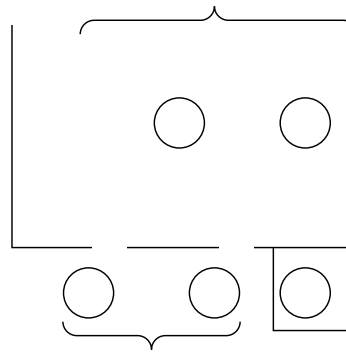
c. Describe the similarities and differences in your solution strategies.

Synthetic Division

Although dividing polynomials through long division is analogous to integer long division, it can still be inefficient and time consuming.

Synthetic division is a more efficient method for dividing a polynomial by a linear expression of the form $(x - r)$. This method requires fewer calculations and less writing by representing the polynomial and the linear factor as a set of numeric values. After the values are processed, you can then use the numeric outputs to construct the quotient and the remainder.

To use synthetic division to divide a polynomial $ax^2 + bx + c$ by a linear factor $x - r$, follow this pattern.



.....

Notice in the form of the linear factor $(x - r)$, the x has a coefficient of 1.

Also, just as in long division, when you use synthetic division, every power of the dividend must have a placeholder.

.....

WORKED EXAMPLE

You can use synthetic division in place of the standard long division algorithm to determine the quotient for $(x^3 + 12x^2 + 41x + 72) \div (x + 8)$.

Long Division	Synthetic Division
$ \begin{array}{r} x^2 + 4x + 9 \\ x + 8 \overline{) x^3 + 12x^2 + 41x + 72} \\ \underline{-(x^3 + 8x^2)} \\ 4x^2 + 41x \\ \underline{-(4x^2 + 32x)} \\ 9x + 72 \\ \underline{-(9x + 72)} \\ 0 \end{array} $	$ \begin{array}{ccccccc} r = -8 & & & & & & \\ -8 & 1 & 12 & 41 & 72 & & \\ & \downarrow & & & & & \\ & 1 & & & & & \\ & \nearrow \text{multiply by } r & \downarrow & & & & \\ & & -8 & & & & \\ & & \downarrow & \text{add} & & & \\ & & 4 & & & & \\ & & \nearrow \text{multiply by } r & \downarrow & & & \\ & & & -32 & & & \\ & & & \downarrow & \text{add} & & \\ & & & 9 & & & \\ & & & \nearrow \text{multiply by } r & \downarrow & & \\ & & & & -72 & & \\ & & & & \downarrow & \text{add} & \\ & & & & 0 & & \end{array} $
$(x^3 + 12x^2 + 41x + 72) \div (x + 8) = x^2 + 4x + 9$	

.....

Think About...

When you use synthetic division, you are multiplying and adding, as opposed to multiplying and subtracting when you use long division.

.....

1. Analyze the Worked Example.

- What is the remainder?
- Write the dividend as the product of its factors.
- Why does the synthetic division algorithm work?

Ask Yourself...

How can you tell by looking at the synthetic division process whether the divisor is a factor of the polynomial?

2. An example of synthetic division is provided. Perform the steps outlined for the problem.

$$\begin{array}{r|rrrrr} 2 & 1 & 0 & -4 & -3 & 6 \\ & & 2 & 4 & 0 & -6 \\ \hline & 1 & 2 & 0 & -3 & 0 \end{array}$$

- Write the dividend.
- Write the divisor.
- Write the quotient.
- Write the dividend as the product of the divisor and the quotient.

3. Complete the synthetic division for $(2x^4 - 4x^3 + 4x^2 - 3x + 6) \div (x + 3)$. Write the dividend as the product of the divisor and the quotient plus the remainder.

Synthetic division works only for linear divisors in the form $(x - r)$. If the divisor has a leading coefficient other than 1, you may need to factor out a constant in order to rewrite the divisor in the form $(x - r)$.

WORKED EXAMPLE

You can use synthetic division to determine the quotient of $\frac{2x^3 - 6x^2 + 4x + 2}{2x - 3}$. Since the divisor is not in the form $x - r$, you must rewrite $2x - 3$ as $2\left(x - \frac{3}{2}\right)$.

$$\begin{array}{r|rrrr}
 r = \frac{3}{2} & 2 & -6 & 4 & 2 \\
 & & 3 & -\frac{9}{2} & -\frac{3}{4} \\
 \hline
 & 2 & -3 & -\frac{1}{2} & \frac{5}{4}
 \end{array}$$

multiply by r add multiply by r add multiply by r add

The numbers in the last row become the coefficients of the quotient.

$$2x^2 - 3x - \frac{1}{2} \text{ R } \frac{5}{4}$$

You can write the dividend as the product of the divisor and the quotient plus the remainder.

$$2x^3 - 6x^2 + 4x + 2 = \left(x - \frac{3}{2}\right)\left(2x^2 - 3x - \frac{1}{2}\right) + \frac{5}{4}$$

To determine the quotient of $\frac{2x^3 - 6x^2 + 4x + 2}{2x - 3}$, you need to return to the original divisor of $(2x - 3)$. To do so, double $\left(x - \frac{3}{2}\right)$ and halve $\left(2x^2 - 3x - \frac{1}{2}\right)$.

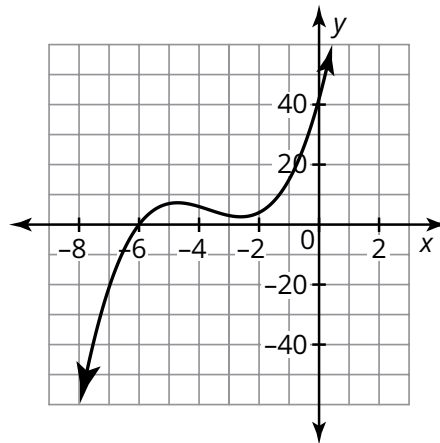
$$\begin{aligned}
 2x^3 - 6x^2 + 4x + 2 &= 2\left(x - \frac{3}{2}\right)\left(\frac{1}{2}\left(2x^2 - 3x - \frac{1}{2}\right)\right) + \frac{5}{4} \\
 &= (2x - 3)\left(x^2 - \frac{3}{2}x - \frac{1}{4}\right) + \frac{5}{4}
 \end{aligned}$$

$$\text{Therefore, } \frac{2x^3 - 6x^2 + 4x + 2}{2x - 3} = x^2 - \frac{3}{2}x - \frac{1}{4} + \frac{5}{4(2x - 3)}.$$

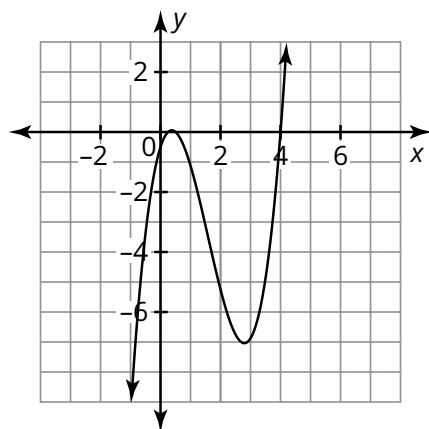
4. Verify $(3x - 2)(x^2 + x + 1) = 3x^3 + x^2 + x - 2$ using synthetic division. Show all work and explain your reasoning.

5. Determine the zeros of each function. Then, rewrite each function as a product of its linear factors. Explain your process.

a. $f(x) = x^3 + 11x^2 + 37x + 42$



b. $f(x) = x^3 - 4.75x^2 + 3.125x - 0.50$





Talk the Talk

A Polynomial Divided

1. Given the information, determine whether each statement is true or false. Explain your reasoning.

$$p(x) = x^3 + 6x^2 + 11x + 6, \text{ and} \\ p(x) \div (x + 4) = x^2 + 2x + 3 \text{ R } -6$$

a. $p(-4) = -6$

b. $p(x) = (x + 4)(x^2 + 2x + 3) - 6$

c. $(x - 3)$ is a factor of $p(x)$

d. $(x + 2)$ is a factor of $p(x)$

2. Explain the difference between the remainder theorem and the factor theorem.

3. Determine each quotient.

a. $(2x^3 - x^2 + 8x + 4) \div (2x - 1)$

b. $(9x^3 - 21x^2 + 3x - 2) \div (3x^2 + 1)$

c. $(8x^4 - 7x^2 - 11x - 1) \div (x - 2)$

d. $(x^4 + x^3 + 2x^2 + 34x - 16) \div (x^2 - x + 8)$

Lesson 3 Assignment

Write

Write an example for each term using the dividend $x^3 + 2x + 4$ and the divisor $x + 1$.

1. Factor theorem
2. Polynomial long division
3. Remainder theorem
4. Synthetic division

Remember

A polynomial function $p(x)$ has $(x - r)$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$. When any polynomial equation or function $f(x)$ is divided by a linear expression of the form $(x - r)$, the remainder is $R = f(r)$, or the value of the equation or function when $x = r$.

Practice

1. Determine each quotient.

a. $(x^3 - 64) \div (x - 4)$

b. $(2x^3 + 14x^2 - 3x - 21) \div (2x^2 - 3)$

c. $(5x^4 - 15x^3 + 2x - 5) \div (x - 3)$

d. $(6x^4 + 19x^2 + 10) \div (2x^2 + 7)$

2. Factor each binomial over the set of real numbers.

a. $4x^2 - 9y^2$

b. $x^3 + 216$

c. $5x^3 - 40$

d. $x^3 + 27y^3$

Lesson 3 Assignment

3. The pool company offers 10 different pool designs numbered 1 through 10. Each pool is in the shape of a rectangular prism. The volume of water in Pool Design x can be determined using the function $V(x) = l(x) \cdot w(x) \cdot d(x) = 2x^3 + 18x^2 + 46x + 30$, where $l(x)$, $w(x)$, and $d(x)$ represent the length, width, and depth of the pool in feet.
- a. Determine the expressions for the functions $w(x)$ and $d(x)$ if $l(x) = 2x + 2$ and the width of each pool is greater than the depth. Do not use a calculator.
- b. Determine the length, width, and depth of Pool Design 9.
4. The function $m(x) = 2x^2 + 6x - 7$ generates the same remainder when divided by $(x - a)$ and $(x - 2a)$ when $a \neq 0$. Calculate the value(s) of a and determine the corresponding factors.

Lesson 3 Assignment

5. The given table of values represents the function $f(x) = x^3 + 9x^2 + 14x - 24$.

x	-2	-1	0	1	2
$f(x)$	-24	-30	-24	0	48

- Determine one of the factors of $f(x)$ without using a calculator.
Explain your reasoning.
- Completely factor $f(x)$ without using a calculator.
- Determine all of the zeros of $f(x)$ without using a calculator.

Prepare

Factor the functions over the set of complex numbers.

1. $f(x) = 2x^2 + 4x + 5$

2. $g(x) = x^3 - 2x^2 + 9x - 18$

3. $h(x) = x^4 - 20x^2 - 125$

4

Rational Root Theorem

OBJECTIVES

- Use the rational root theorem to determine possible roots of a polynomial.
- Use the rational root theorem to factor higher-order polynomials and summarize the process.
- Solve higher-order polynomials.

KEY TERM

- rational root theorem

.....

How can the rational root theorem, synthetic division, factoring and quadratic formula be used to determine all real and complex roots of a polynomial equation?

Getting Started

Ideas Taking Root

Consider the product and sum of each set of roots.

Polynomial	Roots	Product of Roots	Sum of Roots
$x^2 + 4x - 1 = 0$	$-2 \pm \sqrt{5}$	-1	-4
$x^3 + 2x^2 - 5x - 6 = 0$	$-1, 2, -3$	6	-2
$2x^3 + 5x^2 - 8x - 20 = 0$	$\pm 2, -\frac{5}{2}$	10	$-\frac{5}{2}$
$4x^3 - 3x^2 + 4x - 3 = 0$	$\pm i, \frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$36x^3 + 24x^2 - 43x + 86 = 0$	$\frac{2}{3} \pm \frac{\sqrt{3}}{2}i, -2$	$-\frac{43}{18}$	$-\frac{2}{3}$
$4x^4 - 12x^3 + 13x^2 - 2x - 6 = 0$	$1 \pm i, -\frac{1}{2}, \frac{3}{2}$	$-\frac{3}{2}$	3

1. Compare the sums of the roots to the first two coefficients of each polynomial equation. What conclusion can you draw?
2. Compare the products of the roots to the first and last coefficients of each odd degree polynomial equation. What conclusion can you draw?
3. Compare the products of the roots to the first and last coefficients of each even degree polynomial equation. What conclusion can you draw?

.....
 These patterns
 will help you
 factor higher-order
 polynomials.

ACTIVITY 4.1

Introduction to the Rational Root Theorem

In the Getting Started, you discovered some patterns about the relationships between the coefficients of polynomials and their roots. These patterns can be used to create and test the possible rational roots of a polynomial expression.

The **Rational Root Theorem** states that a rational root of a polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0x^0 = 0$ with integer coefficients is of the form $\frac{p}{q}$, where p is a factor of the constant term, a_0 , and q is a factor of the leading coefficient, a_n .

1. Jasmine and Madison each use the rational roots theorem to list all possible rational roots for the polynomial equation they are given.

Jasmine



$$4x^4 - 2x^3 + 5x^2 + x - 10 = 0$$

p could equal any factors of -10 ,
so $\pm 1, \pm 2, \pm 5, \pm 10$

q could equal any factors of 4 , so
 $\pm 1, \pm 2, \pm 4$

Therefore, possible zeros are

$$\frac{p}{q} = \pm 1, \pm 2, \pm 5, \pm 10, \pm \frac{1}{2}, \pm \frac{5}{2}, \pm \frac{1}{4}, \pm \frac{5}{4}$$

Madison



$$6x^5 - 2x^3 + x^2 - 3x - 15 = 0$$

p could equal any factors of -15 ,
so $\pm 1, \pm 3, \pm 5, \pm 15$

q could equal any factors of 6 , so
 $\pm 2, \pm 3, \pm 6$

Therefore, possible zeros are

$$\frac{p}{q} = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{2}, \pm \frac{1}{3}, \pm 1, \pm \frac{5}{3}, \pm 5, \pm \frac{1}{6}, \pm \frac{5}{6}$$

Explain why Madison is incorrect and correct her work.

2. Complete each step to factor and solve $x^3 - x^2 - 16x - 20 = 0$.
- Determine all the possible rational roots.
 - Use synthetic division to determine which of the possible roots are actual roots.
 - Rewrite the polynomial as a product of its quotient and linear factor.
 - Factor completely and solve.

3. Complete each step to factor and solve $x^4 + x^3 - 7x^2 - x + 6 = 0$.

a. Determine all the possible rational roots.

b. Use synthetic division to determine which of the possible roots are actual roots.

c. Rewrite the polynomial as a product of its quotient and linear factor.

d. Repeat steps *a* – *c* for the cubic expression.

e. Factor completely and solve.

4. Determine all roots for $x^4 - 7x^2 - 18 = 0$.

a. Determine the possible roots.

b. Use synthetic division to determine one of the roots.

c. Rewrite the original polynomial as a product.

d. Determine the possible rational roots of the quotient.

e. Use synthetic division to determine one of the roots.

f. Rewrite the original polynomial as a product.

g. Determine the possible rational roots of the quotient.

h. Determine the remaining roots.

i. Rewrite the original polynomial as a product.

Applying the Rational Root Theorem

You have learned many different ways to factor higher-order polynomials. To determine all the roots or solutions of a polynomial equation:

- Determine the possible rational roots.
- Use synthetic division to determine one of the roots.
- Rewrite the original polynomial as a product.
- Determine the possible rational roots of the quotient.
- Repeat the process until all the rational roots are determined.
- Factor the remaining polynomial to determine any irrational or complex roots.
- Recall that some roots may have a multiplicity.

..... 1. Solve each equation over the set of complex numbers.

a. $x^3 + 1 = 0$

.....
If a quadratic is not factorable, use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

.....

b. $x^4 + 3x^2 - 28 = 0$

c. $x^4 - 5x^2 - 6x - 2 = 0$

Ask Yourself . . .

Did you share
your solution with
others?

You can also use the rational roots theorem to determine the linear factors of a polynomial function.

2. Determine the linear factors of $f(x) = x^3 - 3x^2 + 9x - 27$.

a. Rewrite the function as a polynomial equation set equal to zero.

b. Determine the possible roots.

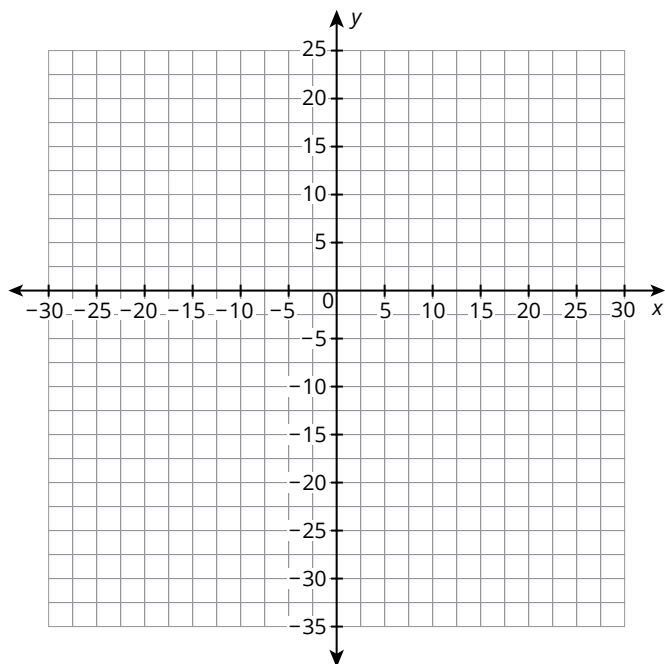
c. Use synthetic division to determine one of the roots.

d. Rewrite the original polynomial as a product.

e. Factor the quadratic factor.

f. Rewrite the polynomial function in factored form.

3. Use the factored form of the polynomial equation to graph the function.



Be prepared to share your solutions and methods.



Talk the Talk

Root Pursuit

1. Factor completely and solve over the set of complex numbers.

$$x^3 + 6x^2 + 13x + 20 = 0$$

Lesson 4 Assignment

Write

Write the step-by-step process in determining all roots of a polynomial equation.

Remember

The rational root theorem states that a rational root of a polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0x^0$ with integer coefficients is of the form $\frac{p}{q}$, where p is a factor of the constant term, a_0 , and q is a factor of the leading coefficient, a_n .

Practice

1. Use the Rational Root Theorem to solve $x^3 - 2x^2 - 5x - 2 = 0$.

Lesson 4 Assignment

2. Use the Rational Root Theorem to solve $2x^3 - 2x^2 - 6x - 18 = 0$.

Lesson 4 Assignment

3. Use the Rational Root Theorem to solve
 $2x^4 - 9x^3 + 11x^2 - 9x + 9 = 0$.

Lesson 4 Assignment

Prepare

Rewrite each fraction in lowest terms.

1. $\frac{12}{3}$

2. $\frac{3}{12}$

3. $\frac{0}{3}$

4. $\frac{3}{0}$

Relating Factors and Zeros

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

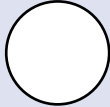
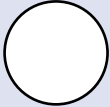
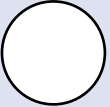
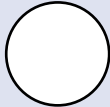
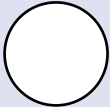
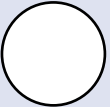
Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Relating Factors and Zeros* topic by:

TOPIC 3: <i>Relating Factors and Zeros</i>	Beginning of Topic	Middle of Topic	End of Topic
adding, subtracting, and multiplying polynomials.			
dividing polynomials.			
extending patterns of factoring to include polynomials.			
factoring polynomials using any method.			
using factoring to identify the zeros of a polynomial function where applicable.			
sketching graphs of polynomials using zeros.			
determining linear and quadratic factors of degree three and degree four polynomial functions.			
using the factor theorem to determine whether a linear expression is a factor of a given polynomial.			

continued on the next page

TOPIC 3 SELF-REFLECTION
continued

TOPIC 3: <i>Relating Factors and Zeros</i>	Beginning of Topic	Middle of Topic	End of Topic
using the remainder theorem to show the relationship between a factor and a zero.			
using the rational root theorem to factor a polynomial completely.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

- Describe a new strategy you learned in the *Relating Factors and Zeros*.

- What mathematical understandings from the topic do you feel you are making the most progress with?

- Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 3 SUMMARY

Relating Factors and Zeros Summary

LESSON

1

Adding, Subtracting, and Multiplying Polynomials

You can add and subtract polynomials by grouping terms that have the same variable and exponent.

To determine the sum $(3x^4 - 2x^2 + 5x + 1) + (2x^4 + x^3 + 3x - 5)$:

Step 1: Use the distributive property to rewrite the expressions without grouping symbols.

$$3x^4 - 2x^2 + 5x + 1 + 2x^4 + x^3 + 3x - 5$$

Step 2: Use the commutative property of addition to rearrange the terms.

$$3x^4 + 2x^4 + x^3 - 2x^2 + 5x + 3x + 1 - 5$$

Step 3: Use the associative property of addition to group like terms.

$$(3x^4 + 2x^4) + x^3 - 2x^2 + (5x + 3x) + (1 - 5)$$

Step 4: Simplify the like terms.

$$5x^4 + x^3 - 2x^2 + 8x - 4$$

To determine the difference $(3x^4 - 2x^2 + 5x + 1) - (2x^4 + x^3 + 3x - 5)$:

Step 1: Use the distributive property to rewrite the expressions without grouping symbols.

$$3x^4 - 2x^2 + 5x + 1 - 2x^4 - x^3 - 3x + 5$$

Step 2: Rewrite the terms using addition.

$$3x^4 + (-2x^2) + 5x + 1 + (-2x^4) + (-x^3) + (-3x) + 5$$

Step 3: Use the commutative property of addition to rearrange the terms.

$$3x^4 + (-2x^4) + (-x^3) + (-2x^2) + 5x + (-3x) + 1 + 5$$

Step 4: Use the associative property of addition to group like terms.

$$(3x^4 + (-2x^4)) + (-x^3) + (-2x^2) + (5x + (-3x)) + (1 + 5)$$

Step 5: Simplify the grouped terms.

$$x^4 - x^3 - 2x^2 + 2x + 6$$

KEY TERMS

- factor theorem [teorema del factor]
- polynomial long division [división (larga) de polinomios]
- remainder theorem
- synthetic division [división sintética]
- rational root theorem

You can use a multiplication table to multiply higher-order polynomials.

Determine the product $(4x^3 - 2x^2 + 3x - 1)(2x^4 - x^2 + 4x)$.

Step 1: Use a multiplication table to determine the product of the factors.

$\cdot$	$4x^3$	$-2x^2$	$3x$	-1
$2x^4$	$8x^7$	$-4x^6$	$6x^5$	$-2x^4$
$-x^2$	$-4x^5$	$2x^4$	$-3x^3$	x^2
$4x$	$16x^4$	$-8x^3$	$12x^2$	$-4x$

Step 2: Combine like terms.

$$\begin{aligned} & (4x^3 - 2x^2 + 3x - 1)(2x^4 - x^2 + 4x) \\ &= 8x^7 - 4x^6 + 2x^5 + 16x^4 - 11x^3 + 13x^2 - 4x \end{aligned}$$

LESSON

2

Factoring Polynomials to Identify Zeros

To begin factoring any polynomial, always look for the greatest common factor (GCF). You can factor out the greatest common factor of the polynomial and then factor what remains. A greatest common factor can be a variable, a constant, or both.

For example, $3x^3 + 12x^2 - 36x$ has a GCF of $3x$: $3x(x^2 + 4x - 12) = 3x(x + 6)(x - 2)$.

Some functions can be factored over the set of real numbers. However, all functions can be factored over the set of complex numbers.

For example, you can use the quadratic formula to determine that the zeros of the function $f(x) = x^2 + 2x + 5$ are $x = -1 + 2i$ and $x = -1 - 2i$. Therefore, the function in factored form is $(x - (-1 + 2i))(x - (-1 - 2i))$.

Some polynomials in quadratic form may have common factors in some of the terms, but not all terms. In this case, it may be helpful to write the terms as a product of 2 terms. You can then substitute the common term with a variable, z , and factor as you would any polynomial in quadratic form. This method of factoring is called *chunking*.

For example, you can use chunking to factor $9x^2 + 21x + 10$. Notice that the first and second terms both contain the common factor, $3x$.

$$\begin{aligned} 9x^2 + 21x + 10 &= (3x)^2 + 7(3x) + 10 && \text{Rewrite terms as a product of common factors.} \\ &= z^2 + 7z + 10 && \text{Let } z = 3x. \\ &= (z + 5)(z + 2) && \text{Factor the quadratic.} \\ &= (3x + 5)(3x + 2) && \text{Substitute } 3x \text{ for } z. \end{aligned}$$

The factored form of $9x^2 + 21x + 10$ is $(3x + 5)(3x + 2)$.

A special form of polynomial is a perfect square trinomial. In a perfect square trinomial, the first and last terms are perfect squares and the middle term is equivalent to 2 times the product of the square root of the first term and the square root of the last term.

Factoring a perfect square trinomial can occur in two forms:

$$a^2 - 2ab + b^2 = (a - b)^2$$

$$a^2 + 2ab + b^2 = (a + b)^2$$

In polynomials with 4 terms, you may notice that although not all terms share a common factor, pairs of terms might share a common factor. In this situation, you can *factor by grouping*.

For example, consider the polynomial $x^3 + 3x^2 - x - 3$.

$(x^3 + 3x^2) + (-x - 3)$ Divide the polynomial into two groups.

$x^2(x + 3) - 1(x + 3)$ Find the common factor in each group.

$(x^2 - 1)(x + 3)$ Regroup the factors.

$(x - 1)(x + 1)(x + 3)$ Factor the first group using the difference of perfect squares.

The factored form of $x^3 + 3x^2 - x - 3$ is $(x - 1)(x + 1)(x + 3)$.

Some degree-4 polynomials, written as trinomials, look very similar to quadratics because they have the same form, $ax^4 + bx^2 + c$. When this is the case, the polynomial may be factored using the same methods you would use to factor a quadratic. This is called *factoring by using quadratic form*.

For example, consider the trinomial $x^4 - 29x + 100$.

$x^4 - 29x + 100$ Determine whether you can factor the given trinomial into 2 factors.

$(x^2 - 4)(x^2 - 25)$ Determine whether you can continue to factor each binomial.

$(x + 2)(x - 2)(x + 5)(x - 5)$

The factored form of $x^4 - 29x + 100$ is $(x + 2)(x - 2)(x + 5)(x - 5)$.

LESSON

3

Polynomial Division

The **factor theorem** states that a polynomial function $p(x)$ has $(x - r)$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$. You can use the Factor Theorem to show that a linear expression is a factor of a polynomial.

For example, consider the graph of the polynomial function $f(x) = x^3 - 3x^2$.

The graph appears to have a zero at (3, 0), so a possible linear factor of the polynomial is $x - 3$.

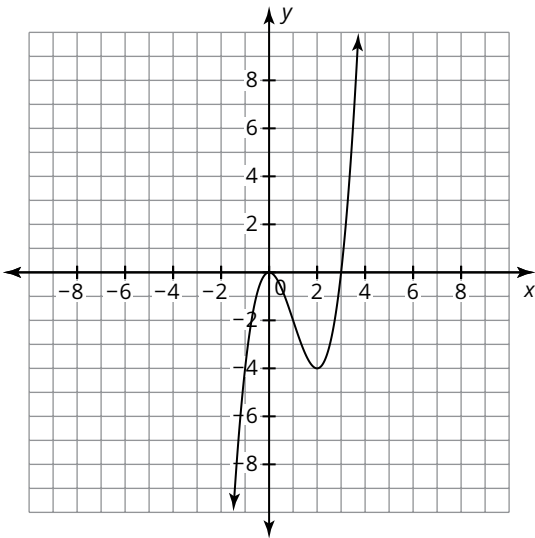
Determine the value of the polynomial at:

$$\begin{aligned} x &= 3, \text{ or } f(3). \\ f(3) &= (3)^3 - 3(3)^2 \\ &= 27 - 3(9) \\ &= 27 - 27 \\ &= 0 \end{aligned}$$

So, $(x - 3)$ is a linear factor of the polynomial function.

The Fundamental Theorem of Algebra states that every polynomial equation of degree n must have n roots. This means that every polynomial can be written as the product of n factors of the form $(ax + b)$. For example, $2x^2 - 3x - 9 = (2x + 3)(x - 3)$.

Just like the integers in a factor tree, the factors of a polynomial divide into that polynomial without a remainder. **Polynomial long division** is an algorithm for dividing one polynomial by another of equal or lesser degree. The process is similar to integer long division.



Integer Long Division	Polynomial Long Division	Description
$3660 \div 12$ or $\begin{array}{r} 3660 \\ 12 \overline{) 3660} \\ \underline{36} \\ 0 \\ \underline{0} \\ 0 \end{array}$	$(x^3 + x^2 + 3x + 3) \div (x + 1)$ or $\begin{array}{r} x^3 + x^2 + 3x + 3 \\ x + 1 \overline{) x^3 + x^2 + 3x + 3} \\ \underline{-(x^3 + x^2)} \\ 0x^2 + 3x + 3 \\ \underline{-(0x^2 + 0x)} \\ 3x + 3 \\ \underline{-(3x + 3)} \\ 0 \end{array}$ Remainder 0	A. Divide $\frac{x^3}{x} = x^2$. B. Multiply $x^2(x + 1)$ and then subtract. C. Bring down $3x$. D. Divide $\frac{0x^2}{x} = 0x$. E. Multiply $0x(x + 1)$ and then subtract. F. Bring down $+3$. G. Divide $\frac{3x}{x} = 3$. H. Multiply $3(x + 1)$ and then subtract.

You can use the difference of squares when you have a binomial of the form $a^2 - b^2 = (a + b)(a - b)$.

The difference of cubes can be rewritten in factored form: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. The sum of cubes can be written in factored form: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Any polynomial $p(x)$ can be written in the form $p(x) = (\text{linear expression})(\text{quotient}) + \text{remainder}$. Generally, the linear expression is written in the form $(x - r)$, the quotient is represented by $q(x)$, and the remainder is represented by R , meaning: $p(x) = (x - r)q(x) + R$.

The **remainder theorem** states that when any polynomial equation or function $f(x)$ is divided by a linear expression of the form $(x - r)$, the remainder is $R = f(r)$, or the value of the equation or function when $x = r$.

You can also use the factor theorem to determine unknown information about a polynomial if you know a linear factor of that polynomial.

For example, given that $x - 5$ is a linear factor of $x^3 - 4x^2 - ax + 10$, you can determine the unknown coefficient, a .

If $(x - 5)$ is a linear factor, then, by the factor theorem, $f(5) = 0$.

$$(5)^3 - 4(5)^2 - a(5) + 10 = 0$$

$$125 - 4(25) - a(5) + 10 = 0$$

$$125 - 100 - a(5) + 10 = 0$$

$$35 - a(5) = 0$$

$$-a(5) = -35$$

$$a(5) = 35$$

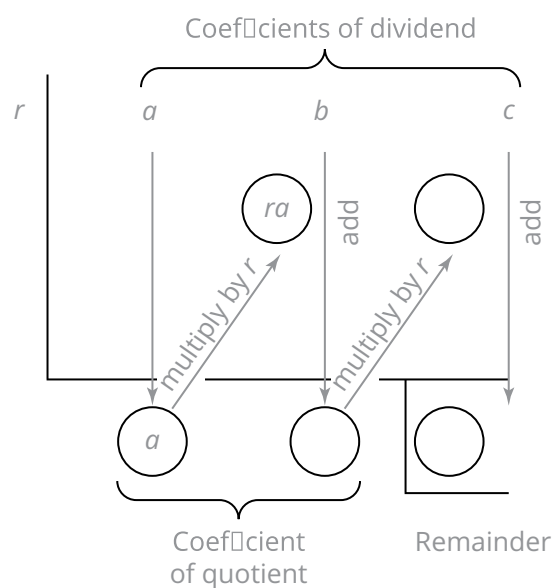
$$a = 7$$

Thus, $f(x) = x^3 - 4x^2 - 7x + 10$.

Synthetic division is a shortcut method for dividing a polynomial by a linear expression of the form $(x - r)$. Notice in the form of the linear factor $(x - r)$, the x has a coefficient of 1. Also, just as in long division, when you use synthetic division, every power of the dividend must have a placeholder.

To use synthetic division to divide a polynomial $ax^2 + bx + c$ by a linear factor $(x - r)$, follow the pattern shown on the right.

You can use synthetic division in place of the standard long division algorithm to determine the quotient for $(x^3 + x^2 + 3x + 3) \div (x + 1)$.



Long Division	Synthetic Division																				
$\begin{array}{r} x^2 + 0x + 3 \\ x + 1 \overline{) x^3 + x^2 + 3x + 3} \\ \underline{-(x^3 + x^2)} \\ 0x^2 + 3x \\ \underline{-(0x^2 + 0x)} \\ 3x + 3 \\ \underline{-(3x + 3)} \\ 0 \end{array}$	$r = -1$ <table><tr><td>-1</td><td>1</td><td>1</td><td>3</td><td>3</td></tr><tr><td></td><td></td><td>-1</td><td>0</td><td>-3</td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr><tr><td></td><td>1</td><td>0</td><td>3</td><td>0</td></tr></table>	-1	1	1	3	3			-1	0	-3							1	0	3	0
-1	1	1	3	3																	
		-1	0	-3																	
	1	0	3	0																	
$(x^3 + x^2 + 3x + 3) \div (x + 1) = (x^2 + 3)$																					

Synthetic division works only for linear divisors in the form $(x - r)$. If the divisor has a leading coefficient other than 1, you may need to factor out a constant in order to rewrite the divisor in the form $(x - r)$.

You can use synthetic division to determine the quotient of $\frac{2x^3 - 6x^2 + 4x + 2}{2x - 3}$. Since the divisor is not in the form $(x - r)$, you can rewrite $2x - 3$ as $2\left(x - \frac{3}{2}\right)$.

$$\begin{array}{r}
 r = \frac{3}{2} \\
 \frac{3}{2} \left| \begin{array}{cccc}
 2 & -6 & 4 & 2 \\
 & 3 & -\frac{9}{2} & -\frac{3}{4} \\
 \hline
 2 & -3 & -\frac{1}{2} & \frac{5}{4}
 \end{array} \right.
 \end{array}$$

The numbers in the last row become the coefficients of the quotient.
 $2x^2 - 3x - \frac{1}{2} R \frac{5}{4}$

To determine the quotient of $\frac{2x^3 - 6x^2 + 4x + 2}{2x - 3}$, you need to return to the original divisor of $(2x - 3)$. To do so, double $\left(x - \frac{3}{2}\right)$ and halve $\left(2x^2 - 3x - \frac{1}{2}\right)$.

$$\begin{aligned}
 2x^3 - 6x^2 + 4x + 23 &= (2x - 3)\left(x^2 - \frac{3}{2}x - \frac{1}{4}\right) + \frac{5}{4} \\
 \text{Therefore, } \frac{2x^3 - 6x^2 + 4x + 2}{2x - 3} &= x^2 - \frac{3}{2}x - \frac{1}{4} + \frac{5}{4(2x - 3)}.
 \end{aligned}$$

The rational root theorem states that a rational root of a polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 x^0$ with integer coefficients will be of the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n .

Given the polynomial: $3x^2 + 2x - 6$

$$p = \pm 1, \pm 2, \pm 3, \pm 6$$

$$q = \pm 1, \pm 3$$

The possible rational roots are $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{3},$ and $\pm \frac{2}{3}$.

To determine all roots of a polynomial:

- Determine the possible rational roots.
- Use synthetic division to determine one of the roots.
- Rewrite the original polynomial as a product.
- Determine the possible rational roots of the quotient.
- Repeat the process until all the rational roots are determined.
- Factor the remaining polynomial to determine any irrational or complex roots.
- Recall that some roots may have a multiplicity.

For example, factor $2x^3 - 9x^2 + 7x + 6$.

The possible rational roots of $2x^3 - 9x^2 + 7x + 6$ are $\pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \pm 3,$ and ± 6 since $p = 6$ and $q = 2$.

Use synthetic division to divide the polynomial by $(x - 3)$.

$$\begin{array}{r|rrrr} 3 & 2 & -9 & 7 & 6 \\ & & 6 & -9 & -6 \\ \hline & 2 & -3 & -2 & 0 \end{array}$$

$$2x^3 - 9x^2 + 7x + 6$$

$$= (x - 3)(2x^2 - 3x - 2)$$

$$= (x - 3)(2x + 1)(x - 2)$$

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