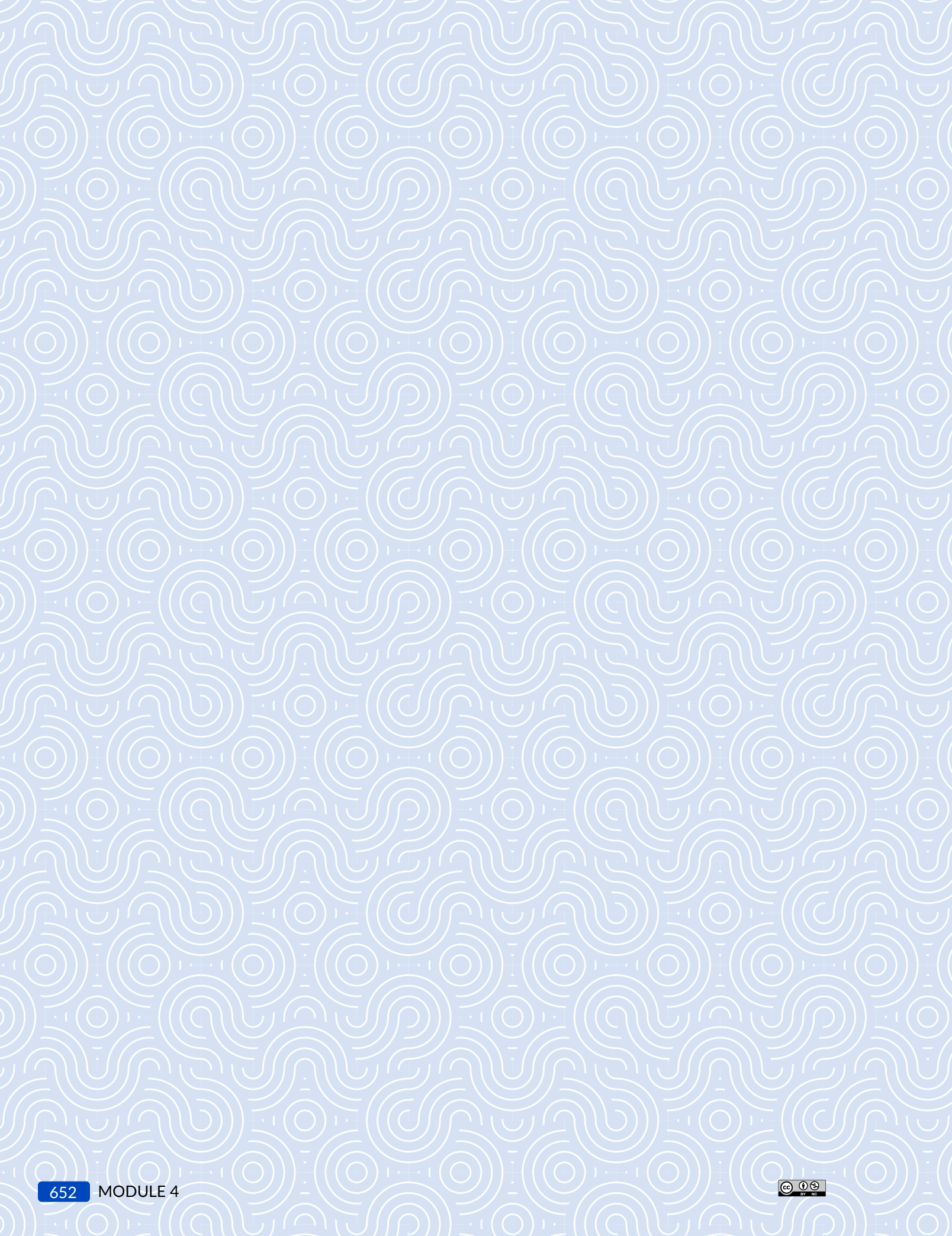
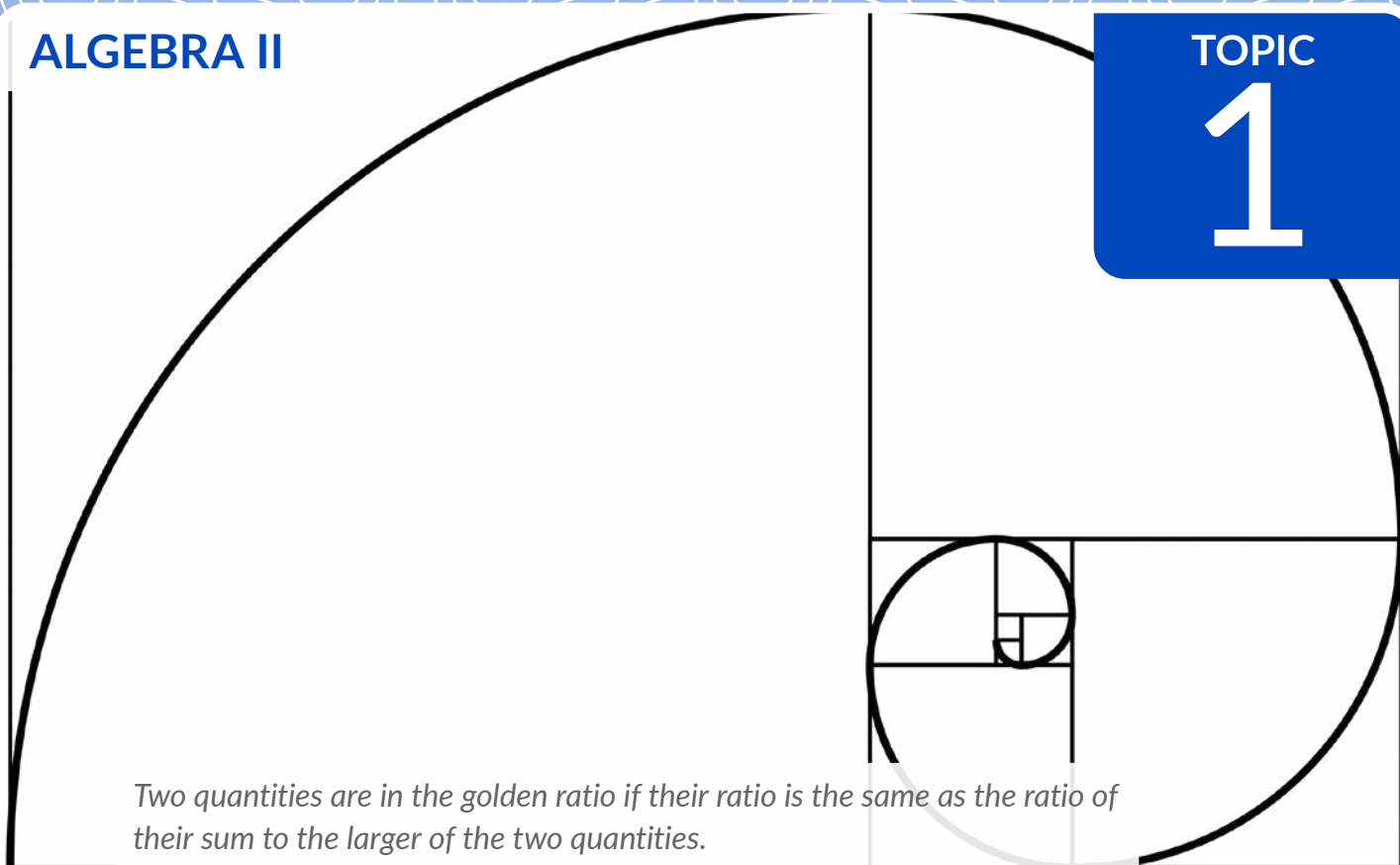


Extending Beyond Polynomials

TOPIC 1	Rational Functions	653
TOPIC 2	Radical Functions.....	801
TOPIC 3	Radical Equations	901

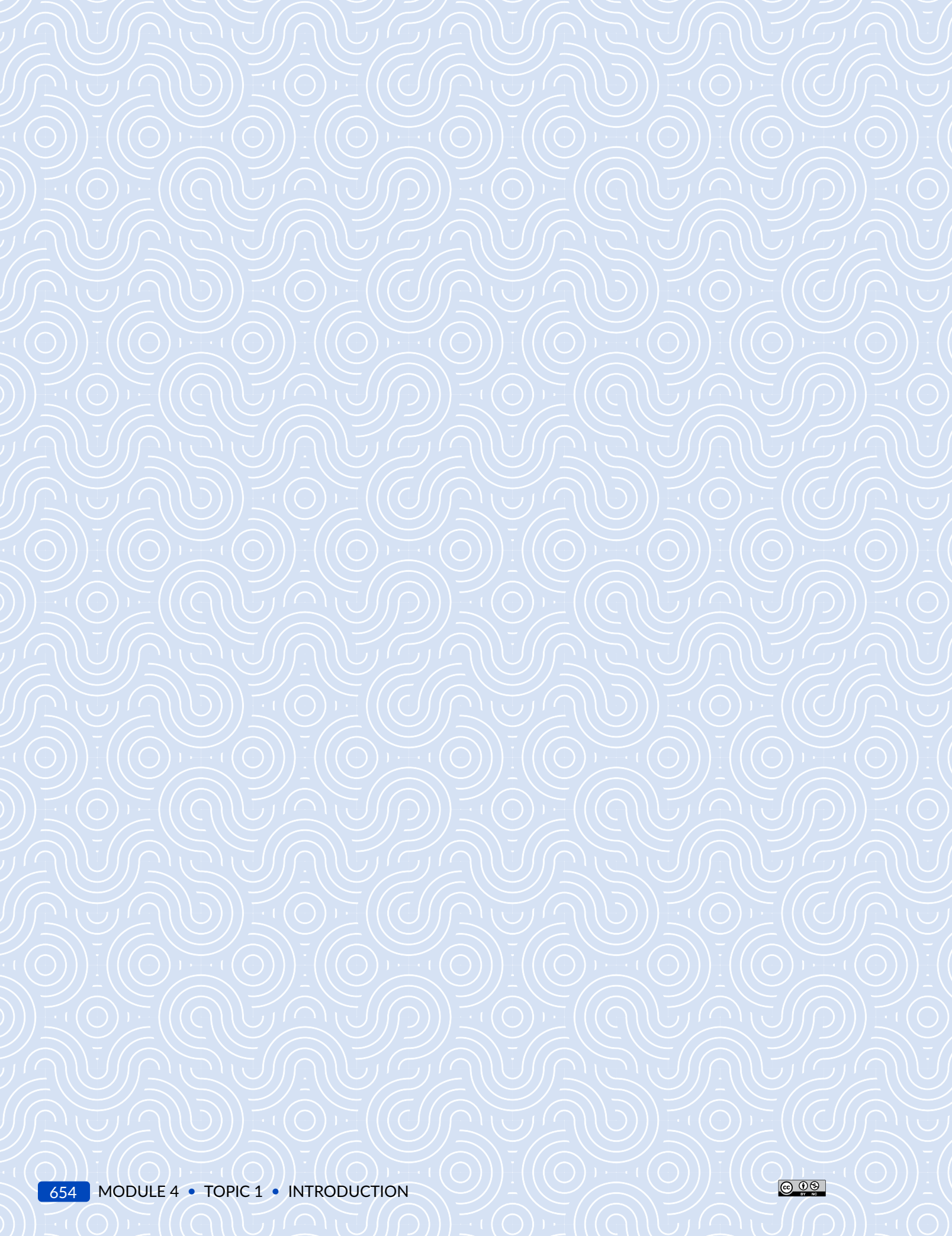




Two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities.

Rational Functions

LESSON 1	Introduction to Rational Functions	655
LESSON 2	Transformations of Rational Functions	677
LESSON 3	Exploring Asymptotes	703
LESSON 4	Operations with Rational Expressions	723
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1

Introduction to Rational Functions

OBJECTIVES

- Graph rational functions.
- Compare rational functions in multiple representations.
- Analyze and verbally describe the key attributes of rational functions.

KEY TERMS

- rational function
- vertical asymptote
- set notation
- horizontal asymptote
- inverse variation function

.....

You have learned that polynomials are related to the integers.
What functions relate to the set of rational numbers?

Getting Started

Remember ...

The reciprocal of any number x is $\frac{1}{x}$. For example, the reciprocal of 5 is $\frac{1}{5}$ and the reciprocal of 0.5 is $\frac{1}{0.5}$, or 2.

Two Integers, Both Alike in Dignity

You have learned about rational numbers in previous courses, and you have used rational numbers throughout this course. Let's review what you know.

1. List some properties of rational numbers, along with examples.
2. Determine the reciprocals of the rational numbers you wrote in Question 1. Explain why these numbers are reciprocals.
3. Explain what happens when the denominator of a rational number is zero.
4. Write a definition of *rational number*.

Graphing Rational Functions

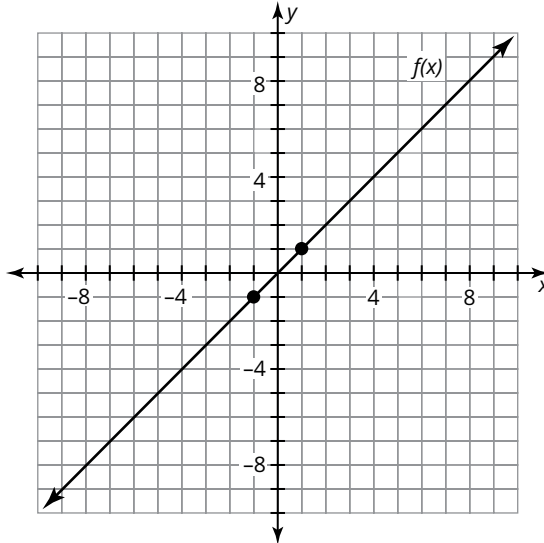
Throughout this course you have studied many connections between polynomial functions and real numbers. It follows then that polynomial functions also have reciprocals. Is the reciprocal also a polynomial? Is it a function? How does $\frac{1}{f(x)}$ compare to the original function $f(x)$?

To begin answering these questions, consider the reciprocal of the basic linear function $f(x) = x$. The reciprocal can be defined as $g(x) = \frac{1}{f(x)}$, or simply $g(x) = \frac{1}{x}$.

1. Consider the graph and table of values for $f(x) = x$. The domain of $f(x)$ is $(-\infty, \infty)$. The points $(-1, -1)$ and $(1, 1)$ are shown and used to create three intervals for analysis.

-6	-5	-4	-3	-2	-1	x
-6	-5	-4	-3	-2	-1	$f(x) = x$
						$g(x) = \frac{1}{x}$

x	1	2	3	4	5	6
$f(x) = x$	1	2	3	4	5	6
$g(x) = \frac{1}{x}$						



- a. Complete the tables of values for $g(x) = \frac{1}{x}$. Then plot the points and draw a smooth curve to graph $g(x)$ on the coordinate plane.

x	-1	$-\frac{1}{2}$	$-\frac{1}{10}$	$-\frac{1}{100}$	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	1
$f(x) = x$	-1	$-\frac{1}{2}$	$-\frac{1}{10}$	$-\frac{1}{100}$	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	1
$g(x) = \frac{1}{x}$									

.....
The point at $g(0)$ is
said to be undefined
because it is
impossible to divide
by 0.
.....

b. Describe the graph of $g(x)$. How is it similar to the graphs of other functions that you've studied? How is it different?

c. Does $g(x)$ have any extrema?

Ask Yourself . . .

How can you
algebraically
verify the
symmetry of $g(x)$?

d. Use patty paper to explore the symmetry of $g(x)$. Does the graph of $g(x)$ have any symmetry?



2. Michael and Olivia disagree about $g(x) = \frac{1}{x}$.

Michael

The graph and table both clearly show that it is a function.

Olivia

It is not a function. Every input doesn't have an output.

Who is correct? Explain your reasoning.

3. Analyze the key attributes of $g(x) = \frac{1}{x}$.

- a. Does the graph ever intersect the horizontal line $y = 0$? Explain your reasoning in terms of the graph, table, and equation.
- b. Does the graph ever intersect the vertical line $x = 0$? Explain your reasoning in terms of the graph, table, and equation.
- c. Describe the domain and range of $g(x)$.

Previously, you have represented the domain and range of functions using words, inequalities, or interval notation. You can also use *set notation* to represent these attributes of functions. In **set notation**, the domain of a function is represented by the set of all possible input values, x , using curly brackets “{” and a description of the allowed values with a vertical bar “|”. The range is the set of all possible output values, y , using the same notation, representing the set of values the function can produce based on the domain inputs.

.....
You can read $\{x|x \neq 4\}$
as “ x such that x does
not equal four”.
.....

WORKED EXAMPLE

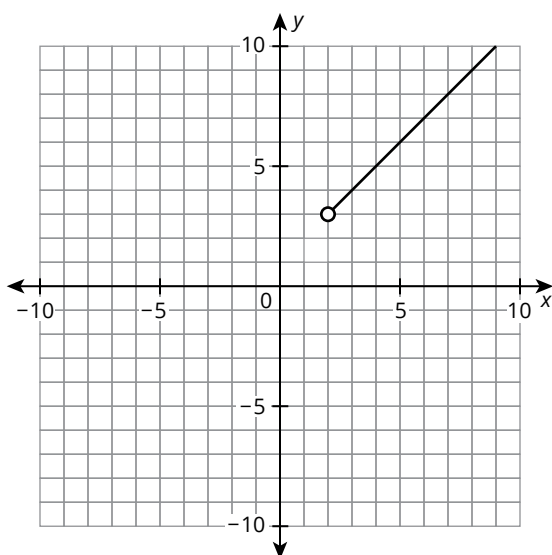
Suppose that both the domain and range of a rational function, $g(x)$, are represented verbally as “all real numbers except for 4.” You can use inequalities, interval notation, and set notation to represent this verbal statement.

	Domain	Range
Inequalities	$x > 4$ or $x < 4$	$g(x) > 4$ or $g(x) < 4$
Interval notation	$(-\infty, 4) \cup (4, \infty)$	$(-\infty, 4) \cup (4, \infty)$
Set notation	$\{x x \neq 4\}$	$\{g(x) g(x) \neq 4\}$

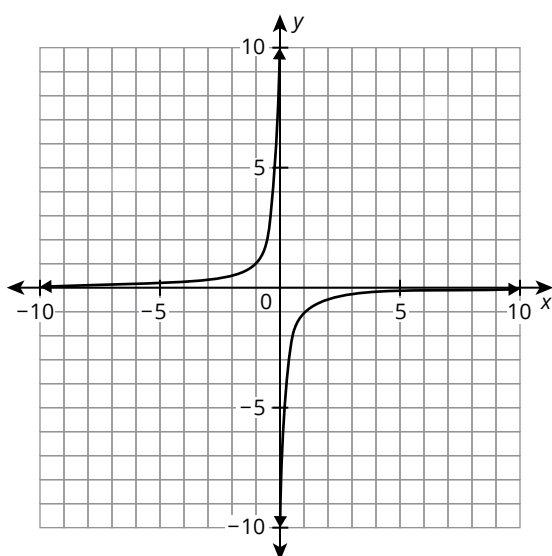
4. Write your answer to Question 3, part (c), using each notation shown.

5. Write the domain and range for each graph using inequalities, interval notation, and set notation.

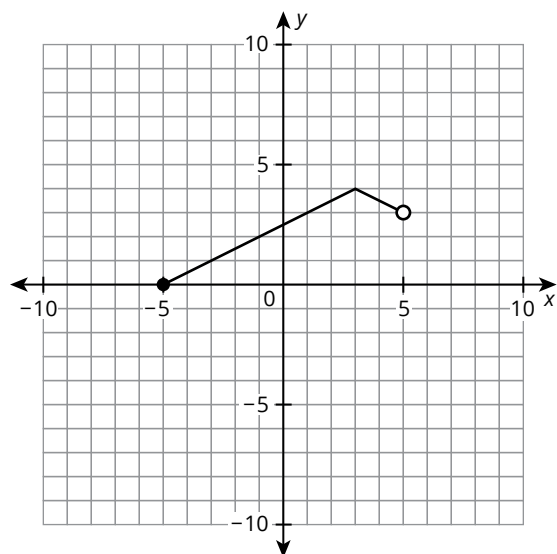
a.



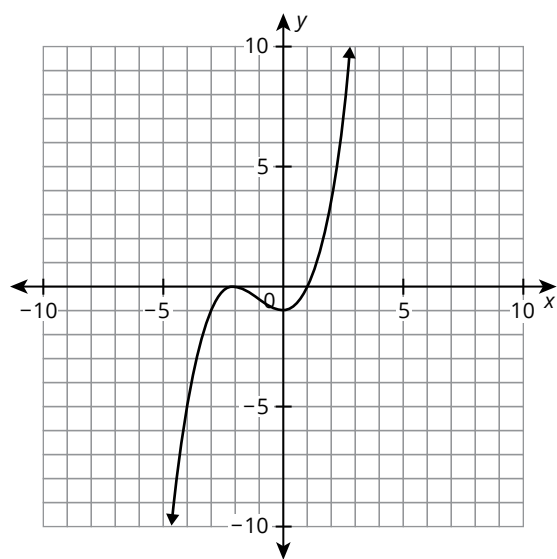
b.



c.



d.



The function $g(x) = \frac{1}{x}$ is an example of a *rational function*. A **rational function** is any function that can be written as the ratio of two polynomials. It can be written in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$. You have already seen some specific types of rational functions. Linear, quadratic, cubic, and higher-order polynomial functions are types of rational functions.

.....
All polynomials are rational functions.
Remember $Q(x)$ can be equal to 1.
.....

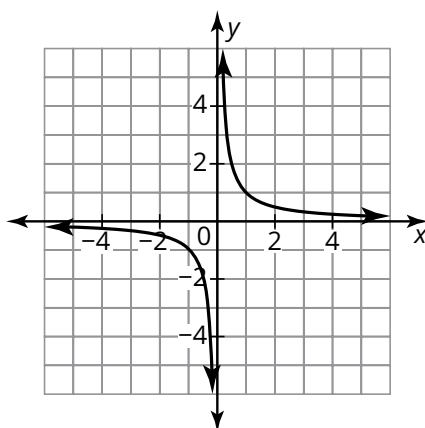
Recall from your study of exponential functions that a **horizontal asymptote** is a horizontal line that a function gets closer and closer to. In most cases, a graph never intersects a horizontal asymptote, but in some cases it does. In this problem, the function $g(x)$ has a horizontal asymptote at $y = 0$.

Given the horizontal asymptote, the range of the rational function $g(x) = \frac{1}{x}$ can be expressed:

- verbally: all real numbers except for 0.
- as a compound inequality: $g(x) > 0$ or $g(x) < 0$.
- in interval notation: $(-\infty, 0) \cup (0, \infty)$.
- in set notation: $\{g(x) \mid g(x) \neq 0\}$.

The function $g(x) = \frac{1}{x}$ has a *vertical asymptote* at $x = 0$. A **vertical asymptote** is a vertical line that a function gets closer and closer to, but never intersects. The asymptote does not represent points on the graph of the function. It represents the output value that the graph approaches. An asymptote occurs for input values that result in a denominator of 0.

.....
A graph cannot cross its vertical asymptote because the function is undefined at that value.
.....



The vertical and horizontal asymptotes are often represented as a dashed or solid line. The convention used in this textbook is to represent asymptotes as dashed lines.

Given the vertical asymptote, the domain of the rational function $g(x) = \frac{1}{x}$ can be expressed:

- verbally: all real numbers except for 0
- as a compound inequality: $x > 0$ or $x < 0$.
- in interval notation: $(-\infty, 0) \cup (0, \infty)$.
- in set notation: $\{x | x \neq 0\}$.

6. Analyze each function.

a. Circle the rational functions.

$$f(x) = x$$

$$g(x) = \frac{3x}{2}$$

$$h(x) = \frac{\sqrt{x}}{2x}$$

$$p(x) = \frac{3}{x} + 2$$

$$k(x) = 12$$

$$n(x) = \frac{2^x}{5}$$

$$j(x) = \frac{4x^2 + 3x + 2}{6x^3 + 10}$$

$$m(x) = \frac{1}{(x+2)(x-3)}$$

b. Explain why the remaining functions are not rational.

c. Do you think the graphs of all rational functions will have a vertical asymptote? Explain your reasoning.

ACTIVITY 1.2

Modeling a Situation with a Rational Function

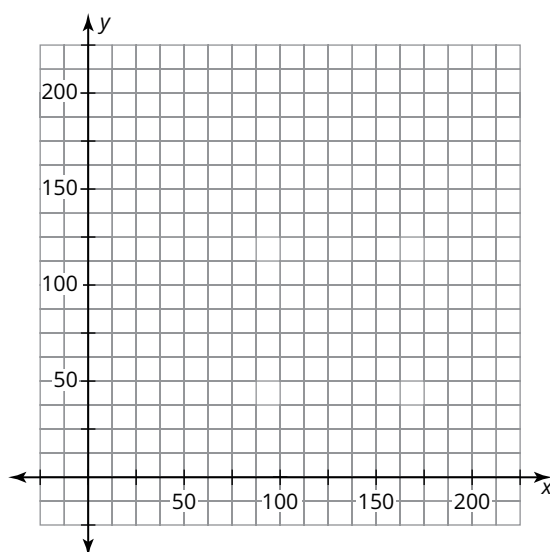
An **inverse variation function** is a function in the form of $y = \frac{k}{x}$, where k is the constant of variation. You can use inverse variation functions to model many real-world situations.

Suppose you want to purchase a new laptop which will cost \$2200.

1. Complete the table to show the amount of time it would take to save \$2200 for different weekly savings amounts.

Weekly Savings (dollars)	Time (weeks)
10	
20	
40	
50	
100	

2. Use the information in the table to construct a graph of the problem situation. Be sure to label the axes.



PROBLEM SOLVING



3. Can this problem situation be modeled by a function?
Explain your reasoning.
4. Write an algebraic equation to model this problem situation.
5. Describe the asymptotic behavior of the graph in this situation.
 - a. What happens to the graph as x approaches zero?
 - b. What happens to the graph as x approaches infinity?
6. How does the domain and range of the problem situation compare to the domain and range of the function?

ACTIVITY
1.3

Writing and Solving Inverse Variation Equations

In our previous discussions, we explored graphing rational functions and how they can be used to model real-world situations. We also introduced the inverse variation function which is of the form $y = \frac{k}{x}$, where k is the constant of variation.

Let us formulate equations to solve real-world problem situations involving inverse variation equations.

1. A team of construction workers is hired to build a wooden deck for a residential home. The time required to complete the job varies inversely with the number of workers on the team. If it takes 12 hours for 6 workers to complete the task, how long would it take if 9 workers are assigned to the task?

2. Mr. Chen is preparing his backyard swimming pool for a party this weekend. The time required to fill the pool varies inversely with the water flow rate. If it takes 10 hours to fill the pool when the water flows at 200 liters per hour, how long will it take if the flow rate is increased to 500 liters per hour?

3. A lightweight sports car weighing 2000 lbs has a fuel efficiency of 30 miles per gallon. The fuel efficiency varies inversely with the weight of the vehicle. What would be the fuel efficiency of a heavier sports car that weighs 3000 lbs?



Talk the Talk

Change in Range. . . and Domain

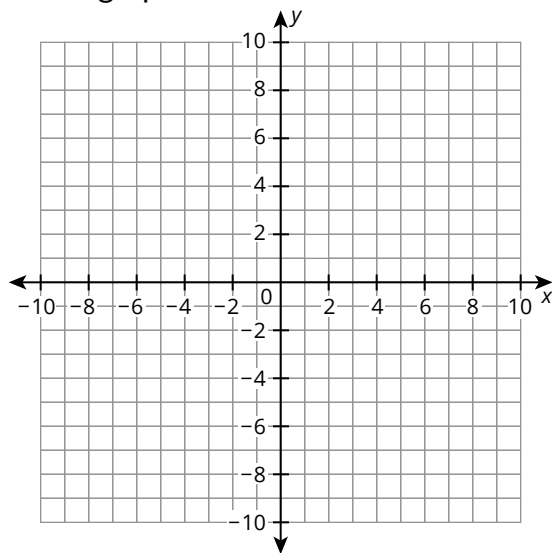
1. How does a vertical asymptote affect the domain of a rational function?
2. How does a horizontal asymptote affect the range of a rational function?
3. How do the asymptotes affect the x- and y-intercepts of the graph of $f(x) = \frac{1}{x}$?

4. Consider the function $f(x) = \frac{3}{x}$.

a. Complete the table.

x	-9	-6	-3	-1	$-\frac{1}{3}$	0	$\frac{1}{3}$	1	3	6	9
$f(x)$											

b. Use the table to graph the function.



c. Analyze the function and the corresponding table and graph. Write the domain and range using interval notation, set notation and inequalities.

d. Determine all of the asymptotes of the function. Explain your reasoning.

5. The workload per student in a collaborative science project varies inversely with the number of students in the group. If a project requires 84 pages of research and there are 6 students, how many pages will each student be responsible for if the group increases to 7 students?

Lesson 1 Assignment

Write

Describe why a rational function never crosses its vertical asymptote.

Remember

A rational function is any function that can be written as the ratio of two polynomials. It can be written in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$.

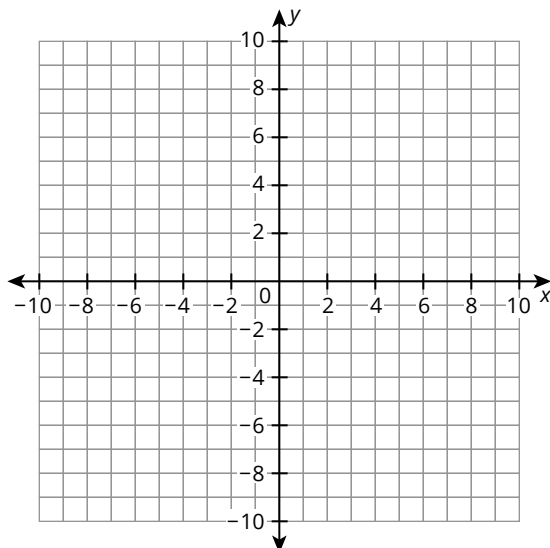
Practice

1. Consider the function $f(x) = \frac{1}{x}$.

a. Complete the table.

x	-8	-4	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4	8
f(x)											

b. Use the table to graph the function.



Lesson 1 Assignment

- c. Write the domain of the function in interval notation, set notation, and inequalities.

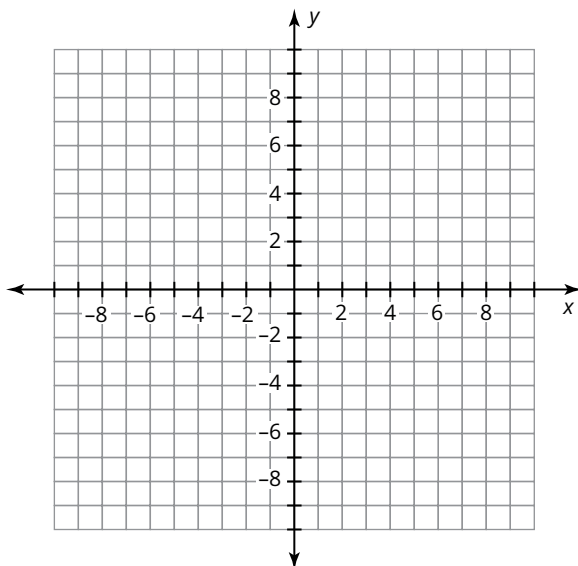
- d. Determine the function's symmetry, horizontal asymptote, vertical asymptote, intercepts, and extrema.

2. Consider the function $f(x) = \frac{4}{x}$.

a. Complete the table.

x	-8	-4	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4	8
$f(x)$											

b. Use the table to graph the function.

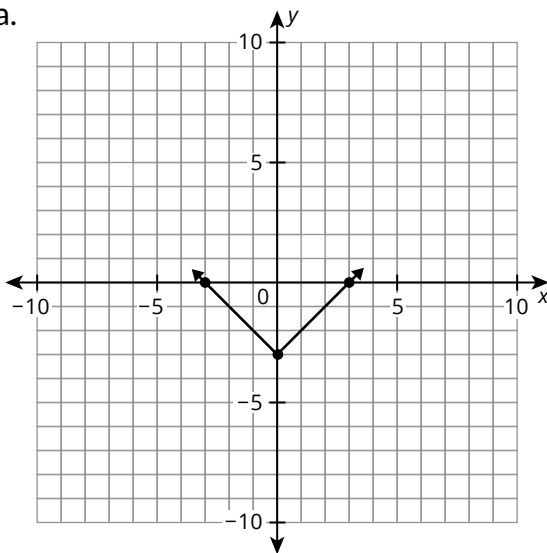


Lesson 1 Assignment

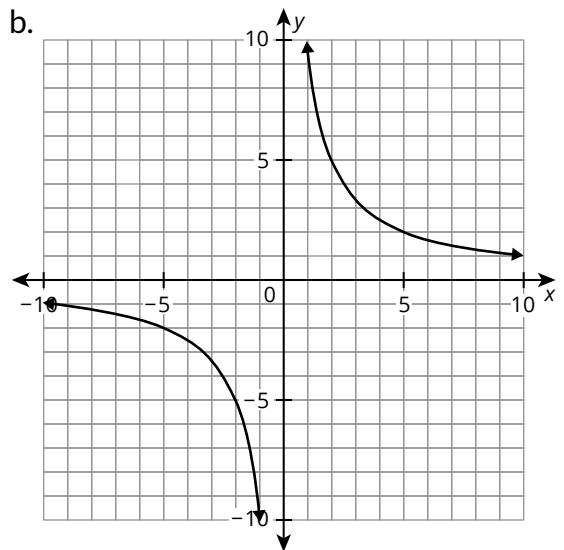
- c. Analyze the function and the corresponding table and graph. Write the domain and range of the function using set notation. Determine all of the asymptotes of the function. Explain your reasoning.

3. Write the domain and range for each graph using interval notation, set notation, and inequalities

a.



b.



Lesson 1 Assignment

4. In a restaurant kitchen, the time per order for preparing meals varies inversely with the number of chefs working. When 2 chefs are preparing meals, each order takes 15 minutes to complete. If the manager adds more chefs to handle the workload, how long will each order take if 5 chefs are working?

Prepare

Write each expression as a product of linear factors.

1. $x^2 + \frac{1}{2}x$

2. $x^2 + 2x - 3$

3. $(2x - 3)^2$

4. $x^3 + 2x^2 - 19x - 20$

2

Transformations of Rational Functions

OBJECTIVES

- Determine the graphical behavior of rational functions from the structure of the equation.
- Analyze and verbally explain transformations of rational functions, including translations and dilations.

.....

You have explored transformations of linear, quadratic, and other polynomial functions.

How can you transform rational functions on the coordinate plane?

Getting Started

That Graph Looks a Little Shifty

.....
Horizontal and
vertical dilations
stretch or compress
the function.
.....

You know that transformations performed on any function $f(x)$ to form a new function $g(x)$ can be described by the transformation function form.

$$g(x) = af(b(x - c)) + d$$

Recall that this transformational function generalizes to any function. Changes to the a - or d -values dilate, translate, or reflect a function vertically. Changes to the b - or c -values dilate, translate, or reflect a function horizontally.

For a rational function, consider the form shown.

$$r(x) = a\left(\frac{1}{b(x - c)}\right) + d$$

1. Cut out the graphs and equations located at the end of the lesson. Match each equation representing a rational function with its graph. If an equation does not match a graph, create the graph of the equation. If a graph does not match an equation, write the equation that matches the graph. Explain how you sorted the equations and graphs.

2. Identify the vertical and horizontal asymptotes of each function.

3. Identify the domain and range of each function using set notation.

ACTIVITY
2.1**Vertical Dilations of Rational Functions**

You can vertically dilate rational functions just like other functions you have studied.

Consider these rational functions.

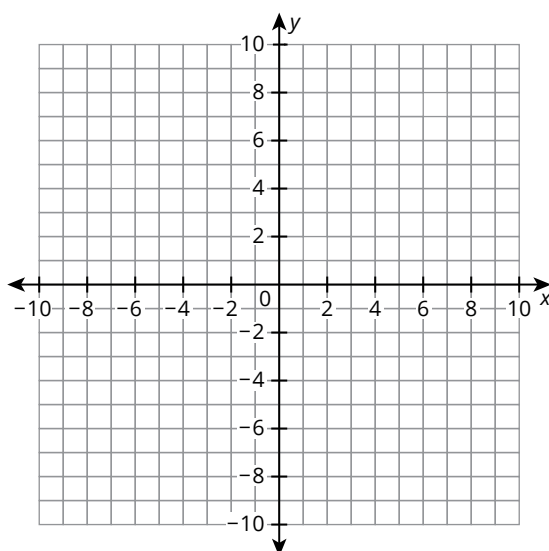
$$f(x) = \frac{1}{x}$$

$$g(x) = 2 \cdot \frac{1}{x}$$

$$h(x) = \frac{1}{2} \cdot \frac{1}{x}$$

$$j(x) = -\frac{1}{x}$$

1. Use graphing technology to graph each function. Then, sketch and label the graph of each function.



2. Write the functions $g(x)$, $h(x)$ and $j(x)$ in terms of the parent function $f(x)$. Then, describe the transformations of each function.

Recall that a function $t(x) = a \cdot f(x)$ is a vertical dilation of the function $f(x)$. The a -value describes the vertical dilation of the graph of the original function.

3. Describe each graph in relation to the parent function $f(x) = \frac{1}{x}$. Then, use coordinate notation to represent the vertical dilation.

a. $g(x) = a \cdot f(x)$ when $a > 1$

b. $g(x) = a \cdot f(x)$ when $a < 0$

c. $g(x) = a \cdot f(x)$ when $0 < a < 1$

- d. Each point (x, y) on the graph of $f(x)$ becomes the point _____ on $g(x)$.



4. Ana and Maria disagree about functions of the form $p(x) = \frac{a}{x}$ where a is a constant.

Ana

The horizontal asymptote will vary depending on the a -value.

Maria

All rational functions of this form will have a horizontal asymptote at $y = 0$.

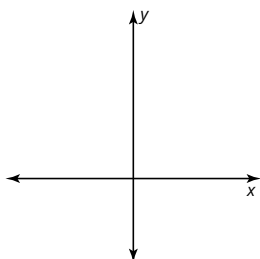
Who is correct? Explain your reasoning.

ACTIVITY 2.2

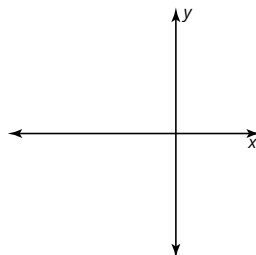
Transforming Rational Functions

1. Given the rational parent function, $f(x) = \frac{1}{x}$, sketch each transformation and explain your process.

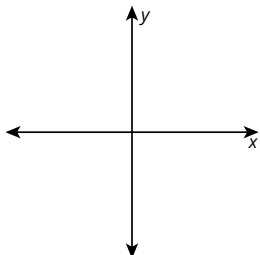
a. $g(x) = f(x) + 5$



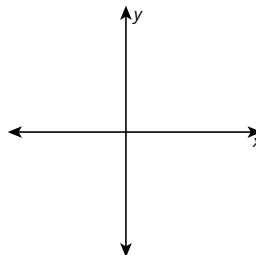
b. $h(x) = f(x + 5)$



c. $j(x) = f(x) - 5$



d. $k(x) = f(x - 5)$



Remember ...

Rational functions are any functions of the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$.

2. Describe the effects of changes to the c -value and d -value on each attribute given.

a. Domain

b. Range

You can horizontally dilate rational functions just like other functions you have studied.

Consider these rational functions.

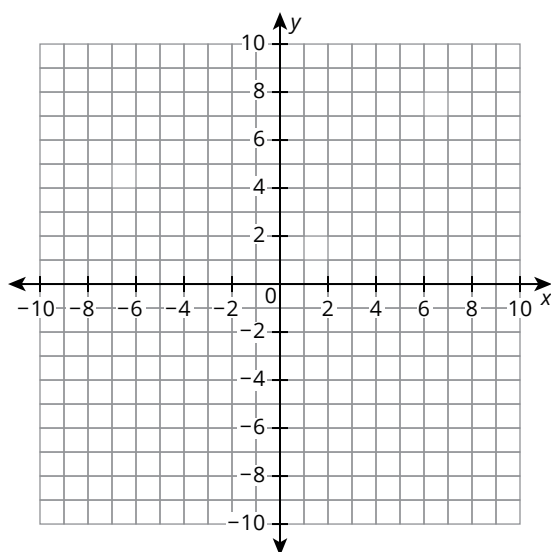
$$f(x) = \frac{1}{x}$$

$$g(x) = \frac{1}{(2x)}$$

$$h(x) = \frac{1}{\left(\frac{1}{2}x\right)}$$

$$j(x) = \frac{1}{(-x)}$$

3. Use graphing technology to graph each function. Then, sketch and label the graph of each function.



4. Write the functions $g(x)$, $h(x)$, and $j(x)$ in terms of the parent function $f(x)$. Then, describe the transformations of each function.

Recall that a function $t(x)$ of the form $t(x) = f(b \cdot x)$ is a horizontal dilation of the function $f(x)$. The b -value describes the horizontal dilation of the graph of the original function.

5. Describe each graph in relation to the parent function $f(x) = \frac{1}{x}$. Then, use coordinate notation to represent the vertical dilation.

a. $g(x) = f(b \cdot x)$ when $b > 1$

b. $g(x) = f(b \cdot x)$ when $b < 1$

c. $g(x) = f(b \cdot x)$ when $0 < b < 1$

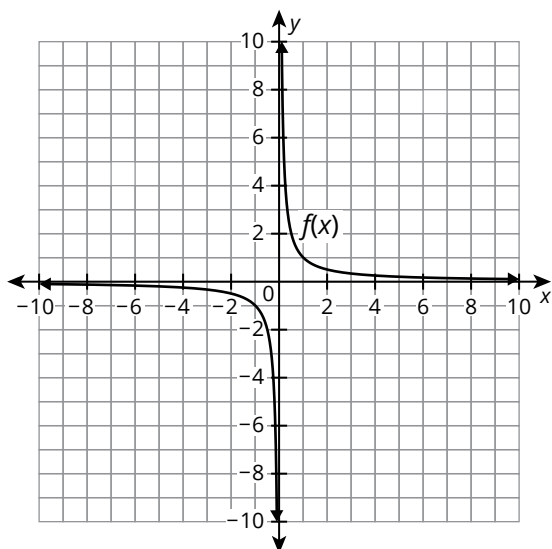
- d. Each point (x, y) on the graph of $f(x)$ becomes the point
_____ on $g(x)$.

ACTIVITY
2.3

Multiple Transformations of Rational Functions

Consider each transformation of $f(x) = \frac{1}{x}$. Sketch the transformed graph. Then, write the domain and range of the function in the indicated form.

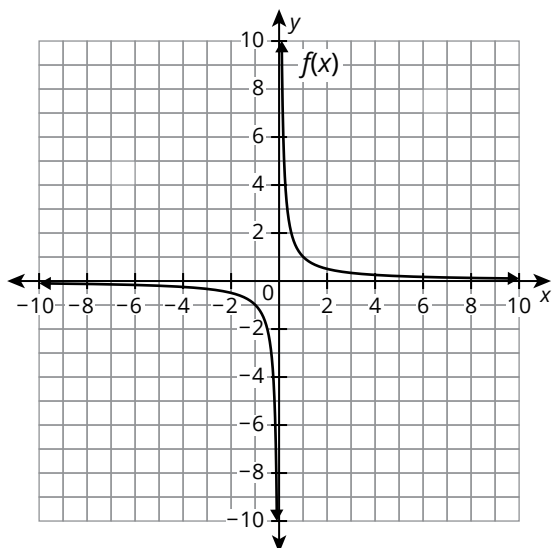
1. $g(x) = 2f(x - 1) + 3$; inequalities



Domain:

Range:

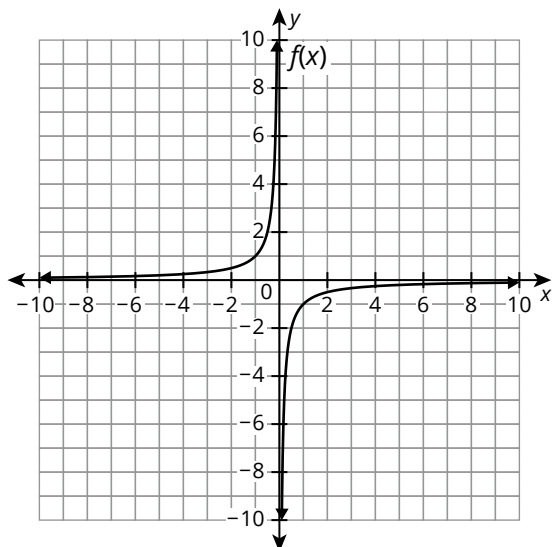
2. $h(x) = f(3(x + 4))$; set notation



Domain:

Range:

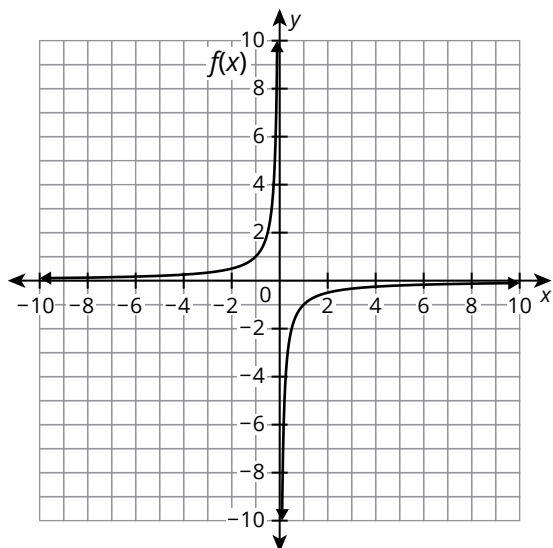
3. $j(x) = -\frac{1}{2}f(x) - 5$; interval notation



Domain:

Range:

4. $k(x) = f(-2(x - 1)) + 4$; set notation



Domain:

Range:



Talk the Talk

Rational Remix

For a rational function, the transformational function can be written as:

$$r(x) = a f\left(\frac{1}{b(x - c)}\right) + d$$

1. Describe how the values of a , b , c , or d transform the graph of $f(x) = \frac{1}{x}$.

a. $g(x) = a\left(\frac{1}{x}\right)$, for positive and negative values of a .

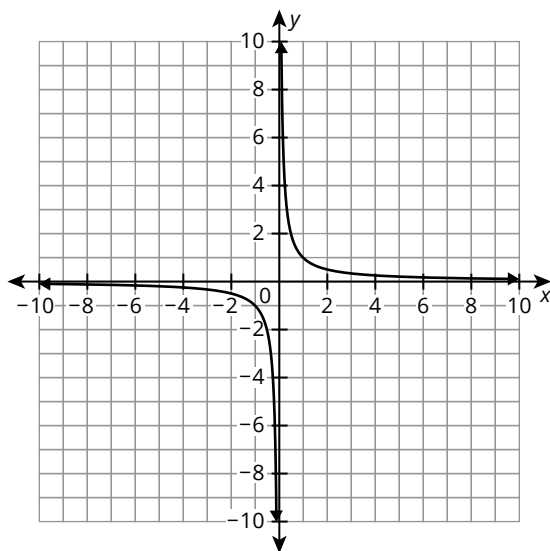
b. $h(x) = \frac{1}{x} + d$, for positive and negative values of d .

c. $j(x) = \frac{1}{x-c}$, for positive and negative values of c .

d. $k(x) = f(x) = \frac{1}{bx}$, for positive and negative values of b .

2. Consider the parent function $f(x) = \frac{1}{x}$ and the transformed function $h(x) = -3f(x + 1) - 4$.

Sketch $h(x)$ on the same coordinate plane as $f(x)$. Write its domain and range using interval notation.



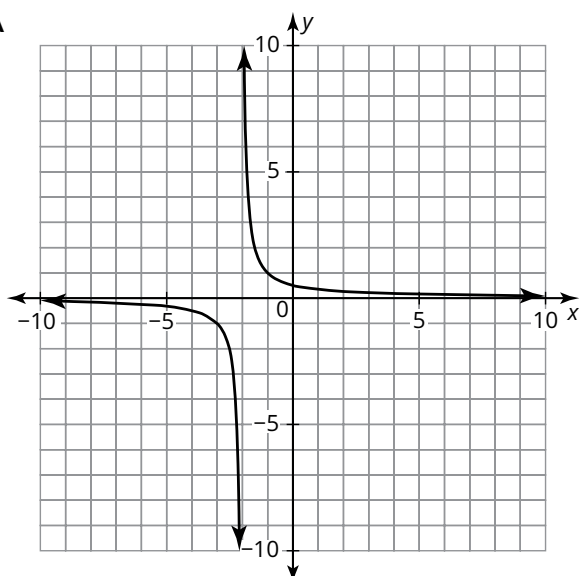
Why is this page blank?

So you can cut out your rectangles on the other side.

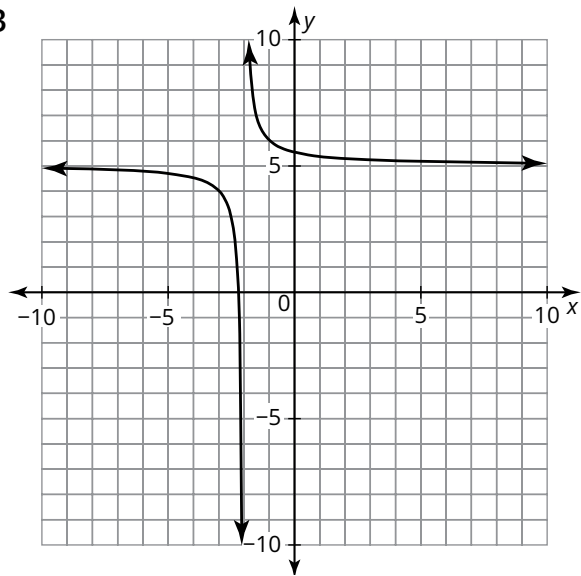
Function Graph Cutouts



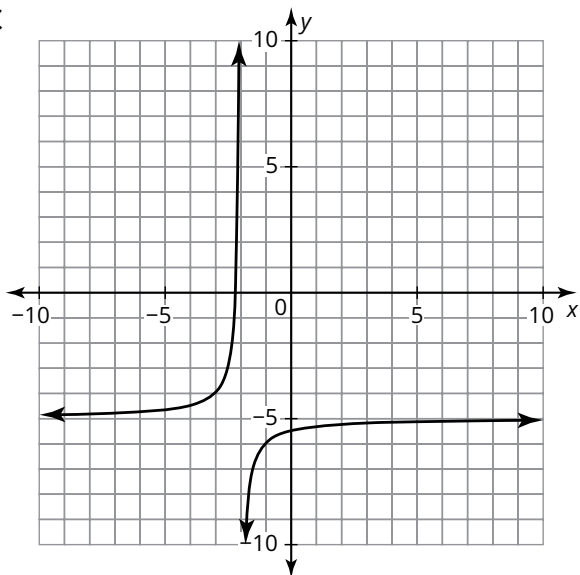
A



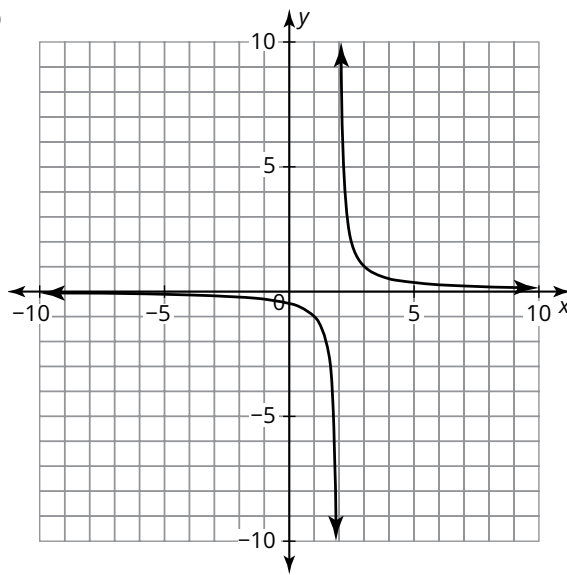
B



C

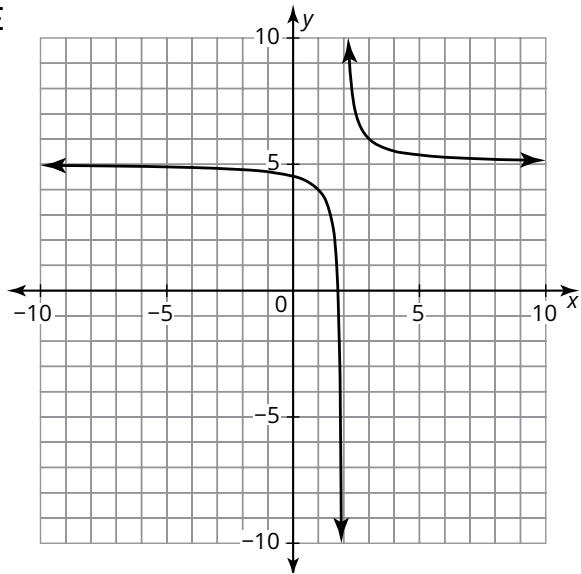
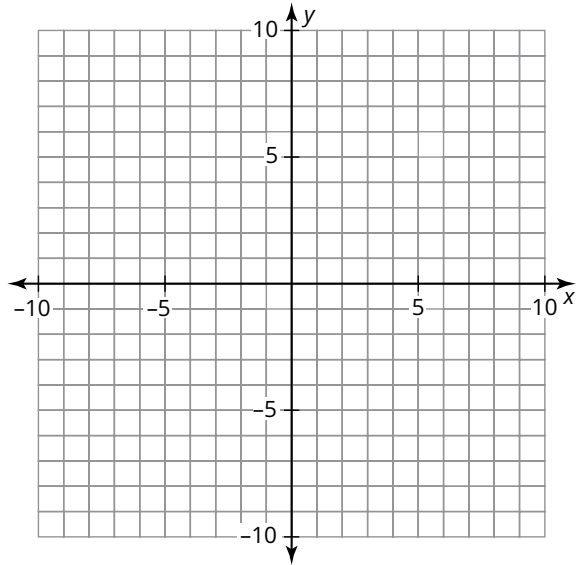
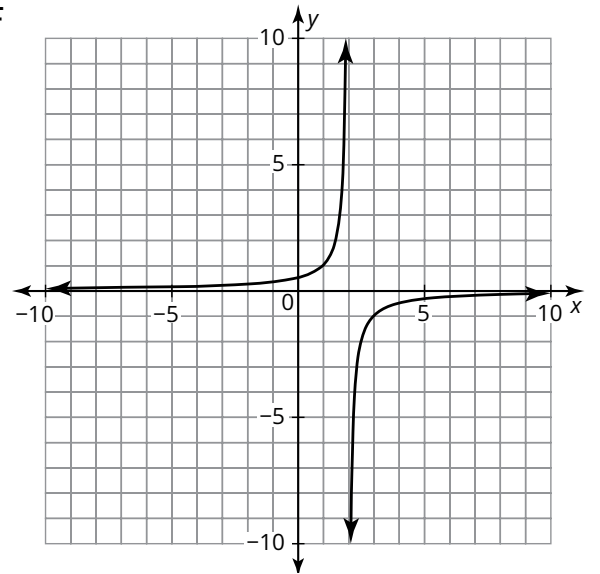
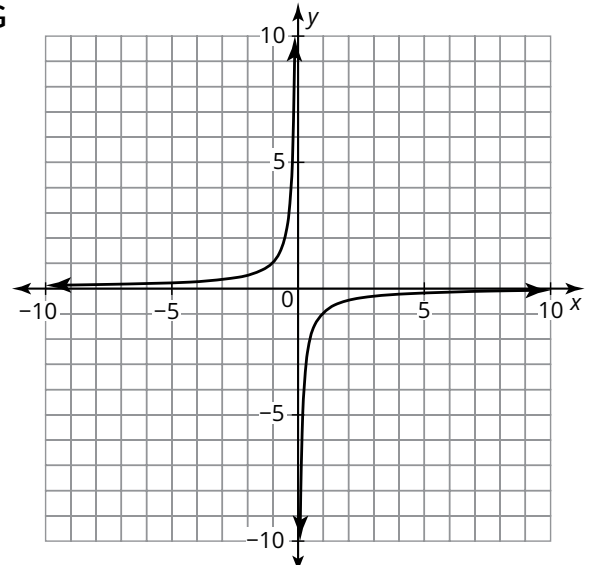


D



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So you can cut out your rectangles on the other side.

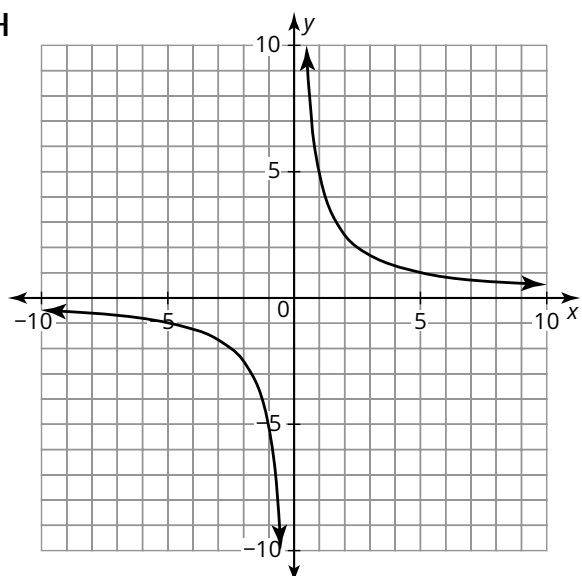
**E****F****G**

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So you can cut out your rectangles on the other side.



H



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So you can cut out your rectangles on the other side.

Function Equation Cutouts



$$f(x) = \frac{1}{x-2} - 5$$

$$f(x) = -\frac{1}{x+2} - 5$$

$$f(x) = \frac{1}{x-2} + 5$$

$$f(x) = \frac{1}{x+2}$$

$$f(x) = \frac{5}{x}$$

$$f(x) = -\frac{1}{x}$$

$$f(x) = -\frac{1}{x-2}$$

$$f(x) = \frac{1}{x-2}$$

$$f(x) = \frac{1}{x+2} + 5$$

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So you can cut out your rectangles on the other side.

Lesson 2 Assignment

Write

Explain how the a -, b -, c -, and d -values of the transformation of the rational function $f(x) = \frac{1}{x}$ affect the horizontal and vertical asymptotes of the function.

Remember

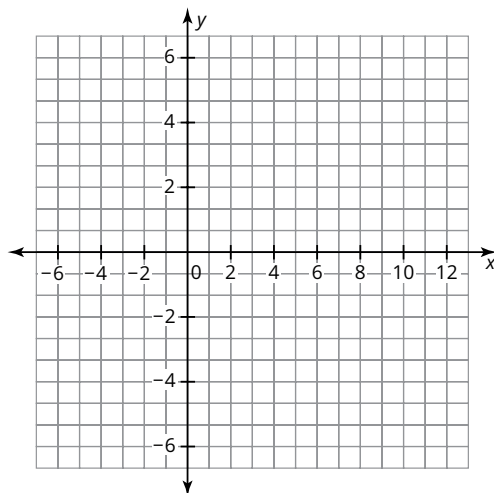
Translations of a rational function $f(x)$ are given in the form $g(x) = af(b(x - c)) + d$, where the d -value translates $f(x)$ vertically, and the c -value translates $f(x)$ horizontally.

Dilations of a rational function $f(x)$ are given in the form $g(x) = af(b(x - c)) + d$, where the a -value vertically stretches $f(x)$ and the b -value horizontally stretches $f(x)$.

Practice

1. Consider the functions $f(x) = x - 6$ and $g(x) = \frac{1}{x - 6}$.

- a. Graph and label the function $f(x) = x - 6$ on a coordinate plane.



- c. Determine the domain, range, vertical asymptote(s), horizontal asymptote(s), and y -intercept of $g(x)$.

- b. Graph and label the function $g(x) = \frac{1}{x - 6}$ on the same coordinate plane.

- d. How do the output values of $f(x)$ and $g(x)$ compare for any given input value?

Lesson 2 Assignment

2. Consider the parent function $f(x) = \frac{1}{x}$. Explain how the graph of each new function compares to the graph of $f(x)$.

a. $g(x) = f(x + 5) - 9$

b. $h(x) = \frac{10}{x} + 8$

c. $m(x) = \frac{4}{x-7} - 1$

d. $n(x) = f(4x)$

e. $p(x) = -3f(x - 4)$

f. $v(x) = f(-2x) + 3$

Lesson 2 Assignment

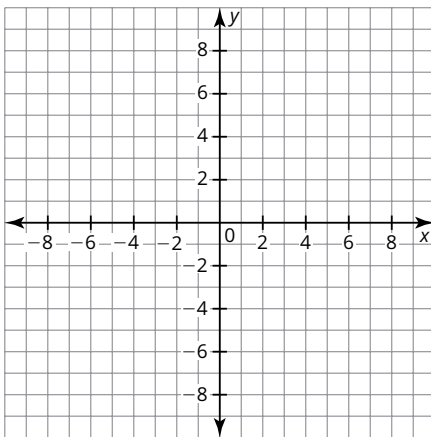
Prepare

1. Sketch the graph of each rational function on the same coordinate plane.

$$f(x) = \frac{1}{x}$$

$$g(x) = \frac{1}{x+5}$$

$$h(x) = \frac{1}{x-5}$$



2. Describe $g(x)$ in terms of $f(x)$.

3. Describe $h(x)$ in terms of $f(x)$.

Lesson 2 Assignment

4. Identify the value(s) of x that make each function undefined.

a. $f(x) = -\frac{3}{4x}$

b. $f(x) = \frac{x}{x-2}$

c. $f(x) = \frac{1}{12-x}$

d. $f(x) = \frac{-6}{5+x}$

3

Exploring Asymptotes

OBJECTIVES

In this lesson, you will:

- Compare rational functions in different forms.
- Explain how to identify vertical asymptotes of rational functions.

.....

You have learned about asymptotes and the behavior of graphs around them.

How do vertical and horizontal asymptotes affect the domain and range of a rational function?

Getting Started

.....
 Compare your
 answers to your
 classmates' and
 discuss the behavior
 of the rational
 functions before
 making a conjecture.

Shifty Behavior

Recall that the parent rational function has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$. The domain is all real numbers except for 0, because division by 0 is undefined.

Let's explore how horizontal translations of the rational parent function affect the domain and range of the function.

Let $g(x) = \frac{1}{x - c}$ for a constant value c .

1. Consider the table shown.
 - a. Identify the vertical asymptote, horizontal asymptote, domain, and range for the given c -values. Write the domain and range using inequalities. Then choose different positive and negative c -values to complete the table.

c-value	$g(x) = \frac{1}{x - c}$	Vertical Asymptote(s)	Horizontal Asymptote(s)	Domain	Range
1	$g(x) = \frac{1}{x - 1}$				
-2	$g(x) = \frac{1}{x + 2}$				

- b. Determine the general formula to identify the vertical asymptote of a rational function in the form $g(x) = \frac{1}{x-c}$. Explain your reasoning.
- c. What generalization(s) can you make about the c -value and the domain? The range?

Identifying Asymptotes

In the Getting Started you explored how horizontal transformations affect the location of the asymptotes and domain and range of a rational function.

1. Without using graphing technology, determine the domain, range, and vertical and horizontal asymptotes of each rational function. Write each domain and range in set notation.

a. $f(x) = \frac{10}{x}$

Domain:

Range:

Vertical Asymptote:

Horizontal Asymptote:

b. $j(x) = \frac{1}{10x}$

Domain:

Range:

Vertical Asymptote:

Horizontal Asymptote:

c. $g(x) = \frac{1}{x + 10}$

Domain:

Range:

Vertical Asymptote:

Horizontal Asymptote:

d. $h(x) = \frac{1}{x - 10}$

Domain:

Range:

Vertical Asymptote:

Horizontal Asymptote:

e. $k(x) = \frac{1}{x} + 10$

Domain:

Range:

Vertical Asymptote:

Horizontal Asymptote:

f. $p(x) = \frac{1}{x} - 10$

Domain:

Range:

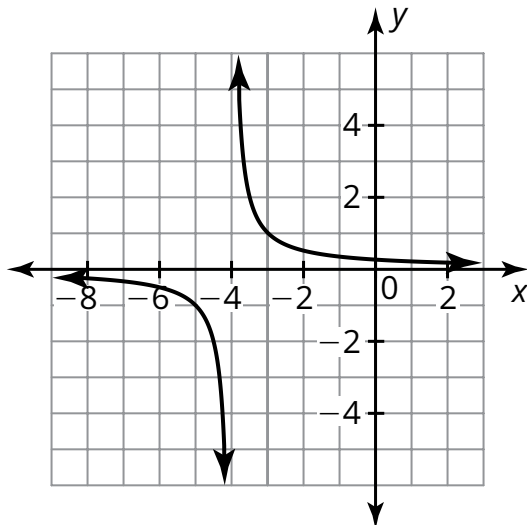
Vertical Asymptote:

Horizontal Asymptote:

You can use what you know about asymptotes and transformations to write the equation of a rational function when given a graph, description, or table of values.

2. Write the rational function(s) from the graph, table, or description provided. Explain your reasoning.

a.



Function: _____

Explanation:

- b. Vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 0$.

Function 1: _____

Function 2: _____

Explanation:

Ask Yourself . . .

You are asked to determine 2 functions. How many functions exist that fit the description given?

- c. Domain: All Real Numbers except $x = 7$
Range: All Real Numbers except $y = 0$

Function 1: _____

Function 2: _____

Explanation:

d.

x	y
-2	-0.5
-1	$-\frac{2}{3}$
0	-1.0
1	-2.0
2	undefined
3	2.0
4	1.0

Function: _____

Explanation:

e.

x	y
-2	-3.0
-1	undefined
0	3.0
1	1.5
2	1
3	0.75
4	0.60

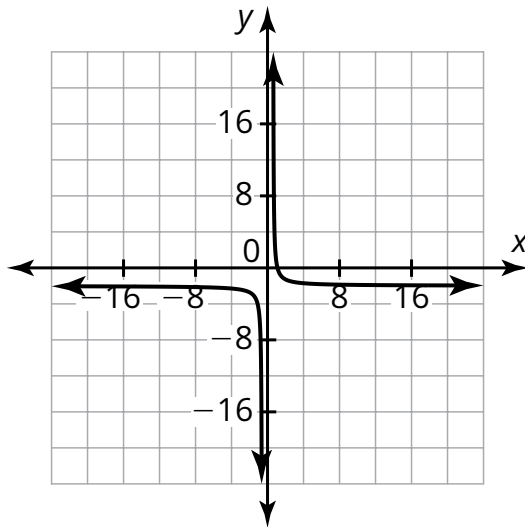
Function: _____

Explanation:

3. Compare the effect that changing the c -value has on the reciprocal function to the effect that changing the c -value has on any polynomial function $f(x)$ to form $g(x) = f(x - c)$.

4. Write a rational function from the graph, table, or description provided. Explain your reasoning.

a.



Function: _____

Explanation:

- b. Vertical asymptote at $x = -2$ and a horizontal asymptote at $y = 4$.

Function: _____

Explanation:

- c. Domain: All Real Numbers except $x = 0$
Range: All Real Numbers except $y = -1$

Function: _____

Explanation:

d.

x	y
-2	undefined
-1	13
0	11.5
1	11
2	10.75
3	10.6
4	10.5

Function: _____

Explanation:

5. Compare the effect that changing the d -value has on the reciprocal function to the effect that the d -value has when performed on any polynomial function $f(x)$ to form $g(x) = f(x) + d$.

ACTIVITY
3.2

Using Algebra to Determine Asymptotes

You have analyzed rational functions with just 1 vertical asymptote. The vertical asymptote occurred at the value for which the denominator was zero.

1. Without graphing, determine the number of vertical asymptotes for each function. Show all work and explain your reasoning.

a. $f(x) = \frac{4}{x^2 + 4}$

b. $g(x) = \frac{4}{x^2 - 4}$

c. $h(x) = \frac{4}{x^2 + 4x + 4}$

.....
Consider factoring the functions. Check your work with graphing technology after you are done.
.....

Recall from the Fundamental Theorem of Algebra that a function of degree n has n zeros. Some of the zeros may be imaginary. Therefore, it follows that the reciprocal of a function of degree n can have at most n vertical asymptotes.

2. Huyen determines the vertical asymptotes for the function

$$f(x) = \frac{1}{2x^2 - 14x - 16}.$$

Huyen

The terms in the denominator have a common factor of 2, so I factored it out first. Then I factored the remaining quadratic.

$$f(x) = \frac{1}{2(x^2 - 7x - 8)} = \frac{1}{2(x - 8)(x + 1)}$$

Vertical asymptotes occur when the denominator is zero. So, the asymptotes will occur when $x - 8 = 0$ and when $x + 1 = 0$.

Therefore, the asymptotes occur at $x = 8$ and $x = -1$.

Is Huyen correct? Explain your reasoning.

3. Write the domain and range of $f(x)$ from Huyen's example using interval notation. How are the domain and range of the graph affected by the asymptotes?

4. Analyze each rational function. Use algebra to determine the vertical asymptotes. Then, write the domain and range of the function using interval notation.

a. $f(x) = \frac{5}{7x - 35}$

b. $g(x) = \frac{1}{x(x - 2)(2x + 3)}$

c. $h(x) = \frac{10}{x^2 - 3x - 10}$

d. $h(x) = \frac{x}{2x^2 + 9x + 4}$

e. $h(x) = \frac{7}{x^4 - 1}$

f. $f(x) = \frac{2}{x^2 + 2}$

5. Determine 2 different rational functions with the attributes given. Then write the domain and range of each function using inequalities.

a. Vertical asymptotes at $x = 3$, and $x = -1$

b. Vertical asymptotes at $x = \frac{1}{2}$ and $x = 2$

6. Consider the rational function with vertical asymptotes at $x = -1$ and $x = 4$ and y -intercept at $(0, -1)$.

a. Write a rational function from the description provided. Explain your reasoning.

- b. Write the domain and range of the function you wrote in part (a) using inequalities, set notation, and interval notation.



Talk the Talk

Asymptote Adventure

Determine a rational function with the attributes given.

1. Vertical asymptotes at $x = -7$, $x = 12$
2. No vertical asymptotes
3. A vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 0$

Determine the asymptotes of each function. Then, write the domain and range of the function using set notation, interval notation, and inequalities.

4. $f(x) = \frac{-1}{2x^2 - 11x + 12}$

5. $g(x) = \frac{6}{x^2 + 4}$

6. $h(x) = \frac{11}{x(x - 3)(x + 1)}$

Lesson 3 Assignment

Write

How do the asymptotes of a rational function affect the domain and range of the function?

Remember

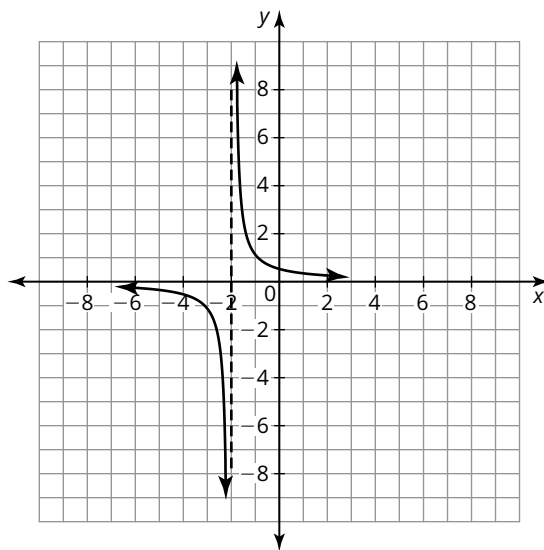
You can use factoring to identify the asymptotes of a rational function.

For a rational function, the vertical asymptotes are determined by input values for which the denominator is 0.

Practice

1. Write a rational function from the graph, table, or description provided. Explain your reasoning.

a.



Function:

Explanation:

b.

x	y
-3	-1
-1	-2
0	-4
1	undefined
2	4
3	2
5	1

Function:

Explanation:

Lesson 3 Assignment

- c. Vertical asymptote at $x = -5$, horizontal asymptote at $y = 0$, and y -intercept $(0, 2)$

Function:

Explanation:

- d. The domain is all real numbers except -6 , the range is all real numbers except 0 , and the y -intercept is $(0, -\frac{1}{2})$.

Function:

Explanation:

2. Use algebra to determine the vertical asymptotes of each function. Then, write the domain and range of each function using inequalities, interval notation, and set notation.

a. $f(x) = \frac{-3}{(x-2)^2}$

Lesson 3 Assignment

b. $g(x) = \frac{5}{(x-3)(x+7)}$

c. $h(x) = \frac{2}{x^2 - 9x + 8}$

Lesson 3 Assignment

d. $j(x) = \frac{5}{x^2 + 25}$

Prepare

1. $\frac{1}{2} + \frac{1}{3}$

2. $\frac{1}{2} - \frac{1}{3}$

3. $\frac{1}{2} \cdot \frac{1}{3}$

4. $\frac{1}{2} \div \frac{1}{3}$

4

Operations with Rational Expressions

OBJECTIVES

- Add and subtract rational expressions.
 - Factor rational expressions to determine a least common denominator.
 - Multiply and divide rational expressions and explain the process orally and in writing.
-

You know how to add, subtract, multiply, and divide rational numbers. How can these processes be extended to include rational expressions?

Light the Corners of My Mind

Remember ...

The remainder when dividing a polynomial function f by a linear expression $x - a$ is equal to $f(a)$.

You can use long division or synthetic division to divide when the divisor is degree-1 polynomial.

In elementary school, you divided rational numbers using long division. You wrote the quotient first with a remainder and then as a mixed number.

For example, $13 \div 3 = 4 \text{ R}1$, and $13 \div 3 = 4\frac{1}{3}$.

1. How are the quotients of rational expressions connected to those you determined in elementary school?
 - a. Write the quotient of $x^3 + 2x^2 - 3x + 1 \div (x - 2)$ in two ways. Show your work.
 - b. How do you know that $x \neq 2$ is a restriction for the quotient?

ACTIVITY
4.1

Calculating Sums and Differences of Rational Expressions

You learned that operating with polynomials is similar to operating with integers. Likewise, operating with rational expressions is similar to operating with rational numbers. Consider the table shown.

	Rational Numbers	Rational Expressions Involving Variables in the Numerator
Example 1	$\frac{1}{6} + \frac{5}{6} - \frac{1}{6} = \frac{5}{6}$	$\frac{1x}{6} + \frac{5x}{6} - \frac{1x}{6} = \frac{5x}{6}$
Example 2	$\frac{3}{2} + \frac{2}{5} - \frac{3}{4} = \frac{3(10)}{2(10)} + \frac{2(4)}{5(4)} - \frac{3(5)}{4(5)}$ $= \frac{30}{20} + \frac{8}{20} - \frac{15}{20}$ $= \frac{23}{20}$	$\frac{3x}{2} + \frac{2y}{5} - \frac{3x}{4} = \frac{3(10)x}{2(10)} + \frac{2(4)y}{5(4)} - \frac{3(5)x}{4(5)}$ $= \frac{30x}{20} + \frac{8y}{20} - \frac{15x}{20}$ $= \frac{15x + 8y}{20}$
Example 3	$\frac{3}{5} + \frac{2}{3} - \frac{2}{15} = \frac{3(3)}{5(3)} + \frac{2(5)}{3(5)} - \frac{2}{15}$ $= \frac{9}{15} + \frac{10}{15} - \frac{2}{15}$ $= \frac{17}{15}$	$\frac{3x}{5} + \frac{2y}{3} - \frac{2}{15} - \frac{2x + 3y}{5} = \frac{3(3)x}{5(3)} + \frac{2(5)y}{3(5)} - \frac{2}{15} - \frac{(2x + 3y)(3)}{5(3)}$ $= \frac{9x}{15} + \frac{10y}{15} - \frac{2}{15} - \frac{6x + 9y}{15}$ $= \frac{9x + 10y - 2 - (6x + 9y)}{15}$ $= \frac{9x + 10y - 2 - 6x - 9y}{15}$ $= \frac{3x + y - 2}{15}$

1. Analyze the examples.

a. Explain the process used to add and subtract each expression.

b. In Example 2, why is $\frac{3}{2} = \frac{3(10)}{2(10)}$ and why is $\frac{3x}{2} = \frac{3(10)x}{2(10)}$?

2. Analyze Abby's work. Explain how Abby could have been more efficient in her work.

Abby

$$\frac{3x}{3} + \frac{2x}{8} - \frac{1}{2}$$

To add and subtract fractions, I need a common denominator. To determine one, I multiply all the denominators together: $3 \cdot 8 \cdot 2 = 48$

$$\begin{aligned}\frac{3x(16)}{3(16)} + \frac{2x(6)}{8(6)} - \frac{1(24)}{2(24)} &= \frac{48x}{48} + \frac{12x}{48} - \frac{24}{48} \\ &= \frac{60x - 24}{48} \\ &= \frac{5x - 2}{4}\end{aligned}$$

3. Calculate each sum and difference.

a. $\frac{7x}{3} + \frac{1}{2}$

b. $\frac{5}{6} - \frac{2y}{3}$

c. $\frac{3}{6} + \frac{5x}{4} - \frac{y}{8}$

d. $\frac{x-2y}{3} + \frac{x}{12} - \frac{z}{4}$

e. $\frac{4x}{6} - \frac{2x}{9} - \frac{x}{18}$

4. Notice that all the variables in the right column of the table on the previous page are in the numerator. Suppose there were variables in the denominator. Do you think the process of adding and subtracting the expressions would change? Explain your reasoning.

Determining the LCD of Rational Expressions

When rational expressions contain variables in the denominator, the process remains the same—you still need to determine the least common denominator (LCD) before adding and subtracting.

1. Consider Method A compared to Method B in both columns of the table.

	Rational Numbers	Rational Expressions Involving Variables in the Denominator
Method A	$\frac{1}{3} + \frac{1}{3^2} = \frac{1(3^2)}{3(3^2)} + \frac{1(3)}{3^2(3)}$ $= \frac{3^2}{3^3} + \frac{3}{3^3}$ $= \frac{3^2 + 3}{3^3}$ $= \frac{3(3 + 1)}{3(3^2)}$ $= \frac{4}{3^2}$	$\frac{1}{x} + \frac{1}{x^2} = \frac{1(x^2)}{x(x^2)} + \frac{1(x)}{x^2(x)}$ $= \frac{x^2}{x^3} + \frac{x}{x^3}$ $= \frac{x^2 + x}{x^3}$ $= \frac{x(x + 1)}{x(x^2)}$ $= \frac{x + 1}{x^2}$ $\frac{1}{x} + \frac{1}{x^2} = \frac{x + 1}{x^2} \text{ for } x \neq 0$
Method B	$\frac{1}{3} + \frac{1}{3^2} = \frac{1(3)}{3(3)} + \frac{1}{3^2} = \frac{4}{3^2}$	$\frac{1}{x} + \frac{1}{x^2} = \frac{1(x)}{x(x)} + \frac{1}{x^2}$ $= \frac{x + 1}{x^2}$ $\frac{1}{x} + \frac{1}{x^2} = \frac{x + 1}{x^2} \text{ for } x \neq 0$

- a. Explain the difference in the methods.

- b. Explain why the statement $\frac{1}{x} + \frac{1}{x^2} = \frac{x + 1}{x^2}$ has the restriction $x \neq 0$.

- c. How would these methods change when the examples involve subtracting the rational expressions: $\frac{1}{3} - \frac{1}{3^2}$ and $\frac{1}{x} - \frac{1}{x^2}$? Explain your reasoning.



2. Maria and Luis determine the LCD for the expression

$$\frac{1}{x^2 - 1} + \frac{1}{x + 1}$$

Maria

$$\frac{1}{x^2 - 1} + \frac{1}{x + 1}$$

$$(x^2 - 1)(x + 1)$$

$$\text{LCD: } x^3 + x^2 - x - 1$$

Luis

$$\frac{1}{x^2 - 1} + \frac{1}{x + 1}$$

$$\frac{1}{(x - 1)(x + 1)} + \frac{1}{x + 1}$$

$$(x - 1)(x + 1)$$

$$\text{LCD: } x^2 - 1$$

Who is correct? Explain your reasoning.

3. Determine the least common denominator for each set of rational expressions.

a. $\frac{3}{x + 4}, \frac{7x}{x - 4}$

b. $\frac{-2}{3x - 2}, \frac{4x}{3x^2 + 7x - 6}$

c. $\frac{-11}{x}, \frac{7}{x - 4}, \frac{x}{x^2 - 16}$

d. $\frac{2x}{x^2 - 5x + 6}, \frac{7x + 11}{x^2 - 6x + 9}$

Make sure to list the restrictions for the variable.

Notice that even when there are binomials in the denominator, subtracting two rational expressions is similar to subtracting two rational numbers. The example shows a difference of two rational expressions of degree 2.

WORKED EXAMPLE

You can determine the difference of two rational expressions with binomials in the denominator.

$$\begin{aligned}\frac{1}{x^2 - 1} - \frac{1}{x^2 + 2x + 1} &= \frac{1}{(x + 1)(x - 1)} - \frac{1}{(x + 1)(x + 1)} \\ &= \frac{1(x + 1)}{(x + 1)(x - 1)(x + 1)} - \frac{1(x - 1)}{(x + 1)(x + 1)(x - 1)} \\ &= \frac{(x + 1) - (x - 1)}{(x + 1)(x + 1)(x - 1)} \\ &= \frac{2}{(x + 1)(x + 1)(x - 1)}, x \neq -1, 1\end{aligned}$$

4. How would the method change when the example shown involved adding the rational expressions? Explain your reasoning. Then add the expressions to determine the sum.

5. Malik and Ana add $\frac{2x+2}{x+1} + \frac{1}{x}$.

Malik



$$\begin{aligned}\frac{2x+2}{x+1} + \frac{1}{x} &= \frac{(2x+2)(x)}{(x+1)(x)} + \frac{1(x+1)}{x(x+1)} \\ &= \frac{2x^2+2x}{(x+1)(x)} + \frac{x+1}{x(x+1)} \\ &= \frac{2x^2+2x+x+1}{x(x+1)} \\ &= \frac{2x^2+3x+1}{x(x+1)} \\ &= \frac{(2x+1)(x+1)}{x(x+1)} \\ &= \frac{2x+1}{x}\end{aligned}$$

Ana



$$\begin{aligned}\frac{2x+2}{x+1} + \frac{1}{x} &= \frac{\cancel{2(x+1)}}{\cancel{(x+1)}} + \frac{1}{x} \\ &= 2 + \frac{1}{x} \\ &= \frac{2(x)}{(x)} + \frac{1}{x} \\ &= \frac{2x+1}{x}\end{aligned}$$

a. Explain the difference in the methods used.

b. Destiny says the only restriction on the variable x in Malik and Ana's problem is $x \neq 0$. Gabriela says $x \neq 0, -1$.

Who is correct? Explain your reasoning.



Patterns in Rational Expression Differences

Consider the Worked Examples.

WORKED EXAMPLE

Determine each difference.

Example 1

$$\frac{3}{x-1} - \frac{3}{x+1}$$

$$\frac{3(x+1)}{(x-1)(x+1)} - \frac{3(x-1)}{(x+1)(x-1)}$$

$$\frac{3x+3-3x+3}{(x-1)(x+1)}$$

$$\frac{6}{(x-1)(x+1)}$$

Example 2

$$\frac{1}{x-1} - \frac{1}{x+1}$$

$$\frac{1(x+1)}{(x-1)(x+1)} - \frac{1(x-1)}{(x+1)(x-1)}$$

$$\frac{x+1-x+1}{(x-1)(x+1)}$$

$$\frac{2}{(x-1)(x+1)}$$

1. Describe the similarities and the differences in the structure of each example.
2. Consider the given expression and the resulting difference. What pattern do you notice?
3. Suppose the numerators in Example 1 of the Worked Example were doubled. What would be the new answer?

4. Can you use this pattern to determine $\frac{4}{x-2} - \frac{4}{x+2}$?
Explain your reasoning.
5. How would the pattern in the Worked Example change when you added the terms together?
6. Calculate each sum or difference. Make sure to list the restrictions for the variable, and rewrite in lowest terms.

a. $\frac{5x-6}{x^2-9} - \frac{4}{x-3}$

b. $\frac{x-7}{x^2-3x+2} + \frac{4}{x^2-7x+10}$

c. $\frac{2x-5}{x} - \frac{4}{5x} - 4$

d. $\frac{3x-5}{4x^2+12x+9} + \frac{4}{2x+3} - \frac{2x}{3}$

e. $\frac{x+2}{x-4} + \frac{2}{x} + \frac{5}{3x-1}$

f. $\frac{x^2+5x+4}{x^2+x-6} + \frac{x^2-11x+30}{x^2+2x-3}$

g. $\frac{x^2+4x-5}{6x^2+7x-3} - \frac{x^2-3x-28}{3x^2+2x-1}$

ACTIVITY
4.4

Calculating Products of Rational Expressions

Previously, you learned that adding and subtracting rational expressions involved the same process as adding and subtracting rational numbers. Now, you will see that multiplying rational expressions involves the same steps as multiplying rational numbers.

1. Consider Method A compared to Method B in both columns of the table.

	Rational Numbers	Rational Expressions Involving Variables	
Method A	$\frac{2}{15} \cdot \frac{5}{8} = \frac{10}{120}$ $= \frac{1}{12}$	$\frac{2x}{15x} \cdot \frac{5x}{8} = \frac{10x^2}{120x}$ $= \frac{x}{12}$	$\frac{2x}{15x^2} \cdot \frac{5x^2}{8} = \frac{10x^3}{120x^2}$ $= \frac{x}{12}$
Method B	$\frac{\frac{1}{2}}{\frac{15}{3}} \cdot \frac{\frac{1}{5}}{\frac{8}{4}} = \frac{1}{12}$	$\frac{\frac{x}{2x}}{\frac{15x}{3}} \cdot \frac{\frac{1}{5x}}{\frac{8}{4}} = \frac{x}{12}$	$\frac{\frac{1}{2x}}{\frac{15x^2}{3}} \cdot \frac{\frac{1}{5x^2}}{\frac{8}{4}} = \frac{x}{12}$

- a. Explain the difference in the methods.

- b. Which method do you prefer?

Remember . . .

When you multiply rational numbers, you can divide out factors at the beginning or the end, and the product is the same. However, dividing out factors earlier saves time and keeps the expressions simpler.



2. Given the equation $\frac{2x}{15x^2} \cdot \frac{5x^2}{8} = \frac{x}{12}$, Michael says that $x \neq 0$. Javier says that there are no restrictions because the answer is $\frac{x}{12}$ and there are no variables in the denominator. Who is correct? Explain your reasoning.

3. Analyze Olivia's work. Explain how Olivia could have multiplied the rational expressions more efficiently.

Olivia


$$\frac{12x^2yz^2}{11} \cdot \frac{33x}{8z} = \frac{\overset{9}{\cancel{36}} x^2 \overset{2}{\cancel{4}} z^2}{\cancel{8} \cancel{8} z}$$
$$= \frac{9x^2yz^2}{2}$$

4. Fernando multiplies $\frac{5x^2}{3x^2 - 75} \cdot \frac{3x - 15}{4x^2}$ without dividing out factors first. Complete the same problem as Fernando, by dividing out common factors first, and then list the restrictions.

Fernando 

$$\begin{aligned} \frac{5x^2}{3x^2 - 75} \cdot \frac{3x - 15}{4x^2} &= \frac{15x^3 - 75x^2}{12x^4 - 300x^2} \\ &= \frac{15x^2(x - 5)}{3x^2(4x^2 - 100)} \\ &= \frac{15x^2\cancel{(x - 5)}}{3x^2(4x^2 - 100)} \\ &= \frac{5(x - 5)}{4(x^2 - 25)} \\ &= \frac{5\cancel{(x - 5)}}{4(x - 5)(x + 5)} \\ &= \frac{5}{4(x + 5)} \end{aligned}$$

5. Multiply each expression. List the restrictions for the variables.

a. $\frac{3ab^2}{4c} \cdot \frac{2c^2}{27ab}$

b. $\frac{3x}{5x - 15} \cdot \frac{x - 3}{9x}$

c. $\frac{x+5}{x^2-4x+3} \cdot \frac{x-3}{4x+20}$

d. $\frac{3x^2+8x-16}{2x^2+3x-14} \cdot \frac{2x^2+19x+42}{x^2+x-12} \cdot \frac{x^2-5x+6}{6x^2-11x+4}$

ACTIVITY
4.5

Patterns in Rational Expression Quotients

Dividing rational expressions is similar to the process you use when dividing rational numbers.

You may recall that, to divide fractions, you multiply the dividend by the multiplicative inverse of the divisor.

	Rational Numbers	Rational Expressions Involving Variables	
Example 1	$\frac{1}{5} \div \frac{3}{10} = \frac{1}{5} \cdot \frac{10}{3}$ $= \frac{2}{3}$	$\frac{xy}{5z} \div \frac{3xy}{10z} = \frac{xy}{5z} \cdot \frac{10z}{3xy}$ $= \frac{2}{3}; x, y, z \neq 0$	$\frac{xy^2}{5z} \div \frac{3xy}{10z^2} = \frac{xy^2}{5z} \cdot \frac{10z^2}{3xy}$ $= \frac{2yz}{3}; x, y, z \neq 0$
Example 2	$\frac{6}{7} \div 4 = \frac{6}{7} \cdot \frac{1}{4}$ $= \frac{3}{14}$	$\frac{6a}{7b} \div 4a = \frac{6a}{7b} \cdot \frac{1}{4a}$ $= \frac{3}{14b}; a, b \neq 0$	$\frac{6a^3}{7b} \div 4a = \frac{6a^3}{7b} \cdot \frac{1}{4a}$ $= \frac{3a^2}{14b}; a, b \neq 0$

1. Analyze the examples shown in the table.

a. Explain the process for dividing rational expressions.

b. In Example 2, explain why $\frac{1}{4a}$ is the multiplicative inverse of $4a$.

.....
Make sure to list
the restrictions for
the variables.
.....

2. Determine the quotients of each expression.

a. $\frac{7a}{b} \div \frac{b+2}{a-8}$

b. $\frac{9ab^2}{4c} \div \frac{18c^2}{5ab}$

c. $\frac{7x^2}{3x^2-27} \div \frac{4x^2}{3x-9}$

d. $\frac{3x^2+15x}{x^2-3x-40} \div \frac{5x^2}{x^2-64}$



Talk the Talk

Closure: Wherever You Go, There You Are

You have compared operating with rational expressions to operating with rational numbers.

1. Determine each sum, difference, product, or quotient.

a. $\frac{1}{x} + \frac{1}{y}$

b. $\frac{1}{x} - \frac{1}{y}$

c. $\frac{1}{x} \cdot \frac{1}{y}$

d. $\frac{1}{x} \div \frac{1}{y}$

e. $\frac{1}{x^2} + \frac{1}{y}$

f. $\frac{1}{x^2} + \frac{1}{y^2}$

g. $\frac{1}{x^2} - \frac{1}{y}$

h. $\frac{1}{x^2} - \frac{1}{y^2}$

i. $\frac{1}{x^2} \cdot \frac{1}{y}$

j. $\frac{1}{x^2} \cdot \frac{1}{y^2}$

k. $\frac{1}{x^2} \div \frac{1}{y}$

l. $\frac{1}{x^2} \div \frac{1}{y^2}$

2. What similarities and differences are there in the methods you used to determine your answers?

Lesson 4 Assignment

Write

In your own words, explain how operating with rational expressions is similar to operating with rational numbers. Use examples to illustrate your reasoning.

Remember

The process for adding and subtracting rational expressions is similar to the process for adding and subtracting rational numbers, and the process for multiplying and dividing rational expressions is similar to the process for multiplying and dividing rational numbers.

Practice

1. Add or subtract each expression. List any restrictions on the variables.

a. $\frac{3y}{4} + \frac{x}{3} + \frac{5y}{6}$

b. $\frac{2}{2x-4} - \frac{5}{x^2-4}$

c. $\frac{60-3x}{x^2+x-20} + \frac{3x+9}{x+3}$

d. $\frac{2}{2x^2+7x+3} - \frac{x}{x^2-2x-15} + 1$

Lesson 4 Assignment

e. $\frac{2x}{x+1} - \frac{x-1}{x+2} - \frac{x}{x-3}$

f. $\frac{x^2 + 3x + 2}{x^2 - 4} + \frac{2x^2 - 3x - 9}{x^2 + x - 6}$

g. $\frac{8x^2 - 18x - 5}{x^2 - 9} - \frac{3x^2 + 7x - 6}{3x^2 - 11x + 6}$

2. Hannah multiplied $\frac{3x-6}{8x-16} \cdot \frac{4x}{9}$. Her work is shown.

Hannah

$$\begin{aligned} \frac{3x-6}{8x-16} \cdot \frac{4x}{9} &= \frac{\overset{1}{\cancel{3x-6}}}{\underset{2}{\cancel{8x-16}}} \cdot \frac{\overset{1}{\cancel{4x}}}{\underset{3}{\cancel{9}}} \\ &= \frac{x-6}{-14} \cdot \frac{1}{3} \\ &= \frac{x-6}{-42}; x \neq 2 \end{aligned}$$

Identify the error(s) in Hannah's work. Then correctly perform the multiplication.

Lesson 4 Assignment

3. Determine each product or quotient. List any restrictions on the variables.

a. $\frac{2x^2}{z^2} \cdot \frac{6z^2}{4y^2}$

b. $\frac{x-4}{5x+10} \cdot \frac{2x+4}{3x-6}$

c. $\frac{2x^2-32}{x^2-10x+24} \cdot \frac{x^2-4x-12}{x^2-5x-14}$

d. $\frac{6x}{x-9} \cdot \frac{x^2-10x+9}{2x+12} \cdot \frac{x+6}{x^4+x^3-2x^2}$

e. $\frac{8xy}{5z} \div \frac{10x}{3z}$

f. $\frac{14x^2}{2x^2+10x} \div \frac{7x^2-21x}{x^2-25}$

g. $\frac{xy^2-6y^2}{2x+18} \div \frac{2xy+y}{8x+24} \div \frac{x^2-3x-18}{2x^2-3x-2}$

Lesson 4 Assignment

Prepare

Solve each proportion.

1. $\frac{x}{12} = \frac{5}{3}$

2. $\frac{x+2}{8} = \frac{6}{3}$

5

Solving Problems with Rational Equations

OBJECTIVES

- Model situations with rational functions.
 - Use rational expressions to solve real-world problems.
 - Orally explain how to solve rational equations in one variable.
-

KEY TERM

- rational equation

You have analyzed and performed operations with rational expressions.

How can you solve equations composed of rational expressions?

Getting Started

.....
Use a ruler to draw
your rectangles with
accurate side lengths.
.....

The Golden Ratio

The ancient Greeks thought certain rectangles in art and architecture were much more pleasing to the eye than others. When the ratio of the sum of the length and width to the length is approximately 1.618, they felt the rectangle was perfectly proportionate. This ratio came to be known as the Golden Ratio.

1. Draw and describe the length and width of several rectangles with dimensions that are in the Golden Ratio.

ACTIVITY
5.1

Methods for Solving Rational Equations

A **rational equation** is an equation that contains one or more rational expressions. There are multiple methods you can use to solve rational equations. Depending on the structure of the equation, some methods will be more efficient than others.

1. Amir used proportional reasoning to solve the equation $\frac{x+5}{x+2} = \frac{x+1}{x-5}$. Explain how he solved it.

Amir

$$\frac{x+5}{x+2} = \frac{x+1}{x-5}$$

Restrictions: $x \neq -2, 5$

$$(x+5)(x-5) = (x+2)(x+1)$$

$$x^2 - 25 = x^2 + 3x + 2$$

$$-25 = 3x + 2$$

$$-27 = 3x$$

$$x = -9$$



Ask Yourself ...

Why is it reasonable that x cannot equal -2 or 5 ?

2. Three students each solved a slightly different equation: $\frac{x+5}{(x-5)(x+2)} = \frac{x+1}{x-5}$. They first recognized the restrictions as $x \neq -2, 5$. Consider their work.

Koda

$$(x+5)(\cancel{x-5}) = (\cancel{x-5})(x+2)(x+1)$$

$$x+5 = x^2 + 3x + 2$$

$$0 = x^2 + 2x - 3$$

$$0 = (x+3)(x-1)$$

$$-3, 1 = x$$



Daniela



$$\frac{x+5}{x+1} = \frac{(x-5)(x+2)}{x-5} \quad \leftarrow \text{Rewrite the proportion.}$$

$$\frac{x+5}{x+1} = x+2$$

$$(x+5) = (x+1)(x+2)$$

$$x^2 + 3x + 2 = x + 5$$

$$x^2 + 2x - 3 = 0$$

$$(x+3)(x-1) = 0$$

$$x = -3, 1$$

William



$$\frac{(x+1)(x+2)}{(x-5)x+2} = \frac{x+5}{(x-5)(x+2)} \quad \leftarrow \text{Write with common denominator.}$$

$$(x+1)(x+2) = (x+5)$$

$$x+5 = x^2 + 3x + 2$$

$$0 = x^2 + 2x - 3$$

$$0 = (x+3)(x-1)$$

$$-3, 1 = x$$

- a. What is different about the structure of this equation compared to the equation in Question 1?
- b. Explain how Daniela rewrote the proportion to solve the equation.

c. How did William determine the common denominator?

d. Which method do you think is more efficient based on the structure of the equation?

3. Consider how Angel begins to solve $\frac{2x+4}{x^2-2x-8} = \frac{x+1}{x-4}$. Finish solving the equation and describe your strategy. Be sure to list the restrictions.

Angel 

$$\frac{2x+4}{x^2-2x-8} = \frac{x+1}{x-4}$$
$$\frac{2(x+2)}{(x-4)(x+2)} = \frac{x+1}{x-4}$$

Restrictions: $x \neq 4, -2$

$$\frac{2\cancel{(x+2)}}{(x-4)\cancel{(x+2)}} = \frac{x+1}{x-4}$$

4. Describe the methods Malik and Huyen each used to solve the rational equation.

Malik



$$\frac{6}{x^2 - 4x} + \frac{4}{x} = \frac{2}{x - 4}$$

$$\frac{6}{x(x - 4)} + \frac{4}{x} \cdot \frac{(x - 4)}{(x - 4)} = \frac{2}{x - 4} \cdot \frac{x}{x}$$

Restrictions: $x \neq 0, 4$

$$\frac{6}{x(x - 4)} + \frac{4x - 16}{x(x - 4)} = \frac{2x}{x(x - 4)}$$

$$\frac{6 + 4x - 16}{x(x - 4)} = \frac{2x}{x(x - 4)}$$

$$\cancel{x(x - 4)} \left[\frac{6 + 4x - 16}{\cancel{x(x - 4)}} = \frac{2x}{\cancel{x(x - 4)}} \right]$$

$$6 + 4x - 16 = 2x$$

$$4x - 10 = 2x$$

$$2x = 10$$

$$x = 5$$

Huyen



$$\frac{6}{x^2 - 4x} + \frac{4}{x} = \frac{2}{x - 4}$$

$$\frac{6}{x(x - 4)} + \frac{4}{x} = \frac{2}{x - 4}$$

Restrictions: $x \neq 0, 4$

$$(x(x - 4)) \cdot \left[\frac{6}{x(x - 4)} + \frac{4}{x} = \frac{2}{x - 4} \right]$$

$$6 + 4(x - 4) = 2x$$

$$6 + 4x - 16 = 2x$$

$$4x - 10 = 2x$$

$$2x = 10$$

$$x = 5$$

.....

Think About ...

How does the structure of this equation inform your strategy?

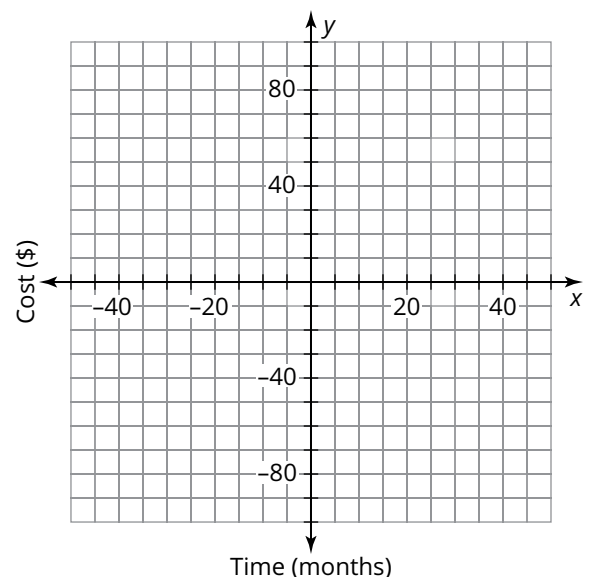
.....

5. Solve the equation $\frac{10}{x^2 - 2x} + \frac{1}{x} = \frac{3}{x - 2}$. Explain why you chose your solution method.

Writing Equations to Solve Problems

A door-to-door salesperson for a cable company offers cable television for only \$55.95 per month. However, there is a one-time installation cost of \$180.

1. Determine the total cost of cable for the first two months. What is the average cost per month over the first two months?
2. Determine the total cost of cable for the first year. What is the average cost per month over the first year?
3. Write an equation to represent the total cost of cable for x months.
4. Write an equation to represent the average monthly cost of cable for x months. Use technology to graph your equation and sketch it on the coordinate plane.
5. A competitor offers a similar product for \$65 per month and no installation charges. Who is offering the better deal? Use your graph to approximate the solution and then verify algebraically. Show all work and explain your reasoning.



A company manufactures a new brand of trail mix containing peanuts, almonds, and chocolate. Each package contains 400 grams of trail mix, with 50% peanuts, 35% almonds, and 15% chocolate. Javier loves chocolate. When he gets home, he wants to add enough chocolate so that the mixture is 50% chocolate. How many grams of chocolate should he add?



6. Daniela and Fernando model the given scenario with a rational function.

Daniela

I must first determine the amount of chocolate in the bag.

$$(0.15) \cdot (400) = 60\text{g}$$

The ratio of chocolate to total trail mix must increase to 50%. Adding chocolate increases the total amount of trail mix, so the new ratio is $\frac{60 + x}{400 + x}$. I can set up the proportion $\frac{60 + x}{400 + x} = 0.50$.

The x-value represents the amount of additional chocolate.

Fernando

The current chocolate to trail mix ratio is $\frac{15}{100}$.

Adding chocolate to get a mixture with 50% chocolate, add x to

the percent chocolate as well as the total, so the rational equation becomes $\frac{15 + x}{100 + x} = 0.50$.

The x-value of the intersection point represents the amount of chocolate Javier must add.

- a. Who is correct? Explain your reasoning. Describe the incorrect reasoning.

- c. Does your answer to part (b) seem reasonable?

Time = $\frac{d}{1.09t + 1050}$, where d is the distance (feet) between the observer and the lightning, and t is the temperature (Fahrenheit).

7. When the temperature outside is 70°F and you count 3 seconds between the thunder and the lightning, approximately how far away is the storm? Show all of your work and explain your reasoning.

Does your solution make sense in terms of the situation?

8. When the temperature is 80°F and you estimate half a second between thunder and lightning, how far away is the storm? Show all of your work and explain your reasoning.

9. On a 60°F day, what is the time between thunder and lightning when the storm is directly overhead? Show all work and explain whether the solution seems reasonable.

In an electrical circuit, when resistors are connected in parallel, the total resistance R_T of the circuit in ohms is given by:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n}$$

where n is the number of resistors and R_1 through R_n are the resistances of each resistor in ohms.

10. Determine the total resistance of an electrical circuit having two resistors connected in parallel if their resistances are 5 ohms and 8 ohms.

11. Three resistors connected in parallel have resistances of 4 ohms, 6 ohms, and 10 ohms. What is the total resistance in this electric circuit?
12. The total resistance in a parallel wiring circuit with two resistors is 12 ohms. If the resistance of one branch is 30 ohms, what is the resistance in the other branch?
13. A three-resistor parallel wiring circuit has a total resistance of 10 ohms. If two of the branches of the circuit have resistances of 20 ohms and 30 ohms, what is the resistance in the third branch?

PROBLEM SOLVING

14. Write and solve a rational equation to answer each question. Show all of your work and explain your reasoning.

- a. Alexander is taking a test that has two different parts to it. His goal is to get a 90%. He finished Part 1, and a quick scan by the teacher reveals that he got 18 out of the 23 questions correct. He begins Part 2. If he answers each consecutive question correctly, how many must he answer correctly for his grade to be higher than a 90%? Show all of your work and explain your reasoning.
- b. Jacob is considering joining a local gym. Joining the gym costs \$30 each month, plus \$2 per hour for using the basketball courts. They also allow people who are not members of the club to use the courts for \$7 per hour. If he joins the gym, how many hours would he have to play before the average cost is less than \$7 per hour? Show all work and explain your reasoning.



Talk the Talk

Structure. Stru- Stru- Structure.

1. Cut out the equations located at the end of the lesson. Sort the equations based on the solution method you intend to use. Finally, solve each equation. Be sure to list any domain restrictions.

Equation Cutouts



a. $\frac{12}{x+5} = -2$

b. $\frac{x-5}{3} = \frac{x-38}{12} - \frac{x}{4}$

c. $\frac{x^2-5x}{4} = \frac{8x}{2}$

d. $\frac{1}{x-5} = \frac{5}{x^2+2x-35}$

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So you can cut out your rectangles on the other side.



$$\text{e. } \frac{3}{x-1} + \frac{2}{5x+5} = \frac{-3}{x^2-1}$$

$$\text{f. } \frac{x-5}{x-2} = \frac{8}{9}$$

$$\text{g. } \frac{5}{x} = 25 + \frac{5}{x}$$

$$\text{h. } \frac{1}{x^2} + \frac{1}{x} = \frac{1}{2x^2}$$

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i. $\frac{-2}{x+3} + \frac{3}{x-2} = \frac{5}{x^2+x-6}$

j. $\frac{7}{x+3} = \frac{8}{x-2}$

k. $\frac{x+3}{x^2-1} + \frac{-2x}{x-1} = 1$

l. $\frac{3}{x^2+2x} = \frac{6}{x^2}$

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So you can cut out your rectangles on the other side.

Lesson 5 Assignment

Write

In your own words, define a rational equation. Use examples to illustrate your reasoning.

Remember

Rational expressions and equations are used to solve problems that involve comparing two quantities of the same unit of measure.

Practice

1. In football, a quarterback's completion percentage is the ratio of the number of complete passes to the total number of pass attempts. A top quarterback completed 66% of his passes over the course of his career in the National Football League. The quarterback after him in completion percentage completed 3843 passes out of 5853 attempts.

- a. Write and solve an equation to determine number of consecutive completed passes the second place quarterback must throw in order to break the record.

- b. Does the exact solution seem reasonable in terms of the problem situation? Explain your reasoning.

Lesson 5 Assignment

2. Ana compares two different refrigerators at the local hardware store.

The sales tags are shown.

Ana does some research online and learns that a kilowatt hour costs approximately \$0.06. She also learns that the average refrigerator lasts about 10 years.

ICY COLD

- Crushed Ice Dispenser
- Price \$699.00
- Uses 75 Kilowatt hours of electricity per month

COOL AS A CUCUMBER

- Gold Star EPA efficiency rating
- Price \$825
- Uses 30 Kilowatt hours of electricity per month

- a. Write a function to represent the average cost of each refrigerator per month.
- b. Which refrigerator will have a lower average monthly cost over the next ten years? Show all of your work and explain your reasoning.

Lesson 5 Assignment

Prepare

Solve for x .

1. $2x + 12 = 1$

2. $\frac{1}{2}(x - 3) = 2$

6

Solving Work, Mixture, Distance, and Cost Problems

OBJECTIVES

- Use rational equations to model, write, and solve work problems, mixture problems, distance problems, and cost problems.
- Model real-world and mathematical situations using rational equations.

.....

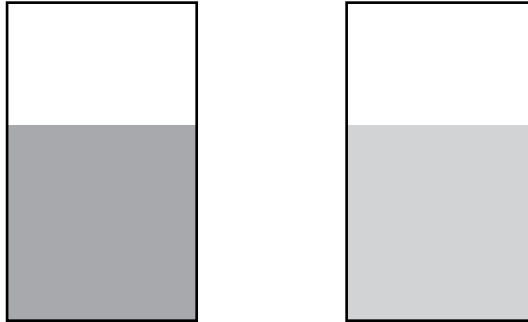
You have solved rational equations.

How can you model real-world situations using rational equations?

Getting Started

Mixing It Up

You have two containers—one holds cranberry juice and one holds ginger ale. The volume of liquid is the same in each container.



Suppose you do the following:

- Take exactly one cup of cranberry juice and pour it into the ginger ale container. Assume that the cranberry juice mixes perfectly into the ginger ale.
- Take exactly one cup from the cranberry juice and ginger ale mixture and pour it back into the cranberry juice container.

.....

Think About...

Suppose the containers were full of marbles—100 red for cranberry juice and 100 yellow for ginger ale.

This can help you solve the original problem.

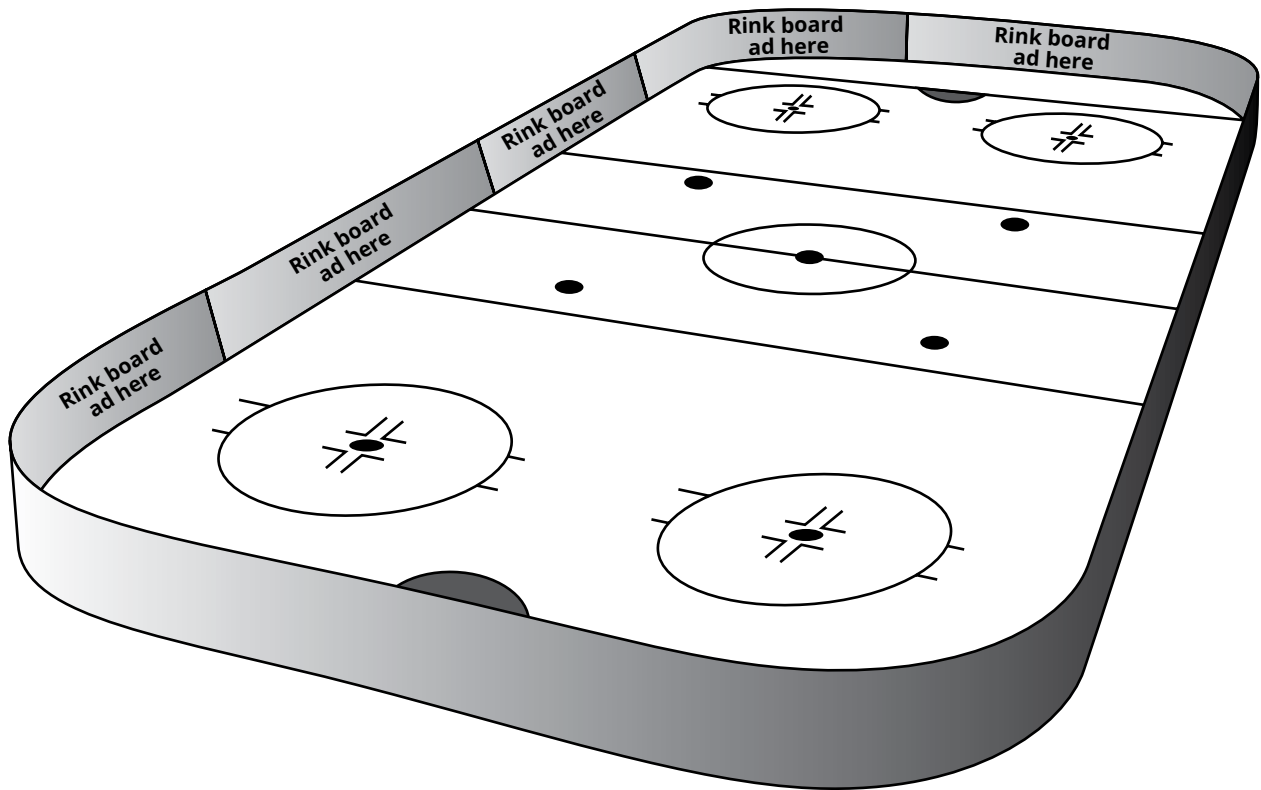
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1. Is there more cranberry juice in the ginger ale or more ginger ale in the cranberry juice? Write a paragraph to make your argument.

Solving Work Problems

A work problem involves the rates of several workers and the time it takes to complete a job.

Madison and William are the assistant managers for the marketing department of a hockey team. This hockey season is fast-approaching, and the rink board ads need to be mounted to the rink boards before the season begins.



Each ad is like a giant vinyl sticker that is stuck to each rink board along the inside of the hockey rink. It takes a team of three people to attach each ad: two people hold the ad while a third person presses it to the rink board, being careful that it does not wrinkle.

Up until last year, Madison's team and William's team have taken turns doing this job—Madison's team attached the rink boards for the first season, William's team attached them for the next season, the following season Madison's team attached them, and so on.

It takes Madison's team 20 hours to attach all of the rink boards, and it takes William's team 30 hours to attach all of the rink boards. This year, however, their boss has asked them to work together to get the job done faster.

1. Determine the portion of the rink each team completes in the given number of hours.

Madison's Team

1 hour:

5 hours:

10 hours:

William's Team

1 hour:

5 hours:

10 hours:

2. Consider the amount of the rink that each team can complete in x hours.
 - a. Write an expression to represent the portion of the rink that Madison's team can complete in x hours.
 - b. Write an expression that represents the portion of the rink that William's team can complete in x hours.
3. Each team's rate of work is defined as the number of jobs completed per hour. In this case, the rate of work is the number of rinks completed per hour.
 - a. Determine Madison's team's rate of work.
 - b. Determine William's team's rate of work.

4. Complete the table.

	Portion of the Rink Completed	Time Spent Working	Rate of Work
	Rinks	Hours	$\frac{\text{Rinks}}{\text{Hour}}$
Madison's Team		x	
William's Team		x	
Entire Job, or 1 Rink		x	

5. Consider the expression from the table that represents the portion of the rink that Madison's and William's teams can complete when working together. If you want to determine the total time it takes the two teams to complete one rink while working together, what should you set this expression equal to?

6. Write and solve an equation to determine the total time it takes the two teams to complete the rink.

.....
Make sure you are
using the appropriate
units of measure.
.....

7. Think about your answer in Question 6. Does your solution seem reasonable? Explain your reasoning.

8. Suppose that the two teams work together attaching rink board ads for 4 hours each day. How many days will it take them to complete the job?

Huyen is a community volunteer. She volunteers by watering the large vegetable garden in her neighborhood. Sometimes, Huyen’s friend Abby also volunteers.

9. It takes Huyen 90 minutes to water the garden. When Huyen and Abby work together, they can complete the job in 40 minutes.
- a. Complete the table. Let x represent the time it takes Abby to water the garden if she works alone.

	Portion of the Garden Watered	Time Spent Watering	Rate of Watering
	Gardens	Minutes	$\frac{\text{Gardens}}{\text{Minute}}$
Huyen		40	
Abby		40	
Entire Job, or 1 Garden		40	

- b. Write and solve an equation to determine the total time it would take Abby to water the garden if she were working alone.
- c. Does your answer to part b seem reasonable?
Explain your reasoning.

ACTIVITY
6.2

Solving Mixture and Density Problems

A mixture problem involves the combination of two or more substances and the concentration, or density, of one substance relative to the other.

Jacob is taking a college chemistry course, and some of his time is spent in the chemistry lab. He is conducting an experiment for which he needs a 2% salt solution. However, all he can find in the lab is 120 milliliters (mL) of 10% salt solution.

1. How many milliliters of salt and how many milliliters of water are in 120 mL of 10% salt solution?
2. What would the concentration of the salt solution be when Jacob added 80 mL of water? 180 mL of water?
3. Write and solve an equation to calculate the amount of water Jacob needs to add to the 120 mL of 10% salt solution to make a 2% salt solution. Let x represent the amount of water Jacob needs to add. Show all your work.

Olivia is working on a chemistry experiment. She has 20 mL of a 20% sulfuric acid solution that she is mixing with a 5% sulfuric acid solution.

4. Describe the range of possible concentrations for the new solution.

5. Suppose that the 20 mL of 20% sulfuric acid solution is mixed with 10 mL of the 5% sulfuric acid solution. What is the concentration of the resulting solution?

6. Write and solve an equation to calculate the amount of 5% sulfuric acid solution Olivia added if the resulting solution is a 12% sulfuric acid solution. Let x represent the amount of 5% sulfuric acid that Olivia added.

The density of a substance is its mass divided by the volume of space it occupies at that mass. Water has a density of 1 g per cubic centimeter, and frozen water has a density of 0.934 g per cubic centimeter.

7. If you place a bottle of water in the freezer long enough, it may explode. How could this happen if the density of frozen water is less than the density of water?

Solving Distance Problems

A distance problem involves distance, rate, and time.

A river barge travels 140 miles from a loading dock to a warehouse to deliver supplies. Then the barge returns to the loading dock. The barge travels with the current to the warehouse and against the current from the warehouse. The barge's total travel time is 20 hours, and it travels in still water at an average speed of 15 miles per hour.

1. Use the given information to complete the table. Let x represent the average speed of the current.

	Distance Traveled	Time Traveled	Average Speed
	Miles	Hours	$\frac{\text{Miles}}{\text{Hours}}$
With the Current			$15 + x$
Against the Current	140		
Round Trip		20	

2. You are given that the barge's total travel time is 20 hours. Write an algebraic expression, in terms of the number of hours the barge travels with the current and the number of hours it travels against the current, that is equivalent to 20 hours.
3. Write and solve an equation to calculate the average speed of the current.

4. Calculate each value.

- a. What is the barge's average speed during its trip to the warehouse?

- b. What is the barge's average speed during its trip back to the loading dock?

- c. How long does it take the barge to get from the loading dock to the warehouse?

- d. How long does it take the barge to return to the loading dock from the warehouse?

- e. Use your answers to parts (a) and (b) to calculate the average speed of the barge in still water. Verify that your answer matches the given information.

- f. Use your answers to parts (c) and (d) to calculate the barge's total travel time. Does your answer seem reasonable?

Solving Cost Problems

A cost problem involves the cost of ownership of an item over time.

Daniela has decided that it is time to replace her old refrigerator. She purchases a new Energy Star certified refrigerator. Energy Star certified refrigerators use less electricity than those that are not certified. In the long run, the Energy Star refrigerator should cost Daniela less to operate.

Daniela purchases a new Energy Star refrigerator for \$2000. The refrigerator costs \$46 per year to operate. Assume that the refrigerator is reliable and its only costs of ownership are the purchase price and the cost of operation.

1. Determine Daniela's average annual cost of owning the new refrigerator for the given number of years.
 - a. 1 year
 - b. 5 years
 - c. 10 years

2. Write an expression to represent Daniela's average annual cost of owning the new refrigerator for x years.

3. When Daniela's average annual cost of owning the refrigerator is less than \$400, she plans to shop for a new television. When can Daniela shop for a new television? Show all your work.

Daniela is curious to know how much money the Energy Star certified refrigerator will save her, compared to one that is not certified. A comparable non-certified model costs \$1900 to purchase and \$60 per year to operate. Assume that this non-certified refrigerator's only costs of ownership are the purchase price and the operation cost.

4. Determine the average annual cost of owning this refrigerator in the given number of years.
 - a. 1 year

 - b. 5 years

 - c. 10 years

5. Write an expression to represent the average annual cost of owning the non-certified refrigerator for x years.
6. In how many years will the average annual cost of owning the Energy Star certified refrigerator be less than the average annual cost of owning the non-certified refrigerator? Show all of your work.



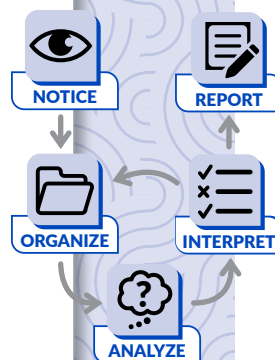
Talk the Talk

Why You Salty?

Use a rational expression to solve this problem situation.

1. A saline solution of 150 mL contains 8% salt. How much water should be added to the solution for it to contain only 3% salt? Does your answer seem reasonable?

PROBLEM SOLVING



2. Consider the problem situation. Describe and correct the error in the equation.

In a printing company, Koda and Destiny, are tasked with printing marketing materials for a new marketing campaign. Koda can complete the job in 6 hours. If Koda and Destiny work together, they can finish the job 2 hours. To find how long will it take Destiny to complete the job if she works alone, solve for x from this equation:

Let x = the number of hours Destiny can complete the job working alone.

$$\frac{2}{6} + \frac{x}{2} = 1$$

3. Write the correct equation for Question 2 and solve. Determine the reasonableness of the solution.

Lesson 6 Assignment

Write

In your own words, explain how you use rational expressions to solve work, mixture, distance, and/or cost problems.

Remember

A work problem is a type of problem that involves the rates of several workers and the time it takes to complete a job.

A mixture problem is a type of problem that involves the combination of two or more liquids and the concentrations of those liquids.

A distance problem is a type of problem that involves distance, rate, and time.

A cost problem is a type of problem that involves the cost of ownership of an item over time.

Practice

1. Alexander and Amir have been hired to paint the interior of a school during summer break. Alexander can paint the entire school in 80 hours by himself. Working together, it takes Alexander and Amir 50 hours to paint the entire school.
 - a. Write and solve an equation to determine how long it would take Amir to paint the interior of the school by himself. Show your work.
 - b. Does your answer to part (a) seem reasonable? Explain your reasoning.

Lesson 6 Assignment

2. Michael buys a 500 mL bottle of rubbing alcohol. The bottle contains a solution of 70% alcohol and 30% water. The instructions for his toy train set say he should use a solution of 40% alcohol and 60% water to clean the tracks. Write and solve an equation to determine how much water Michael must add to the solution he purchased to have the correct solution for cleaning the tracks. Show your work.
3. Javier is an aerial photographer. He photographs a 200-mile long oil pipeline that runs straight from east to west. He flies the 200 miles from west to east with an eastward tailwind. He returns over the same route and flies 200 miles against an eastward headwind. The total time of the 400-mile flight is 180 minutes. The plane travels at a speed of 150 miles per hour without a tailwind or headwind. Assume the wind speed is constant during the entire flight. Write and solve an equation to determine the wind speed during Javier's flight. Show your work.

Lesson 6 Assignment

4. Gabriela is trying to decide between 2 satellite Internet providers. ProSat Internet charges a \$250 installation fee and a monthly fee of \$50. SuperSat Internet charges a \$90 installation fee and a monthly fee of \$58. Write and solve an equation to determine when the average monthly cost (including installation and monthly fees) of each service would be the same. Show your work.

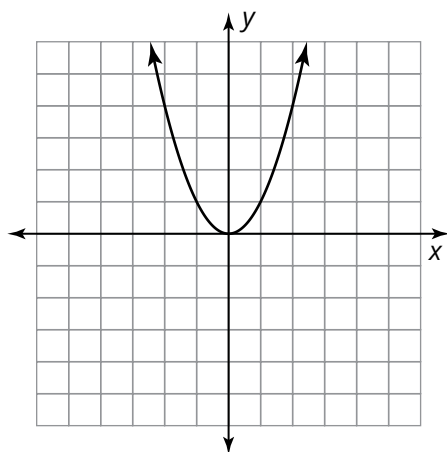
Lesson 6 Assignment

Prepare

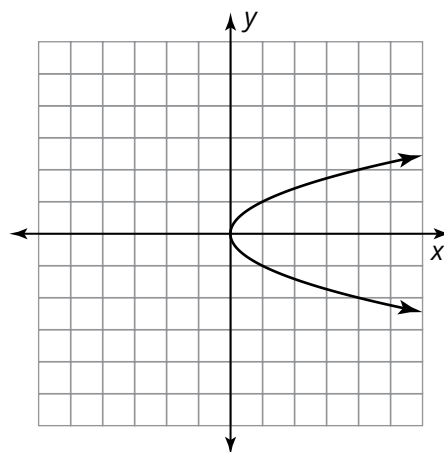
Determine whether each graph represents a function.

Explain your reasoning.

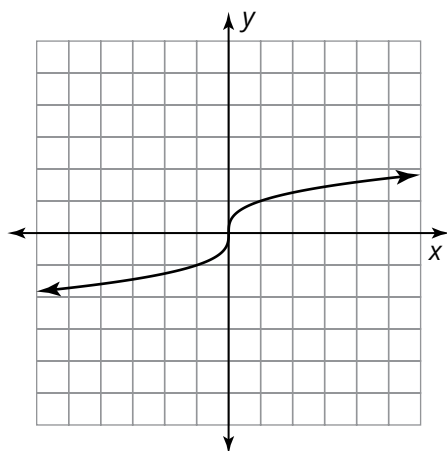
1.



2.



3.



Rational Functions

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

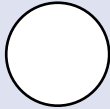
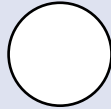
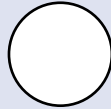
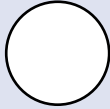
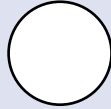
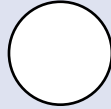
Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Rational Functions* topic by:

TOPIC 1: <i>Rational Functions</i>	Beginning of Topic	Middle of Topic	End of Topic
graphing rational functions to identify key attributes of the function.			
applying transformations to graphs of rational functions, including horizontal and vertical asymptotes.			
identifying the domain and range of rational functions using inequalities, set notation, and interval notation.			
adding, subtracting, multiplying, and dividing rational expressions.			
rewriting rational equations to solve for a specified variable.			
creating and solving rational equations, identifying any domain restrictions.			
explaining each step in solving a rational equation as following from the properties of equality.			
recognizing the situations when extraneous solutions may arise when solving rational equations.			

continued on the next page

TOPIC 1 SELF-REFLECTION
continued

TOPIC 1: <i>Rational Functions</i>	Beginning of Topic	Middle of Topic	End of Topic
modeling a situation with a rational function and apply the model to solve problems.			
using algebra to identify vertical asymptotes of rational functions.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

- Describe a new strategy you learned in the *Rational Functions* topic.
- What mathematical understandings from the topic do you feel you are making the most progress with?
- Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

continued on the next page

TOPIC 1 SUMMARY

Rational Functions Summary

LESSON

1

Introduction to Rational Functions

The function $g(x) = \frac{1}{x}$ is an example of a *rational function*. A **rational function** is any function that can be written as the ratio of two polynomials. It can be written in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$. Linear, quadratic, cubic, and higher-order polynomial functions are types of rational functions.

Given the horizontal asymptote, the range of the rational function $g(x) = \frac{1}{x}$ can be expressed:

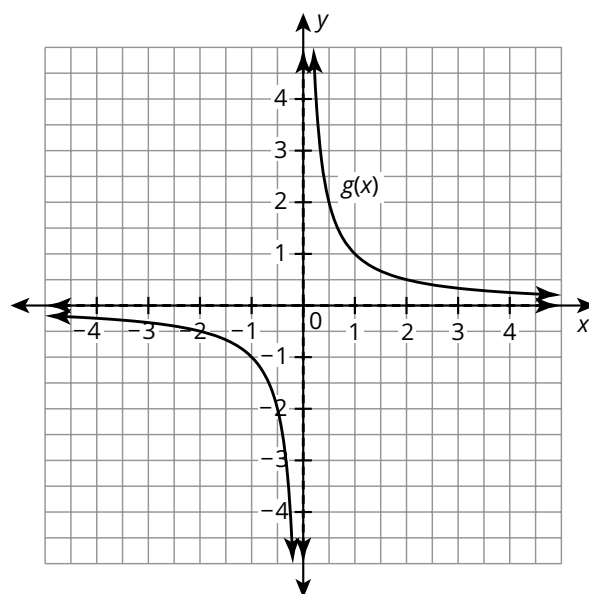
- Verbally: all numbers except for 0
- As a compound inequality: $g(x) > 0$ or $g(x) < 0$
- In interval notation: $(-\infty, 0) \cup (0, \infty)$
- In set notation: $\{g(x) \mid g(x) \neq 0\}$

by the set of all possible input values, x , using curly brackets “{” and a description of the allowed values with a vertical bar “|”. The range is the set of all possible output values, y , using the same notation, representing the set of values the function can produce based on the domain inputs.

The function $g(x) = \frac{1}{x}$ has a *vertical asymptote* at $x = 0$. A **vertical asymptote** is a vertical line that a function gets closer and closer to, but never intersects. The asymptote does not represent points on the graph of the function. It represents the output value that the graph approaches. An asymptote occurs for input values that result in a denominator of 0. The function $g(x)$ also has a horizontal asymptote at $y = 0$. A **horizontal asymptote** is a horizontal line that a function gets closer

KEY TERMS

- rational function
[función racional]
- vertical asymptote
[asíntota vertical]
- horizontal asymptote
[asíntota horizontal]
- set notation
- inverse variation function
[función de variación inversa]



and closer to. In most cases, a graph never intersects a horizontal asymptote, but in some cases it does.

The vertical and horizontal asymptotes are often represented as a dashed or solid line.

An **inverse variation function** is a function in the form of $y = \frac{k}{x}$, where k is the constant of variation. You can use inverse variation functions to model many real-world situations.

LESSON

2

Transformations of Rational Functions

Consider the functions $y = f(x)$ and $g(x) = af(b(x - c)) + d$. Recall that adding a constant d translates $f(x)$ vertically, while adding a constant c translates $f(x)$ horizontally. Multiplying by the constant a dilates $f(x)$ vertically, while multiplying by the constant b dilates $f(x)$ horizontally.

Rational functions are transformed in the same manner.

For a rational function, the transformation function form can be written as:

$$r(x) = a\left(\frac{1}{b(x - c)}\right) + d$$

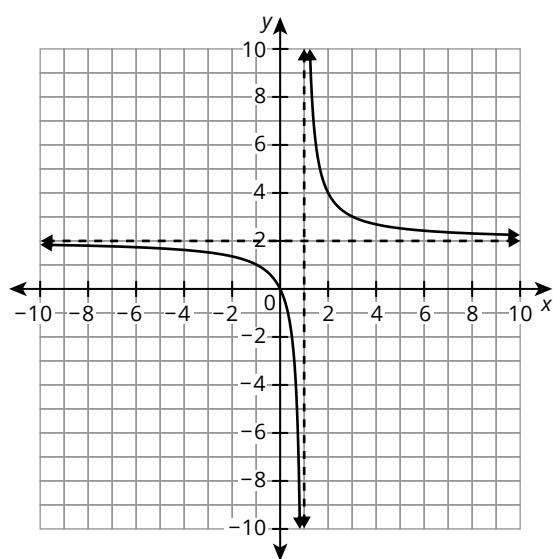
Vertical asymptotes are determined by input values for which the denominator is 0. The vertical asymptotes will limit the domain of a function.

For example, $g(x) = \frac{1}{x - 1} + 2$ is created by transforming the graph of $f(x) = \frac{1}{x}$ right one unit and up two units.

LESSON

3

Exploring Asymptotes



Vertical asymptotes for a rational function occur at the value(s) for which the denominator is zero. Horizontal translations affect the location of the vertical asymptote(s), while vertical translations affect the location of the horizontal asymptotes.

For example, $f(x) = \frac{2}{x - 1} + 2$ has a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 2$.

Sometimes algebra is needed to help identify the vertical asymptote(s).

$$f(x) = \frac{1}{x^2 + 2x - 8}$$

$$x^2 + 2x - 8 = 0$$

$$(x + 4)(x - 2) = 0$$

$$x = -4, x = 2$$

Vertical asymptotes exist at $x = -4$ and $x = 2$.

Recall that rational numbers are numbers that can be written as the ratio of two integers, $\frac{a}{b}$, such that the denominator b is not 0.

Performing operations on rational expressions involving variables is just like performing operations on rational numbers. Consider the table shown.

	Rational Numbers	Rational Expressions Involving Variables in the Numerator
Example 1	$\frac{1}{6} + \frac{5}{6} - \frac{1}{6} = \frac{5}{6}$	$\frac{1x}{6} + \frac{5x}{6} - \frac{1x}{6} = \frac{5x}{6}$
Example 2	$\begin{aligned} \frac{3}{2} + \frac{2}{5} - \frac{3}{4} &= \frac{3(10)}{2(10)} + \frac{2(4)}{5(4)} - \frac{3(5)}{4(5)} \\ &= \frac{30}{20} + \frac{8}{20} - \frac{15}{20} \\ &= \frac{23}{20} \end{aligned}$	$\begin{aligned} \frac{3x}{2} + \frac{2y}{5} - \frac{3x}{4} &= \frac{3(10)x}{2(10)} + \frac{2(4)y}{5(4)} - \frac{3(5)x}{4(5)} \\ &= \frac{30x}{20} + \frac{8y}{20} - \frac{15x}{20} \\ &= \frac{15x + 8y}{20} \end{aligned}$
Example 3	$\begin{aligned} \frac{3}{5} + \frac{2}{3} - \frac{2}{15} &= \frac{3(3)}{5(3)} + \frac{2(5)}{3(5)} - \frac{2}{15} \\ &= \frac{9}{15} + \frac{10}{15} - \frac{2}{15} \\ &= \frac{17}{15} \end{aligned}$	$\begin{aligned} \frac{3x}{5} + \frac{2y}{3} - \frac{2}{15} - \frac{2x + 3y}{5} &= \frac{3(3)x}{5(3)} + \frac{2(5)y}{3(5)} - \frac{2}{15} \\ &\quad - \frac{(2x + 3y)(3)}{5(3)} \\ &= \frac{9x}{15} + \frac{10y}{15} - \frac{2}{15} - \frac{6x + 9y}{15} \\ &= \frac{9x + 10y - 2 - 6x - 9y}{15} \\ &= \frac{3x + y - 2}{15} \end{aligned}$

When rational expressions contain variables in the denominator, the process remains the same—you still need to determine the least common denominator (LCD) before adding and subtracting.

For example, consider the difference of two rational expressions of degree 2.

$$\begin{aligned} \frac{1}{x^2 - 1} - \frac{1}{x^2 + 2x + 1} &= \frac{1}{(x + 1)(x - 1)} - \frac{1}{(x + 1)(x + 1)} \\ &= \frac{1(x + 1)}{(x + 1)(x - 1)(x + 1)} - \frac{1(x - 1)}{(x + 1)(x + 1)(x - 1)} \\ &= \frac{(x + 1) - (x - 1)}{(x + 1)(x + 1)(x - 1)} \\ &= \frac{2}{(x + 1)(x + 1)(x - 1)}, x \neq -1, 1 \end{aligned}$$

Multiplying rational expressions involves the same steps as multiplying rational numbers. Remember that when you multiply rational numbers, you can divide out factors at the beginning or the end, and the product is the same; however, dividing out earlier saves time and will keep the expressions simpler.

For example, consider both methods. In Method A, the process is to multiply and then divide out common factors. In Method B, common factors are divided out first.

	Rational Numbers	Rational Expressions Involving Variables	
Method A	$\frac{2}{15} \cdot \frac{5}{8} = \frac{10}{120} = \frac{1}{12}$	$\frac{2x}{15x} \cdot \frac{5x}{8} = \frac{10x^2}{120x} = \frac{x}{12}$	$\frac{2x}{15x^2} \cdot \frac{5x^2}{8} = \frac{10x^3}{120x^2} = \frac{x}{12}$
Method B	$\frac{2}{15} \cdot \frac{5}{8} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$	$\frac{2x}{15x} \cdot \frac{5x}{8} = \frac{1}{3} \cdot \frac{1x}{4} = \frac{x}{12}$	$\frac{2x}{15x^2} \cdot \frac{5x^2}{8} = \frac{1x}{3} \cdot \frac{1}{4} = \frac{x}{12}$

Dividing rational expressions is similar to the process you use when dividing rational numbers.

Notice that when you multiply $\frac{4}{5}$ by a form of 1, in this case $\frac{3}{3}$, you maintain equivalent fractions;

$$\begin{aligned}\frac{4}{5} \cdot \frac{3}{3} &= \frac{12}{15} \\ \frac{4}{5} &= \frac{12}{15}\end{aligned}$$

This same process works when the numerator and denominator are fractions.

Consider $\frac{\frac{4}{5}}{\frac{3}{7}}$. When you multiply by a form of 1, in this case $\frac{7}{3}$, you maintain equivalent fractions;

$$\begin{aligned}\frac{\frac{4}{5}}{\frac{3}{7}} \cdot \frac{7}{3} &= \frac{\frac{28}{15}}{\frac{21}{21}} = \frac{28}{15} = \frac{28}{15} \\ \frac{\frac{4}{5}}{\frac{3}{7}} &= \frac{28}{15}\end{aligned}$$

You may recall that, to divide fractions, you multiply the dividend by the multiplicative inverse of the divisor.

	Rational Numbers	Rational Expressions Involving Variables	
Example 1	$\frac{1}{5} \div \frac{3}{10} = \frac{1}{5} \cdot \frac{10}{3} = \frac{2}{3}$	$\frac{xy}{5z} \div \frac{3xy}{10z} = \frac{xy}{5z} \cdot \frac{10z}{3xy} = \frac{2}{3}$	$\frac{xy^2}{5z} \div \frac{3xy}{10z^2} = \frac{xy^2}{5z} \cdot \frac{10z^2}{3xy} = \frac{2yz}{3}; x, y, z \neq 0$
Example 2	$\frac{6}{7} \div 4 = \frac{6}{7} \cdot \frac{1}{4} = \frac{3}{14}$	$\frac{6a}{7b} \div 4a = \frac{6a}{7b} \cdot \frac{1}{4a} = \frac{3}{14b}$	$\frac{6a^3}{7b} \div 4a = \frac{6a^3}{7b} \cdot \frac{1}{4a} = \frac{3a^2}{14b}; a, b \neq 0$

Solving Problems with Rational Equations

A **rational equation** is an equation that contains one or more rational expressions. Rational equations are proportions. There are multiple methods you can use to solve rational equations. Depending on the structure of the equation, some methods will be more efficient than others.

For example, consider the equation:

$$\frac{x+5}{x+2} = \frac{x+1}{x-5}$$

This equation is already set up as a proportion, so rewriting the equation as the product of the means equal to the product of the extremes is the most efficient method to solve.

$$\begin{aligned}(x-5)(x+5) &= (x+1)(x+2) \\ x^2 - 25 &= x^2 + 3x + 2 \\ -25 &= 3x + 2 \\ -27 &= 3x \\ x &= -9\end{aligned}$$

It is more efficient to solve equations that have been factored.

For example, consider the equation:

$$\begin{aligned}\frac{2x+4}{x^2-2x-8} &= \frac{x+1}{x-4} \\ \frac{2(x+2)}{(x-4)(x+2)} &= \frac{x+1}{x-4} \\ \frac{2}{x-4} &= \frac{x+1}{x-4} \\ 2(x-4) &= (x+1)(x-4) \\ 2 &= x+1 \\ x &= 1\end{aligned}$$

Solving Work, Mixture, Distance, and Cost Problems

A work problem is a type of problem that involves the rates of several workers and the time it takes to complete a job.

For example, consider the work problem.

Huyen is a community volunteer. She volunteers by watering the large vegetable garden in her neighborhood. Sometimes, Huyen's friend William also volunteers. It takes Huyen 90 minutes to water the garden. When Huyen and William work together, they can complete the job in 40 minutes.

	Portion of the Garden Watered	Time Spent Watering	Rate of Watering
	Gardens	Minutes	Gardens/Minute
Huyen	$40\left(\frac{1}{90}\right) = \frac{4}{9}$	40	$\frac{1}{90}$
William	$40\left(\frac{1}{x}\right) = \frac{40}{x}$	40	$\frac{1}{x}$
Entire Job, or 1 Garden	$\frac{4}{9} + \frac{40}{x}$	40	

A mixture problem is a type of problem that involves the combination of two or more liquids and the concentrations of those liquids.

For example, consider the mixture problem.

Jacob is taking a college chemistry course, and some of his time is spent in the chemistry lab. He is conducting an experiment for which he needs a 2% salt solution. However, all he can find in the lab is 120 milliliters (mL) of 10% salt solution.

Calculate the amount of water Jacob would need to add to the 120 mL of 10% salt solution to make a 2% salt solution.

	Liquid	% Salt	Amount of Salt
Water	x	0	0
10% Solution	120	10	$0.10(120) = 12$
2% Solution	$120 + x$	2	$0.02(120 + x)$

$$12 = 0.02(120 + x)$$

$$12 = 2.4 + 0.02x$$

$$9.6 = 0.02x$$

$$x = 480$$

Jacob would need to add 480 mL of water to make a 2% salt solution.

A distance problem is a type of problem that involves distance, rate, and time.

For example, consider the distance problem.

A river barge travels 140 miles from a loading dock to a warehouse to deliver supplies. Then the barge returns to the loading dock. The barge travels with the current to the warehouse and against the current from the warehouse. The barge's total travel time is 20 hours, and it travels in still water at an average speed of 15 miles per hour.

	Distance Traveled	Time Traveled	Average Speed
	Miles	Hours	Miles/Hours
With the Current	140	$\frac{140}{15 + x}$	$15 + x$
Against the Current	140	$\frac{140}{15 - x}$	$15 - x$
Round Trip	280	20	$\frac{280}{20} = 14$

If the total time the barge travels is 20 hours then the equation to represent the total time traveled is $\frac{140}{15 + x} + \frac{140}{15 - x}$.

A cost problem is a type of problem that involves the cost of ownership of an item over time.

For example, consider the cost problem.

Olivia has decided that it is time to replace her old refrigerator. She purchases a new Energy Star certified refrigerator. Energy Star certified refrigerators use less electricity than those that are not certified. In the long run, the Energy Star refrigerator should cost Olivia less to operate.

Olivia purchases a new Energy Star refrigerator for \$2000. The refrigerator costs \$46 per year to operate. Assume that the refrigerator is reliable and its only costs of ownership are the purchase price and the cost of operation. Then Olivia's average annual cost of ownership for 1 year would be \$2046.

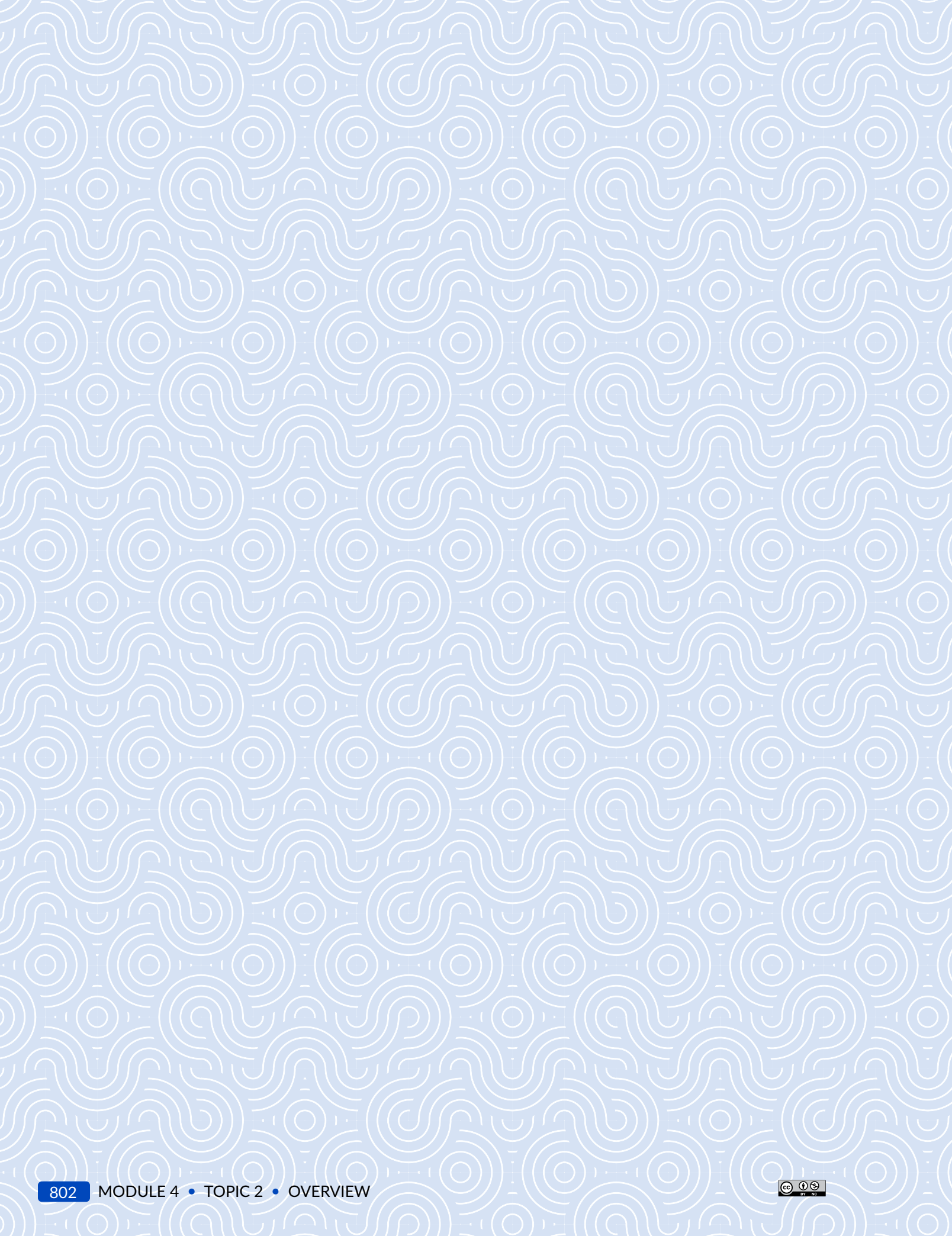
The average annual cost of operating the refrigerator over x years would be $\frac{2000 + 46x}{x}$.



The most basic radical function, $y = \sqrt{x}$, climbs steeply at first, then more shallowly. These breaching whales have a similar trajectory.

Radical Functions

LESSON 1	Inverses of Power Functions	803
LESSON 2	Radical Functions	821
LESSON 3	Transformations of Radical Functions	853
LESSON 4	Compositions of Functions	875



1

Inverses of Power Functions

OBJECTIVES

- Sketch the graphs of the inverses of power functions.
- Use the Vertical Line Test to determine whether an inverse relation is a function.
- Use graphs to determine and write an explanation of whether a function is invertible.
- Use the Horizontal Line Test to determine whether a function is invertible.

KEY TERMS

- invertible function
- Horizontal Line Test

.....

You have analyzed the graphs of power functions, like $y = x^2$ and $y = x^3$.

What do the graphs of these functions look like when the independent and dependent variables are switched?

Getting Started

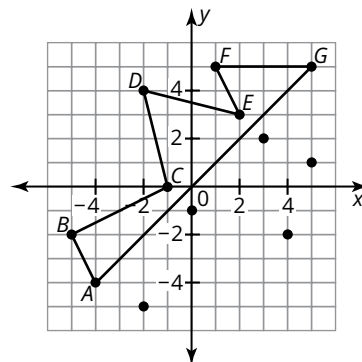
Reflect on It

Consider the polygon drawn on the coordinate plane.

1. Trace the polygon and the axes onto a piece of patty paper.
 - a. Describe how you can use your tracing to show a reflection of the polygon across the line $y = x$.

- b. Describe the location of the x - and y -axis on your patty paper in the reflection.

2. Draw the reflection of the polygon across $y = x$ on the same coordinate plane shown.



3. Write the coordinates of the named vertices of the polygon after a reflection across $y = x$.
4. Compare the coordinates of the vertices of the original polygon with the coordinates you wrote in Question 3.
5. Predict what the graph of the function $f(x) = x^2$ would look like when reflected across $y = x$. Explain your reasoning.

Switching x and y

The graphs located at the end of this lesson show these 3 power functions.

$$L(x) = x \quad Q(x) = x^2 \quad C(x) = x^3$$

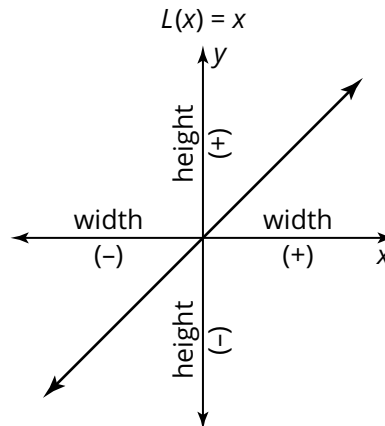
Remember...

A power function is a polynomial function of the form $P(x) = ax^n$, where n is a non-negative integer.

Think About...

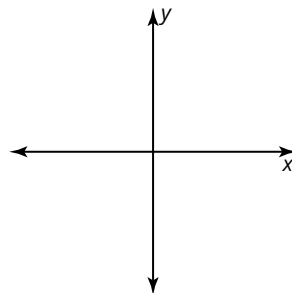
Which part or parts of each graph don't make sense in terms of the quantities in each situation?

1. Trace each graph onto a separate piece of patty paper and label the axes as shown on the original graph. Label each function to help you identify them.
2. The graph of the linear function $L(x) = x$ models the width of a square as the independent quantity and the height of the square as the dependent quantity.



- a. Use the patty paper image of $L(x)$ to transform the graph so that it shows the height as the independent quantity on the horizontal axis and the width as the dependent quantity on the vertical axis. Then, sketch the resulting graph and label the axes.

Transformed Graph of $L(x)$



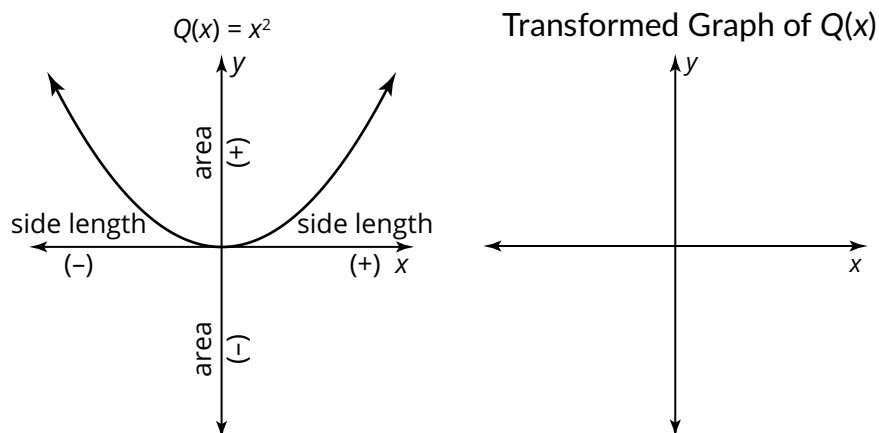
Ask Yourself...

How do you know when you have the correct graph?

- b. Describe the transformations you used to transpose the independent and dependent quantities.
- c. Is the resulting graph a function? Explain your reasoning.
- d. Compare the graph of $L(x) = x$ to the resulting graph. Interpret both graphs in terms of the width and height of a square.

3. The graph of the quadratic function $Q(x) = x^2$ models the side length of a square as the independent quantity and the area of the square as the dependent quantity.

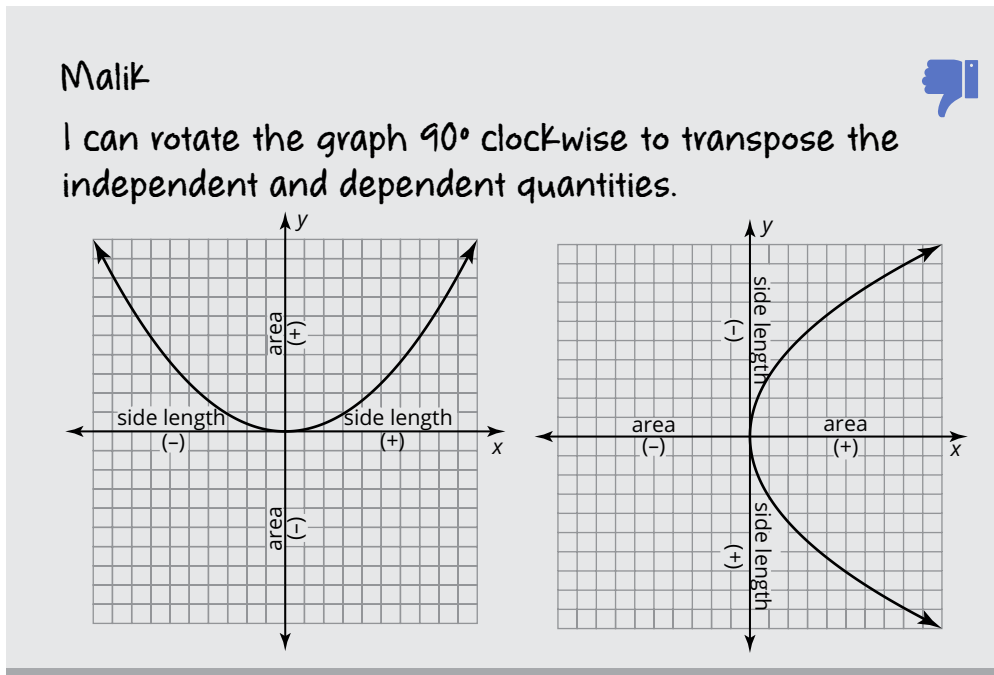
- a. Use the patty paper image of $Q(x)$ to transform the graph so that it shows the area as the independent quantity on the horizontal axis and the side length as the dependent quantity on the vertical axis. Then, sketch the resulting graph and label the axes.



- b. Describe the transformations you used to transpose the independent and dependent quantities.

- c. Is the resulting graph a function? Explain your reasoning.

- d. Malik used an incorrect strategy to transpose the independent and dependent quantities. Describe why Malik's strategy is incorrect.

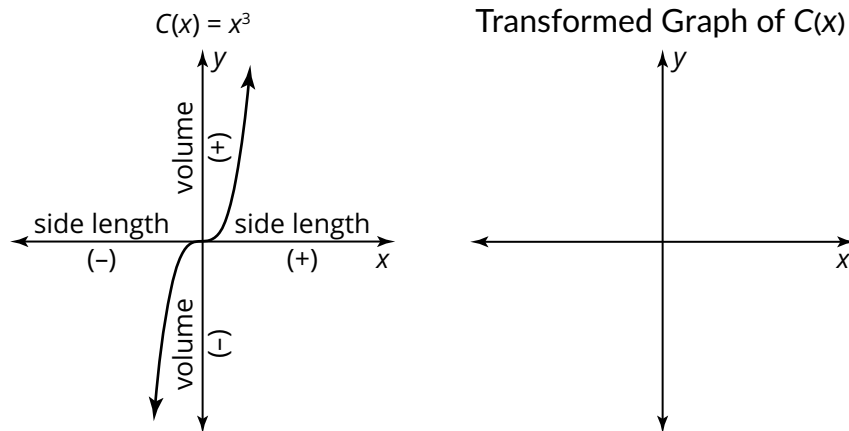


- e. Compare the graph of $Q(x) = x^2$ to the transformed graph you sketched. Interpret both graphs in terms of the side length and area of a square.

Ask Yourself . . .

What units are used to describe area?

4. The graph of the cubic function $C(x) = x^3$ models the side length of a cube as the independent quantity and the volume of the cube as the dependent quantity.
- a. Use the patty paper image of $C(x)$ to transform the graph so that it shows the volume as the independent quantity on the horizontal axis and the side length as the dependent quantity on the vertical axis. Then, sketch the resulting graph and label the axes.



- b. Describe the transformations you used to transpose the independent and dependent quantities.

c. Is the resulting graph a function? Explain your reasoning.

d. Compare the graph of $C(x) = x^3$ to the resulting graph. Interpret both graphs in terms of the side length and volume of a cube.

Remember ...

A function f is the set of all ordered pairs (x, y) , or $(x, f(x))$, where for every value of x there is one and only one value of y , or $f(x)$. The inverse of a function is the set of all ordered pairs (y, x) , or $(f(x), x)$.

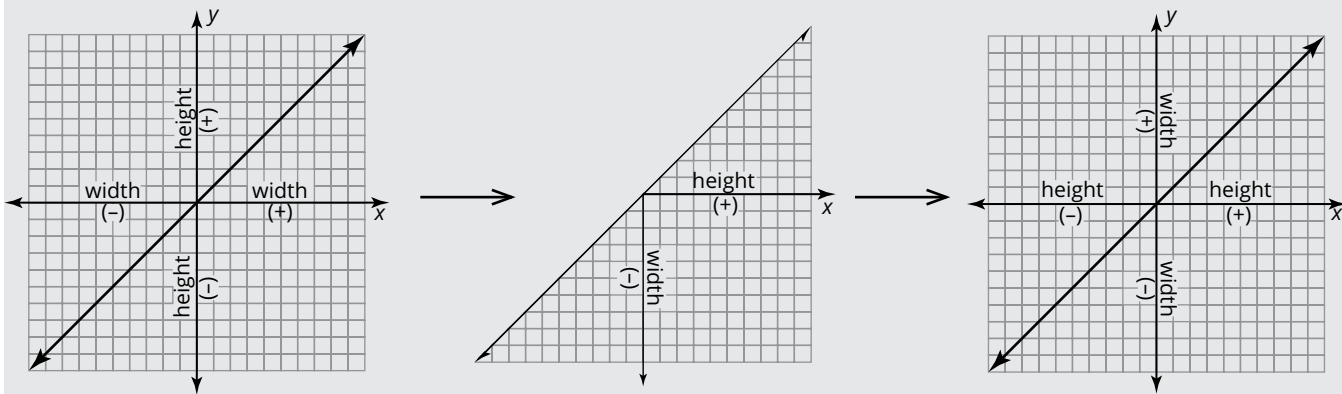
By transforming the graphs you traced in the previous activity, you were able to see and sketch the inverses of the functions $L(x) = x$, $Q(x) = x^2$, and $C(x) = x^3$.

Huyen discovered a way to use just one reflection to transpose the independent and dependent quantities.

Huyen



I can reflect the graph across the line $y = x$ by folding it diagonally to switch the independent and dependent quantities.

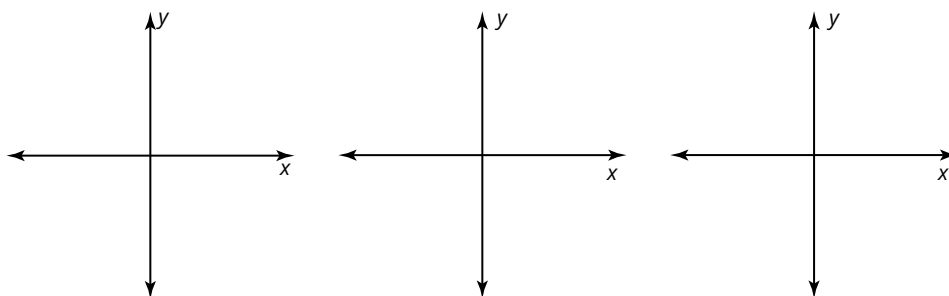


1. Use your traced graphs to verify Huyen's strategy.

inverse of $L(x) = x$

inverse of $Q(x) = x^2$

inverse of $C(x) = x^3$



If the inverse of a function f is also a function, then f is an **invertible function**, and its inverse is written as $f^{-1}(x)$.

2. Which of the 3 power functions that you explored are invertible functions? Explain your reasoning.

.....
To be an invertible function, the function must be a one-to-one function.
.....

You know that the Vertical Line Test can be used to determine whether the graph of a relation represents a function. The **Horizontal Line Test** is a visual method to determine whether a function has an inverse that is also a function. To apply the Horizontal Line Test, consider all the horizontal lines that could be drawn on the graph of the function. When any of the horizontal lines intersect the graph of the function at more than one point, the inverse of the function is not a function.

3. When a graph passes both the Horizontal Line Test and the Vertical Line Test, what can you conclude about the graph?

Ask Yourself ...

What tools or strategies can you use to solve this problem?

4. When a graph passes the Vertical Line Test but not the Horizontal Line Test, what can you conclude about the graph?



Talk the Talk

Power Lines

1. How does the graph of a power function and the graph of its inverse demonstrate symmetry? Explain your reasoning.

2. Determine whether each function is an invertible function.

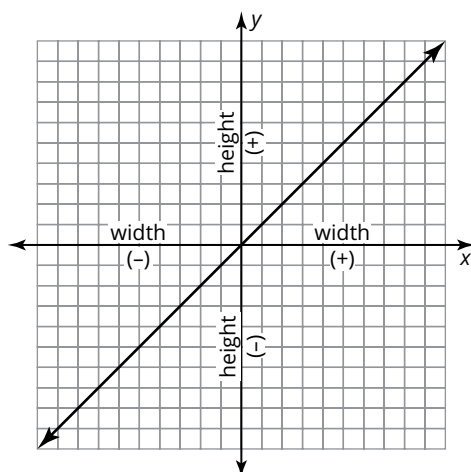
a. $L(x) = -x$

b. $Q(x) = -x^2$

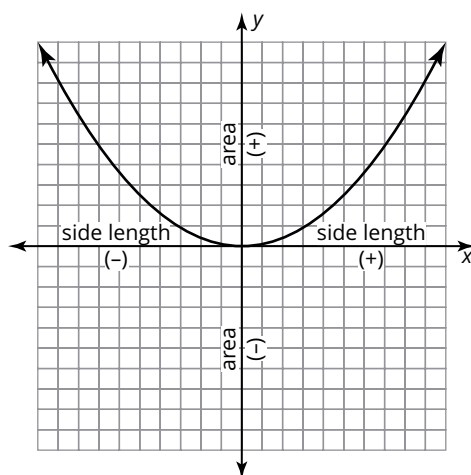
c. $C(x) = -x^3$

Power Function Graphs

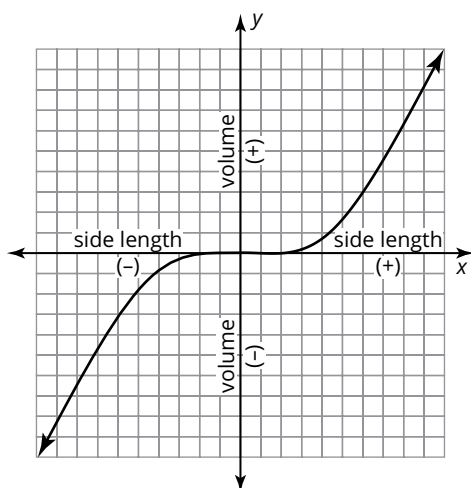
$$L(x) = x$$



$$Q(x) = x^2$$



$$C(x) = x^3$$



Lesson 1 Assignment

Write

Describe how to create the inverse of a function graphically.

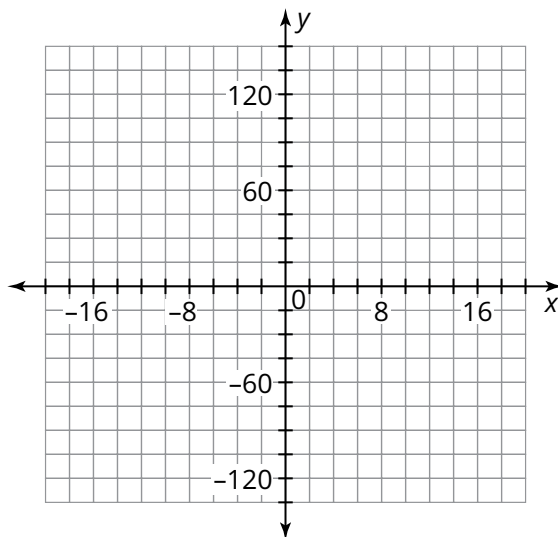
Remember

The inverse of a function is the set of all ordered pairs (y, x) , or $(f(x), x)$. If the inverse of a function is also a function, the function is said to be an invertible function, and its inverse is written as $f^{-1}(x)$. To apply the Horizontal Line Test, consider all the horizontal lines that could be drawn on the graph of the function. When any of the horizontal lines intersect the graph of the function at more than one point, then the inverse of the function is not a function.

Practice

1. Consider the power function, $f(x) = x^3$.

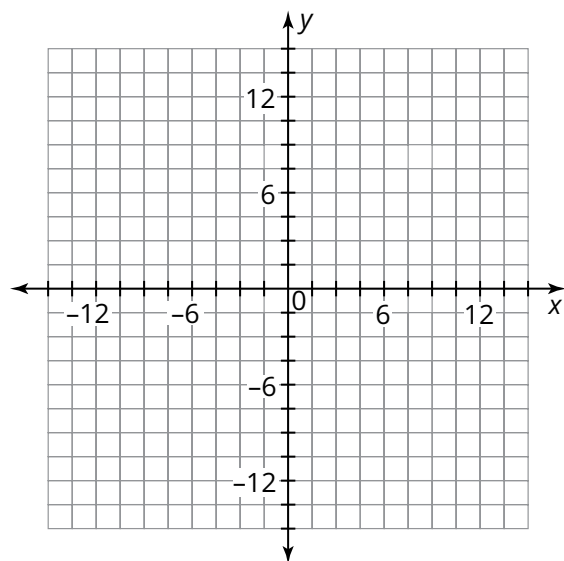
a. Sketch the graph of $f(x)$.



b. Is $f(x)$ invertible? Explain your reasoning.

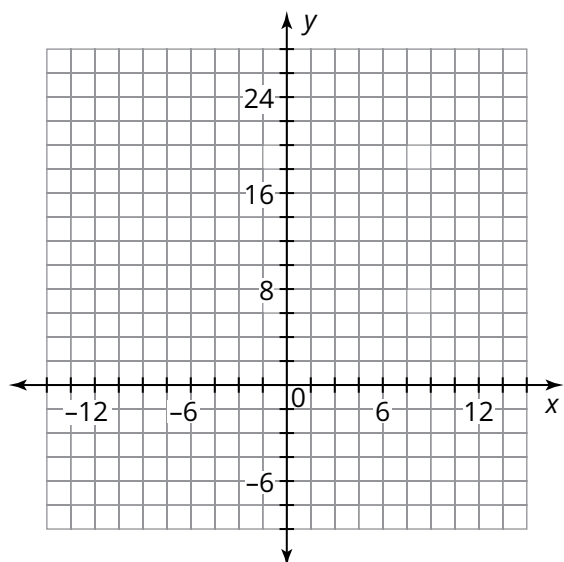
Lesson 1 Assignment

c. If $f(x)$ is invertible, sketch the graph of $f^{-1}(x)$.



2. Consider the power function, $g(x) = x^2$.

a. Sketch the graph of $g(x)$.



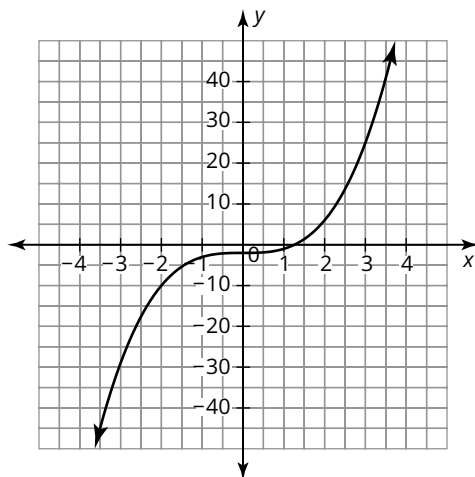
Lesson 1 Assignment

b. Is $g(x)$ invertible? Explain your reasoning.

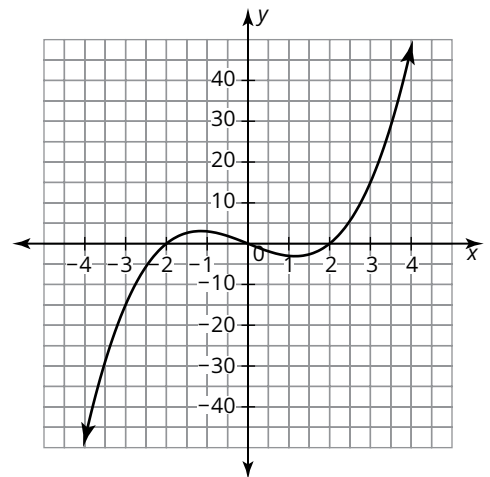
c. If $g(x)$ is invertible, sketch the graph of $g^{-1}(x)$.

3. Determine whether the inverse of each graphed function is a function. Explain your reasoning.

a.

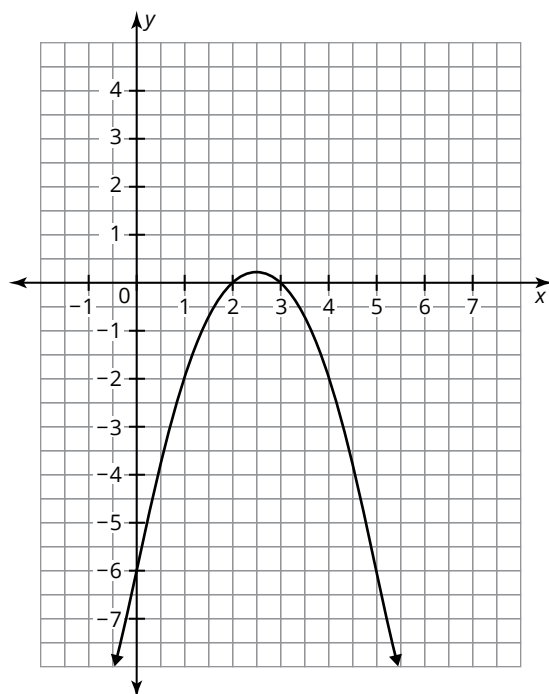


b.



Lesson 1 Assignment

c.



Prepare

Solve each equation for x .

1. $y = x^2 - 1$

2. $y = \frac{1}{4}x^2$

3. $y = (x + 4)^3$

2

Radical Functions

OBJECTIVES

- Restrict the domain of $f(x) = x^2$ to graph the square root function.
- Determine equations for the inverses of power functions.
- Identify and verbally describe attributes of square root and cube root functions, such as domain and range.
- Use composition of functions to determine whether two functions are inverses of each other.
- Solve real-world problems using the square root and cube root functions.

KEY TERMS

- square root function
- cube root function
- radical function
- composition of functions
- restrict the domain

.....

You have explored the graphs of the inverses of power functions.

How can you write the inverse of a power function algebraically?

How can you write the inverse of any power function using the notation $f^{-1}(x)$?

Getting Started

The Root of the Matter

You know that the inverse of a power function defined by the set of all points (x, y) , or $(x, f(x))$ is the set of all points (y, x) , or $(f(x), x)$. Thus, to determine the equation of the inverse of a power function, you can transpose x and y in the equation and solve for y .

WORKED EXAMPLE

Determine the inverse of the power function $f(x) = x^2$, or $y = x^2$. First, transpose x and y .

$$y = x^2 \longrightarrow x = y^2$$

Then, solve for y .

$$\begin{aligned}\sqrt{x} &= \sqrt{y^2} \\ y &= \pm\sqrt{x}\end{aligned}$$

The inverse of $f(x) = x^2$ is $y = \pm\sqrt{x}$.

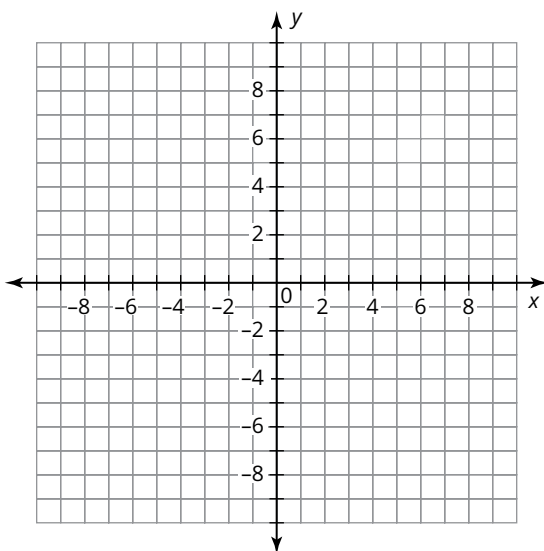
1. Why must the symbol $\pm$ be written in front of the radical to write the inverse of the function $f(x) = x^2$?
2. Notice that the inverse of the function $f(x) = x^2$ is not written with the notation $f^{-1}(x)$. Explain why not.

The Square Root Function

You can use the horizontal line test to recognize that the function $f(x) = x^2$ is not an invertible function. Are there any conditions for which the function $f(x) = x^2$ could pass the horizontal line test?

1. The table shows several coordinates of the function $f(x) = x^2$.
 - a. Use the ordered pairs in the table and what you know about inverses to graph the function and the inverse of the function, $y = \pm\sqrt{x}$. Explain your reasoning.

x	$f(x) = x^2$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



Ask Yourself . . .

How does each point (x, y) of the function map to the inverse?

b. What point or points do the two graphs have in common? Why?

2. Describe the key attributes of each graph. Write the domain and range using interval notation.

Function: $f(x) = x^2$

Inverse of $f(x)$: $y = \pm\sqrt{x}$

Domain: _____

Domain: _____

Range: _____

Range: _____

x-intercept(s): _____

x-intercept(s): _____

y-intercept(s): _____

y-intercept(s): _____

The graph in Question 1 shows that every positive real number has 2 square roots—a positive square root and a negative square root. For example, 9 has 2 square roots because $3^2 = 9$ and $(-3)^2 = 9$. The two square roots of 9 are 3 and -3 .

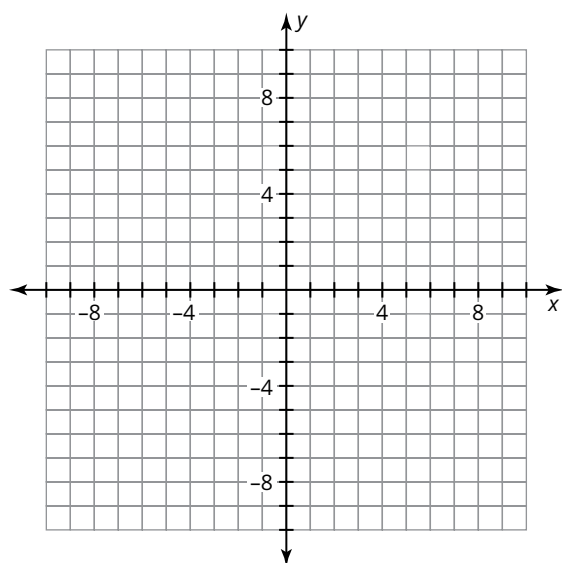
You know that the inverse of $f(x) = x^2$ is not a function. However, you can *restrict the domain* of this function so that the inverse is also a function. To **restrict the domain** of a function means to define a new domain for the function that is a subset of the original domain.

When you restrict the domain of the power function $f(x) = x^2$ to values greater than or equal to 0, the inverse of the function is called the **square root function** and is written as:

$$f^{-1}(x) = \sqrt{x}, \text{ for } x \geq 0.$$

3. Why can you use the notation $f^{-1}(x)$ to describe the inverse function $\sqrt{x}$, for $x \geq 0$?

4. Draw dashed line segments between the plotted points on the function for the restricted domain $x \geq 0$ and the corresponding inverse points.

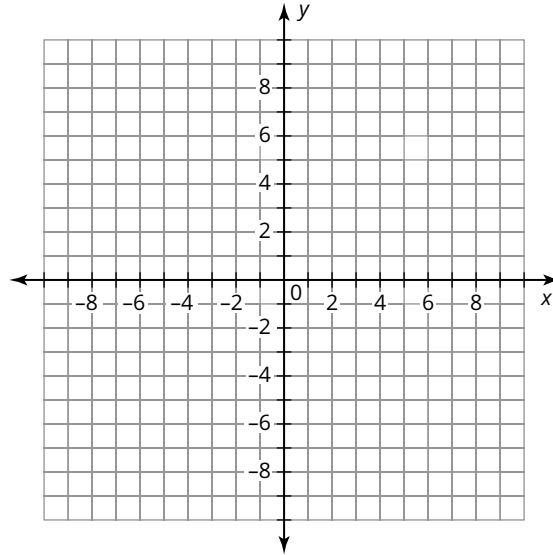


- a. List the ordered pairs of the points you connected.
- b. List the ordered pairs of the points that you did not connect.
Explain why these points are not connected.

5. Graph the square root function $f^{-1}(x) = \sqrt{x}$ by restricting the domain of $f(x) = x^2$.

Ask Yourself . . .

Does restricting the domain of the function restrict the range of the inverse?



6. Describe the key attributes of each function. Write the domain and range using interval notation.

Function: $f(x) = x^2$, for $x \geq 0$

Inverse function: $f^{-1}(x) = \sqrt{x}$

Domain: _____

Domain: _____

Range: _____

Range: _____

x-intercept(s): _____

x-intercept(s): _____

y-intercept(s): _____

y-intercept(s): _____

7. Does the inverse function $f^{-1}(x) = \sqrt{x}$ have an asymptote? Explain your reasoning.

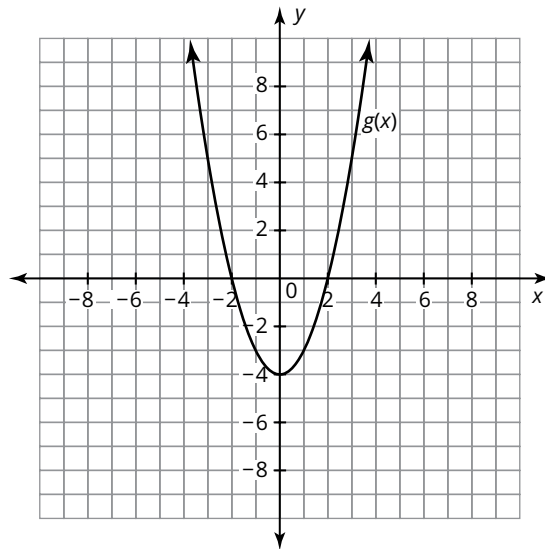
You've explored the relationship between the function $f(x) = x^2$ and its inverse, both with a domain restriction and without a domain restriction.

8. How does the domain and range of the inverse relate to the restricted domain and range of the original function?

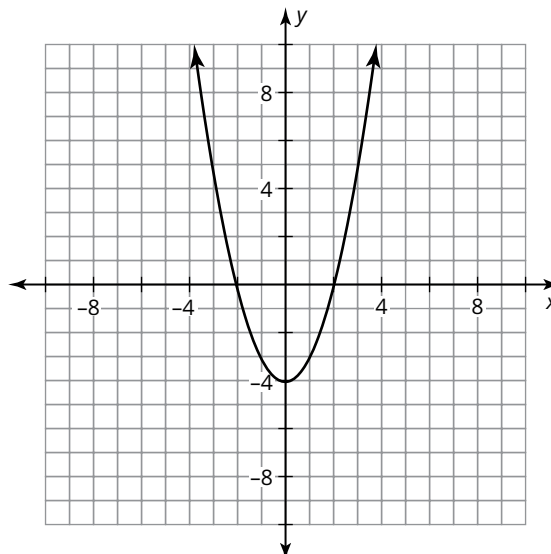
Domain and Range Restrictions

Let's look at more quadratic functions to explore domain restrictions and the relationship between the domain and range of a quadratic function and its inverse.

1. Consider the function $g(x) = x^2 - 4$, shown on the coordinate plane.



- a. How is $g(x)$ transformed from the basic quadratic function $f(x) = x^2$?
- b. Write the equation for the inverse of $g(x)$ and sketch its graph.



c. Is the inverse of $g(x)$ a function? Explain your reasoning.

d. How is the inverse of $g(x)$ transformed from the parent square root relation $y = \pm\sqrt{x}$?

e. List the domain and range of $g(x)$ and the inverse of $g(x)$ using set notation.

Function: $g(x) = x^2 - 4$ Inverse of: $g(x)$: $y =$ _____

Domain: _____ Domain: _____

Range: _____ Range: _____

f. What conclusion can you make about the relationship between the domain and range of a quadratic function and its inverse when the domain is not restricted?

g. How can you restrict the domain of $g(x)$ so that its inverse is also a function? How does this affect the range of the function?

- h. List the domain and range for both the quadratic function with the domain and range restrictions and the inverse function. Write the domain and range using set notation.

Function: $g(x) = x^2 - 4$ Inverse of $g(x)$: $g^{-1}(x) = \underline{\hspace{2cm}}$

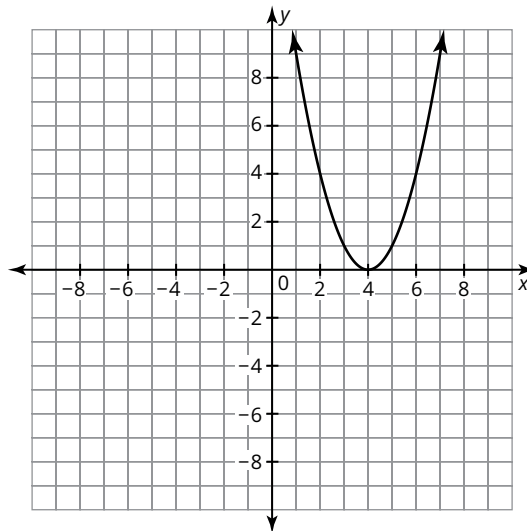
Domain restriction: $\underline{\hspace{2cm}}$

Domain: $\underline{\hspace{2cm}}$ Domain: $\underline{\hspace{2cm}}$

Range: $\underline{\hspace{2cm}}$ Range: $\underline{\hspace{2cm}}$

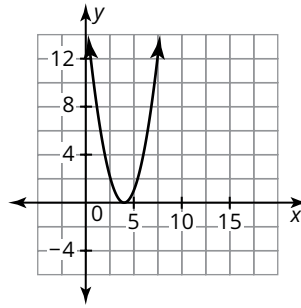
- i. What conclusion can you make about the relationship between the domain and range of a quadratic function and its inverse when the domain is restricted?

2. Consider the function $h(x) = (x - 4)^2$, shown on the coordinate plane.



- a. How is $h(x)$ transformed from the basic quadratic function $f(x) = x^2$?

- b. Write the equation for the inverse of $h(x)$ and sketch its graph.



- c. Is the inverse of $h(x)$ a function? Explain your reasoning.

- d. How is the inverse of $h(x)$ transformed from the basic square root relation $y = \pm \sqrt{x}$?

- e. List the domain and range of $h(x)$ and the inverse of $h(x)$ using inequalities.

Function: $h(x) = (x - 4)^2$ Inverse of $h(x)$: $y = \underline{\hspace{2cm}}$

Domain: $\underline{\hspace{2cm}}$ Domain: $\underline{\hspace{2cm}}$

Range: $\underline{\hspace{2cm}}$ Range: $\underline{\hspace{2cm}}$

- f. What conclusion can you make about the relationship between the domain and range of a quadratic function and its inverse when the domain is not restricted?

g. How can you restrict the domain of $h(x)$ so that its inverse is also a function? How does this affect the range of the function?

h. List the domain and range for both the quadratic function with the domain and range restrictions and the inverse function. Write the domain and range using inequalities.

Function: $h(x) = (x - 4)^2$ Inverse of $h(x)$: $h^{-1}(x) = \underline{\hspace{2cm}}$

Domain restriction: $\underline{\hspace{2cm}}$

Domain: $\underline{\hspace{2cm}}$ Domain: $\underline{\hspace{2cm}}$

Range: $\underline{\hspace{2cm}}$ Range: $\underline{\hspace{2cm}}$

i. What conclusion can you make about the relationship between the domain and range of a quadratic function and its inverse when the domain is restricted?

3. When the domain of a quadratic function is restricted to create an inverse function, what is the lower bound of the domain? Explain your reasoning.

4. Complete the table to describe the effect of each transformation on the inverse of the quadratic function.

Transformation of Quadratic Function, $f(x)$	Transformation of Inverse Function, $f^{-1}(x)$
translation up d units	
translation down d units	
translation right c units	
translation left c units	

5. Write the equation for the inverse of each quadratic function and identify the appropriate domain restrictions. Then, describe the domain and range of each function and its inverse without graphing the functions.

a. $f(x) = x^2 - 2$

b. $f(x) = (x + 2)^2$



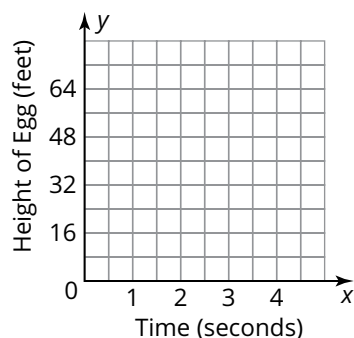


More with Inverses of Quadratics

Olivia is competing in the Egg Drop Competition at her school's Science Fair. Competitors in the Egg Drop Competition are required to create a container in which they place a raw egg, and then drop the container from various heights to see if the egg breaks. The winner of the contest is the person whose container is dropped from the greatest height without breaking the egg.

Olivia is testing a container she built for the competition. She placed an egg in her container and dropped it from the roof of a building. The height of the egg can be modeled by the function $f(x) = -16x^2 + 64$, where x represents the time in seconds.

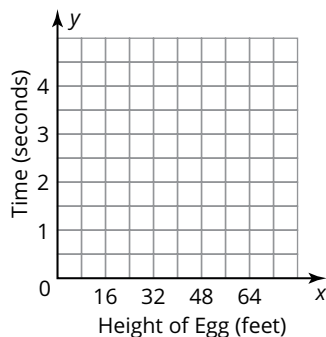
1. Define the independent and dependent quantities of $f(x)$.
2. Identify the domain and range of $f(x)$ based on its equation using inequalities.
3. Determine any restrictions on the domain of $f(x)$ based on this problem situation. Explain your reasoning.
4. Graph $f(x) = -16x^2 + 64$ with the restricted domain based on this problem situation.



5. Define the independent and dependent quantities of the inverse of $f(x)$.

6. Write the domain and range of the inverse of $f(x)$ with the restricted domain using inequalities. How does the domain and range of the inverse relate to the restricted domain and range of the original function?

7. Graph the inverse of $f(x)$ with the restricted domain.



8. Explain why the inverse of $f(x)$ with the restricted domain is a function. Then, write an equation for the inverse.

9. Explain what the inverse models in terms of this problem situation.
10. After 1.5 seconds, what is the egg's height? Explain which function you used and how you determined your answer.
11. After how many seconds is the egg at a height of 55 feet? Explain which function you used and how you determined your answer.

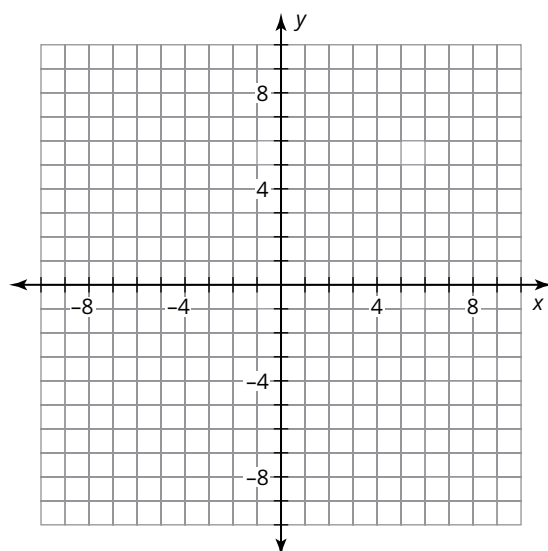
ACTIVITY
2.4

The Cube Root Function

The **cube root function** is the inverse of the power function $f(x) = x^3$ and can be written as $f^{-1}(x) = \sqrt[3]{x}$.

1. The table shows several coordinates of the function $f(x) = x^3$.
 - a. Use these points to graph the function and the inverse of the function, $f^{-1}(x)$.

x	$f(x) = x^3$
-2	-8
-1	-1
0	0
1	1
2	8



- b. Explain how you determined the coordinates for the points on the inverse of the function.

- c. What point or points do the two graphs have in common? Why?
2. Why is the symbol $\pm$ not written in front of the radical to write the inverse of the function $f(x) = x^3$?
3. Do you need to restrict the domain of the function $f(x) = x^3$ to write the inverse with the notation $f^{-1}(x)$? Explain your reasoning.
4. Describe the key attributes of each function. Write the domain and range using interval notation.
- | | |
|------------------------|---|
| Function: $f(x) = x^3$ | Inverse function: $f^{-1}(x) = \sqrt[3]{x}$ |
| Domain: _____ | Domain: _____ |
| Range: _____ | Range: _____ |
| x-intercept(s): _____ | x-intercept(s): _____ |
| y-intercept(s): _____ | y-intercept(s): _____ |
5. Does the inverse function $f^{-1}(x) = \sqrt[3]{x}$ have an asymptote? Explain your reasoning.

The inverses of power functions with exponents greater than or equal to 2, such as the square root function and the cube root function, are called **radical functions**. Radical functions are used in many areas of science, including physics and computer science.

Inverse by Composition

You know that when the domain is restricted to $x \geq 0$, the function $f(x) = \sqrt{x}$ is the inverse of the power function $g(x) = x^2$. You also know that the function $h(x) = \sqrt[3]{x}$ is the inverse of the power function $q(x) = x^3$.

The process of evaluating one function inside of another function is called the **composition of functions**. For two functions f and g , the composition of functions uses the output of one function as the input of the other. It is expressed as $f(g(x))$ or $g(f(x))$.

WORKED EXAMPLE

To write a composition of the functions $g(x) = x^2$ and $f(x) = \sqrt{x}$ when the domain of $g(x)$ is restricted to $x \geq 0$, substitute the value of one of the functions for the argument, x , of the other function.

$$f(x) = \sqrt{x}$$

$$g(x) = x^2$$

$$f(g(x)) = \sqrt{x^2} = x, \text{ for } x \geq 0$$

You can write the composition of these two functions as $f(g(x)) = x$ for $x \geq 0$.

1. Determine $g(f(x))$ for the functions $g(x) = x^2$ and $f(x) = \sqrt{x}$ for $x \geq 0$.

If $f(g(x)) = g(f(x)) = x$, then $f(x)$ and $g(x)$ are inverse functions.

2. Are $f(x)$ and $g(x)$ inverse functions? Explain your reasoning.

3. Algebraically determine whether the functions in each pair are inverses. Show your work.

a. $h(x) = \sqrt[3]{x}$ and $q(x) = x^3$

b. $k(x) = 2x^2 + 5$ and $j(x) = -2x^2 - 5$



4. Michael said that all linear functions are inverses of themselves because $f(x) = x$ is the inverse of $g(x) = x$.

Is Michael correct? Explain your reasoning.

ACTIVITY 2.6

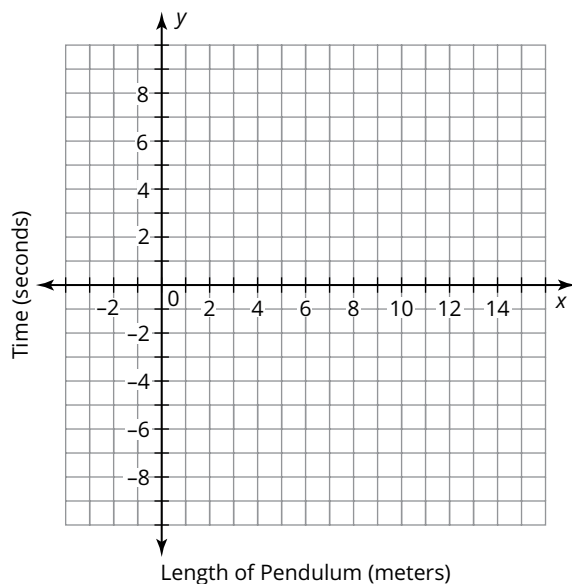
Pendulums

The time it takes for one complete swing of a pendulum depends on the length of the pendulum and the acceleration due to gravity.

The formula for the time it takes a pendulum to complete one swing is $T = 2\pi\sqrt{\frac{L}{g}}$, where T is time in seconds, L is the length of the pendulum in meters, and g is the acceleration due to gravity in meters per second squared.

1. Write a function $T(L)$ that represents the time of one pendulum swing.

2. Use technology to sketch a graph of the function $T(L)$.



Acceleration due to gravity is the force acting on an object falling freely under the influence of Earth's gravitational pull. Acceleration due to gravity is approximately equal to $9.8 \frac{m}{s^2}$.

Think About...

Notice that the time for one swing does not depend on the mass of the pendulum.

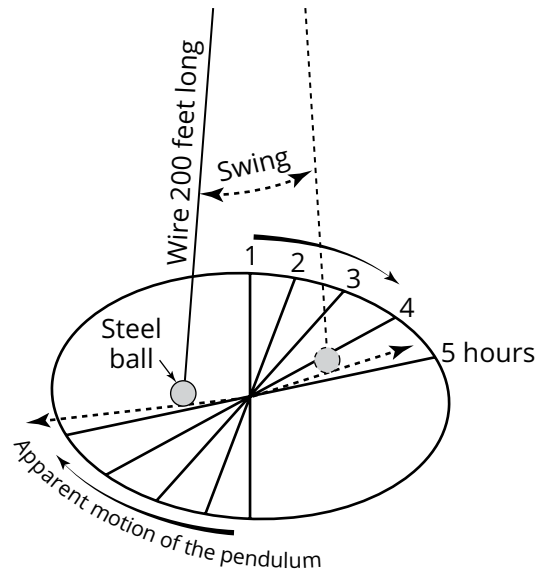
PROBLEM SOLVING



3. Describe the key attributes of the function, such as its domain, range, and intercepts. Write the domain and range using inequalities. Explain your reasoning.

4. How long does it take for one complete swing when the length of the pendulum is 0.5 meter?

5. Many museums display what is known as a Foucault pendulum. As a Foucault pendulum swings back and forth throughout the day, the Earth's rotation causes it to appear to move in a circular direction. If the pendulum takes 1 day to come back to its initial position on the circle, approximately how many full pendulum swings occur each day?



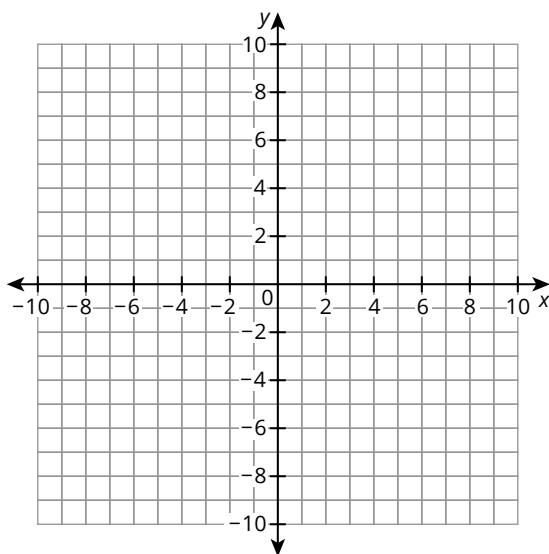


Talk the Talk

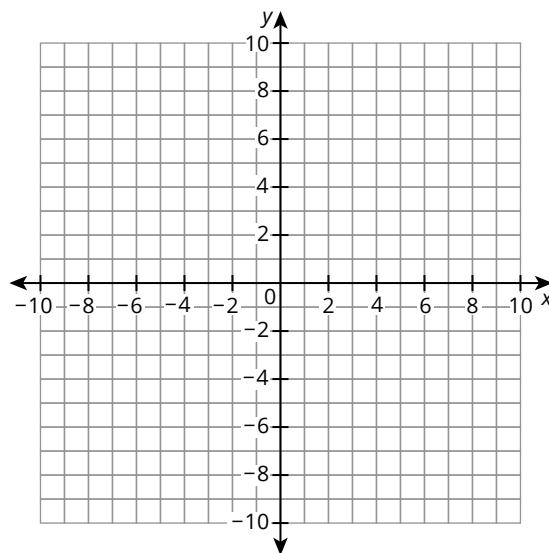
Masters of the Inverse

1. Graph the functions on the given coordinate plane.

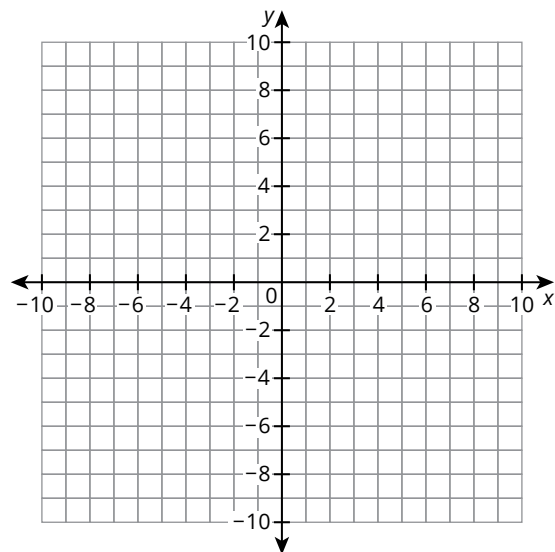
a. $f(x) = x^2$, for $x \geq 0$



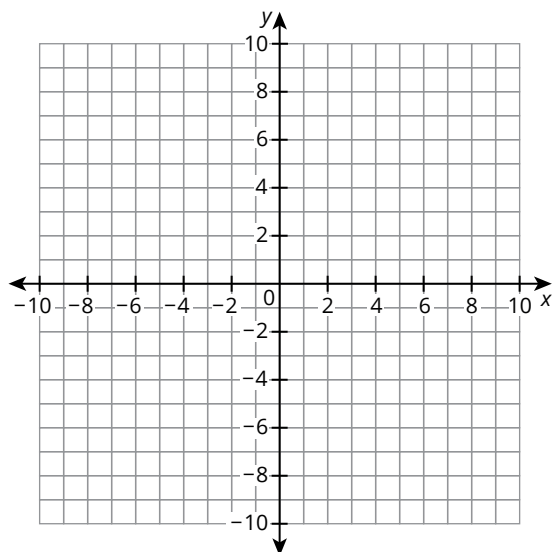
b. $f^{-1}(x) = \sqrt{x}$



c. $g(x) = x^3$



d. $g^{-1}(x) = \sqrt[3]{x}$



2. Use the graphs you created in question 1 to complete the table.
Write the domain and range using interval notation.

Function	Domain	Range	x-Intercept(s)	y-Intercept(s)
$f(x) = x^2$, for $x \geq 0$				
$f^{-1}(x) = \sqrt{x}$				
$g(x) = x^3$				
$g^{-1}(x) = \sqrt[3]{x}$				

3. Determine whether each statement is true or false. If false, explain the error in the statement.
- The inverses of all radical functions are functions.
 - The inverses of all quadratic functions are always functions.
 - The inverses of all cubic functions are always functions.
4. Algebraically determine whether the functions $p(x) = 3x^2 + 1$ and $q(x) = -3x^2 + 1$ are inverses. Show your work.

Lesson 2 Assignment

Write

Provide an example of each term.

1. square root function
2. cube root function
3. radical function
4. composition of functions

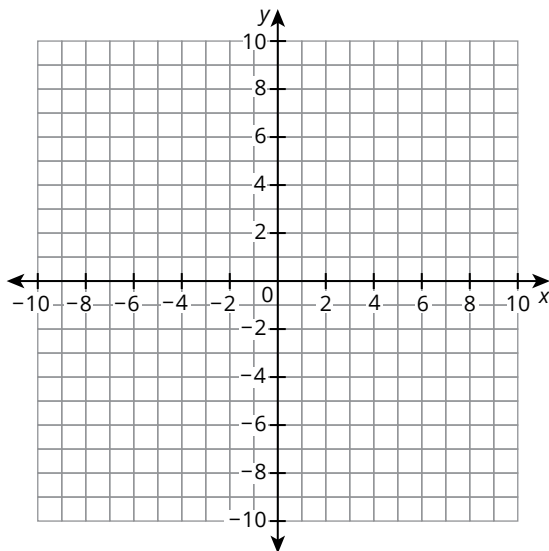
Remember

Radical functions are inverses of even-degree power functions if the domain of the even-degree power function is restricted to $x \geq 0$.

For two functions f and g , the composition of functions uses the output of one function as the input of the other. If $f(g(x)) = g(f(x)) = x$, then $f(x)$ and $g(x)$ are inverse functions.

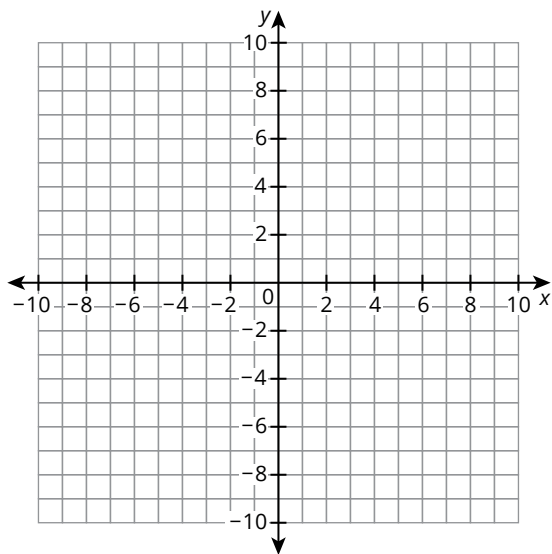
Practice

1. Graph the square root parent function $f(x) = \sqrt{x}$. Determine its key attributes such as its domain and range using set notation, extrema, and intercepts.



Lesson 2 Assignment

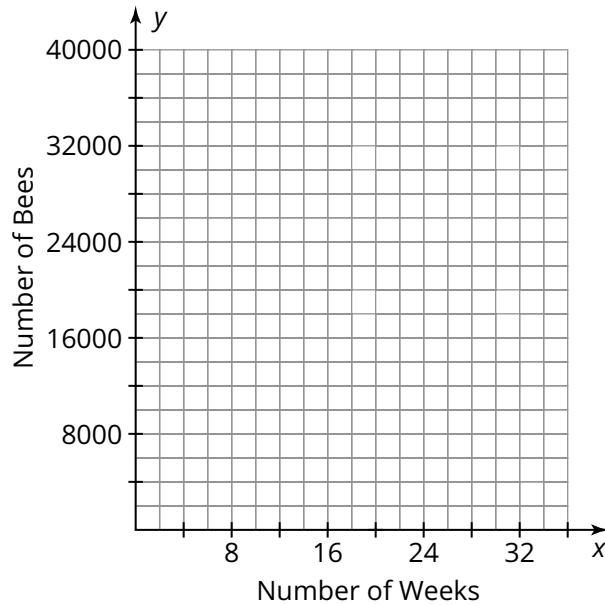
2. Graph the cube root parent function $g(x) = \sqrt[3]{x}$. Determine its key attributes such as its domain and range using set notation, extrema, and intercepts.



3. Javier has a new beehive. The number of bees in the hive after x weeks can be modeled by the function $b(x) = 36x^2$ for $1 \leq x \leq 30$.
- a. Determine the corresponding range of $b(x)$ for the given domain using inequalities. Describe what the domain and range represent in this problem.

Lesson 2 Assignment

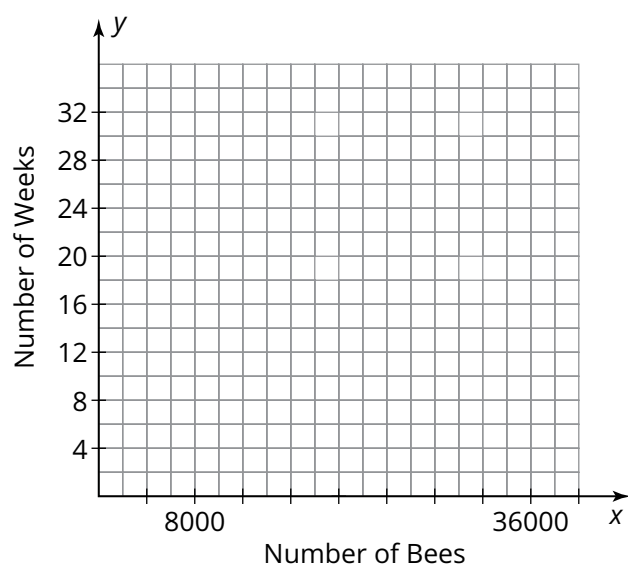
- b. Sketch the function $b(x)$ with the given domain restrictions.



- c. Use the function $b(x)$ to predict the bee population after 10 weeks.
- d. Use the function $b(x)$ to predict the bee population after 20 weeks.
- e. Write the inverse function $b^{-1}(x)$.

Lesson 2 Assignment

- f. Use compositions to verify that $b(x)$ and $b^{-1}(x)$ are inverse functions. Show your work.
- g. Determine the domain and range of $b^{-1}(x)$ using inequalities. Describe what the domain and range represent in this problem.
- h. Sketch the graph of the inverse function $b^{-1}(x)$.

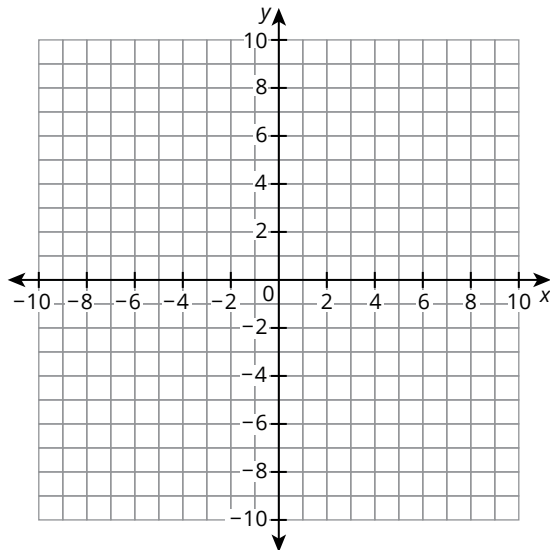


Lesson 2 Assignment

- i. Use the inverse function to determine when the bee population will be 25,000.

4. Consider the function $f(x) = -x^2 + 2$.

- a. Graph the function and its inverse.



- b. How can you restrict the domain and range of $f(x)$ so the inverse is a function?
- c. How does the restricted domain and range of $f(x)$ affect the domain and range of the inverse?

Lesson 2 Assignment

Prepare

1. Describe the similarities and differences between the graphs of

$h(x)$ and $g(x)$.

$$h(x) = f(x - 5)$$

$$g(x) = f(x) + 5$$

2. Describe the similarities and differences between the graphs of $m(x)$ and $n(x)$.

$$m(x) = f(2x)$$

$$n(x) = 2f(x)$$

3

Transformations of Radical Functions

OBJECTIVES

- Graph transformations of radical functions.
- Analyze transformations of radical functions using transformational function form.
- Describe transformations of radical functions using algebraic, graphical, and verbal representations.
- Generalize the effects of transformations on power functions and their inverses.

.....

You have transformed many different function types.

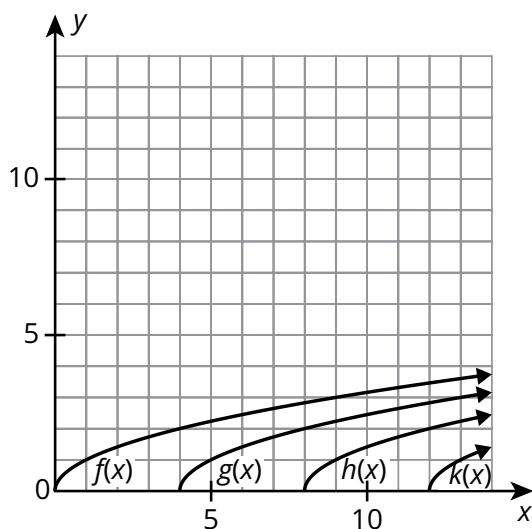
How does the transformational function form apply to the transformations of radical functions?

Getting Started

A Sea Change

A group of art students had the idea to use transformations of radical functions to create a logo for a surfing school.

To start, they graphed the function $f(x) = \sqrt{x}$, for $0 \leq x \leq 14$, and shifted copies of the curve to create the waves $g(x)$, $h(x)$, and $k(x)$.



Remember ...

The transformation function form is
 $g(x) = af(b(x - c)) + d$.

Ask yourself ...

Do the transformations of $f(x)$ shown on the graph take place inside the function or outside the function?

1. What value(s) in the transformation function form were changed to create these curves? Explain your reasoning.
2. Do the graphs of $g(x)$, $h(x)$, and $k(x)$ have the same domain as $f(x)$? Explain your reasoning.

Review the definition of **explain your reasoning** in the Academic Glossary.

Transformations of Radical Functions

Consider the transformations of the radical function $f(x) = \sqrt{x}$ graphed by the art students in the previous activity.



1. Fernando, Alexander, and Daniela each wrote an equation for one of the functions that was added to the graph, first in terms of $f(x)$, and then in terms of x .

- Fernando's equation: $g(x) = f(x) - 4$
 $= \sqrt{x} - 4$

- Alexander's equation: $h(x) = f(x - 8)$
 $= \sqrt{x - 8}$

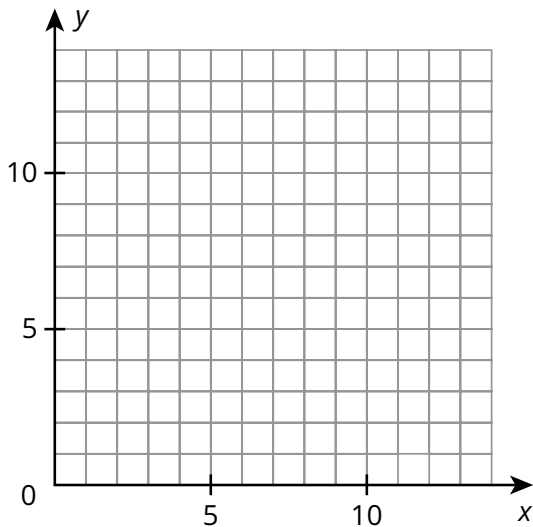
- Daniela's equation: $k(x) = f(x + 12)$
 $= \sqrt{x + 12}$

Explain why each student's equation is either correct or incorrect. If it is incorrect, write the correct equation, first in terms of $f(x)$, and then in terms of x . Finally, state the domain of each.

.....
Remember . . .
You can use the prime symbol (') to indicate that a function is a transformation of another function.
.....

The students decide that reflecting each curve, $g(x)$, $h(x)$, and $k(x)$, across the respective lines where $x = c$ will make them look more like waves crashing on the beach.

2. Consider the graph of the reflections.
- a. Graph the resulting functions $f'(x)$, $g'(x)$, $h'(x)$, and $k'(x)$. Write each function, first in terms of its transformation of $f(x)$, $g(x)$, $h(x)$, or $k(x)$, and then in terms of x . Finally, use set notation to state the domain and range of each function.



- b. Describe how you can use the transformation function form to determine the equations of the new functions.
- c. How did the domain of each transformed function change as a result of the reflection across $x = c$?
- d. Why does your graph show only 3 curves when the original graph had 4? Explain your reasoning.

Ask Yourself . . .

How does representing mathematics in multiple ways help to communicate reasoning?

Suppose the students wanted to reflect the 3 new waves $g'(x)$, $h'(x)$, and $k'(x)$ across the line $y = 0$.

3. Consider the equations for the reflected functions.
 - a. Describe how you can use transformation function form to determine the equations of the reflected functions.

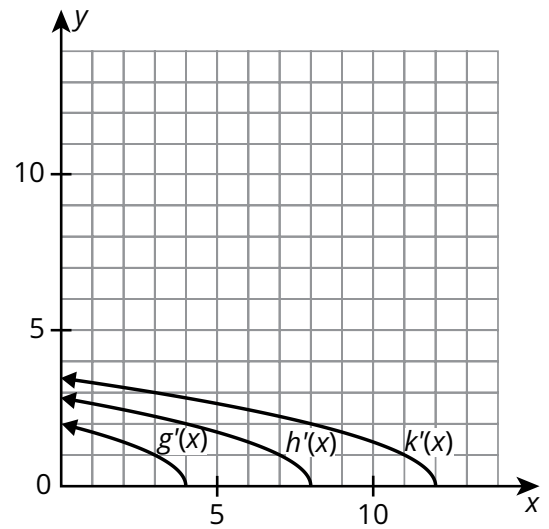
.....
Use the double prime symbol (") to indicate each transformed function.
.....

- b. Write three new functions using transformational form to represent each reflection of $g'(x)$, $h'(x)$, and $k'(x)$, and then each in terms of x . Finally, use interval notation to write the domain and range of each transformed function.

The art students want to add waves below the 3 waves, as shown. These waves are copies of $g'(x)$, $h'(x)$, and $k'(x)$, except half as high and shifted to the left 2 units.

4. Consider the equations for the waves.

- a. Write 3 new functions $q(x)$, $r(x)$, and $s(x)$ in terms of $g'(x)$, $h'(x)$, and $k'(x)$ to create the waves that the art students want. Write the domain and range of each transformed function using inequalities.

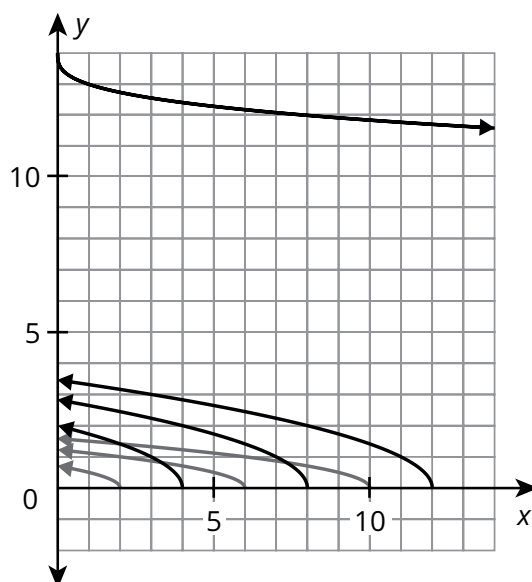


- b. Describe how you used what you know about the transformation function form to determine your answer to part (a).

The art students want to add some clouds to the top of the logo. For the clouds, they use the inverses of cubic functions. They start with the function $c(x) = -\sqrt[3]{x} + 14$.

5. How is the function $c(x) = -\sqrt[3]{x} + 14$ different than the parent cube root function, $\sqrt[3]{x}$?

6. Transform this function and write 2 more equations to create the clouds the students want. Graph the results.



In many graphic design programs, a trace path can be created. A trace path is an invisible line or curve that acts as the baseline of text that is added to the design. When you insert text on a trace path, the text follows the line or curve.

.....
The text shown, for example, follows the curve $f(x) = -x^2$.
.....

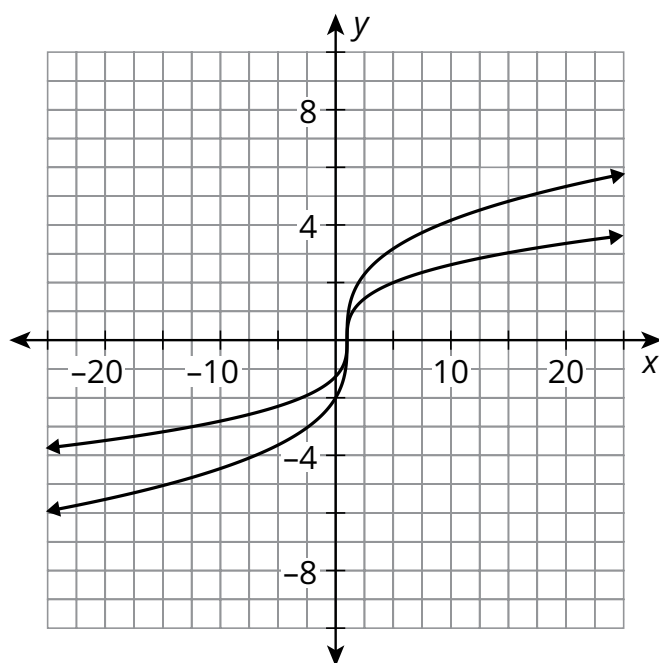
Your Text Here
.....

7. The art students are experimenting with different radical function graphs to use as trace paths for a surfing school's name: They have narrowed their trace paths down to 2 choices. The graphs of the functions are shown.

$$h(x) = \sqrt[3]{2(x-1)}$$

$$j(x) = 2\sqrt[3]{x-1}$$

- a. Use technology to sketch the graph of the function $f(x) = \sqrt[3]{x}$ and list its domain, range, x-intercept, and y-intercept. Write the domain using interval notation.



Domain: _____

Range: _____

x-intercept: _____

y-intercept: _____

b. Compare and contrast the graphs of the functions $h(x)$ and $j(x)$ to $f(x) = \sqrt[3]{x}$. What do you notice?

c. Compare the effects of increasing the a -value with increasing the b -value in a radical function. What do you notice?

d. State the domain of each transformed function using set notation.

8. Choose one of the cube root functions as a trace path for the title of the surfing school. Or, write a different radical function to use as a trace path. Graph the function on the coordinate plane in Question 5, and write the title of the school on the trace path.

ACTIVITY
3.2

More Transformations of Radical Functions

You have already explored transformations with functions in many function families. You learned that transformations performed on any function $f(x)$ to form a new function $g(x)$ can be described by the transformational function:

$$g(x) = af(b(x - c)) + d$$

Recall that this transformational function generalizes to any function. Changes to the a - or d -values dilate, translate, or reflect a function vertically. Changes to the b - or c -values dilate, translate, or reflect a function horizontally.

For the square root function and cube root function respectively, the transformational function can be written as:

$$s(x) = a\sqrt{b(x - c)} + d \quad c(x) = a\sqrt[3]{b(x - c)} + d$$

1. Determine how the values of a , b , c , or d transform the graph of $f(x) = \sqrt{x}$ or $q(x) = \sqrt[3]{x}$.

a. $g(x) = a\sqrt{x}$ or $r(x) = a\sqrt[3]{x}$, for positive and negative values of a .

b. $h(x) = \sqrt{x} + d$ or $u(x) = \sqrt[3]{x} + d$, for positive and negative values of d .

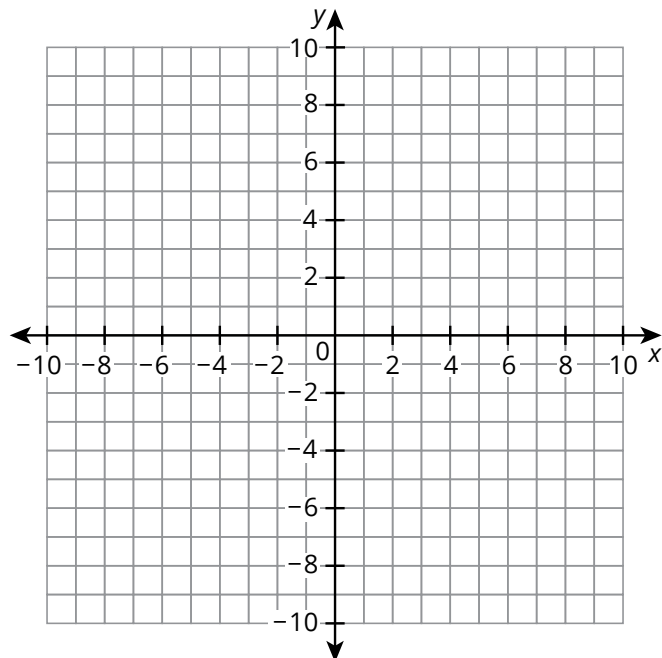
c. $j(x) = \sqrt{x - c}$ or $v(x) = \sqrt[3]{x - c}$, for positive and negative values of c .

d. $k(x) = \sqrt{bx}$ or $w(x) = \sqrt[3]{bx}$, for positive and negative values of b .

2. Graph each function, and describe the similarities and differences between the graphs of the two functions:

$$f(x) = \sqrt[3]{4(x - 5)}$$

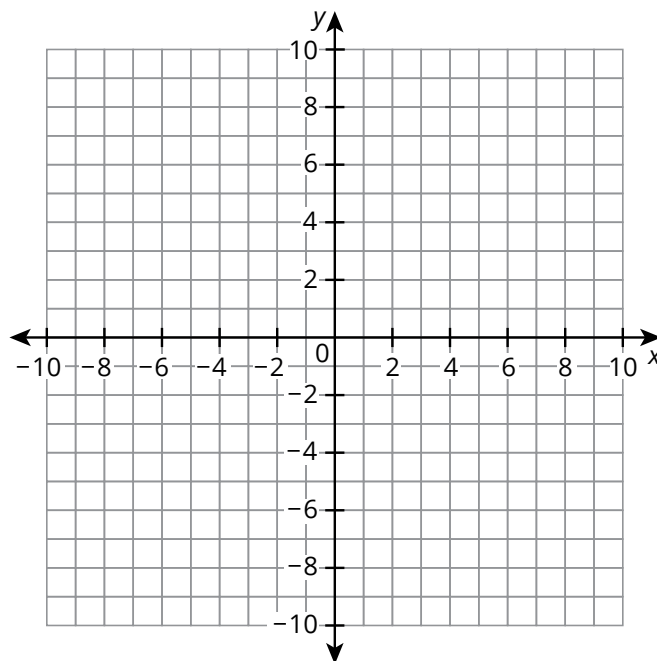
$$g(x) = 4\sqrt[3]{x + 5}$$



3. Graph each function and describe the similarities and differences between the graphs of the two functions:

$$h(x) = -2\sqrt[3]{x} + 1$$

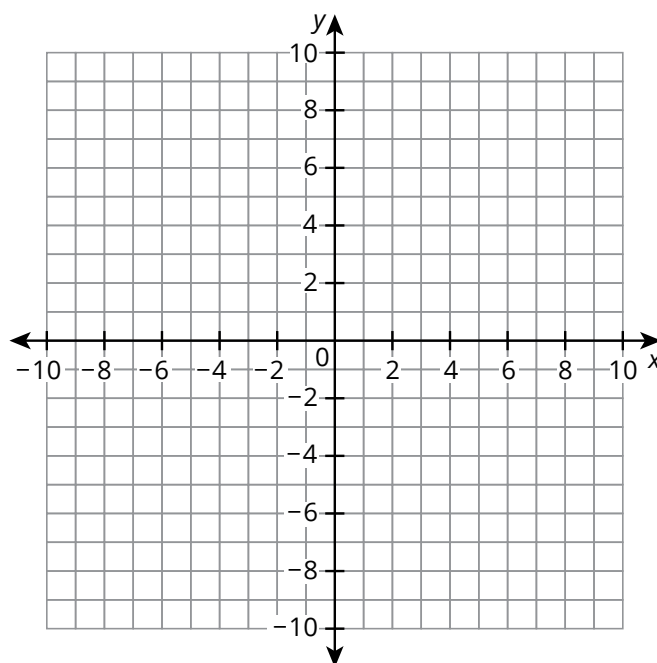
$$j(x) = \sqrt[3]{-2x} - 1$$



4. Graph each function and describe the similarities and differences between the graphs of the two functions:

$$l(x) = \frac{1}{2}\sqrt{x-3} + 1$$

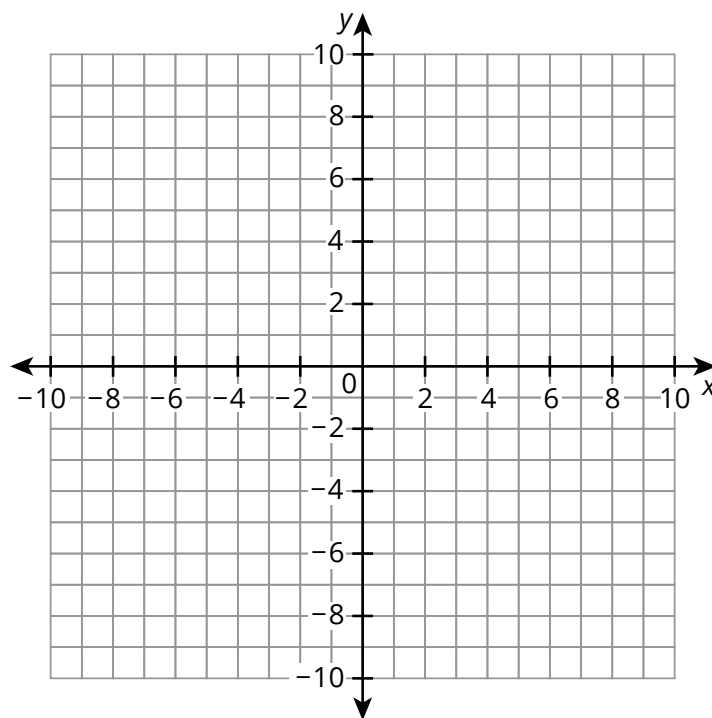
$$m(x) = \sqrt{\frac{1}{2}(x+3)} + 1$$



5. Graph each function and describe the similarities and differences between the graphs of the two functions:

$$n(x) = \sqrt{-x + 3} - 4$$

$$p(x) = -\sqrt{x - 3} - 4$$

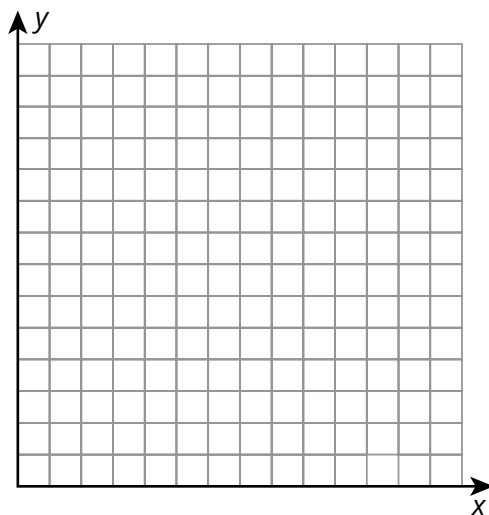


Talk the Talk

Now it's your turn to design a logo!

Logo to Town

1. Use technology to design a logo that includes the transformations of $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$.
 - Graph at least two transformed functions that each include at least two different a -, b -, c -, or d -value transformations.
 - Write each function you graphed first in terms of its transformations of $f(x)$ or $g(x)$, and then in terms of x .
 - State the domain of each transformed function using set notation.



Ask Yourself . . .

How can you use graphs of square root and cube root functions in everyday life?

Lesson 3 Assignment

Write

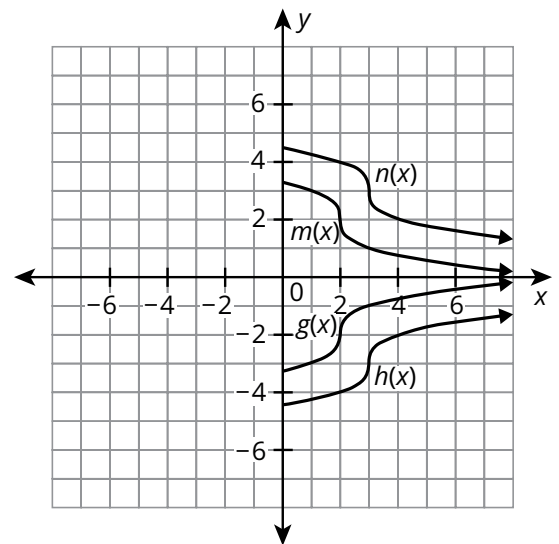
Describe how transformations affect the domain and range of radical functions.

Remember

For the square root function and cube root function, respectively, the transformation function can be written as $s(x) = a\sqrt{b(x - c)} + d$ or $c(x) = a\sqrt[3]{b(x - c)} + d$. Changes to the a - or d -values dilate, translate, or reflect a function vertically. Changes to the b - or c -values dilate, translate, or reflect a function horizontally.

Practice

- Jacob, a graphic designer, designed the logo shown for a local ranch. Each curve in the design is a transformation of the cube root function $f(x) = \sqrt[3]{x}$ with a restricted domain.
 - Describe each of the four transformations of $f(x) = \sqrt[3]{x}$ that were used to create the four functions in the design.

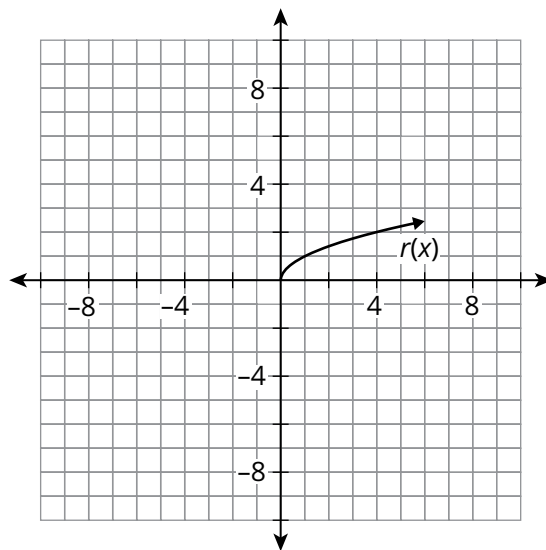


- Write each function used in the design. For each function, write the domain as an inequality.

Lesson 3 Assignment

2. Jacob is working on a logo for a publishing company. He starts by graphing the function $r(x) = \sqrt{x}$ with the restricted domain $0 \leq x \leq 6$. He plans to add the graphs of 5 more functions to complete the design.

a. The next function Jacob adds is $s(x)$, which is the square root function after a vertical stretch by a factor of 2 and a translation 1 unit up. Write the function $s(x)$, and graph $s(x)$ with the domain $0 \leq x \leq 6$.



b. Next, Jacob adds the function $t(x)$, which is the square root function after a vertical stretch by a factor of 3 and a translation 2 units up. Write the function $t(x)$, and graph $t(x)$ with the domain $0 \leq x \leq 6$.

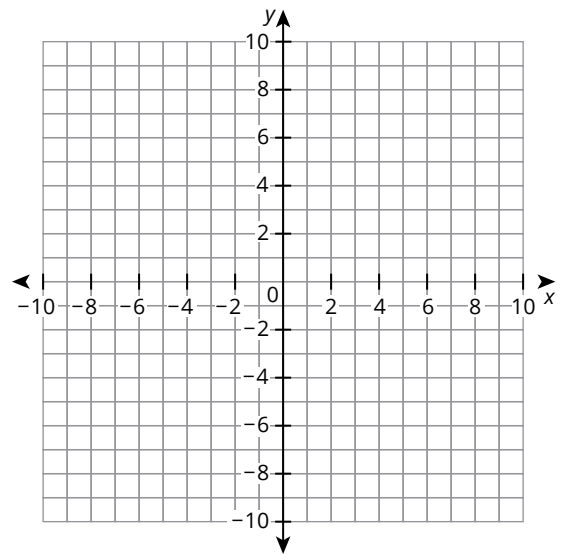
c. To complete the design, Jacob adds the functions $r'(x)$, $s'(x)$, and $t'(x)$, which are reflections of the original 3 functions across the y -axis. Write the functions $r'(x)$, $s'(x)$, and $t'(x)$ and graph each function with the domain $-6 \leq x \leq 0$.

Lesson 3 Assignment

3. Graph each function, and describe the similarities and differences between the graphs of the two functions:

$$f(x) = \sqrt{3(x - 1)}$$

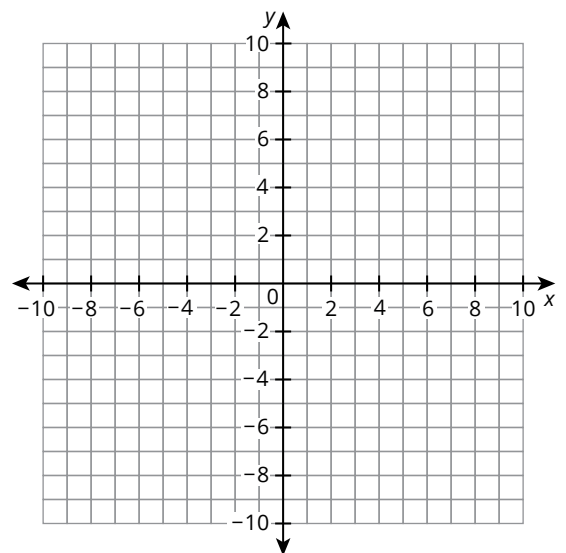
$$g(x) = 3\sqrt{x + 1}$$



4. Graph each function, and describe the similarities and differences between the graphs of the two functions:

$$h(x) = -\sqrt{x - 4} - 2$$

$$j(x) = \sqrt{-(x + 4)} - 2$$

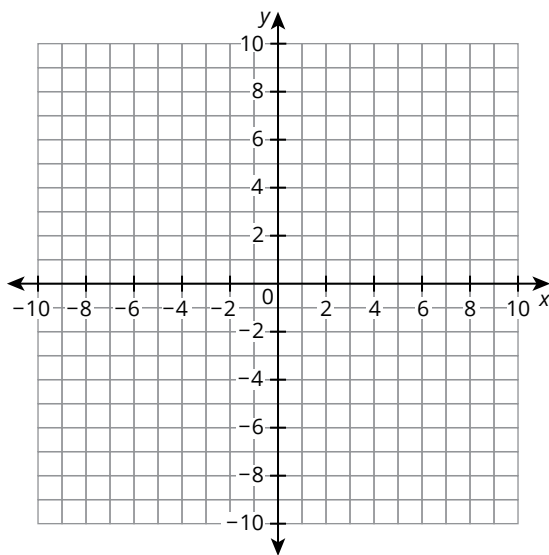


Lesson 3 Assignment

5. Graph each function, and describe the similarities and differences between the graphs of the two functions:

$$m(x) = -\frac{1}{3}\sqrt[3]{x-1}$$

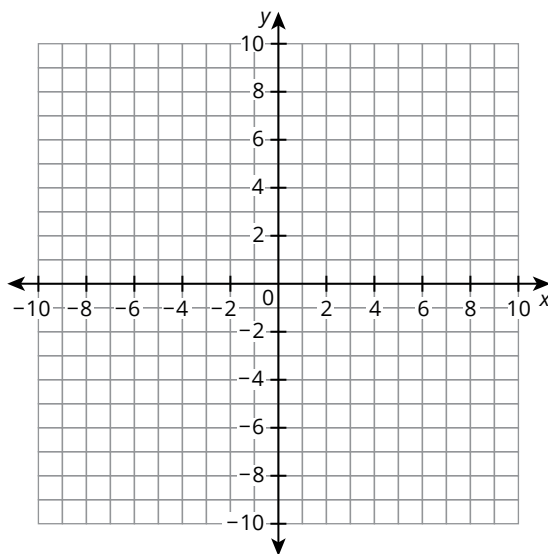
$$n(x) = 3\sqrt[3]{x+1} - 2$$



6. Graph each function, and describe the similarities and differences between the graphs of the two functions:

$$p(x) = \sqrt[3]{2x} + 1$$

$$q(x) = \sqrt[3]{-2(x+1)} + 1$$



Lesson 3 Assignment

Prepare

Consider the function $f(x) = 2x^2 + 4x - 3$. Determine each value.

1. $f(0)$

2. $f(2)$

3. $f(-2)$

4

Compositions of Functions

OBJECTIVES

- Perform the composition of two functions algebraically.
- Use the composition of functions to determine whether two functions are inverses.
- Determine the domain and range of a composite function and explain the reasoning in written form.

KEY TERM

- identity function

.....

You have learned about functions and their inverses.

How can you use composition of functions to determine whether two functions are inverses?

Getting Started

Extreme Couponing

Koda is in the market for a new flat screen TV. She has two coupons that she can use to purchase the television. One coupon is for 20% off the original price and the other coupon is for \$100 off the original price.

1. Consider the situation.
 - a. Write a function $p(x)$ to represent the discounted price of the television after the application of the 20% off coupon.
 - b. Write a function $d(x)$ to represent the discounted price of the television after the application of the \$100 off coupon.
 - c. Which coupon should Koda use? Explain your reasoning.

Ask Yourself . . .

Is one choice
always better
than the other?
Are they ever
the same?

2. Suppose Koda is able to use both coupons. If the television that she wants costs \$350, in which order should Koda apply the coupons to yield the lowest final price? Show your work.
3. Suppose Koda uses the 20% off coupon first, followed by the \$100 off coupon. Write the function $t(x)$ that represents the discounted price of a television that costs x dollars.
4. Suppose Koda uses the \$100 off coupon first, followed by the 20% off coupon. Write the function $m(x)$ that represents the discounted price of a television that costs x dollars.

Recall that the process of evaluating one function inside of another function is called composition of functions. For two functions f and g , the composition of functions uses the output of $g(x)$ as the input of $f(x)$. It is notated as $(f \circ g)(x)$ or $f(g(x))$. The resulting function is a composite function.

You can also write the two functions $t(x)$ and $m(x)$ that you wrote in Questions 3 and 4 as compositions of the original functions $p(x)$ and $d(x)$.

5. Rewrite $t(x)$ as $d(p(x))$ and $m(x)$ as $p(d(x))$.

6. Based on the functions $t(x)$ and $m(x)$, what can you conclude about the order to apply a percentage off coupon and a flat rate coupon? Explain your reasoning.

Evaluating Function Compositions

You have learned about composing functions to create new functions. You can write these functions using what you know about algebra.

1. Consider the functions $f(x) = 3x - 1$ and $g(x) = x^2 + 4$. Evaluate each composition of functions.

a. $f(g(2x))$

b. $f(g(x))$

c. $g(f(x))$

2. Destiny was asked to determine the composite function $(m \circ p)(x)$ given the functions $m(x) = 5 - x$ and $p(x) = 3x^2 + x$.

- a. Describe the error in Destiny's method.

Destiny



$$\begin{aligned}(m \circ p)(x) &= (5 - x)(3x^2 + x) \\ &= 15x^2 + 5x - 3x^3 - x^2 \\ &= -3x^3 + 14x^2 + 5x\end{aligned}$$

b. Determine the correct composite function $(m \circ p)(x)$.

c. Is $(p \circ m)(x)$ equivalent to $(m \circ p)(x)$? Explain your reasoning.

3. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions $f(x)$ and $g(x)$.

a. $f(x) = 2x^2 - x$
 $g(x) = \sqrt{x}$

b. $f(x) = 0.5x + 1.5$
 $g(x) = 2x - 3$

4. Compare the compositions $(f \circ g)(x)$ and $(g \circ f)(x)$ from Question 3.
- Identify the pair(s) of functions in which $f(g(x))$ and $g(f(x))$ are equivalent functions.
 - Determine the relationship between the functions you identified in part (a).

Consider these two statements:

- The additive identity is 0. It results when two quantities that are additive inverses are added.
 - The multiplicative identity is 1. It results when two quantities that are multiplicative inverses are multiplied.
5. How would you describe the composition identity? Provide an example to support your claim.

The *identity function* is the function that maps every real number x to the same real number x . The composition identity, also known as the **identity function**, is defined as $f(x) = x$.

6. Use composition of functions to determine whether the given functions are inverses of each other.

a. $c(x) = \frac{1}{x} - 6$

$$d(x) = \frac{1}{x + 6}$$

b. $z(x) = \frac{5}{4}x + 3$

$$w(x) = \frac{4}{5}x - 3$$

7. Gabriela is asked to determine whether $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x} + 1$ are inverse functions.

Gabriela



The two functions are equivalent at the value $x = 0$, so they must be inverse functions.

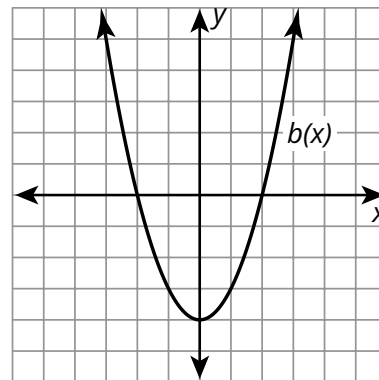
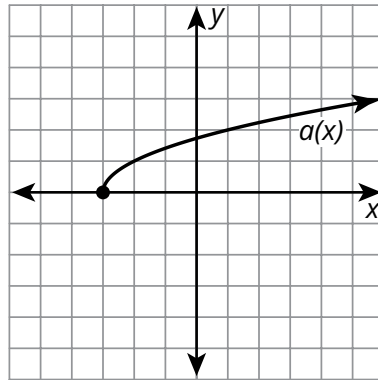
$$\begin{aligned}(f \circ g)(0) &= f(g(0)) \\ &= f(\sqrt[3]{0} + 1) \\ &= f(1) \\ &= 1^3 + 1 \\ &= 2\end{aligned}$$

$$\begin{aligned}(g \circ f)(0) &= g(f(0)) \\ &= g(0^3 + 1) \\ &= g(1) \\ &= \sqrt[3]{1} + 1 \\ &= 2\end{aligned}$$

Explain why Gabriela's method is insufficient, and provide a counterexample.

Domain and Range of Composite Functions

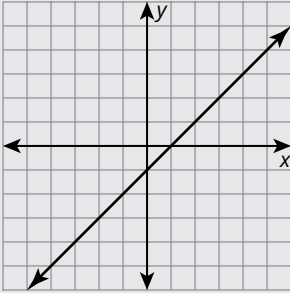
Consider the functions $a(x) = \sqrt{x + 3}$ and $b(x) = x^2 - 4$.



1. State the domain and range of $a(x)$ and of $b(x)$ using interval notation.
2. Write the composite function $(b \circ a)(x)$ and state the domain and range using interval notation.

3. Amir simplified the composition function algebraically and then graphed the result. Describe the error Amir made.

Amir

$$\begin{aligned}(b \circ a)(x) &= b(a(x)) \\&= b(\sqrt{x+3}) \\&= (\sqrt{x+3})^2 - 4 \\&= x + 3 - 4 \\&= x - 1\end{aligned}$$


4. Determine $(a \circ b)(x)$. Show your work.
5. Are the $a(x)$ and $b(x)$ inverse functions? Explain your reasoning.
6. State the domain and range of $(a \circ b)(x)$ using interval notation. Is it the same as the domain and range of $(b \circ a)(x)$? Explain your reasoning.

.....
You can graph the unsimplified version of a composition function in your graphing calculator to help you see domain restrictions.
.....

The domain of $(f \circ g)(x)$ is the set of all real numbers x in the domain of g such that $g(x)$ is in the domain of f . In order for an element to be in the domain of the function $(f \circ g)(x)$, it must first be defined in the domain of $g(x)$.

7. Hannah and Madison were given the functions $m(x) = \frac{1}{x}$ and $n(x) = \sqrt{x+5}$ and asked to determine the domain of $m(n(x))$. Determine whose method is correct and explain your reasoning.

Hannah

I know that $m(n(x))$ means that $n(x) = \sqrt{x+5}$ is the input function and it has a domain of $[-5, \infty)$.

The output function is $m(x) = \frac{1}{x}$ and I know that the denominator can't be 0.

Therefore, the domain of $m(n(x))$ is $x \geq -5, x \neq 0$.

Madison

I know that $m(n(x))$ means $m(\sqrt{x+5}) = \frac{1}{\sqrt{x+5}}$.

The input function is $n(x) = \sqrt{x+5}$, and it has a domain of $[-5, \infty)$, but because it's in the denominator and the denominator can't be zero, -5 is not a part of the domain.

Therefore, the domain of $m(n(x))$ is $(-5, \infty)$.



Talk the Talk

Compose and Conquer

1. Determine whether each set of equations are inverses.

a. $f(x) = (x - 2)^2, x \geq 2$
 $g(x) = \sqrt{x} + 2$

b. $m(x) = 3x + 5$
 $n(x) = \frac{(x + 5)}{3}$

c. $p(x) = \frac{(2x + 3)}{(x - 1)}$
 $q(x) = \frac{(x + 3)}{(x - 2)}$

2. Consider the functions $v(x) = x^2 + 3$ and $w(x) = \sqrt{x - 2}$.

- a. State the domain and range of $v(x)$ and $w(x)$ using interval notation.

- b. Determine $(v \circ w)(x)$ and $(w \circ v)(x)$. Show your work.

- c. State the domain and range of $(v \circ w)(x)$ using interval notation. Is it the same as the domain and range of $(w \circ v)(x)$? Explain your reasoning.

- d. Are $v(x)$ and $w(x)$ inverses of each other? Why or why not?

Lesson 4 Assignment

Write

How can you use compositions of functions to determine whether two functions are inverses?

Remember

When two functions are composed to make a new function, one function's output is used as the other function's input.

Practice

Determine $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions. Then, state the domain and range of the composite function using interval notation. Explain your reasoning.

1. $f(x) = \frac{1}{2}x + 4$
 $g(x) = -6x - 2$

Lesson 4 Assignment

2. $f(x) = -x^2$
 $g(x) = 1 - x$

3. Determine whether the functions are inverses.

a. $f(x) = 3x - 4$

$$g(x) = \frac{x}{3} + 2$$

Lesson 4 Assignment

b. $h(x) = \frac{(3x - 4)}{(2x + 5)}$

$$j(x) = \frac{(4 + 5x)}{(3 - 2x)}$$

c. $p(x) = (x - 5)^2, x \geq 5$

$$q(x) = \sqrt{x} + 5$$

Lesson 4 Assignment

Prepare

Extract roots to rewrite each radical expression.

1. $\sqrt{48}$

2. $\sqrt{27}$

3. $\sqrt{32}$

Radical Functions

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Radical Functions* topic by:

TOPIC 2: <i>Radical Functions</i>	Beginning of Topic	Middle of Topic	End of Topic
graphing square root and cube root functions.			
identifying the key attributes of the graph of a radical function.			
transforming cubic and square root functions.			
defining inverse function.			
determining the inverse of a function algebraically and graphically.			
identifying and describing the domain of a radical function.			
restricting the domain of a function to produce an invertible function from a non-invertible function.			
composing two simple functions to determine whether they are inverses.			

continued on the next page

TOPIC 2 SELF-REFLECTION
continued

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Radical Functions* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 2 SUMMARY

Radical Functions Summary

LESSON

1

Inverses of Power Functions

Recall that a power function is a polynomial function of the form $P(x) = ax^n$, where n is a non-negative integer.

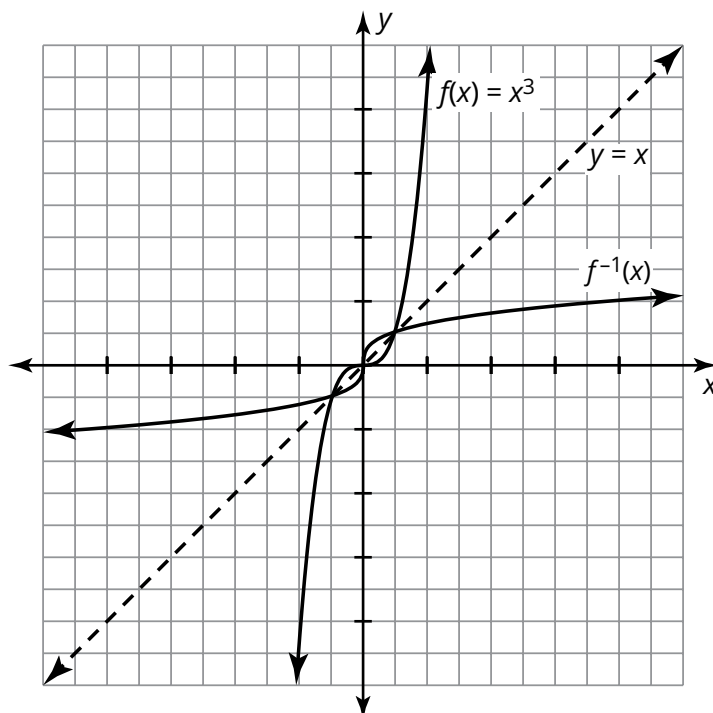
A function f is the set of all ordered pairs (x, y) , or $(x, f(x))$, where for every value of x there is one and only one value of y , or $f(x)$. Recall the inverse of a function is the set of all ordered pairs (y, x) , or $(f(x), x)$.

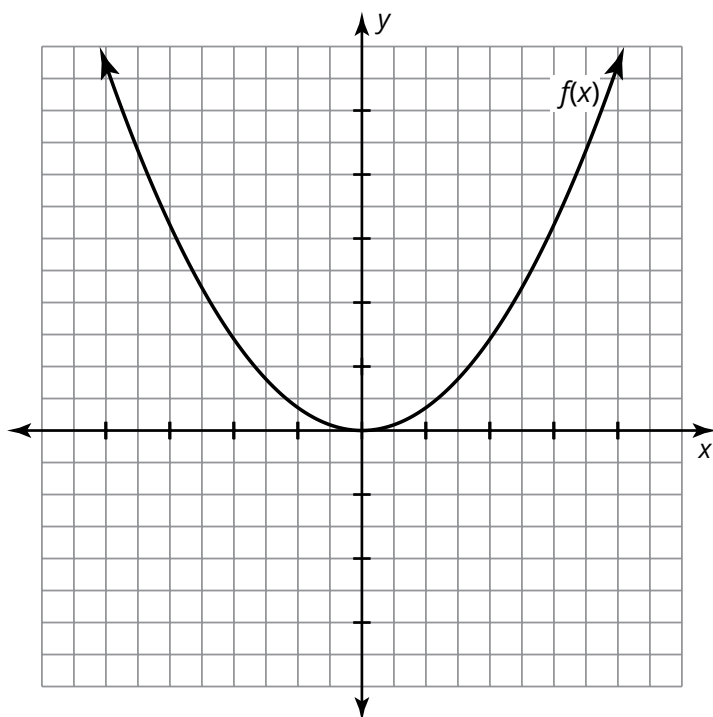
For example, the graph shows $f(x) = x^3$ and its inverse, $f^{-1}(x)$.

If the inverse of a function f is also a function, then f is an **invertible function**, and its inverse is written as $f^{-1}(x)$.

KEY TERMS

- invertible function [función invertible]
- Horizontal Line Test
- square root function
- restrict the domain [restringir el dominio]
- cube root function
- radical function [función radical]
- composition of functions [composición de funciones]
- identity function [función identidad]





The Vertical Line Test is used to determine whether or not the inverse of a power function is also a function. The **Horizontal Line Test** is a visual method to determine whether a function has an inverse that is also a function. If any of the horizontal lines intersect the graph of the function at more than one point, then the inverse of the function is not a function.

For example, $f(x) = \frac{x^2}{56}$ is not invertible because it fails the Horizontal Line Test.

LESSON

2

Radical Functions

To determine the equation of the inverse of a power function, transpose x and y in the equation and solve for y .

For example, consider the function $y = x^3$. Determine the inverse of the function.

$$y = x^3$$

$$x = y^3$$

$$\sqrt[3]{x} = y$$

The equation for the inverse of the function $y = x^3$ is $y = \sqrt[3]{x}$.

The function $f(x) = x^2$ fails the Horizontal Line Test. Therefore, it is not an invertible function. But there are conditions that would allow $f(x) = x^2$ to pass the Horizontal Line Test. When you **restrict the domain** of the power function $f(x) = x^2$ to values greater than or equal to 0, the inverse of the function is called the **square root function** and is written as:

$$f^{-1}(x) = \sqrt{x}, \text{ for } x \geq 0$$

When determining the inverse of a function, you transpose x and y which are the inputs and outputs of a function. Thus, you are also switching the domain and range of the function. The domain of the function becomes the range of the inverse and the range of the function becomes the domain of the inverse. Restricting the domain of the function will restrict the range of the inverse.

Transformations of a function will also transform the inverse.

Transformation of Quadratic Function, $f(x)$	Transformation of Inverse Function, $f^{-1}(x)$
translation up d units	translation right d units
translation down d units	translation left d units
translation right c units	translation up c units
translation left c units	translation down c units

The **cube root function** is the inverse of the power function $f(x) = x^3$ and can be written as $f^{-1}(x) = \sqrt[3]{x}$. Because $f(x) = x^3$ is an invertible function the domain does not have to be restricted.

The inverses of power functions with exponents greater than or equal to 2, such as the square root function and the cube root function, are called **radical functions**.

The process of evaluating one function inside of another function is called the **composition of functions**. For two functions f and g , the composition of functions uses the output of $g(x)$ as the input of $f(x)$. It is notated as $f(g(x))$ or $g(f(x))$.

For example, to write a composition of the functions $g(x) = x^2$ and $f(x) = \sqrt{x}$ when the domain of $g(x)$ is restricted to $x \geq 0$, substitute the value of one of the functions for the argument, x , of the other function. This can be done in two ways.

$$\begin{aligned} f(g(x)) &= f(x^2) \\ &= \sqrt{x^2} \\ &= x \end{aligned}$$

$$f(g(x)) = x, \text{ for } x \geq 0$$

$$\begin{aligned} g(f(x)) &= g(\sqrt{x}) \\ &= (\sqrt{x})^2 \\ &= x \end{aligned}$$

$$g(f(x)) = x, \text{ for } x \geq 0$$

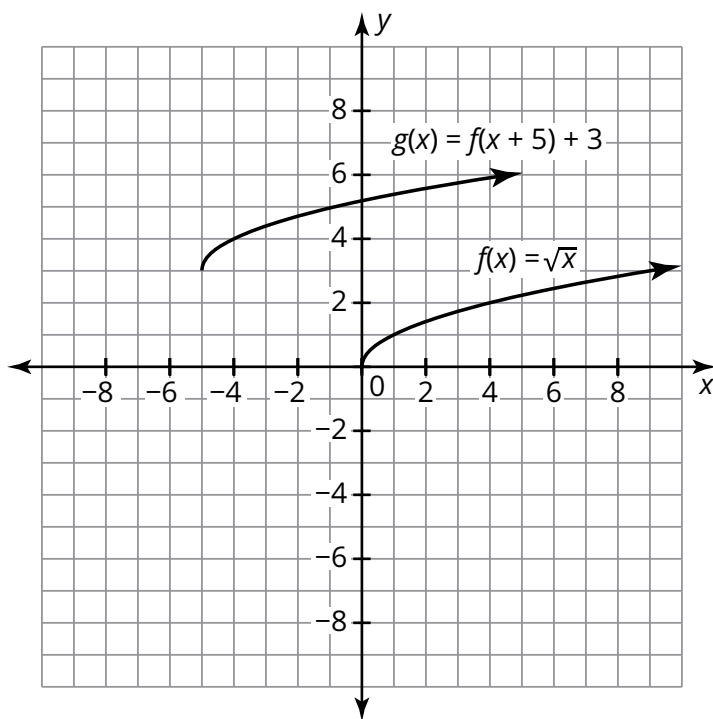
If $(g(x)) = g(f(x)) = x$, then $f(x)$ and $g(x)$ are inverse functions.

LESSON

3

Transformations of Radical Functions

For the square root function and cube root function respectively, the transformation function form can be written as $s(x) = a\sqrt{b(x - c)} + d$ or $c(x) = a\sqrt[3]{b(x - c)} + d$. Transformations that take place inside the radical shift the function left or right. Transformations that take place outside the radical shift the function up or down.



For example, consider the graph of $f(x) = \sqrt{x}$ and its transformation represented by the equation $g(x) = f(x+5) + 3$.

The domain of $f(x)$ is $[0, \infty)$.

The range of $f(x)$ is $[0, \infty)$.

The range of $g(x)$ is $[3, \infty)$.

The domain of $g(x)$ is $[-5, \infty)$.

LESSON

4

Compositions of Functions

In a composition of two functions f and g , written as $(f \circ g)(x)$, the output of $g(x)$ becomes the input of $f(x)$. That is, the function $g(x)$ is substituted for x into the function $f(x)$.

Consider the functions $f(x) = 2x - 3$ and $g(x) = x^2 + 3x - 7$.

Determine $(f \circ g)(x)$.

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(x^2 + 3x - 7) \\ &= 2(x^2 + 3x - 7) - 3 \\ &= 2x^2 + 6x - 14 - 3 \\ &= 2x^2 + 6x - 17 \end{aligned}$$

Two functions f and g are inverses of each other if the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$ both yield the identity function. That is, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

Use function composition to determine whether the functions $q(x) = \sqrt{x-1}$ and $h(x) = x^2 + 1, x \geq 0$, are inverses of each other.

$$(q \circ h)(x) = q(h(x))$$

$$= q(x^2 + 1)$$

$$= \sqrt{(x^2 + 1) - 1}$$

$$= \sqrt{x^2}$$

$$= x$$

$$(h \circ q)(x) = h(q(x))$$

$$= h(\sqrt{x-1})$$

$$= (\sqrt{x-1})^2 + 1$$

$$= (x-1) + 1$$

$$= x$$

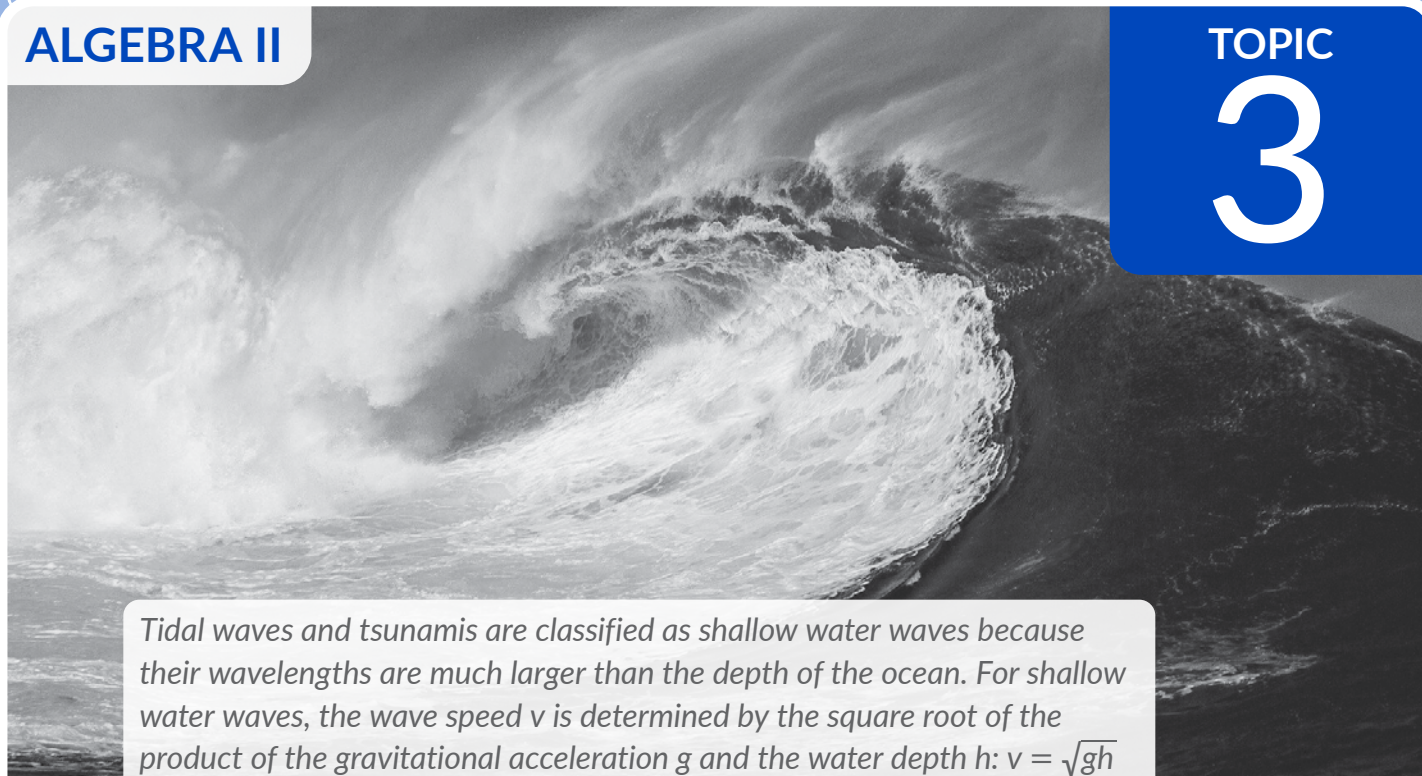
The functions $q(x) = \sqrt{x-1}$ and $h(x) = x^2 + 1, x \geq 0$, are inverses of each other, because $(q \circ h)(x) = x$ and $(h \circ q)(x) = x$.

The domain of $(f \circ g)(x)$ is the set of all real numbers x in the domain of g such that $g(x)$ is in the domain of f . In order for an element to be in the domain of the function $(f \circ g)(x)$ it must first be defined in the domain of $g(x)$.

Let $r(x) = \frac{3}{6-x}$ and $s(x) = \sqrt{x-4}$. State the domain of $(r \circ s)(x)$. Explain how you determined your answer.

The composition of the functions $r(x)$ and $s(x)$ is $(r \circ s)(x) = \frac{3}{6-\sqrt{x-4}}$.

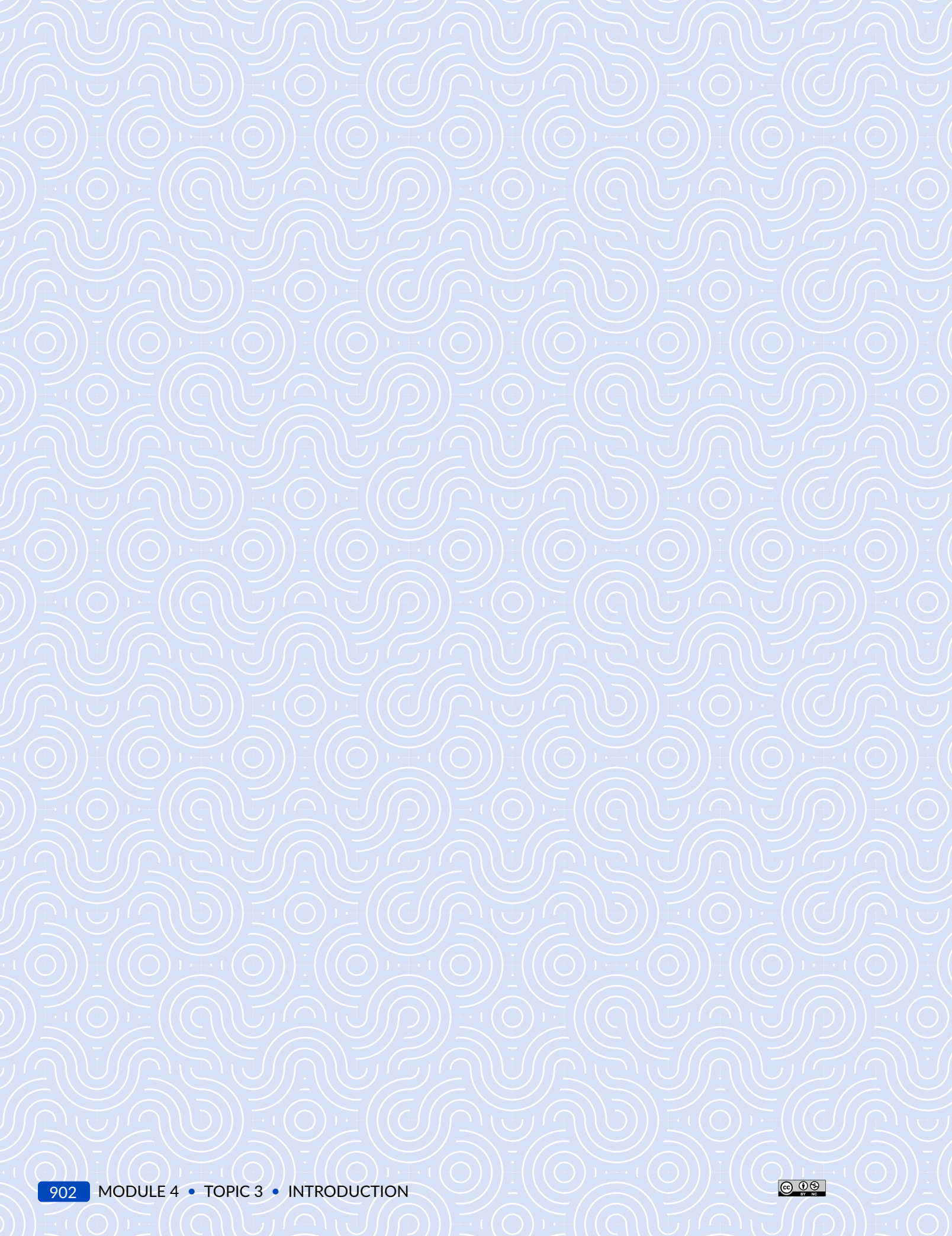
The domain of the input is $[4, \infty)$, but because the denominator cannot be equal to zero, the domain of the composite function must be $[4, \infty), x \neq 40$.



Tidal waves and tsunamis are classified as shallow water waves because their wavelengths are much larger than the depth of the ocean. For shallow water waves, the wave speed v is determined by the square root of the product of the gravitational acceleration g and the water depth h : $v = \sqrt{gh}$

Radical Equations

LESSON 1	Rewriting Radical Expressions	903
LESSON 2	Solving Radical Equations	929
LESSON 3	Modeling with Radical Functions	947



1

Rewriting Radical Expressions

OBJECTIVES

- Extract roots from radicals.
 - Rewrite radicals as powers that have rational exponents.
 - Rewrite powers that have rational exponents as radicals.
 - Rewrite radicals by extracting roots.
 - Multiply, divide, add, and subtract radicals.
-

You know how to rewrite radical expressions that have only numbers in the radicand.

How do you rewrite radical expressions that have variables in the radicand?

Getting Started

The Power of Positivity

You know how to use the properties of powers to rewrite expressions.

1. Match each expression to an equivalent expression in the box.
For each given expression, $a \neq 0$.

a. $a^3 \cdot a$

b. $\frac{a^3}{a^7}$

c. $a^4 \cdot a^{-4}$

d. $(ab^2)^2$

e. $(a^2)^2$

f. $\frac{a^6}{a^6}$

g. $(a^8 b^4)^{\frac{1}{2}}$

h. $a^2 \cdot a^{-6}$

1

a^4

a^{-4}

$a^2 b^4$

$a^4 b^2$



2. William says that the expression $\frac{a^6}{a^6}$ is equivalent to 1 because any number, except 0, divided by itself is 1. Ana says $\frac{a^6}{a^6}$ is equal to 1 because $a^{6-6} = a^0$, and anything to the zero power, except zero, equals 1. Who's correct? Explain your reasoning.

3. Consider each expression in Question 1. When a and b are real numbers, what do you know about the value of each expression? Explain your reasoning.

Extracting Variables from Radicals

Recall, the symbol $\sqrt{\quad}$ is called a radical and it is used to indicate square roots. The radicand is the quantity under a radical.

radical $\swarrow$ $\sqrt{25}$ $\nwarrow$ radicand

Previously, you have rewritten radicals by extracting roots involving numbers. In this lesson, you will explore how to extract roots for expressions of the form $\sqrt[n]{x^n}$. To determine how to extract a variable from a radical, let's consider several different values of n .

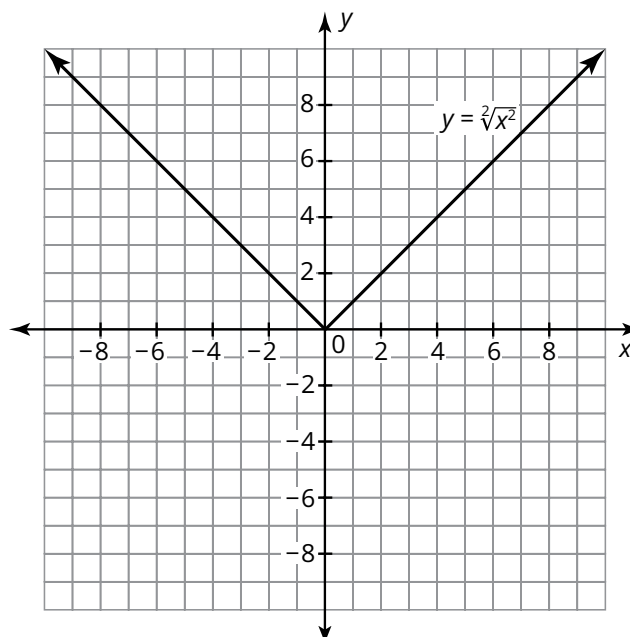
Remember...

You can write $\sqrt[2]{x^2}$ as $\sqrt{x^2}$. Since the root index is 2.

- For each value of n for the expression $\sqrt[n]{x^n}$, analyze the table and graph. Identify the function family associated with the graph, and write the corresponding equation.

a. Let $n = 2$.

x	$x^n = x^2$	$\sqrt[n]{x^n} = \sqrt[2]{x^2}$
-2	4	2
-1	1	1
0	0	0
1	1	1
2	4	2

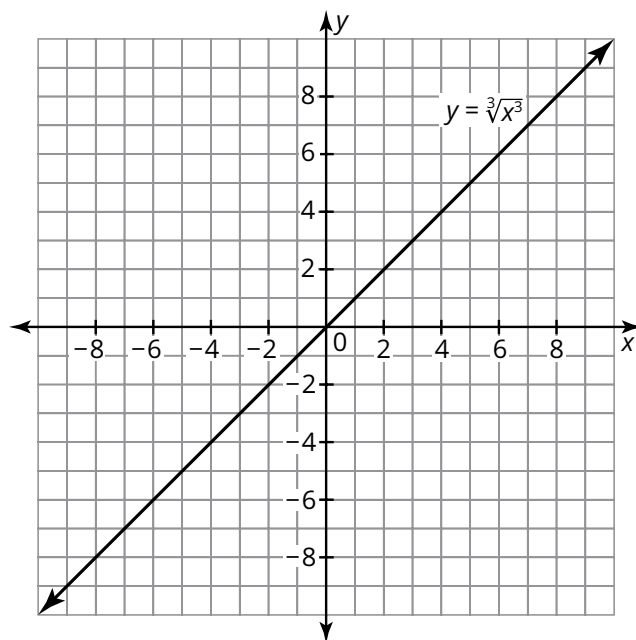


Function family of the graph: _____

Equation of the graph: _____

b. Let $n = 3$.

x	$x^n = x^3$	$\sqrt[n]{x^n} = \sqrt[3]{x^3}$
-2	-8	-2
-1	-1	-1
0	0	0
1	1	1
2	8	2

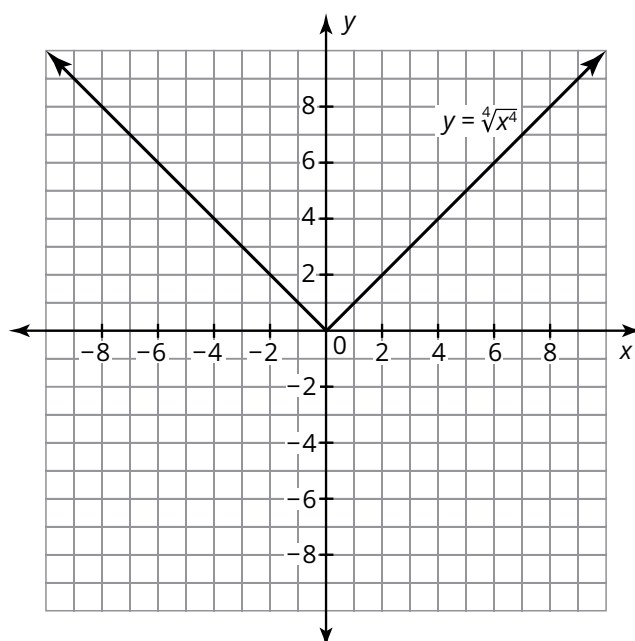


Function family of the graph: _____

Equation of the graph: _____

c. Let $n = 4$.

x	$x^n = x^4$	$\sqrt[n]{x^n} = \sqrt[4]{x^4}$
-2	16	2
-1	1	1
0	0	0
1	1	1
2	16	2

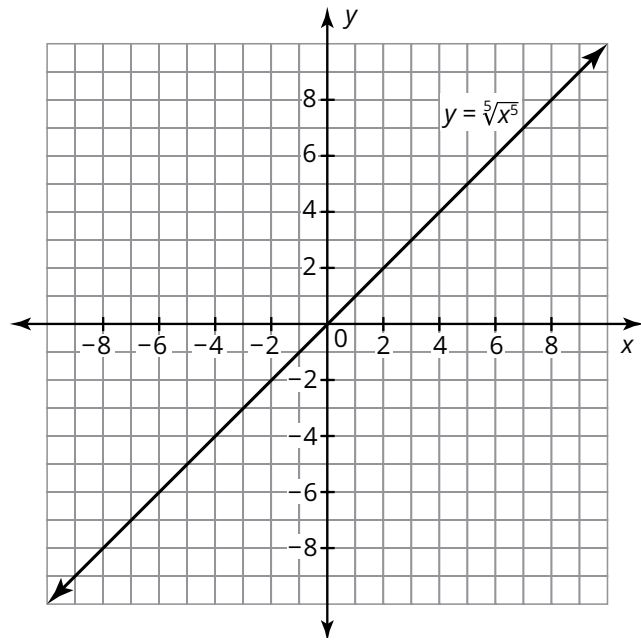


Function family of the graph: _____

Equation of the graph: _____

d. Let $n = 5$.

x	$x^n = x^5$	$\sqrt[n]{x^n} = \sqrt[5]{x^5}$
-2	-32	-2
-1	-1	-1
0	0	0
1	1	1
2	32	2



Function family of the graph: _____

Equation of the graph: _____

.....
 Review the definition
 of **analyze** in the
 Academic Glossary.

e. Analyze the representations for each value of n . What do you notice?

To extract a variable from a radical, the expression $\sqrt[n]{x^n}$ can be written as:

$$\sqrt[n]{x^n} = \begin{cases} |x|, & \text{when } n \text{ is even} \\ x, & \text{when } n \text{ is odd} \end{cases}$$

2. Explain why $\sqrt[7]{x^7} = |x|$ is incorrect for real values of x .

.....
One way to say $\sqrt[7]{x^7}$ is “the seventh root of x to the seventh power.”
.....



3. Hannah and Gabriela shared their work for extracting the root from $\sqrt{x^4}$, for real values of x .

Hannah

$$\sqrt{x^4} = |x^2|$$

Gabriela

$$\begin{aligned}\sqrt{x^4} &= \sqrt{x^2 \cdot x^2} \\ &= x^2\end{aligned}$$

Who's correct? Explain your reasoning.

Rewriting Radicals as Powers

In a previous course, you have learned how to rewrite a radical as a power with a rational exponent and to rewrite a power with a rational exponent as a radical. You can use the rules of exponents to verify the relationship between powers and roots.

WORKED EXAMPLE

You can rewrite an expression in radical form as an expression with a rational exponent. Solve the equation $\sqrt{x} = x^a$ for a , given $x \geq 0$, to determine the exponential form of $\sqrt{x}$.

Square each side of the equation.

The bases are the same, so set the exponents equal to each other and solve for a .

$$\sqrt{x} = x^a$$

$$(\sqrt{x})^2 = (x^a)^2$$

$$x = x^{2a}$$

$$1 = 2a$$

$$a = \frac{1}{2}$$

The exponential form of the square root of x , given $x \geq 0$, is x to the one-half power.

$$\sqrt{x} = x^{\frac{1}{2}}, \text{ given } x \geq 0$$

1. Why is the restriction “given $x \geq 0$ ” stated at the beginning of the Worked Example?
2. How do you know when the initial x -value can be any real number, or when the initial x -value should be restricted to a subset of the real numbers?

3. Rewrite each expression in radical form as an expression with a rational exponent.

a. The cube root of x

b. The cube root of x squared

4. Complete the cells in each row. In the last column, write “ $x \geq 0$ ” or “all real numbers” to describe the restrictions that result in equal terms for each row.

Radical Form	Radical to a Power Form	Exponential Form	Restrictions
$\sqrt[4]{x^2}$	$(\sqrt[4]{x})^2$		
		$x^{\frac{3}{4}}$	
		$x^{\frac{2}{5}}$	
$\sqrt[5]{x}$			

You can rewrite a radical expression $\sqrt[n]{x^a}$ as an exponential expression $x^{\frac{a}{n}}$:

- For all real values of x when the index n is odd.
- For all real values of x greater than or equal to 0 when the index n is even.

Extracting Roots to Rewrite Radicals

You can extract roots to rewrite radicals using radicals or powers.

Remember...

The root of a product is equal to the product of the roots of each factor: $\sqrt[p]{a^m b^n} = \sqrt[p]{a^m} \cdot \sqrt[p]{b^n}$.

WORKED EXAMPLE

Extract the roots and rewrite $\sqrt[3]{8x^6}$ using radicals and using powers.

Using Radicals

$$\begin{aligned}\sqrt[3]{8x^6} &= \sqrt[3]{2^3 \cdot x^6} \\ &= \sqrt[3]{2^3} \cdot \sqrt[3]{(x^2)^3} \\ &= \sqrt[3]{2^3} \cdot \sqrt[3]{(x^2)^3} \\ &= 2x^2\end{aligned}$$

Using Powers

$$\begin{aligned}\sqrt[3]{8x^6} &= (8x^6)^{\frac{1}{3}} \\ &= (2^3 \cdot x^6)^{\frac{1}{3}} \\ &= 2^{\frac{3}{3}} \cdot x^{\frac{6}{3}} \\ &= 2^1 \cdot x^2 \\ &= 2x^2\end{aligned}$$

1. Which method do you prefer?
2. Destiny and Amir shared their work for extracting roots from $\sqrt[4]{j^8 k^4}$.

Destiny



$$\begin{aligned}\sqrt[4]{j^8 k^4} &= \sqrt[4]{j^8} \cdot \sqrt[4]{k^4} \\ &= \sqrt[4]{(j^2)^4} \cdot \sqrt[4]{k^4} \\ &= j^2 |k|\end{aligned}$$

Amir



$$\begin{aligned}\sqrt[4]{j^8 k^4} &= (j^8 k^4)^{\frac{1}{4}} \\ &= (j^8)^{\frac{1}{4}} (k^4)^{\frac{1}{4}} \\ &= j^{\frac{8}{4}} k^{\frac{4}{4}} \\ &= j^2 |k|\end{aligned}$$

Explain why it is not necessary to use the absolute value symbol around j^2 and why it is necessary to use the absolute value symbol around k .



3. Daniela, Olivia, and Koda each extracted roots and rewrote the radical $\sqrt{x^2y^2}$.

Daniela

$$\begin{aligned}\sqrt{x^2y^2} &= \sqrt{x^2 \cdot y^2} \\ &= \sqrt{x^2} \cdot \sqrt{y^2} \\ &= |x| \cdot |y|\end{aligned}$$

Olivia

$$\begin{aligned}\sqrt{x^2y^2} &= \sqrt{x^2 \cdot y^2} \\ &= \sqrt{x^2} \cdot \sqrt{y^2} \\ &= |xy|\end{aligned}$$

Koda

$$\begin{aligned}\sqrt{x^2y^2} &= \sqrt{x^2 \cdot y^2} \\ &= \sqrt{x^2} \cdot \sqrt{y^2} \\ &= xy\end{aligned}$$

Who's correct? Explain your reasoning.

4. Rewrite each radical involving a quotient.

a. $\sqrt{\frac{m^4}{n^6}}$

b. $\sqrt[3]{\frac{j^{12}}{k^3}}$

.....
Remember ...

The quotient of powers rule states terms of the same base are divided, the exponents are subtracted: $\sqrt[n]{\frac{a^m}{a^n}} = a^{m-n}$

For some radicals, you may not be able to extract the entire radicand.

5. Madison and Jacob extracted the roots from $\sqrt[3]{16x^8}$.

Madison 

$$\begin{aligned}\sqrt[3]{16x^8} &= 2^{\frac{4}{3}} \cdot x^{\frac{8}{3}} \\ &= 2^{\frac{3}{3}} \cdot 2^{\frac{1}{3}} \cdot x^{\frac{6}{3}} \cdot x^{\frac{2}{3}} \\ &= 2 \cdot \sqrt[3]{2} \cdot x^2 \cdot \sqrt[3]{x^2} \\ &= 2x^2 \sqrt[3]{2x^2}\end{aligned}$$

Jacob 

$$\begin{aligned}\sqrt[3]{16x^8} &= \sqrt[3]{2^4 \cdot x^8} \\ &= \sqrt[3]{2^3 x^6} \cdot \sqrt[3]{2x^2} \\ &= 2x^2 \sqrt[3]{2x^2}\end{aligned}$$

a. In the last line of work, why was $2x^2$ not extracted from the radical?

b. Compare and contrast the methods.

.....
When the power and root are equal and even numbers, remember to use absolute value for the principal root.
.....

6. Rewrite each radical by extracting all possible roots. Write the final answer in radical form.

a. $\sqrt{16x^6}$

b. $-\sqrt{8v^3}$

c. $\sqrt{d^3f^4}$

d. $\sqrt{h^4j^6}$

e. $\sqrt{25a^2b^8c^{10}}$

f. $\sqrt[4]{81x^5y^{12}}$

g. $\sqrt[3]{(x+3)^9}$

h. $\sqrt{(x+3)^2}$

Operating with Radicals

Huyen and Abby multiplied $\sqrt{18a^2} \cdot 4\sqrt{3a^2}$ and extracted all roots.

1. Compare Huyen's and Abby's solution methods. Explain the difference in their solution methods.

Huyen



$$\begin{aligned}\sqrt{18a^2} \cdot 4\sqrt{3a^2} &= \sqrt[4]{54a^4} \\ &= 4\sqrt{9 \cdot 6 \cdot a^4} \\ &= 4 \cdot \sqrt{9} \cdot \sqrt{6} \cdot \sqrt{a^4} \\ &= 4 \cdot 3 \cdot \sqrt{6} \cdot a^2 \\ &= 12a^2\sqrt{6}\end{aligned}$$

Abby



$$\begin{aligned}\sqrt{18a^2} \cdot 4\sqrt{3a^2} &= \sqrt{9 \cdot 2 \cdot a^2} \cdot 4 \cdot \sqrt{3 \cdot a^2} \\ &= \sqrt{9} \cdot \sqrt{2} \cdot \sqrt{a^2} \cdot 4 \cdot \sqrt{3} \cdot \sqrt{a^2} \\ &= 3 \cdot \sqrt{2} \cdot |a| \cdot 4 \cdot \sqrt{3} \cdot |a| \\ &= 12a^2\sqrt{6}\end{aligned}$$

You have seen that the root of a product is equal to the product of the roots of each factor. This concept applies to quotients as well. The root of a quotient is equal to the quotient of the roots of the dividend and divisor: $\sqrt[p]{\frac{a^m}{b^n}} = \frac{\sqrt[p]{a^m}}{\sqrt[p]{b^n}}$.

WORKED EXAMPLE

Extract the roots to rewrite the quotient $\frac{\sqrt{18a^2}}{4\sqrt{3a^2}}$.

$$\begin{aligned}\frac{\sqrt{18a^2}}{4\sqrt{3a^2}} &= \frac{\sqrt{6 \cdot 3 \cdot \cancel{a^2}}}{4\sqrt{3 \cdot \cancel{a^2}}} \\ &= \frac{\sqrt{6}}{4}\end{aligned}$$

2. Would you get the same quotient when you extract the roots from each radical first, and then divide out common factors? Explain your reasoning.
3. Maria shared her solution for extracting roots and rewriting the quotient $\frac{\sqrt{25bc}}{\sqrt[3]{b^2c^2}}$ given $b > 0$ and $c > 0$. Explain why Maria's work is incorrect.

Maria



$$\frac{\sqrt{25bc}}{\sqrt[3]{b^2c^2}} = \frac{\sqrt{25}}{\sqrt[3]{bc}}$$

$$= \frac{5}{\sqrt[3]{bc}}$$

4. Perform each operation and extract all roots. Write your final answer in radical form.

a. $2\sqrt{x} \cdot \sqrt{x} \cdot 5\sqrt{x}$, given $x \geq 0$

b. $2(\sqrt[3]{k})(\sqrt[3]{k})$

Ask Yourself . . .

Why are the restrictions $b > 0$ and $c > 0$, instead of $b \geq 0$ and $c \geq 0$?

c. $7\sqrt{h}(3\sqrt{h} + 4\sqrt{h^3})$,
given $h \geq 0$

d. $\sqrt{a} \cdot \sqrt[3]{a}$, given $a \geq 0$

e. $(n)(\sqrt[3]{4n})(\sqrt[3]{2n^2})$

f. $\sqrt{\frac{4x^4}{x^2}}$, given $x \neq 0$

In some cases, you can rewrite the sum or difference of two terms as one term.

5. Consider how Ana, Hannah, Malik, and Gabriela rewrote the sum or difference of their original expression as one term.

Ana

$$2\sqrt{x} + 6\sqrt{x} = 8\sqrt{x}$$



Hannah

$$16\sqrt[3]{x^2} - 10\sqrt[3]{x^2} = 6\sqrt[3]{x^2}$$



Malik

$$0.1\sqrt{x} + 3.6\sqrt[3]{x} = 3.7\sqrt[3]{x}$$



Gabriela

$$\sqrt{x} + \sqrt{y} = 2\sqrt{xy}$$



a. Explain why Ana and Hannah were each able to rewrite their original expression as one term.

b. Explain why Malik's answer is incorrect.

c. Explain why Gabriela's answer is incorrect.

When adding or subtracting radicals, you can combine like terms and write the result using fewer terms.

For example, the two terms $3\sqrt{x}$ and $\sqrt{x}$ are like terms because their radicand and index, $\sqrt{x}$, are the same. The coefficients do not have to be the same.

On the other hand, the terms $-8\sqrt[3]{x^4}$ and $7\sqrt[3]{x}$ are not like terms. The indices are the same, 3, but the radicands are different, x^4 and x . Consider the exponential forms, $x^{\frac{4}{3}}$ and $x^{\frac{1}{3}}$, and notice that the bases are the same and the denominators in the exponent are the same, but the numerators in the exponents are different.

WORKED EXAMPLE

To determine the sum or difference of like radicals, add or subtract the coefficients.

$$3\sqrt{x} + \sqrt{x} = 4\sqrt{x}, \text{ given } x \geq 0$$

You can also write an equivalent expression using powers.

$$3x^{\frac{1}{2}} + x^{\frac{1}{2}} = 4x^{\frac{1}{2}}, \text{ given } x \geq 0$$

6. Luis considered whether or not $4\sqrt{x}$ and $-5x^{\frac{1}{2}}$ are like terms, given $x \geq 0$.

Luis

They are not like terms because they are not written in the same form.



Explain the error in Luis' reasoning.

7. Combine like terms, when possible, and write your final answer in radical form.

a. $\sqrt{y} - \sqrt{y}$, given $y \geq 0$

b. $9\sqrt{a} + 5\sqrt{b}$, given $a \geq 0, b \geq 0$

c. $2\sqrt{x} + \sqrt{x} + 5\sqrt{x}$, given $x \geq 0$

d. $7\sqrt{h} - 4.1\sqrt{h} + 2.4\sqrt{h}$, given $h \geq 0$

e. $3\sqrt{t}(\sqrt{t} - 8\sqrt{t}) + 4t$, given $t \geq 0$

f. $5\sqrt{g} + 2\sqrt[3]{g}$, given $g \geq 0$

Ask Yourself . . .

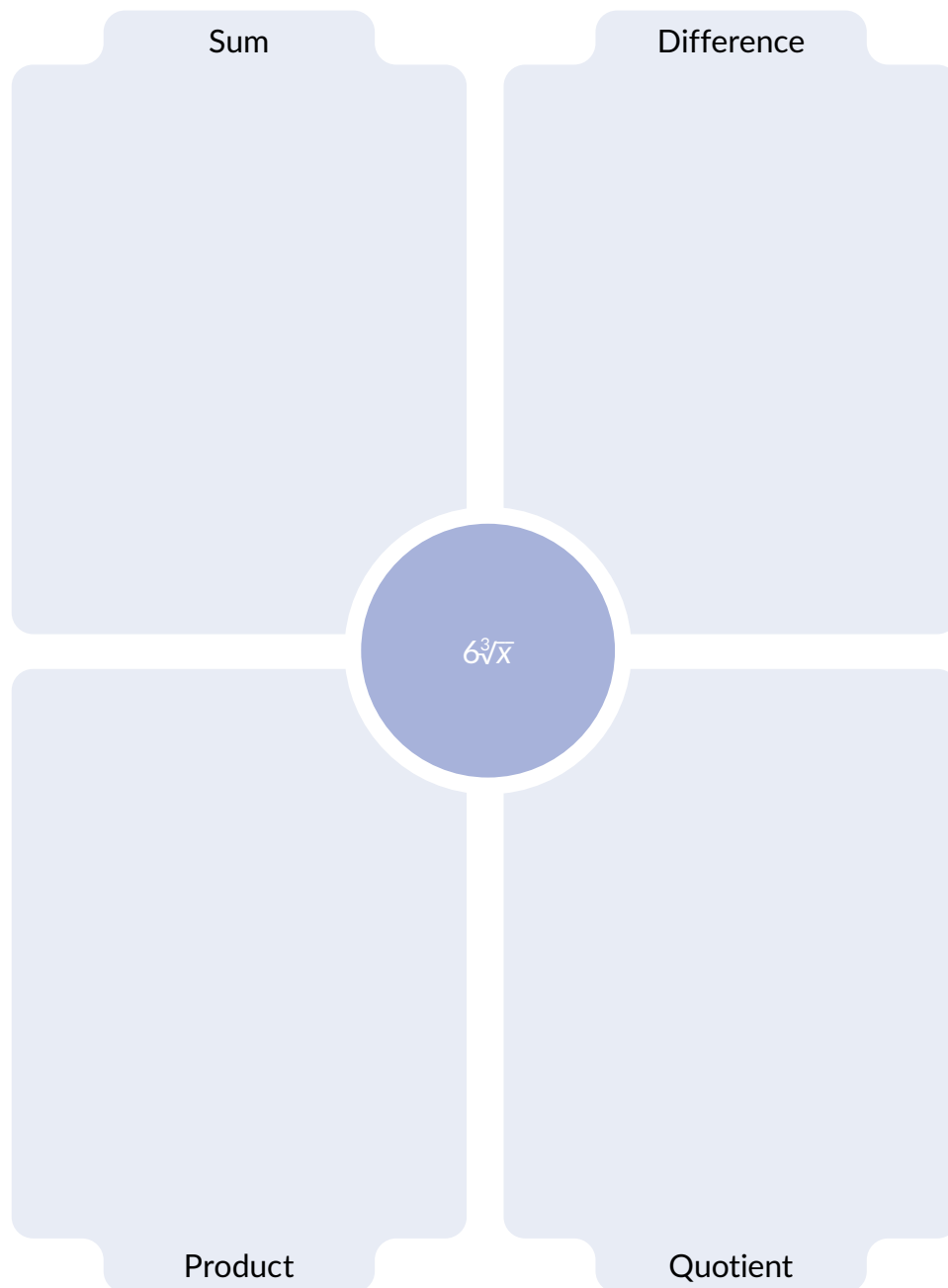
How can you organize and record your mathematical ideas?



Talk the Talk

Time to Operate!

1. Complete the graphic organizer. Write two radicals whose sum, difference, product, and quotient are each equivalent to $\sqrt[6]{x}$.





2. Identify, describe, and correct any error(s), when possible. When there are no errors, write Correct next to the problem.

a. $\sqrt{4x^2} = \sqrt{4} \cdot \sqrt{x^2}$
 $= \sqrt{2^2} \cdot \sqrt{x^2}$
 $= 2|x|$

b. $\sqrt[3]{3x^3} = (3x^3)^{\frac{1}{2}}$
 $= |x|\sqrt{3}$

c. $\sqrt[4]{16x^8y^{12}} = \sqrt[4]{16} \cdot \sqrt[4]{x^8} \cdot \sqrt[4]{y^{12}}$
 $= 4|x^2y^3|$

d. $\sqrt{81x^2} = (81x^2)^{\frac{1}{2}}$
 $= 9x$

e. $\sqrt{16x^4y} = (16x^4y)^{\frac{1}{2}}$
 $= 4|x^2|y$

Lesson 1 Assignment

Write

Explain two different instances when the expression $\sqrt[n]{x^n}$ can be written as x .

Remember

To extract a variable from a radical, the expression $\sqrt[n]{x^n}$ can be written as $|x|$ when n is even, or as x when n is odd.

A radical expression $\sqrt[n]{x^a}$ can be rewritten as an exponential expression $x^{\frac{a}{n}}$ for all real values of x if the index n is odd and for all real values of x greater than or equal to 0 if the index n is even.

Practice

1. Analyze Luis' and Abby's incorrect work. Identify the error(s) and correctly rewrite each radical after extracting all possible roots.

Luis



$$\begin{aligned}\sqrt{9a^5b^2} &= \sqrt{9} \cdot \sqrt{a^5} \cdot \sqrt{b^2} \\ &= \sqrt{9} \cdot \sqrt{a^4} \cdot \sqrt{a} \cdot \sqrt{b^2} \\ &= 3a^2b\sqrt{a}\end{aligned}$$

Abby



$$\begin{aligned}\sqrt[3]{24x^3y^6} &= \sqrt[3]{24} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{y^6} \\ &= \sqrt[3]{8} \cdot \sqrt[3]{3} \cdot |x| \cdot y^2 \\ &= 2 \cdot 1 \cdot |x| \cdot y^2 \\ &= 2|x|y^2\end{aligned}$$

2. Extract all possible roots to rewrite each expression.

a. $\sqrt{50a^3b^4}$



b. $\sqrt[4]{48x^5y^{16}}$



c. $\sqrt{(x-5)^2}$

Lesson 1 Assignment

3. Analyze Angel's and Maria's incorrect work. Identify the error(s) and correctly rewrite each radical after performing each operation and extracting all possible roots.

Angel



$$\begin{aligned}4\sqrt[3]{x^2} \cdot 5\sqrt{x^3} &= 4x^{\frac{2}{3}} \cdot 5x^{\frac{3}{2}} \\&= 20x^{\frac{6}{6}} \\&= 20x\end{aligned}$$

Maria



$$\begin{aligned}12\sqrt[3]{m^2} + 5\sqrt{m^2} - 8\sqrt[3]{m^2} &= 9\sqrt{m^2}\sqrt[3]{m^2} \\&= 9|m|\sqrt[3]{m^2}\end{aligned}$$

Lesson 1 Assignment

4. Rewrite $6\sqrt[3]{x^2}(15\sqrt[3]{x} - 13\sqrt[3]{x}) + 8x$ with the fewest terms possible.

Prepare

Determine the value of x , if possible.

1. $\sqrt{x} = 2$

2. $\sqrt[3]{x} = -2$

3. $\sqrt{x} = -2$

2

Solving Radical Equations

OBJECTIVES

- Read and apply the properties of equality when solving radical equations.
- Identify extraneous roots when solving radical equations.

KEY TERM

- extraneous solutions

.....

You have used the properties of equality to isolate the variable in many types of equations in order to solve for an unknown quantity.

How can you use these same properties to solve for an unknown quantity when the variable is under the radicand?

Getting Started

Step to It

Strategies for solving equations using the properties of equality to isolate the term containing the unknown are applicable when solving radical equations.

Let's compare the algebraic solution of a two-step quadratic equation to a two-step radical equation.

WORKED EXAMPLE

Solution Steps for a Quadratic Equation

$$2x^2 - 5 = 13$$

$$2x^2 = 18$$

$$x^2 = 9$$

$$\sqrt{x^2} = \sqrt{9}$$

$$x = \pm 3$$

Solution Steps for a Radical Equation

$$2\sqrt{x} - 5 = 13$$

$$2\sqrt{x} = 18$$

$$\sqrt{x} = 9$$

$$(\sqrt{x})^2 = (9)^2$$

$$x = 81$$

1. Analyze the Worked Example.

a. Describe the similarities in the first two steps of each solution.

b. Describe the differences in the remaining steps of each solution.

2. How would the strategy shown in the Worked Example change for cube and cube root equations? Provide an example to explain your reasoning.

Solving Radical Equations

Let's look at different strategies for solving radical equations.

You can solve a radical equation with a single variable term under the radicand, or a radical equation with more than one term under the radicand.

.....
 You should always check your answers when solving equations. However, with radical equations, it's even more important to check your answers. You'll soon learn why.

WORKED EXAMPLE

Solve $3\sqrt{x} + 7 = 25$.

$$3\sqrt{x} + 7 = 25$$

$$3\sqrt{x} = 18$$

$$\sqrt{x} = 6$$

$$(\sqrt{x})^2 = (6)^2$$

$$x = 36$$

Check:

$$3\sqrt{36} + 7 \stackrel{?}{=} 25$$

$$3(6) + 7 \stackrel{?}{=} 25$$

$$25 = 25 \checkmark$$

Solve $\sqrt{x+1} = 8$.

$$\sqrt{x+1} = 8$$

$$(\sqrt{x+1})^2 = (8)^2$$

$$x+1 = 64$$

$$x = 63$$

Check:

$$\sqrt{63+1} \stackrel{?}{=} 8$$

$$\sqrt{64} \stackrel{?}{=} 8$$

$$8 = 8 \checkmark$$

1. Analyze the Worked Examples.

a. How does the form of the equation $3\sqrt{x} + 7 = 25$ compare to the form of the equation $\sqrt{x+1} = 8$?

b. How does the form of each equation relate to the solution strategy shown?

- c. How would these strategies change for equations that are cube root equations instead of square root equations?

Raising both sides of an equation to a power may introduce an *extraneous solution*, so it is important to check your answers.

Extraneous solutions are solutions that result from the process of solving an equation, but which are not valid solutions to the equation.

2. Solve and check each equation.

a. $\sqrt{2x} = 3$

b. $\sqrt[3]{2x - 3} = 2$

c. $\sqrt{2x + 1} = 5$

d. $5\sqrt[3]{x - 6} + 2 = 7$

Remember ...

You can rewrite a radical expression $\sqrt[n]{x^a}$ as an exponential expression $x^{\frac{a}{n}}$

- For all real values of x when the index n is odd.
- For all real values of x greater than or equal to 0 when the index n is even.

e. $\sqrt{3x - 1} + 9 = 8$

3. Angel solved the equation $x - \sqrt{x} = 2$. Explain Angel's error. Give the correct solution in your explanation.

Angel



$$x - \sqrt{x} = 2$$

$$-\sqrt{x} = -x + 2$$

$$\sqrt{x} = x - 2$$

$$(\sqrt{x})^2 = (x - 2)^2$$

$$x = x^2 - 4x + 4$$

$$0 = x^2 - 5x + 4$$

$$0 = (x - 4)(x - 1)$$

$$x = 4 \text{ or } x = 1$$

The solution is $x = 4$ or $x = 1$.

4. Solve the equation $x - 1 = \sqrt{x + 1}$.

Recall that you can write radicals as fractional exponents. For example, $\sqrt{x} = x^{\frac{1}{2}}$.

Malik solved the equation $(x - 2)^{\frac{1}{3}} = 4$.

Malik

$$\begin{aligned}(x - 2)^{\frac{1}{3}} &= 4 \\ \left((x - 2)^{\frac{1}{3}}\right)^3 &= 4^3 \\ x - 2 &= 64 \\ x &= 66\end{aligned}$$

Check:

$$\begin{aligned}(66 - 2)^{\frac{1}{3}} &\stackrel{?}{=} 4 \\ (64)^{\frac{1}{3}} &\stackrel{?}{=} 4 \\ \sqrt[3]{64} &\stackrel{?}{=} 4 \\ 4 &= 4 \checkmark\end{aligned}$$



5. How is solving equations with rational exponents similar to solving radical equations?

6. Solve each equation. Check for extraneous solutions.

a. $4(x - 6)^{\frac{1}{2}} = 8$

b. $2x^{\frac{1}{3}} + 16 = 0$

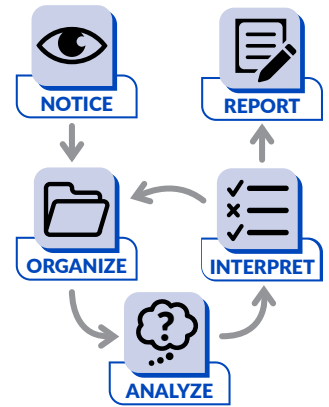
c. $x^{\frac{2}{3}} + 4 = 8$

Radical Equations in Context

The Beaufort scale is a system that measures wind speed and describes conditions at sea and on land. The scale's range is from 0 to 12. A zero on the Beaufort scale means that the wind speed is less than 1 mile per hour and the conditions at sea and on land are calm. A twelve on the Beaufort scale represents hurricane conditions with wind speeds greater than 74 miles per hour, resulting in greater than 50-foot waves at sea and severe damage to structures and landscape.

1. Consider the equation $V = 1.837B^{\frac{3}{2}}$ that models the relationship between wind speed in miles per hour, V , and the Beaufort number, B .
 - a. Solve the equation for B .
 - b. Determine the Beaufort number for a wind speed of 20 miles per hour.

PROBLEM SOLVING



In medicine, Body Surface Area, BSA, is used to help determine proper dosage for medications.

2. The equation $BSA = \frac{\sqrt{W \cdot H}}{60}$ models the relationship between BSA in square meters, the patient's weight, W , in kilograms, and the patient's height, H , in centimeters.

a. Solve the equation for H .

b. Determine the height of a patient who weighs 90 kilograms and has a BSA of 2.1.

3. The radius of a sphere is given by the equation $r = \sqrt[3]{\frac{3}{4\pi}} \cdot V$, where r is the radius of the sphere and V is the volume of the sphere. Determine the volume of a sphere with a radius of 3 inches.

The Philadelphia City Hall Clock Tower is a famous landmark in Philadelphia, Pennsylvania, standing 548 feet tall. The mechanism includes a 750-pound pendulum that continually swings back and forth.

4. The relationship between the length of the pendulum L in feet and the time it takes for a pendulum to swing back and forth one time, or its period T , is modeled by the equation $T = 2\pi\sqrt{\frac{L}{32}}$.

a. Solve the equation for L .

- b. When the pendulum's period is 4 seconds, determine the pendulum's length. Be sure to check for extraneous solutions.

A pilot is flying a plane high above the earth. She has clear vision to the horizon ahead.

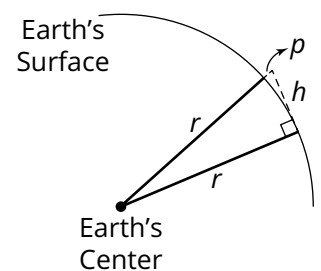
5. Consider the diagram shown. The variable r represents the Earth's radius (miles), p represents the plane's height above the earth, or altitude (miles), and h represents the distance from the pilot to the horizon (miles).

- a. Derive an equation to show the relationship between the three sides of the triangle. Then, solve the equation for the distance from the pilot to the horizon, h .

.....

The Earth is not actually a perfect sphere, but it is very close. The solution will give you a very good estimate of the distance from the pilot to the horizon.

.....



.....

Can others follow
and understand your
process and reasoning
for each situation?

.....

- b. Use your equation from part (a) to calculate the plane's altitude if the distance from the pilot to the horizon is 225 miles. The earth's radius is 3959 miles. Be sure to check for extraneous solutions.

6. A wind farm is testing a new wind turbine with a blade span that creates a swept area of 1200 m^2 . The air density at the site is 1.225 kg/m^3 , and the turbine has a power coefficient $C_p = 0.42$ meaning it converts 42% of the wind's energy into electrical power. The wind speed during a test is measured at 14 m/s.

The wind speed, v , required to generate the power output, P , in watts can be calculated using the formula $v = \sqrt[3]{\frac{2P}{1.225 \cdot 1200 \cdot 0.42}}$. Calculate the power output of the turbine during the test.



Talk the Talk

Balances and Checks

1. Solve each equation for the unknown. Check for extraneous solutions.

a. $\sqrt{2x - 1} = 7$

b. $\sqrt[3]{x} + 15 = 0$

c. $(2x + 2)^{\frac{1}{4}} + 5 = 3$

d. $(4x + 1)^{\frac{1}{3}} = 3$

Lesson 2 Assignment

Write

Explain in your own words why it is important to check all solutions when solving radical equations.

Remember

Strategies to solve equations using the properties of equality to isolate the term containing the unknown are applicable when solving radical equations. Increasing the power of the variable during the solution process may introduce extraneous solutions.

Practice

1. Analyze William's incorrect work. Identify the error(s) and correctly solve the radical equation.

William



$$\sqrt[3]{2x - 5} = 5$$

$$(\sqrt[3]{2x - 5})^3 = (5)^3$$

$$2x^3 - 5^3 = 125$$

$$2x^3 - 125 = 125$$

$$2x^3 = 250$$

$$x^3 = 125$$

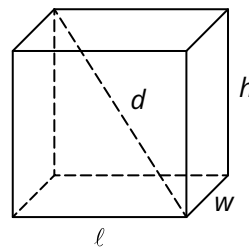
$$x = 5$$

Lesson 2 Assignment

2. Solve $x + \sqrt{x} - 9 = -3$ and check your solution.

3. Solve $9.7x^{\frac{2}{3}} = 38.8$ and check your solution.

4. The length, d , of the diagonal in a right rectangular prism can be determined using the equation $d = \sqrt{\ell^2 + w^2 + h^2}$, where ℓ represents the length, w represents the width, and h represents the height. Determine the height of a right rectangular prism with a length of 8 inches, a width of 4 inches, and a diagonal length of 12 inches. Check your solution.



Lesson 2 Assignment

Prepare

1. Use graphing technology to write a quadratic equation that represents the data given in each table.
 - a. The data shows the annual population from 2013 to 2022 of bottlenose dolphins off the coast of Texas in a certain area.

Year	Number of Bottlenose Dolphins
2013	80
2014	72
2015	77
2016	95
2017	121
2018	158
2019	196
2020	254
2021	318
2022	392

Lesson 2 Assignment

- b. The data shows the monthly cost to maintain the community garden in a certain neighborhood x number of months after its initial creation.

Number of Months	Monthly Cost (\$)
0	215
1	229
2	236
3	242
4	241
5	234
6	225
7	212
8	201
9	183

3

Modeling with Radical Functions

OBJECTIVES

- Use graphing technology to formulate square root and quadratic functions given a table of data.
- Use quadratic and square root functions to represent real-world situations, expressing reasoning in both verbal and written form..

.....

You have used graphing technology to write linear and quadratic regression functions.

How can you use graphing technology and your knowledge of inverse functions to write a square root function for a table of data?

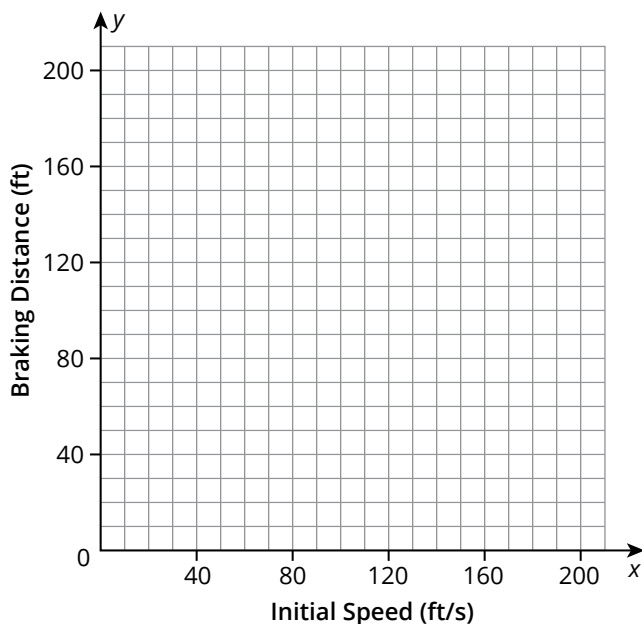
Getting Started

Brake Fast

When you ride in a fast-moving vehicle, you know that they do not stop on a dime. In other words, when the driver applies the brakes to slow the car, the car still travels some distance before coming to a stop. This distance is known as the braking distance.

Sometimes, when a car accident occurs, authorities collect evidence such as braking distance and road conditions to determine who is at fault.

On a rainy day, a scientist collected data on braking distance. He drove the same car on the same road at different speeds and then measured the distance that the car moves after braking before coming to a stop. The data is listed in the table.



Initial Speed (ft/s)	Braking Distance (ft)
0	0
10	4
20	16
30	36
40	64
50	100
60	144
70	196

1. Graph the data in the table.

2. Examine your graph. What type of function do you think would best fit the data?
Explain your answer.
3. Use graphing technology to write a regression function, $f(x)$, for the braking distance given the initial speed.
4. Use your regression function to predict the braking distance when the initial speed is 80 ft/s.
5. Use your regression function to predict the initial speed when the braking distance is 150 ft.

Writing Square Root Functions

Think about the situation from the Getting Started.

1. What are the independent and dependent quantities?
2. Identify the domain and range of the regression function based on the type of function using set notation.
3. Determine any restrictions on the domain of the regression function based on the problem situation. Write the domain of the problem situation using set notation.

The scientist wants to write a new function to determine the vehicle's initial speed given the braking distance. To do so, write the inverse function of the regression function.

4. Define the independent and dependent quantities of the inverse of $f(x)$.

5. How does the domain restriction of the problem situation affect the inverse function?

6. Algebraically determine $f^{-1}(x)$.

7. Use the table from the Getting Started to create a new table for the inverse function.

8. Use graphing technology to graph the new table of data and the inverse function $f^{-1}(x)$. Does the graph of the data match your inverse function from Question 6?

9. Determine whether each table of values represents a square root or quadratic function. Then, use graphing technology to formulate an function for each table of values. Show your work when the table of values represent a square root function.

a.

x	$g(x)$
-2	9
-1	6
0	5
1	6
2	9

b.

x	$h(x)$
0	5
1	6
4	7
9	8
16	9

c.

x	$j(x)$
0	1
4	5
9	7
16	9
25	11

d.

x	$k(x)$
-5	-84
-2	-12
-1	0
1	6
2	0



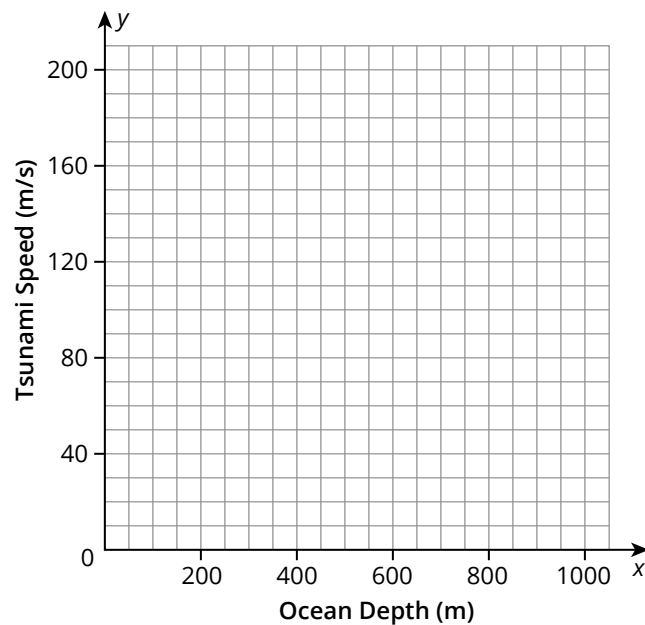
Talk the Talk

Totally Radical

A tsunami is a massive ocean wave usually caused by a seismic event, such as an underwater earthquake. A hydrologist gathers data to compare the speed of a tsunami to the depth of the water in which it is traveling. Her data is shown in the table.

Ocean depth (m)	Tsunami speed (m/s)
900	93
100	31
25	15.5
4	6.2

1. Graph the data on the coordinate plane.



2. Use graphing technology to formulate a square root function to represent the data in the table. Sketch the graph of the function on the graph.
3. According to your model, at what speed is a tsunami traveling when the depth of the ocean is 400 meters?
4. When a tsunami is traveling at a speed of 78 m/s, what is the depth of the ocean at that point?

Lesson 3 Assignment

Write

When you are using graphing technology that determines polynomial function regressions such as quadratic but not radical function regressions such as square root, how can you determine radical function regressions?

Remember

You can use quadratic regression functions to model real-world situations.

The inverse of a quadratic function with a restricted domain is a square root function.

Practice

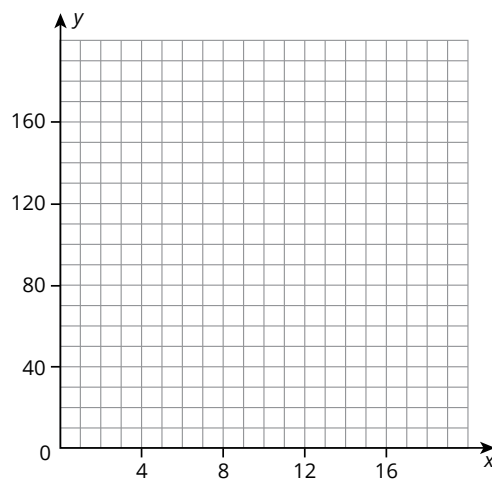
1. The number of catfish in a lake is growing as shown in the table.

Number of Months	Number of Catfish
0	50
1	52
2	58
3	68
4	82
5	100
6	122
7	148
8	178

- a. Determine any restrictions on the domain of $c(x)$ based on the problem situation. Explain your reasoning.

Lesson 3 Assignment

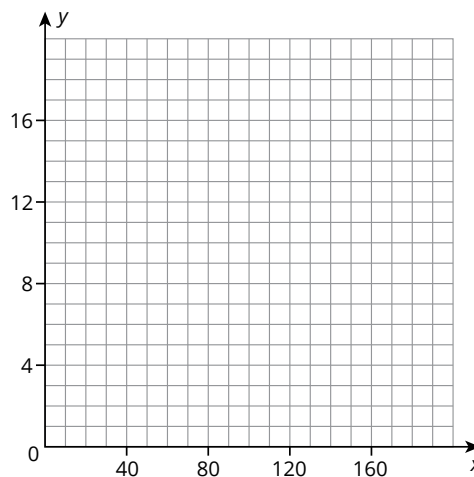
- b. Graph the data. Be sure to label your graph.



- c. Use graphing technology to formulate a quadratic function for the data. Sketch the quadratic function on the graph.

- d. What is the domain and range of the inverse of $c(x)$ with the restricted domain?

- e. Graph the inverse of $c(x)$ with the restricted domain. Be sure to label your graph.



Lesson 3 Assignment

- f. Explain why the inverse of $c(x)$ with the restricted domain is a function. Then, write an function for its inverse.

2. Determine whether each table of values represents a square root or quadratic function. Then, use graphing technology to formulate an function for each table of values. Show your work when the table of values represent a square root function.

a.

x	$f(x)$
-2	24
-1	14
0	10
1	12
2	20

b.

x	$g(x)$
0	-1
1	0
4	1
9	2
16	3

Lesson 3 Assignment

Prepare

Define each term and provide an example.

1. Arithmetic sequence
2. Geometric sequence
3. Product rule of powers

Radical Equations

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Radical Equations* by:

TOPIC 3: <i>Radical Equations</i>	Beginning of Topic	Middle of Topic	End of Topic
using radical notation to represent rational exponents and vice versa.	☐	☐	☐
identifying the appropriate unit of measure for each variable or quantity.	☐	☐	☐
explaining the properties of radical exponents using the rules of integer exponents.	☐	☐	☐
rewriting radical expressions by extracting roots.	☐	☐	☐
solving one-variable radical equations and equations with rational exponents.	☐	☐	☐
identifying extraneous solutions.	☐	☐	☐
giving examples showing how extraneous solutions may arise when solving a radical equation or an equation with rational exponents.	☐	☐	☐
using graphing technology to write quadratic and square root equations given a table of values.	☐	☐	☐

continued on the next page

TOPIC 3 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Radical Equations* topic.
2. What mathematical understandings from the topic do you feel you are making the most progress with?
3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

Radical Equations Summary

KEY TERM

- extraneous solution [solución extraña]

LESSON

1

Rewriting Radical Expressions

To rewrite a radical expression, extract the roots by using the rational exponents and the properties of powers. To extract a variable from a radical, the expression $\sqrt[n]{x^n}$ can be written as $|x|$ when n is even, and x when n is odd.

For example, you can rewrite the expression $\sqrt[4]{625x^8y^5z}$ as shown.

$$\begin{aligned}\sqrt[4]{625x^8y^5z} &= \sqrt[4]{625 \cdot x^8 \cdot y^5 \cdot z} \\ &= \sqrt[4]{625} \cdot \sqrt[4]{x^8} \cdot \sqrt[4]{y^5} \cdot \sqrt[4]{z} \\ &= 5x|y|\sqrt[4]{yz}\end{aligned}$$

You can rewrite a radical as a power with a rational exponent, and rewrite a power with a rational exponent as a radical. If n is an integer greater than 1, and, $x \geq 0$, then $\sqrt[n]{x} = x^{\frac{1}{n}}$.

For example, $\sqrt[4]{b} = b^{\frac{1}{4}}$ and $6^{\frac{1}{5}} = \sqrt[5]{6}$.

You can extract roots to rewrite radicals, using radicals or powers.

For example, you can extract the roots from $\sqrt[3]{8x^6}$ to rewrite the expression using radicals and using powers.

Using Radicals

$$\begin{aligned}\sqrt[3]{8x^6} &= \sqrt[3]{2^3 \cdot x^6} \\ &= \sqrt[3]{2^3 \cdot (x^2)^3} \\ &= \sqrt[3]{2^3} \cdot \sqrt[3]{(x^2)^3} \\ &= 2x^2\end{aligned}$$

Using Powers

$$\begin{aligned}\sqrt[3]{8x^6} &= (8x^6)^{\frac{1}{3}} \\ &= (2^3 \cdot x^6)^{\frac{1}{3}} \\ &= 2^{\frac{3}{3}} \cdot x^{\frac{6}{3}} \\ &= 2^1 \cdot x^2 \\ &= 2x^2\end{aligned}$$

The root of a product is equal to the product of its roots, $\sqrt[p]{a^m b^n} = \sqrt[p]{a^m} \cdot \sqrt[p]{b^n}$.

For example, you can extract roots to rewrite the product $\sqrt{18a^2} \cdot 4\sqrt{3a^2}$.

$$\begin{aligned}\sqrt{18a^2} \cdot 4\sqrt{3a^2} &= \sqrt[4]{54a^4} \\ &= 4\sqrt{9 \cdot 6 \cdot a^4} \\ &= 4 \cdot 3 \cdot a^2 \sqrt{6} \\ &= 12a^2 \sqrt{6}\end{aligned}$$

The root of a quotient is equal to the quotient of its roots, $\sqrt[p]{\frac{a^m}{b^n}} = \frac{\sqrt[p]{a^m}}{\sqrt[p]{b^n}}$.

For example, you can extract the roots to rewrite the quotient $\frac{\sqrt{18a^2}}{4\sqrt{3a^2}}$.

$$\begin{aligned}\frac{\sqrt{18a^2}}{4\sqrt{3a^2}} &= \frac{\sqrt{6 \cdot 3a^2}}{4\sqrt{3a^2}} \\ &= \frac{\sqrt{6}}{4}\end{aligned}$$

In some cases, you can rewrite the sum or difference of two terms as one term. When adding or subtracting radicals, you can combine like terms and write the result using fewer terms.

For example, the two terms, $3\sqrt{x}$ and $\sqrt{x}$ are like terms because their radicand and index, $\sqrt{x}$, are the same. The coefficients do not have to be the same. On the other hand, the terms $-8\sqrt[3]{x^4}$ and $7\sqrt[3]{x}$ are not like terms. The indices are the same, 3, but the radicands are different, x^4 and x^3 .

To determine the sum or difference of like radicals, add or subtract the coefficients.

$$3\sqrt{x} + \sqrt{x} = 4\sqrt{x}, \text{ given } x \geq 0$$

You can also write an equivalent expression using powers.

$$3x^{\frac{1}{2}} + x^{\frac{1}{2}} = 4x^{\frac{1}{2}}, \text{ given } x \geq 0$$

LESSON

2

Solving Radical Equations

To solve a radical equation, isolate the radical term if possible. Then, raise the entire equation to the power that will eliminate the radical. Finally, follow the steps necessary to solve the equation and check for *extraneous solutions*.

Extraneous solutions are solutions that result from the process of solving an equation, but which are not valid solutions to the equation.

For example, consider the solution steps to solve $\sqrt{x+2} + 10 = x$.

$$\sqrt{x+2} + 10 = x$$

Check:

Check:

$$\sqrt{x+2} = x - 10$$

$$\sqrt{14+2} + 10 \stackrel{?}{=} 14$$

$$\sqrt{7+2} + 10 \stackrel{?}{=} 14$$

$$(\sqrt{x+2})^2 = (x-10)^2$$

$$\sqrt{16} + 10 \stackrel{?}{=} 14$$

$$\sqrt{9} + 10 \stackrel{?}{=} 14$$

$$x+2 = x^2 - 20x + 100$$

$$14 = 14 \checkmark$$

$$13 \neq 14$$

$$0 = x^2 - 21x + 98$$

Extraneous solution

$$0 = (x-14)(x-7)$$

$$x = 14 \text{ or } x = 7$$

There is one solution, $x = 14$.

To solve a problem with radical equations, identify what the problem is asking. Then, determine how to use the given equation to solve the problem. Finally, follow the process for solving radical equations.

For example, consider the following problem situation.

Big Ben is the nickname of a well-known clock tower in London, England, that stands 316 feet tall. The clock is driven by a 660-pound pendulum in the tower that continually swings back and forth. The relationship between the length of the pendulum, L , in feet and the time it takes for a pendulum to swing back and forth one time, or its period T , is modeled by the equation

$T = 2\pi\sqrt{\frac{L}{32}}$. If the pendulum's period is 4 seconds, solve for L .

$$4 = 2\pi\sqrt{\frac{L}{32}}$$

$$\frac{4}{2\pi} = \sqrt{\frac{L}{32}}$$

$$\left(\frac{2}{\pi}\right)^2 = \left(\sqrt{\frac{L}{32}}\right)^2$$

$$\frac{4}{\pi^2} = \frac{L}{32}$$

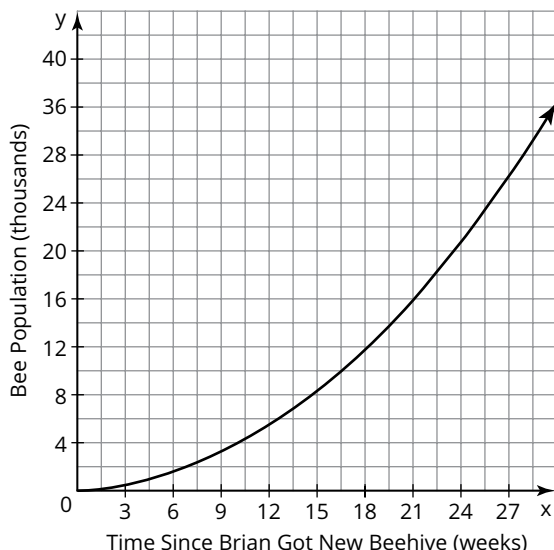
$$4 \cdot 32 = \pi^2 \cdot L$$

$$L = \frac{128}{\pi^2} \approx 12.97$$

The length of the pendulum is approximately 12.97 feet.

You can use graphing technology and your knowledge of inverses to write quadratic and square root equations given a table of values.

For example, Jacob has a new beehive. He records the number of bees in the hive after x weeks and uses graphing technology to write an equation to represent the number of bees in the hive. The regression function $b(x) = 36x^2$ represents the data shown on the graph.



You can write the inverse function $b^{-1}(x)$ to create a function that will determine the number of weeks given the bee population.

$$y = 36x^2 \rightarrow x = 36y^2$$

$$\frac{x}{36} = y^2$$

$$\sqrt{\frac{x}{36}} = \sqrt{y^2}$$

$$\pm \frac{\sqrt{x}}{6} = y$$

$$b^{-1}(x) = \frac{\sqrt{x}}{6}$$

You can verify your inverse function is correct by reversing the independent and dependent quantities and using graphing technology to write the equation.

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