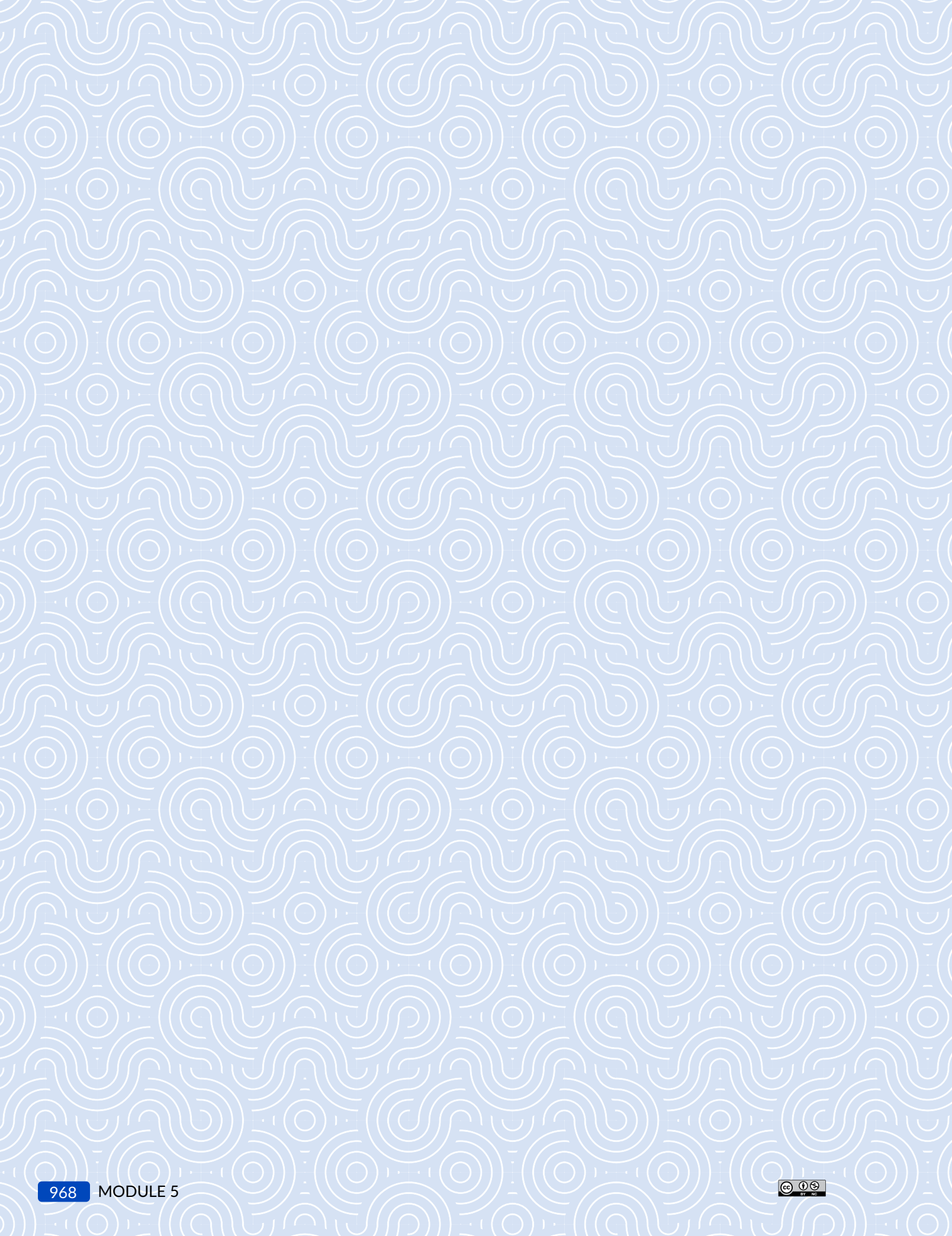
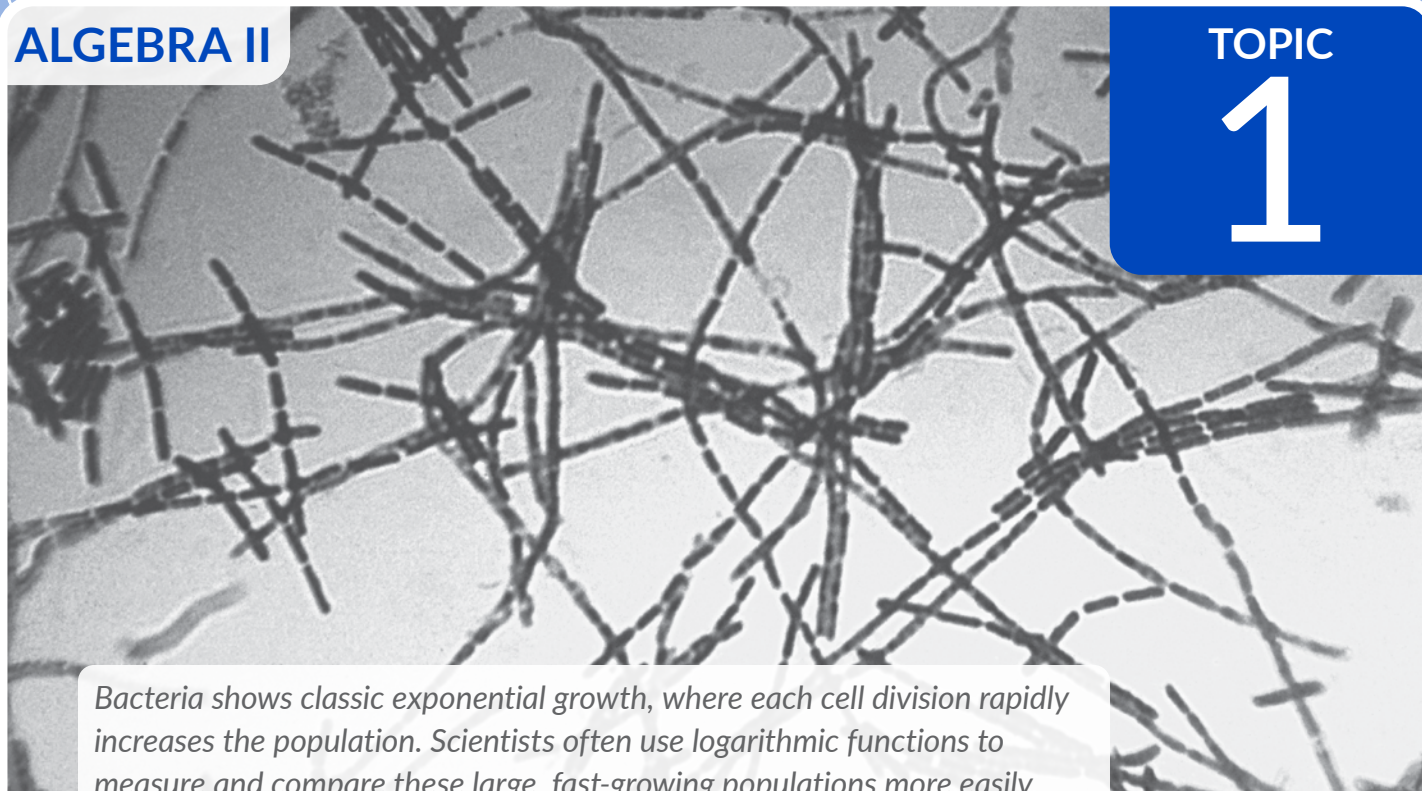


Exploring Exponentials and Logarithmics

TOPIC 1	Exponential and Logarithmic Functions	969
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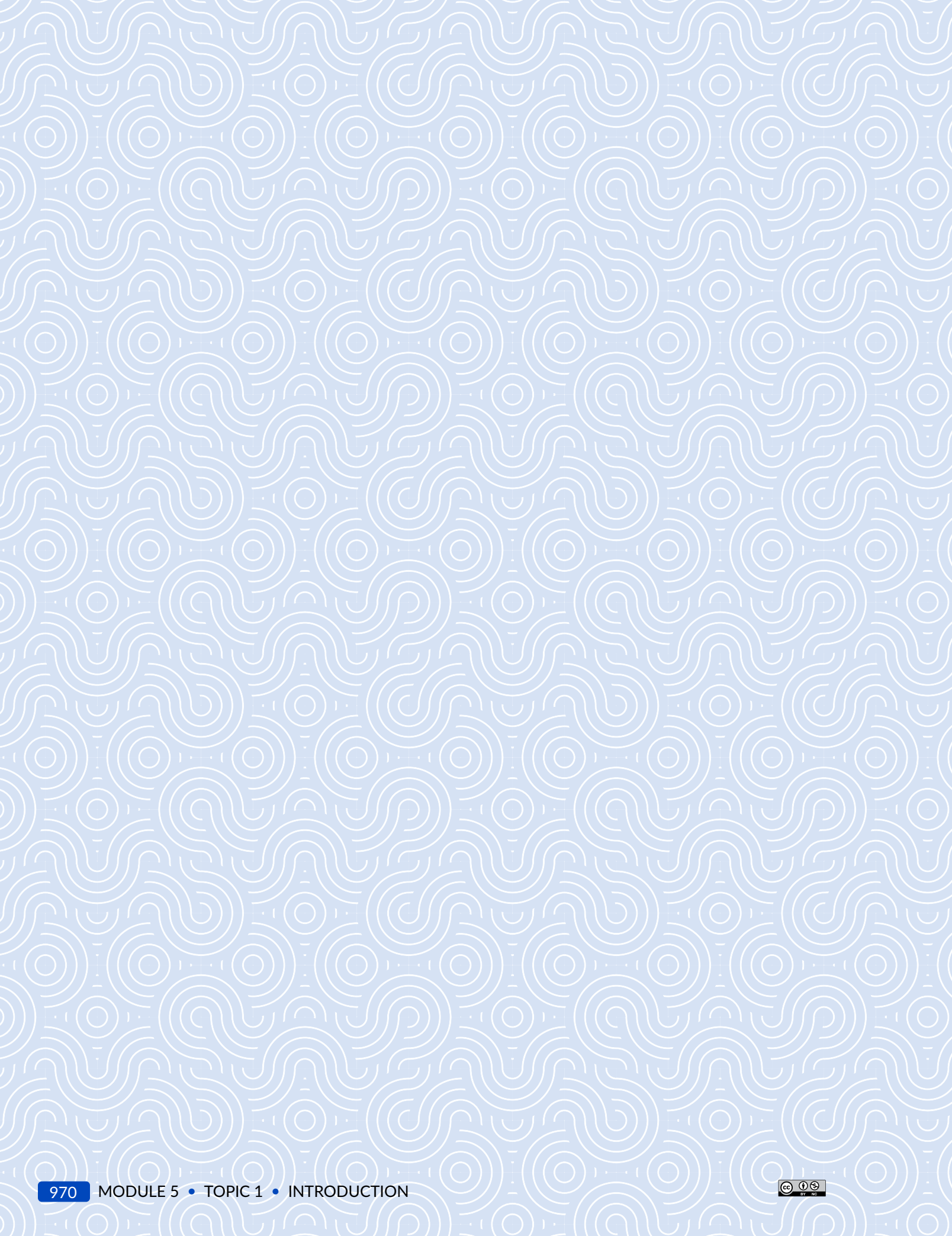




Bacteria shows classic exponential growth, where each cell division rapidly increases the population. Scientists often use logarithmic functions to measure and compare these large, fast-growing populations more easily.

Exponential and Logarithmic Functions

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1

Comparing Linear and Exponential Functions

OBJECTIVES

- Construct an exponential function from a geometric sequence.
- Classify functions as exponential growth or decay.
- Compare tables, graphs, and equations of exponential functions.
- Compare linear and exponential functions in real-world contexts and write an explanation of how they differ.

KEY TERM

- half-life

.....

You have studied exponential functions in previous courses.

How can you use exponential functions to model real-world situations?

Getting Started

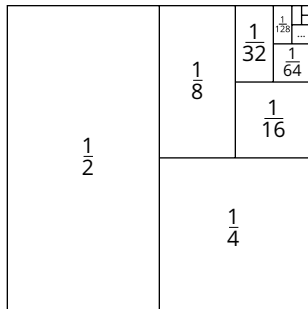
Asymptotic, Don't You Think?

When you work with sequences, you often use something called *inductive reasoning*. This means that you notice when a pattern repeats and think about how a pattern will extend when repeated.

Remember ...

Inductive reasoning is a process of arriving at a conclusion by generalizing a statement from observing patterns.

1. A square is shown. Use a ruler to draw a line segment that divides the square in half. Then draw a second line segment in just one of the two halves you created to divide that section in half. Continue like this until the divisions become too small to make.



Remember ...

The explicit formula for a geometric sequence is given by $a_n = a_1 \cdot r^{(n-1)}$, where a_n represents the n th term of the sequence, a_1 represents the first term of the sequence, and r represents the common ratio, or constant multiplier.

2. Compare your drawing with your classmates' drawings. What is the area of the smallest section shown in your square if A is the area of the original square?
3. Write the explicit formula you could use to determine the area of the smallest section. Explain your reasoning.
4. What would happen to the area of the smallest section if you could keep dividing the square forever?

ACTIVITY 1.1

Writing Exponential Equations Using Recursive Notation

Recall that a *geometric sequence* is a sequence of numbers in which the ratio between any two consecutive terms is a constant. In other words, it is a sequence of numbers in which you multiply each term by a constant to determine the next term. This integer or fraction constant is called the *common ratio*.

Under ideal conditions, a certain type of bacteria doubles every hour.

1. The sequence shown represents the population of bacteria.

1, 2, 4, 8, 16, ...

- a. Describe why this sequence is geometric and identify the common ratio.

- b. Complete the table of values. Use the number of hours to identify the term number and the total number of bacteria after each hour.

Number of Doubling Hours	Term Number (n)	Total Number of Bacteria
0	1	
1		
2		
3		
4		
5		
6		
7		
8		
9		

.....
Notice that the 1st term in this sequence is the total number of bacteria after 0 hours.
.....

- c. Explain how you can calculate the tenth term based on the ninth term.

In Algebra I, you used the recursive formula for a geometric sequence. The recursive formula to determine the n th term of a geometric sequence is:

$$\begin{array}{ccc}
 \begin{array}{c} \text{nth} \\ \text{term} \end{array} & & \begin{array}{c} \text{common} \\ \text{ratio} \end{array} \\
 \swarrow & & \swarrow \\
 a_n = \underbrace{a_{n-1}}_{\text{previous term}} \cdot r
 \end{array}$$

Consider the number of hours and the total number of bacteria.

2. Write an exponential equation in recursive form for the sequence and use the equation to determine the 12th term in the sequence. Explain what your result means in terms of this problem situation.

Recall from the Getting Started, the explicit formula to determine the n th term of a geometric sequence is:

$$a_n = a_1 \cdot r^{n-1}$$

Diagram illustrating the components of the explicit formula for the n th term of a geometric sequence:

- a_n : n th term
- a_1 : 1st term
- r : common ratio
- $n-1$: previous term number

3. Write an exponential equation in explicit form and use the formula to determine the number of bacteria after 11 hours.

4. Mario is competing in a giant cake-cutting challenge at a local festival. Each contestant starts with the biggest cake they can bake, and Mario's cake has an 8-foot diameter! After the contest, he decides to slice the cake into pieces to share with the crowd. He starts with 1 whole cake. First, he cuts it into thirds. Then, he cuts each of those slices into thirds again. He continues this pattern, cutting each existing slice into three smaller pieces every time.
- a. Write an exponential equation in recursive and an explicit forms for the sequence representing the fraction of the original cake in each piece after n cuts.
- b. Determine the size of the slice compared to the whole cake after 4 cuts.

Arithmetic and Geometric Sequences

Natalia and Kaya each receive \$10 per week for doing chores for their neighbor. One day, Natalia decides to increase her income using her knowledge of exponential growth. She proposes to her neighbor that her payment change to a penny for the first week, and then double each week thereafter.

1. Complete the table to represent the amount that Natalia and Kaya will earn each week.

Week	Natalia's Income (dollars)	Kaya's Income (dollars)
1	0.01	10.00
2		
3		
4		
5		
6		
7		
8		

.....

Recall an *arithmetic sequence* is a sequence of numbers in which the difference between any two consecutive terms is a constant. In other words, it is a sequence of numbers in which a constant is added to each term to produce the next term. This constant is called the common distance.

.....

2. How does Natalia's income change as the number of weeks increases?
3. Does Natalia's income represent an arithmetic or geometric sequence? Explain your reasoning.
4. Write an equation to represent Natalia's income, g_n , after n weeks in recursive and explicit forms.

5. What is the value of g_n for $n = 0$? Does this value make sense in the problem situation?
6. If the pattern were to continue, how many weeks would it take for Natalia to have a greater weekly income than Kaya?

Week	Natalia's Income (dollars)	Kaya's Income (dollars)

WORKED EXAMPLE

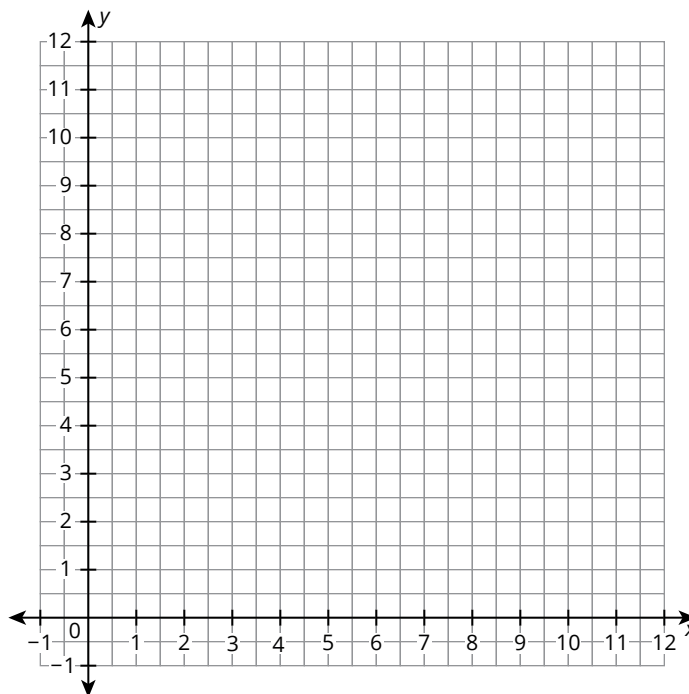
You can write the explicit formula for the geometric sequence $g_n = 0.01 \cdot 2^{(n-1)}$ as a function in the form $f(x) = ab^x$, using the properties of powers.

Statement	Reason
$g_n = 0.01 \cdot 2^{(n-1)}$	Explicit formula for the geometric sequence
$f(n) = 0.01 \cdot 2^{(n-1)}$	Rewrite in function notation
$f(n) = 0.01 \cdot 2^n \cdot 2^{-1}$	Product of powers rule
$f(n) = 0.01 \cdot 2^n \cdot \frac{1}{2}$	Definition of negative exponents
$f(n) = 0.01 \cdot \frac{1}{2} \cdot 2^n$	Commutative property of multiplication
$f(n) = 0.005 \cdot 2^n$	Perform multiplication
So, $g_n = 0.01 \cdot 2^{(n-1)}$ can be written as $f(n) = 0.005 \cdot 2^n$.	

Remember ...

Some geometric sequences, when written in function notation, are exponential functions. The function gets its name from the variable in the exponent.

7. Calculate the income that Natalia would earn per week in each week given:
- a. 15th week
 - b. 20th week
 - c. 24th week
8. Predict the shape and key attributes of the graph that models Natalia's income as a function of the number of weeks.



Linear and Exponential Models

Kaya is amazed at how quickly Natalia was able to make a lot of money and decides that she wants in on the action. She asks her two friends, Andrew and Samantha, to help her come up with a plan.

1. Consider Andrew's and Samantha's plans. Whose plan should Kaya choose? Complete the table and graph to justify your reasoning. Round to the nearest hundredth.

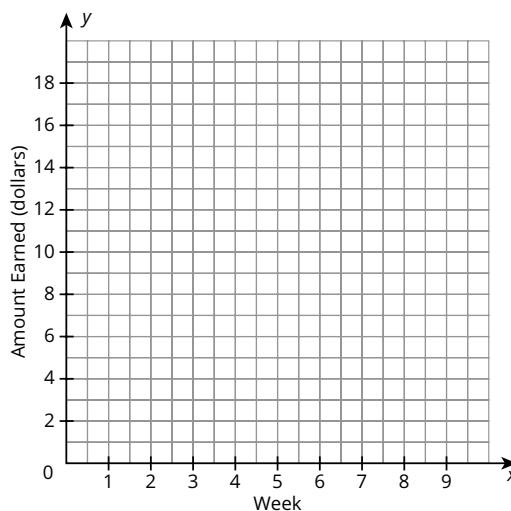
Andrew

You could start with a dollar and ask for 50% more each week.

Samantha

You could start with a dollar and add another dollar each week.

Week	Andrew's Plan (dollars)	Samantha's Plan (dollars)
1		
2		
3		
4		
5		
6		
7		



2. Write functions to represent Andrew's plan, $q(x)$, and Samantha's plan, $a(x)$.

3. Use your choice from Question 1 to determine how much Kaya will earn in Week 10.

4. If Kaya and Natalia both start using their exponential model to earn income at the same time, who will earn a higher income in Week 12?

5. Use technology to determine when Natalia's and Kaya's incomes will be equal. Does this make sense in the problem situation? Explain your reasoning.

6. Compare Natalia's and Kaya's function models.
 - a. As the number of weeks continues to increase, whose model will generate more per week?

 - b. Consider the a - and b -values of the exponential functions. How do they further support your claim?

.....
Remember...
 The standard form
 of an exponential
 equation is $y = a \cdot b^x$.

ACTIVITY
1.4

Modeling Situations with Exponential Functions

Alejandro is studying for a big test and is trying to stay awake. He drank a cup of coffee that contained 80 milligrams of caffeine. He is wondering how long the caffeine will stay in his system if the caffeine has a *half-life* of 5 hours.

A **half-life** is the amount of time it takes a substance to decay to half of its original amount.

- How much caffeine remains in Alejandro's system after 5 hours? After 10 hours? Explain your reasoning.
- Complete the table to determine the amount of caffeine in Alejandro's system at each time interval.

Time Elapsed (hours)	0	5	10	15	20
Caffeine in System (mg)					
Number of Half-Life Cycles					

- What is the initial amount of caffeine in Alejandro's system? What is the rate of decay?

PROBLEM SOLVING



.....

The term *half-life* does not just apply to radioactive material. It is also used in medicine to describe how long a chemical exists in the human body.

.....

4. Trung, Eduardo, and Paola were asked to write an exponential function $A(t)$ to represent the amount of caffeine remaining in Alejandro's system after t hours.

Trung
 $A(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{5}}$



The variable t represents the number of hours, and the half-life occurs in 5 hour cycles, so I divided my exponent by 5.

Eduardo
 $A(t) = 80\left(\frac{1}{2}\right)^{-\frac{t}{5}}$



The variable t represents the number of hours, and since it's a decay function, I made my exponent negative.

Paola

$$A(t) = 80\left(\frac{1}{2}\right)5t$$



The variable t represents the number of hours, and I multiplied it by 5 to represent the half-life cycle of 5 hours.

.....

Think about...

It may be helpful to substitute the values from the table to check each student's function.

.....

- a. Why is Eduardo's reasoning incorrect?

- b. Why is Paola's reasoning incorrect?

5. How much caffeine remains in Alejandro's system after 2 hours?



6. Elena suggests that she can calculate the amount of caffeine remaining by rewriting the equation as $A(t) = 40^{\frac{t}{5}}$. Is Elena correct? Explain your reasoning.

7. Use technology to predict when the caffeine will be completely out of Alejandro's system. Does this make sense, given what you know about exponential functions? Explain your reasoning.
8. Approximately when will the amount of caffeine remaining in Alejandro's system be less than 1 milligram?
9. Use the properties of exponents to rewrite your function so that only the variable t is in the exponent. What percentage of caffeine remains after each hour?
10. Suppose Alejandro is taking an antibiotic that extends the half-life of caffeine to 8 hours. Write a function $B(t)$ that models the amount of caffeine remaining under these new conditions.
11. Complete the table for the new half-life. Round to the nearest hundredth.

Time Elapsed (hours)	0	5	10	15	20
Caffeine in System (mg)					
Number of Half-Life Cycles					

12. How does the half-life of 8 hours rather than 5 hours affect the amount of caffeine remaining in Alejandro's system?
13. Under these new conditions, approximately when will the amount of caffeine remaining in Alejandro's system be less than 1 milligram?
14. What generalization can you make about the effect of greater or lesser half-lives on substances?



Talk the Talk

Exponentials and Dating

Carbon-14 is a radioactive isotope used to date the age of plants and animals. It has a half-life of about 5730 years. While a plant or animal is alive, the amount of carbon-14 present is constant, but when the plant or animal dies, the amount of carbon-14 begins to decay. To determine the length of time since the plant or animal died, you measure the percent of carbon-14 remaining and apply the decay function.

1. Write the decay function for carbon-14, using the value 100 as the initial amount present. The unit for 100 is not important because you will express the amount of remaining carbon-14 as a percent.

2. Use your function to calculate the percentage of carbon-14 remaining after:
 - a. 100 years.

 - b. 1000 years.

 - c. 10,000 years.

3. A park ranger is monitoring the growth of a newly planted oak tree in a national park. In its first year, the tree grows to 2 feet tall. Due to ideal growing conditions, the tree grows to 1.5 times its height from the previous year.
- a. Write an exponential equation in both recursive and explicit forms to represent the height of the tree after n years.
- b. Determine the height of the tree after 10 years.

Lesson 1 Assignment

Write

Define the term *half-life* in your own words.

Remember

The recursive formula for a geometric sequence is $g(n) = g_{n-1} \cdot r$, where g_n is the n th term and r is the common ratio.

The standard form of an exponential equation is $y = a \cdot b^x$ where $b > 0$ and $b \neq 1$.

Practice

1. Wildlife biologists are studying the coyote populations on two wildlife preserves to better understand the role climate plays in population change.

The table displays the coyote populations on both preserves for each year of the study. The Year 0 corresponds to the date of the biologists' initial observations.

Year	Alaska Preserve	Tennessee Preserve
0	200	80
1	220	100
2	242	125
3		
4		

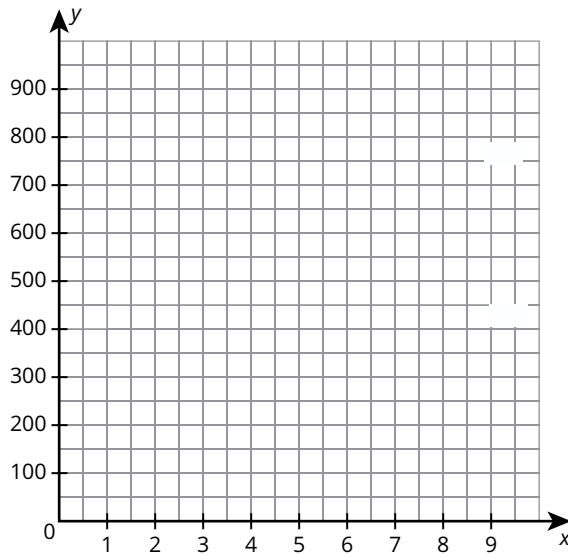
- a. Can the population be represented by an arithmetic or a geometric sequence? Explain your reasoning.

Lesson 1 Assignment

- b. For each wildlife preserve, write a sequence to represent the coyote population in a given year.
- c. For each wildlife preserve, write an exponential equation in recursive and explicit form to represent the coyote population as an exponential equation in recursive and explicit form of the year of the study.
- d. Use the functions you wrote in part (c) to complete the table. Round all answers to the nearest whole number.

Lesson 1 Assignment

e. Graph and label the functions you wrote in part (c).



- f. Do the populations represent examples of exponential decay, exponential growth, or neither? Explain your reasoning.
- g. Will the coyote population on the Tennessee preserve ever exceed the coyote population on the Alaska preserve? If so, when will this occur?
- h. Make a hypothesis about the role the climate plays on coyote populations based on the results of the study, assuming all other population growth factors are equal. Explain your reasoning.

Lesson 1 Assignment

Prepare

Juno is opening her first savings account and is depositing \$100. The bank offers 3% annual interest to be calculated at the end of each year.

1. Write a function $A(t)$ to model the amount of money in Samantha's savings account after t years.
2. Calculate the amount of money in Samantha's savings account at the end of 1 year.
3. Calculate the amount of money in Samantha's savings account at the end of 5 years.

2

Properties of Exponential Graphs

OBJECTIVES

- Identify the domain and range of exponential functions.
- Investigate and describe graphs of exponential functions through intercepts, asymptotes, and intervals of increase and decrease.
- Explore the irrational number e .

KEY TERM

- natural base e

.....

You have calculated interest compounded annually using an exponential equation in the form $A = P \cdot (1 + r)^t$.

How does this equation change when interest is compounded at more frequent intervals? Can interest be compounded continuously?

Getting Started

You Have the Power

1. Cut out the exponential graphs and equations located at the end of the lesson. Match each equation to its graph. Sort them into growth or decay functions, and attach them onto the graphic organizer located at the end of the lesson. Then, complete each table. Write the domain and range using inequalities.

.....

Think about...

How does the base
affect the graph of an
exponential function?

.....

2. Analyze the exponential growth and decay functions.
 - a. What point do the graphs have in common? Why?
 - b. Compare the equations of the six functions you just sorted. What differentiates an exponential growth function from an exponential decay function?

ACTIVITY
2.1

Key Attributes of Exponential Functions

Consider the equations of the six functions you sorted in the Getting Started.

1. Mariana and Chris's teacher asked them to each write a rule that determines whether a function represents exponential growth or decay, based on its equation.

Mariana



For exponential growth functions, b is a value greater than 1, but for exponential decay functions, b is a fraction or decimal between 0 and 1.

Chris



For exponential growth functions, b is greater than 1.
For exponential decay functions, b is less than 1.

Why is Chris's reasoning incorrect? Provide a counterexample that would disprove his claim and explain your reasoning.

2. Which b -values for the function $f(x) = b^x$ produce neither growth nor decay? Provide examples to support your answer.

Ask Yourself . . .

Do you have enough information to write a unique exponential function?

3. Write an exponential function that satisfies the given attributes.

a. Increasing over $(-\infty, \infty)$

Reference point $(1, 6)$

b. Decreasing over $(-\infty, \infty)$

Reference point $(-1, 4)$

4. Summarize the key attributes for the parent exponential growth and exponential decay functions.

	Exponential Parent Function	
	Exponential Growth $f(x) = b^x, b > 1$	Exponential Decay $f(x) = b^x, 0 < b < 1$
Domain in interval notation		
Range in interval notation		
Asymptote		
Intercepts		
Intervals of Increase or Decrease		

ACTIVITY 2.2

The Natural Base e

Ashley is opening her first savings account and is depositing \$500. Suppose she decides on a bank that offers 6% annual interest to be calculated at the end of each year.

1. Write a function $A(t)$ to model the amount of money in Ashley's savings account after t years, and determine the amount in the account at the end of 1 year and 5 years.
2. Suppose that the bank decides to start compounding interest at the end of every 6 months. If they still want to offer 6% per year, how much interest would they offer per 6-month period?

Remember...

The formula for compound interest is $A = P \cdot \left(1 + \frac{r}{k}\right)^{kt}$, where P represents the initial amount, r is the interest rate, k is the number of compounding periods per year, and t represents time in years.

3. Jorge, Tiara, and Lizzie were each asked to calculate the amount of money Ashley would have in her savings at the end of the year if interest was compounded twice a year. Who's correct? Explain your reasoning.

Jorge

1st 6 months:

$$A(t) = 500(1 + 0.03)$$

$$A(t) = 515$$

2nd 6 months:

$$A(t) = 515(1 + 0.03)$$

$$A(t) = 530.45$$

Tiara

$$A(t) = 500 \left(1 + \frac{0.06}{2}\right)^2$$

$$A(t) = 530.45$$

Lizzie

$$A(t) = 2(500(1 + 0.03))$$

$$A(t) = 1030$$



4. Write a function to model the amount of money in Ashley's savings account at the end of t years, compounded k times during the year.

5. Determine the amount of money in Ashley's account at the end of 3 years if it is compounded:
 - a. twice a year.

 - b. monthly.

 - c. daily.

6. What effect does the frequency of compounding have on the amount of money in her savings account?

Recall that in Question 4 the variable k represented the number of compounding periods per year. Let's examine what happens as the interest is compounded more frequently.

7. Imagine that Ashley finds a different bank that offers her 100% interest. Complete the table to calculate how much Helen would accrue in 1 year for each period of compounding if she starts with \$1.

.....
Think about ...
 Even though you are working with money, to see the pattern, it is helpful to compute the compound interest to at least six decimal places.

Period of Compounding	k	Formula	Amount
Yearly	1	$1\left(1 + \frac{1}{1}\right)^{1 \cdot 1}$	2.00
Semi-Annually	2	$1\left(1 + \frac{1}{2}\right)^{2 \cdot 1}$	2.25
Quarterly	4	$1\left(1 + \frac{1}{4}\right)^{4 \cdot 1}$	
Monthly	12		
Weekly			
Daily			
Hourly			
Every Minute			
Every Second			

8. Make an observation about the frequency of compounding and the amount that Ashley earns. What amount is it approaching?

.....

You have worked with π , an irrational number that was approximated as 3.14159265... Pi is an incredibly important part of many geometric formulas and occurs so frequently that, rather than write out "3.14159265..." each time, you use the symbol π .

.....

The amount that Ashley's earnings approach is actually an irrational number called e .

$$e < 2.718281828459045 \dots$$

It is often referred to as the **natural base e** .

The symbol e is used to represent the constant 2.718281... It is often used in models of population changes as well as radioactive decay of substances, and it is vital in physics and calculus.

The symbol for the natural base e was first used by Swiss mathematician Leonhard Euler in 1727 as part of a research manuscript he wrote at age 21. In fact, he used it so much, e became known as Euler's number. The constant e represents continuous growth and has many other mathematical properties that make it unique, which you will study further in calculus.

In Question 7, you showed that the expression $\left(1 + \frac{1}{k}\right)^k$ approaches e as k approaches infinity. You can use this fact to derive the formula for compound interest with continuous compounding.

WORKED EXAMPLE

For compounded interest, the amount in an account, A , can be represented with the equation $A = P \cdot \left(1 + \frac{r}{k}\right)^{kt}$, where P represents the principal, or starting amount, r represents the interest rate, k represents the number of compounding periods per year, and t represents time in years. You can rewrite the compound interest formula using algebraic reasoning.

$$\begin{aligned}
 P \cdot \left(1 + \frac{r}{k}\right)^{kt} &= P \cdot \left(\left(1 + \frac{r}{k}\right)^k\right)^t && \text{Reasoning } a^{mn} = (a^m)^n \\
 &= P \cdot \left(\left(1 + \frac{r}{k}\right)^{\frac{k}{r}}\right)^{rt} && (a^m)^n = (a^{\frac{m}{b}})^{bn} \\
 &= P \cdot \left(\left(1 + \frac{1}{\frac{k}{r}}\right)^{\frac{k}{r}}\right)^{rt} && \frac{a}{b} = \frac{1}{\frac{b}{a}}
 \end{aligned}$$

Consider the expression $\left(1 + \frac{1}{\frac{k}{r}}\right)^{\frac{k}{r}}$. If $\frac{k}{r} = n$, then the expression can be written as $\left(1 + \frac{1}{n}\right)^n$, which approaches e as n approaches infinity.

Thus, the formula for compound interest with continuous compounding can be written as:

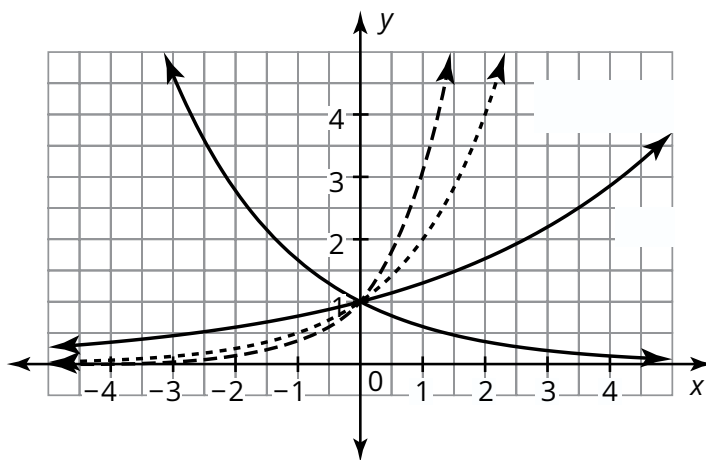
$$A = Pe^{rt}$$

9. Suppose Ashley's investment of \$500 in her savings account is compounded continuously at a rate of 6%. How much would be in her account at the end of 1 year and 5 years?

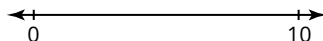
10. The given functions are sketched on the coordinate plane shown.

$$f(x) = 2^x \quad g(x) = 3^x \quad j(x) = \left(\frac{3}{5}\right)^x \quad k(x) = 1.3^x$$

- a. Label each function.



- b. Consider the function $m(x) = e^x$. Use your knowledge of the approximate value of e to sketch its graph. Explain your reasoning.
- c. Using the functions $f(x) = 2^x$, $g(x) = 3^x$, and $m(x) = e^x$, approximate the values of $f(2)$, $g(2)$, and $m(2)$ on the number line. Explain your reasoning.



- d. Identify the key attributes of $m(x) = e^x$. Write the domain and range in set notation.

Domain:

Range: {

y-intercept: (

Horizontal asymptote:

increasing/decreasing:

maximum/minimum:

ACTIVITY 2.3

Modeling Population Change With e

In the previous activity, you explored the continuous growth of a principal amount using the value e . How can e be used to model population changes?

1. The formula for population growth is $N(t) = N_0 e^{rt}$. Describe the contextual meaning of each quantity.

Quantity	Contextual Meaning
N_0	
r	
t	
$N(t)$	

2. Why is e used as the base?

3. How could this formula be used to represent a decline in population?

PROBLEM SOLVING



4. The population of the city of Round Rock, Texas was approximately 119,468 in 2020 and has been continuously growing at a rate of approximately 2.32% each year.
 - a. Use the formula for population growth to write a function to model this growth.
 - b. Use your function model to predict the population of Round Rock in 2040.
 - c. What value does your function model give for the population of Round Rock in the year 2000?
5. Use technology to estimate the number of years it would take Round Rock to grow to 340,000 people, assuming that the population trend continues.



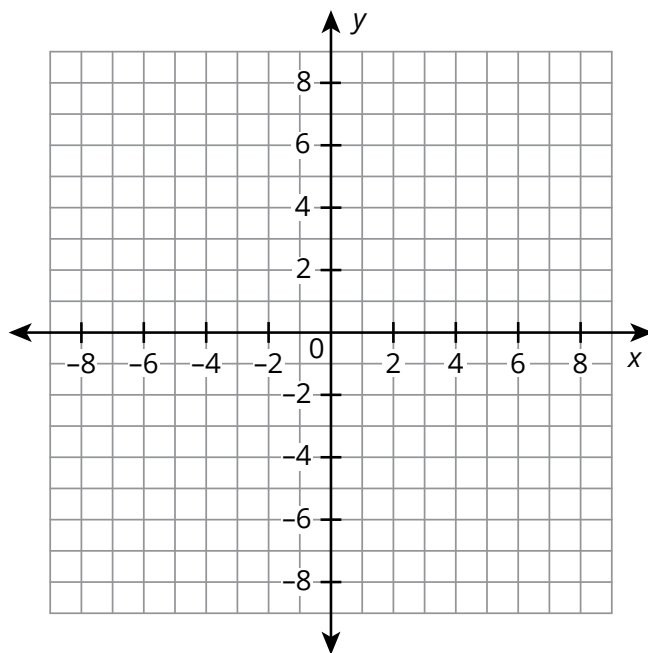
Talk the Talk

Easy E

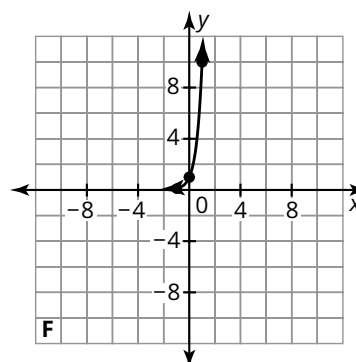
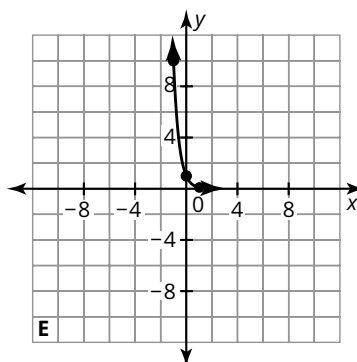
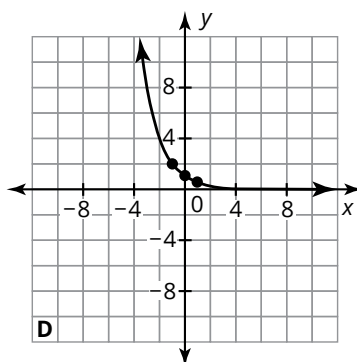
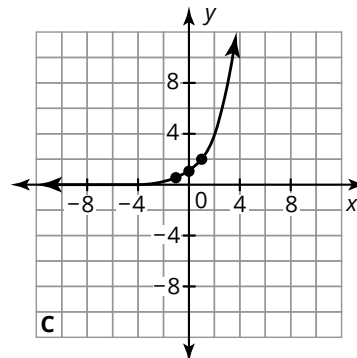
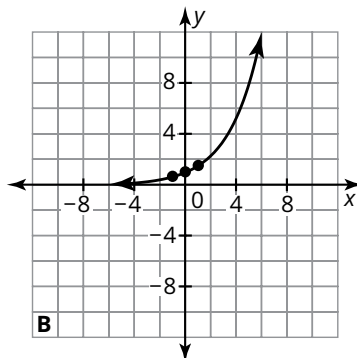
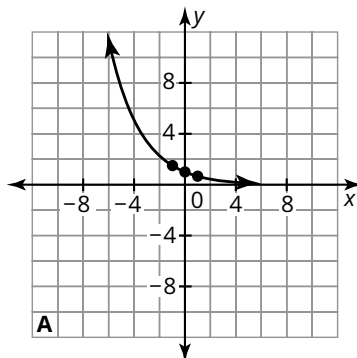
In this lesson, you learned about the natural base e and derived the formula for interest compounded continuously. You also looked at key attributes of graphs of exponential functions.

1. Alyssa invested \$5000 in a savings account for 6 years. The bank pays 2% interest, compounded monthly. How much more would Alyssa's investment be worth at the end of 6 years if the interest were compounded continuously? Show your work.

2. Graph the function $f(x) = e^x$. Describe the key attributes of function.
Write the domain and range of the function using set notation.



Exponential Graphs and Equations Cutouts



$$f(x) = \left(\frac{2}{3}\right)^x$$

$$f(x) = 10^x$$

$$f(x) = 1.5^x$$

$$f(x) = 0.1^x$$

$$f(x) = 2^x$$

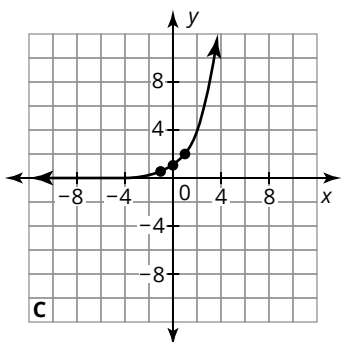
$$f(x) = \left(\frac{1}{2}\right)^x$$

Why is this page blank?

So you can cut out the graphs and equations on the other side.

Exponential Graphs and Equations Graphic Organizer

Growth



x	$f(x)$
-1	
0	
1	

Domain:

Range:

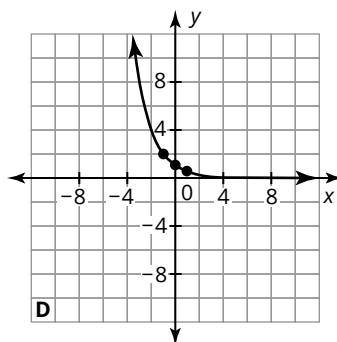
Horizontal Asymptote:

y-intercept:

Increasing/decreasing:

maximum/minimum:

Decay



x	$f(x)$
-1	
0	
1	

Domain:

Range:

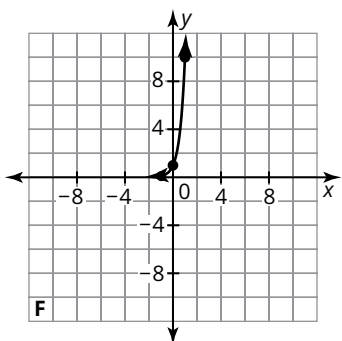
Horizontal Asymptote:

y-intercept:

Increasing/decreasing:

maximum/minimum:

Growth



x	$f(x)$
-1	
0	
1	

Domain:

Range:

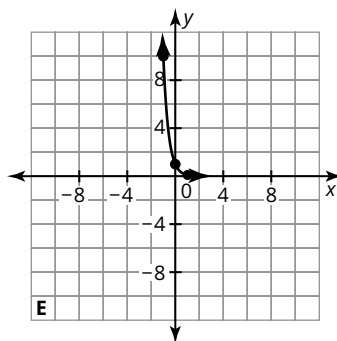
Horizontal Asymptote:

y-intercept:

Increasing/decreasing:

maximum/minimum:

Decay



x	$f(x)$
-1	
0	
1	

Domain:

Range:

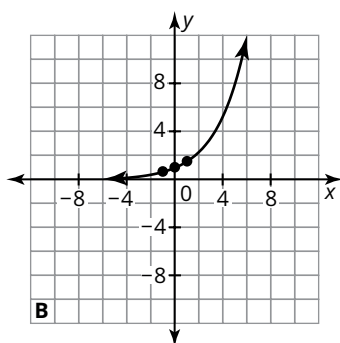
Horizontal Asymptote:

y-intercept:

Increasing/decreasing:

maximum/minimum:

Growth



x	$f(x)$
-1	
0	
1	

Domain:

Range:

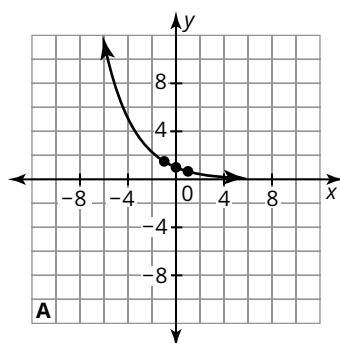
Horizontal Asymptote:

y-intercept:

Increasing/decreasing:

maximum/minimum:

Decay



x	$f(x)$
-1	
0	
1	

Domain:

Range:

Horizontal Asymptote:

y-intercept:

Increasing/decreasing:

maximum/minimum:

Lesson 2 Assignment

Write

Explain how the natural base e is similar to and different from π .

Remember

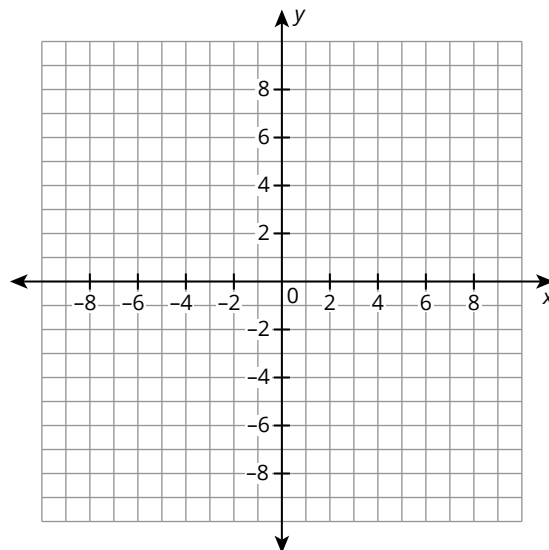
The natural base $e \approx 2.7182818$ is an irrational number that represents continuous growth and is used to model population changes, as well as continuously compounded interest.

For continuous compounding, the compound interest formula is $A = Pe^{rt}$.

Practice

1. Graph the functions and describe their key attributes. Write their domain and range using interval notation.

a. $f(x) = 2^x$



Domain:

Range:

y-intercept:

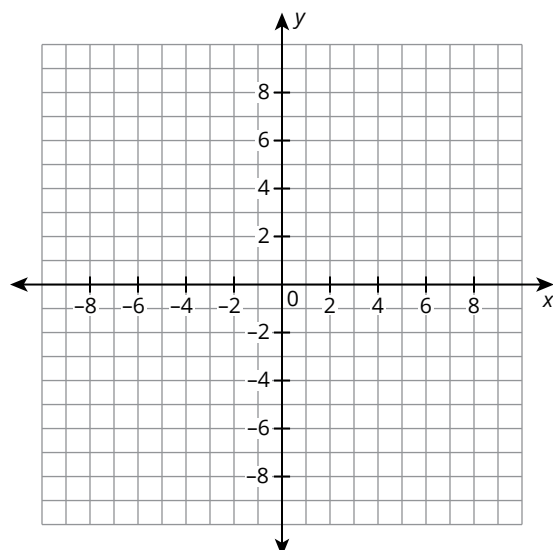
Horizontal Asymptote:

increasing/decreasing:

maximum/minimum:

Lesson 2 Assignment

b. $g(x) = 10^x$



Domain:

Range:

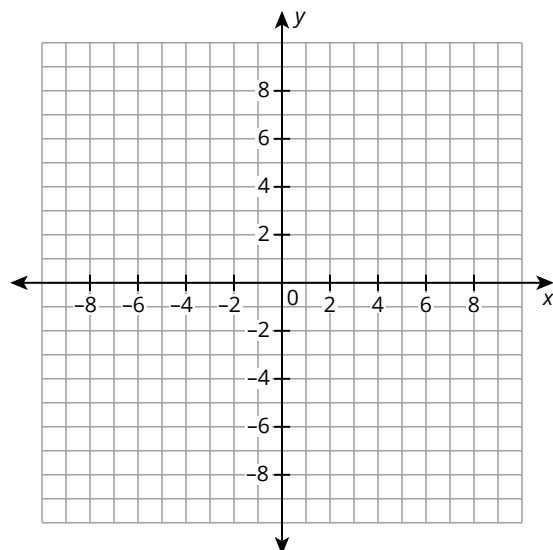
y-intercept:

Horizontal Asymptote:

increasing/decreasing:

maximum/minimum:

c. $h(x) = e^x$



Domain:

Range:

y-intercept:

Horizontal Asymptote:

increasing/decreasing:

maximum/minimum:

Lesson 2 Assignment

2. Jaylen wants to invest \$1000 in a savings account. Lincoln Federal Bank is offering 6% interest compounded yearly. Washington National Bank is offering 5.5% interest compounded daily.
- For each bank, write an exponential function to represent the amount of money Jaylen would have in the account after t years.
 - Determine which bank Jaylen should choose if he plans to invest his money for 5 years. If Jaylen decides to leave the money in the bank for a longer period of time, will the other bank be a better deal in the long run? Explain your reasoning.
 - Discuss the domain, range, asymptotes, intercepts, and intervals of increase and decrease for each function as they relate to this problem situation. Write the domain and range using inequalities.

Lesson 2 Assignment

3. In 2010, Bolivia had a population of 10.5 million people and an annual growth rate of 1.6%.
- a. Write a function to model Bolivia's population with respect to t , the number of years since 2010. Write your function in the form $N(t) = N_0 e^{rt}$.
- b. Use your model to predict Bolivia's population in the year 2030.
- c. Use your model to estimate Bolivia's population in the years 1990 and 1970.
- d. Use technology to estimate when Bolivia's population will reach 20 million people.

Lesson 2 Assignment

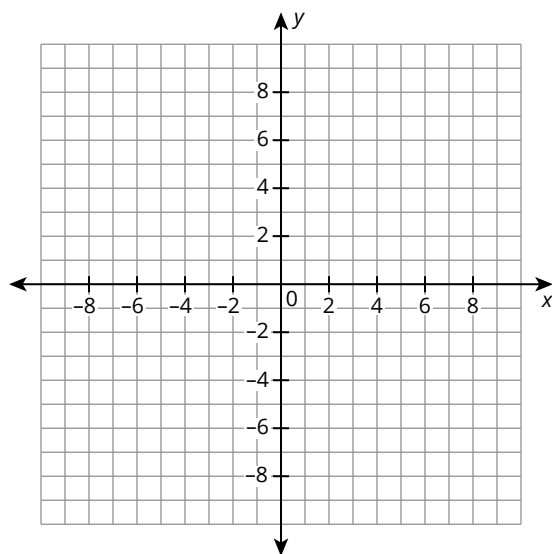
- e. Discuss the domain, range, asymptotes, intercepts, and intervals of increase and decrease for your population model as they relate to this problem situation. Write the domain and range using set notation.

Lesson 2 Assignment

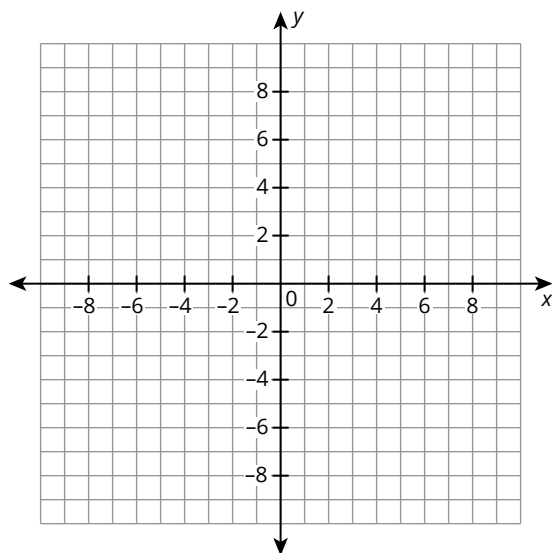
Prepare

1. Consider the parent rational function $f(x) = \frac{1}{x}$. Graph the transformed function.

a. $g(x) = 2f(x - 3) + 1$

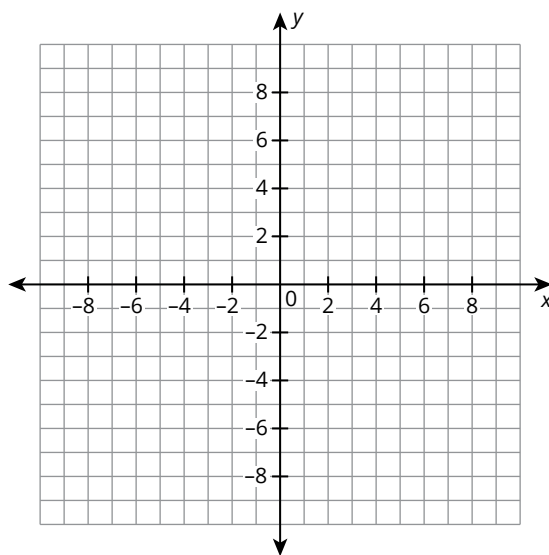


b. $h(x) = f(3(x + 1))$

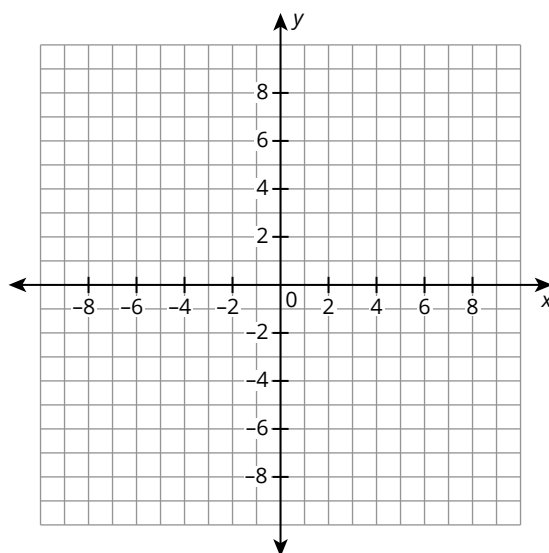


Lesson 2 Assignment

c. $j(x) = -3f(x + 1) - 2$



d. $k(x) = f(-2x) + 3$



3

Transformations of Exponential Functions

OBJECTIVES

- Dilate, reflect, and translate exponential functions using reference points and transformation function form.
- Investigate graphs of exponential functions through intercepts, asymptotes, and intervals of increase and decrease.
- Verbally describe how transformations of exponential functions affect their key attributes.

KEY TERM

- none

.....

You have explored transformations with functions in many function families.

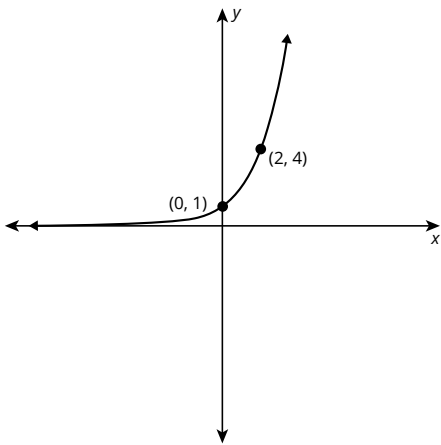
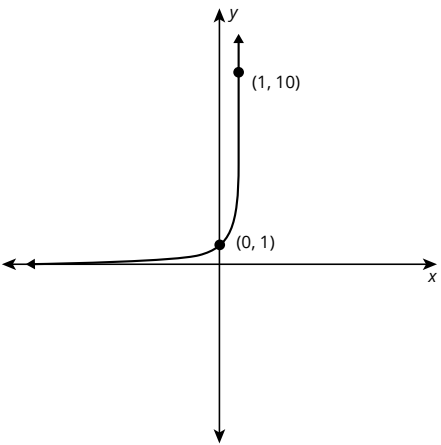
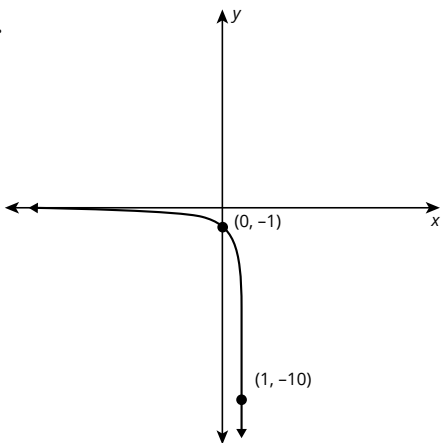
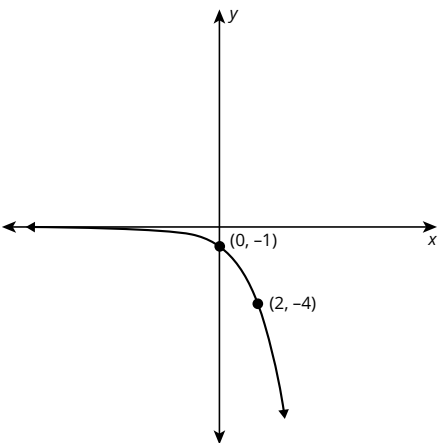
How are the transformations of exponential functions similar to the transformations of other functions you have studied?

Getting Started

Don't Flip Out

Consider the two tables that show four exponential functions represented as equations and as graphs.

Exponential Equations	
$f(x) = 10^x$	$g(x) = -10^x$
$h(x) = 2^x$	$j(x) = -2^x$

Exponential Graphs	
A. 	B. 
C. 	D. 

Remember ...

All transformations can be written in transformation function form in terms of the original function.

1. Analyze the graphs.

- Match each exponential equation to its corresponding graph, and write the equation under the graph it represents.

- b. Write an equation for each transformation in terms of its parent function.

2. Complete the table to describe the key attributes of each function.

Function	Asymptote	Intervals of Increase and Decrease	y-intercept	Domain and Range in Interval Notation
$f(x) = 10^x$				
$g(x) = -10^x$				
$h(x) = 2^x$				
$j = -2^x$				

3. Analyze the key attributes in the table of each exponential function. What similarities exist among the four functions?

4. What generalizations can you make about the effects of these transformations on the domain and range of an exponential function?

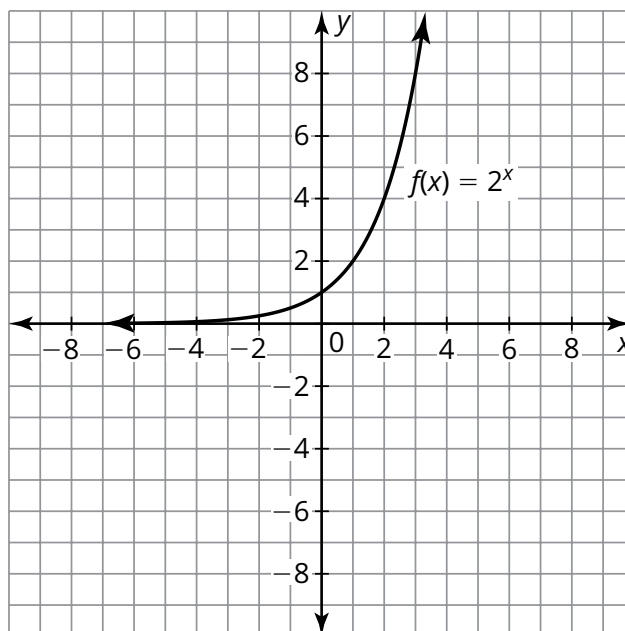
Graphing Transformations

In the Getting Started, you analyzed transformations of exponential functions that were a reflection across the x -axis.

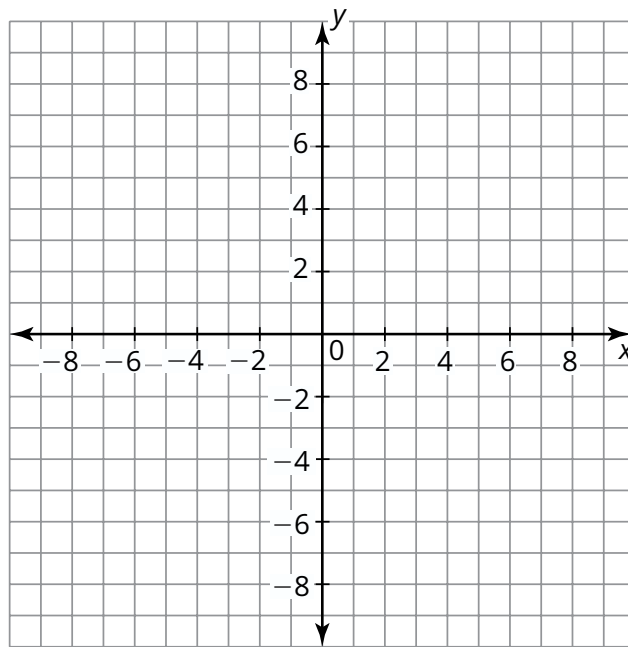
You know that transformations performed on any function $f(x)$ to form a new function $g(x)$ can be described by the transformation function form.

$$g(x) = af(x - c) + d$$

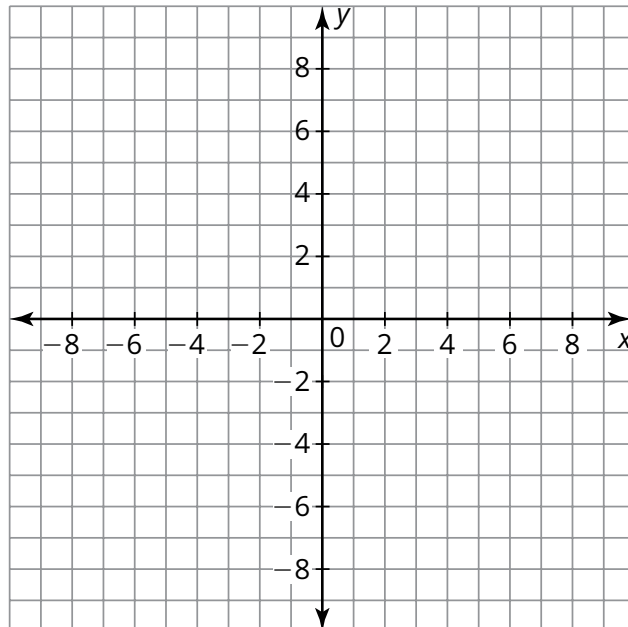
1. Consider each transformation to $f(x) = 2^x$. Describe the effects of each transformation, including effects on the key attributes of the graph. Then, sketch the transformed graph.



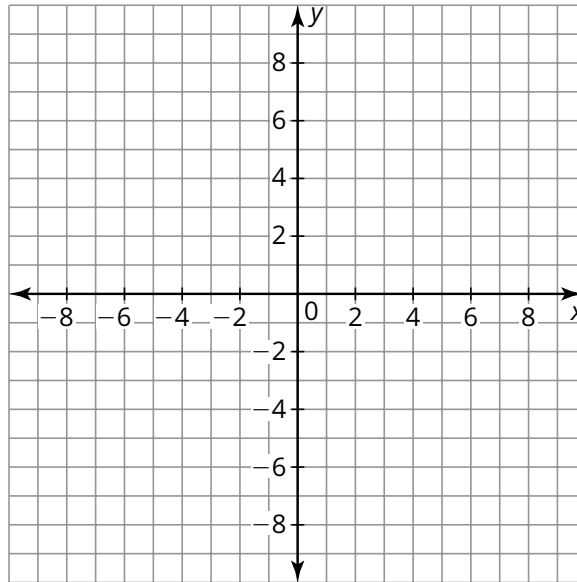
a. $g(x) = 5 \cdot 2^x$



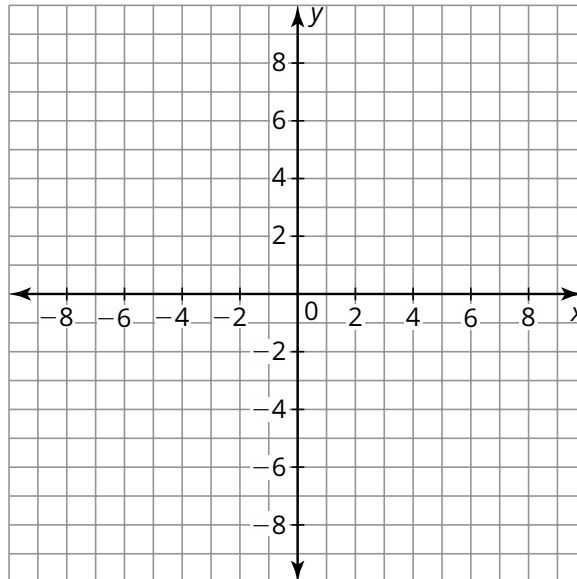
b. $p(x) = 2^{x-4}$



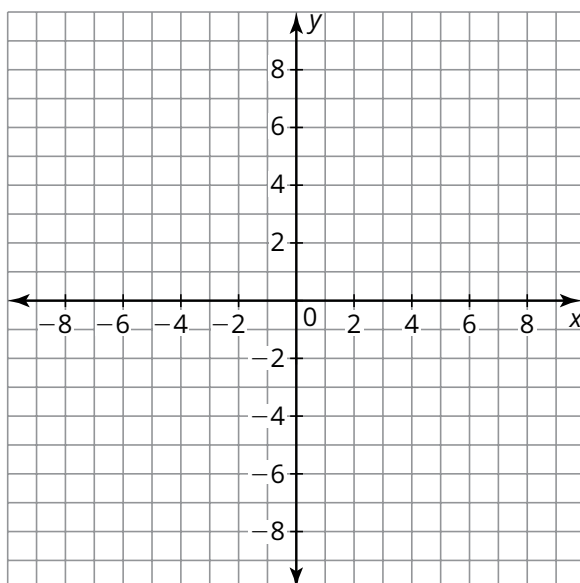
c. $k(x) = 2^{x-3}$



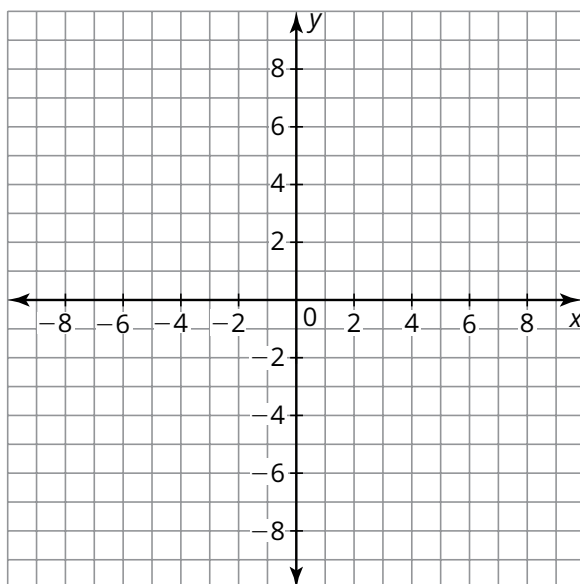
d. $q(x) = 2^{x+4}$



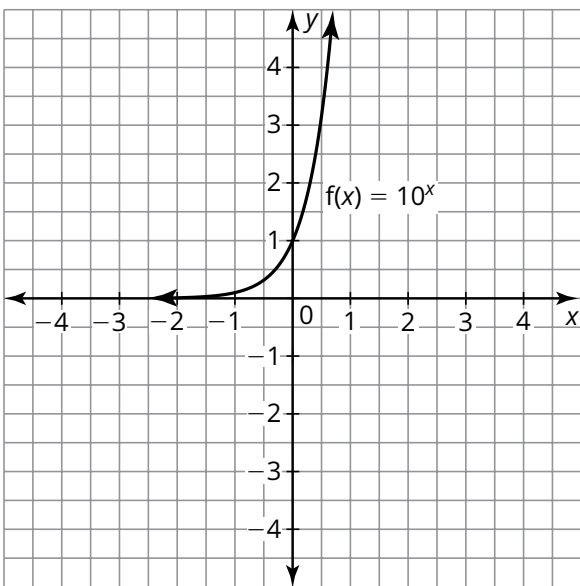
e. $j(x) = 2^x + 3$



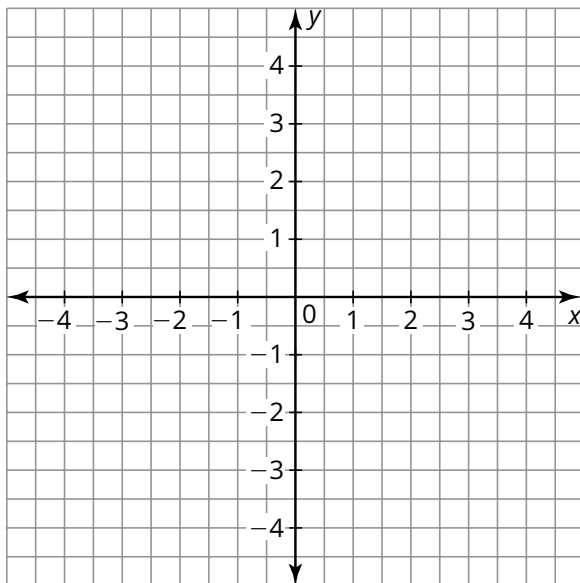
f. $h(x) = -5 \cdot 2^x$



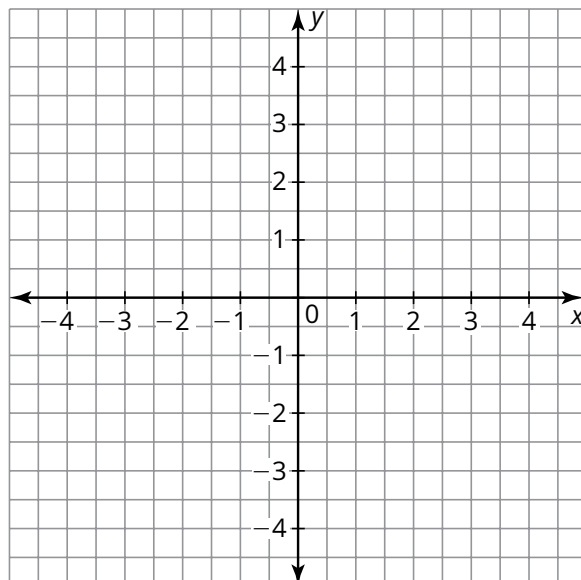
2. Consider the transformations to $f(x) = 10^x$. Describe the effects of the multiple transformations, including effects on the key attributes of the graph. Then, sketch the transformed graph.



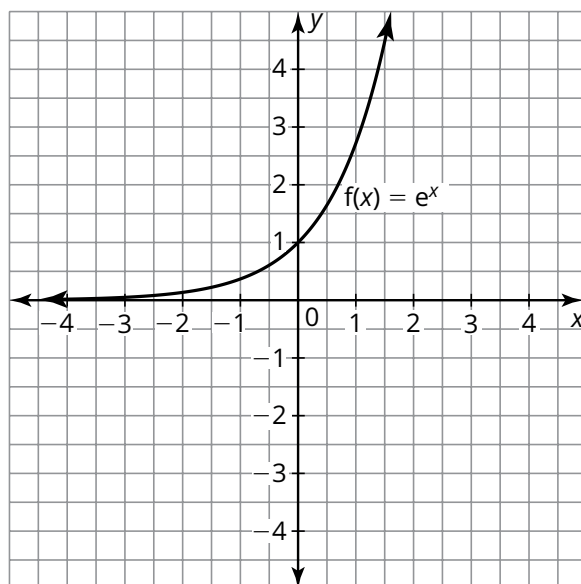
a. $s(x) = -3 \cdot 10^{x+2} - 1$



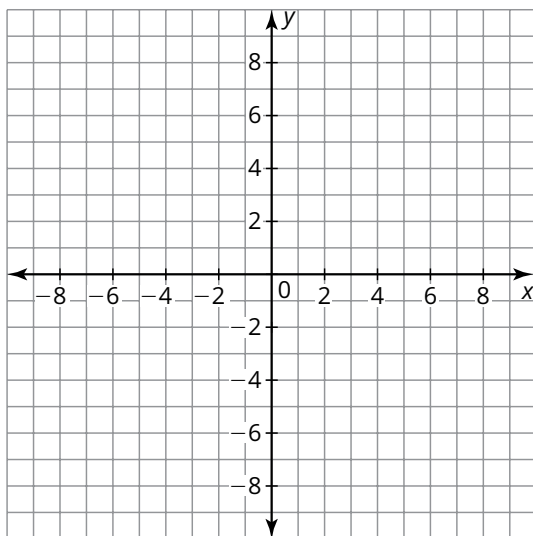
b. $t(x) = 2 \cdot 10^{x-1} + 3$



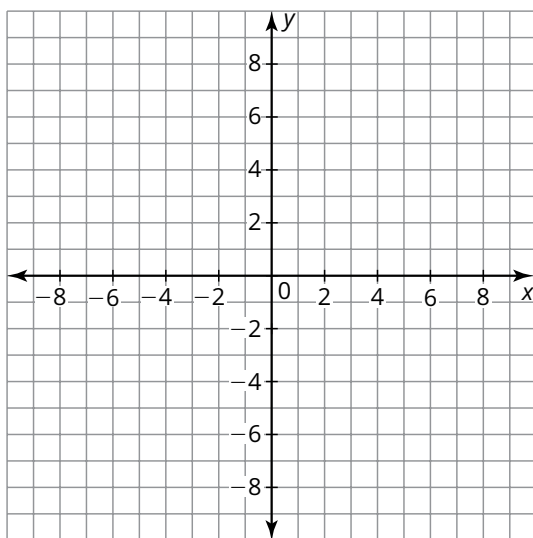
3. Consider the transformations to $f(x) = e^x$. Describe the effects of the multiple transformations, including effects on the key attributes of the graph. Then, sketch the transformed graph.



a. $w(x) = 4 \cdot e^{x+3} + 1$



b. $z(x) = -8 \cdot e^{x-2} - 5$



4. Analyze the transformations performed on $f(x)$ in Questions 1 through 3.
- Which, if any, of these transformations affected the domain, range, and asymptotes?
 - What generalizations can you make about the effects of transformations on the domain, range, and asymptotes of exponential functions?

Describing Transformations

You can describe a transformation in more than one way—in terms of the original function, as an equation in transformation function form, using key attributes, or using words. Is it possible to describe the same transformation using different notation?



1. Paola and Jamal each described the effects of transforming the graph of $f(x) = 3^x$, such that $p(x) = 3f(x)$.

Paola

$$p(x) = 3f(x)$$

The a -value is 3,
so the graph is
stretched vertically
by a scale factor of 3.

Jamal

$$p(x) = 3f(x)$$

$$p(x) = 3 \cdot 3^x$$

$$p(x) = 3^{1+x}$$

$$p(x) = f(x + 1)$$

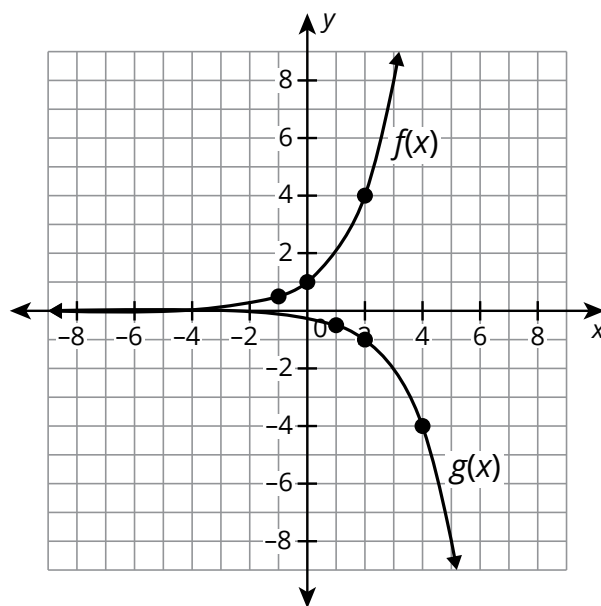
The C -value is -1 , so
the graph is horizon-
tally translated
1 unit to the left.

Who is correct? Explain your reasoning.

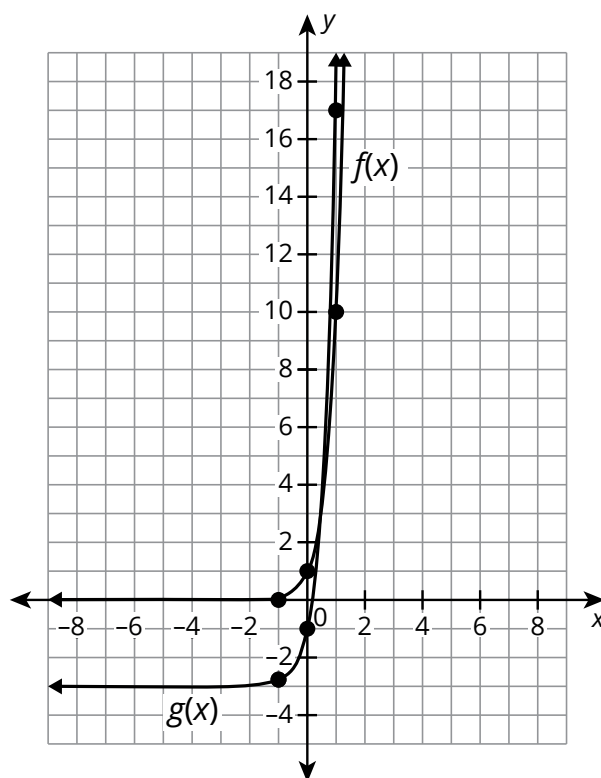
You can describe the transformation(s) performed on a function by comparing the graphs or equations of the original function and the transformed function.

2. Analyze the graphs of $f(x)$ and $g(x)$. Describe the transformations performed on $f(x)$ to create $g(x)$. Then, write an equation for $g(x)$ in terms of $f(x)$. For each set of points shown on $f(x)$, the corresponding points are shown on $g(x)$.

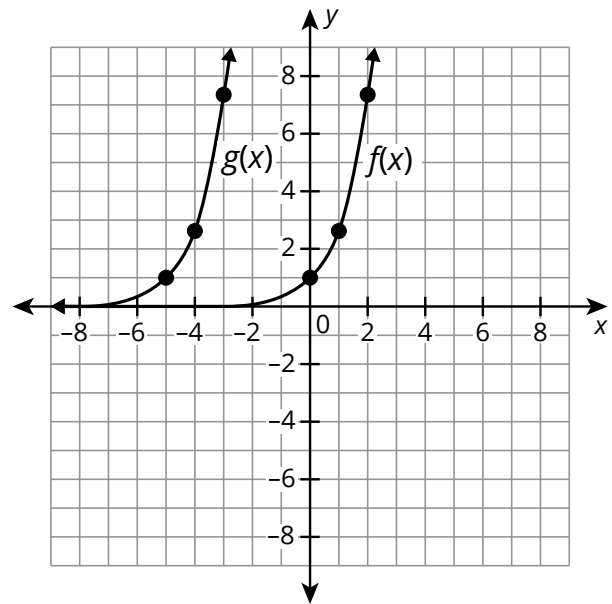
a. $g(x) =$ _____



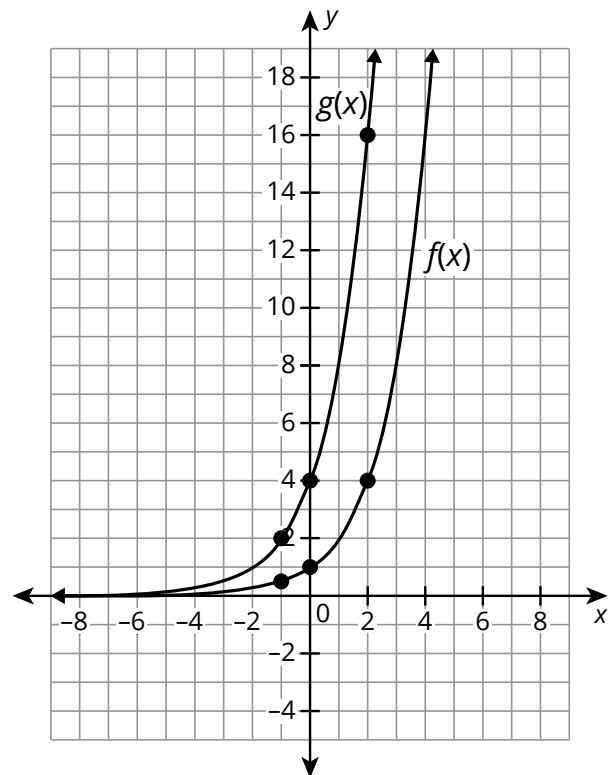
b. $g(x) =$ _____



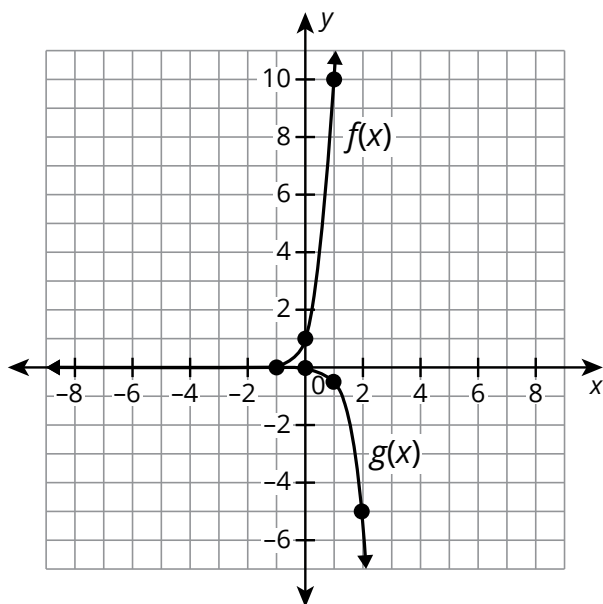
c. $g(x) =$ _____



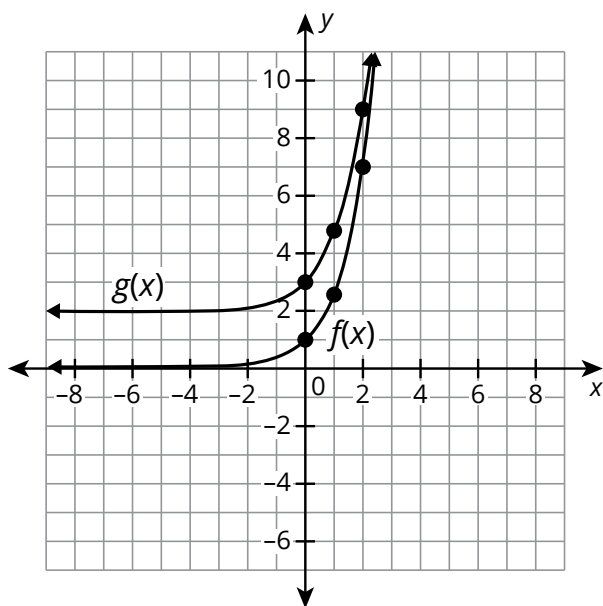
d. $g(x) =$ _____



e. $g(x) =$ _____



f. $g(x) =$ _____



3. The equation for an exponential function $m(x)$ is given. The equation for the transformed function $t(x)$ in terms of $m(x)$ is also given. Describe the graphical transformation(s) on $m(x)$ that produce(s) $t(x)$. Then, write an exponential equation for $t(x)$.

a. $m(x) = 2^x$
 $t(x) = 0.5m(x + 3)$

b. $m(x) = e^x$
 $t(x) = -m(x) - 1$

c. $m(x) = 10^x$
 $t(x) = 2m(x) + 2$

d. $m(x) = 2^x$
 $t(x) = m(x - 4) + 2$

e. $m(x) = e^x$
 $t(x) = \frac{1}{3}m(x + 5)$

f. $m(x) = 10^x$
 $t(x) = -5m(x) - 4$



Talk the Talk

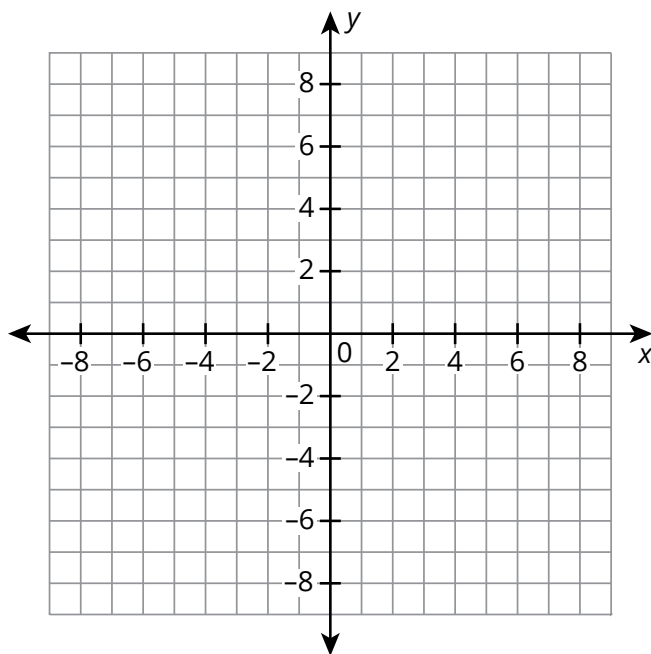
Moving Right Along

In this lesson, you graphed and analyzed transformations of exponential functions.

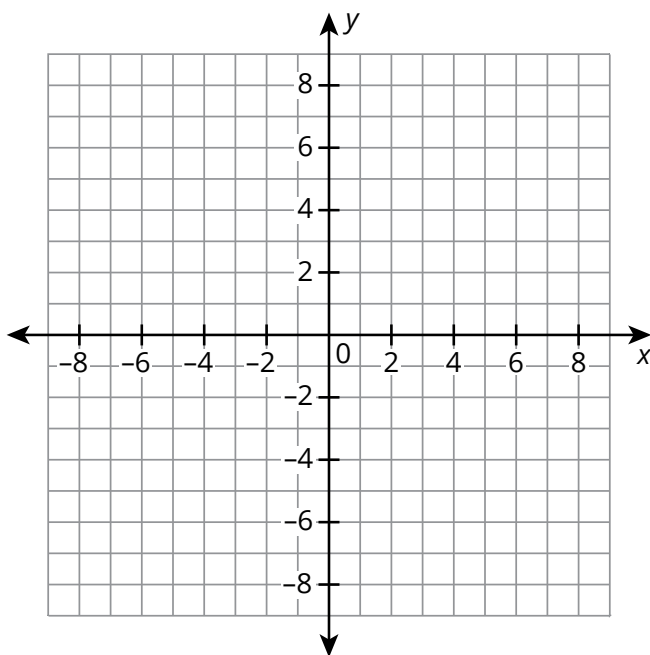
1. Determine whether each statement is true or false. If the statement is false, rewrite the statement so that it is true.
 - a. Vertical translations do not affect the range and the horizontal asymptote of exponential functions.
 - b. The domain of exponential functions is not affected by translations or dilations.
 - c. Horizontal dilations do not affect the domain of exponential functions.

2. Graph each function.

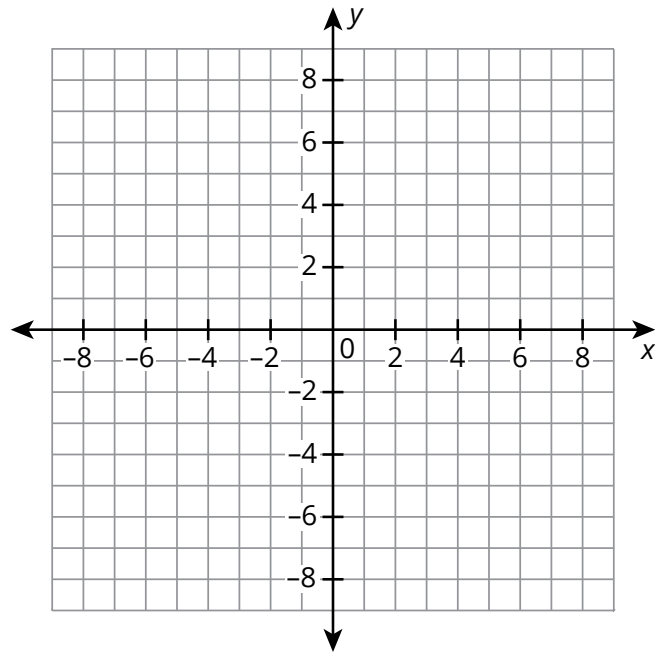
a. $f(x) = -2^x$



b. $g(x) = 2e^x - 1$



c. $h(x) = -10^{x+4}$



Lesson 3 Assignment

Write

Explain how a , c , and d -value transformations affect the domain, range, and asymptotes of an exponential function.

Remember

The transformation function form $g(x) = af(b(x - c)) + d$ can be applied to exponential and logarithmic functions.

Practice

1. Given $p(x) = 2^x$ and $t(x) = 2p(x - 1) - 4$
 - a. Describe the transformation of $p(x)$ that produces $t(x)$.
 - b. Write $t(x)$ as an exponential function.

2. Given $m(x) = 10^x$ and $k(x) = \frac{1}{3}m(x + 2) - 5$
 - a. Describe the transformation of $m(x)$ that produces $k(x)$.
 - b. Write $k(x)$ as an exponential function.

Lesson 3 Assignment

3. Given $n(x) = e^x$ and $r(x) = -3n(x + 4) + 1$.

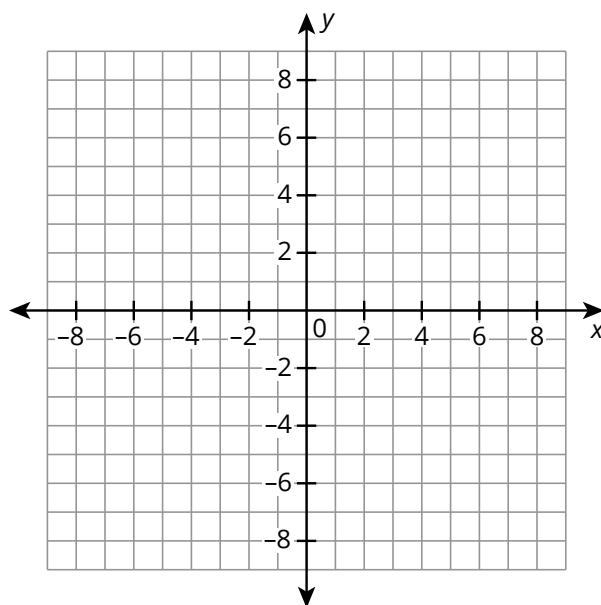
a. Describe the transformation of $n(x)$ that produces $r(x)$.

b. Write $r(x)$ as an exponential function.

4. Consider $p(x) = -\frac{1}{2} 2^{x+1} + 3$ which is a transformation of the function $f(x) = 2^x$.

a. Describe the transformation(s) on the graph of $f(x)$ to produce $p(x)$.

b. Graph and label the transformed function $p(x)$.

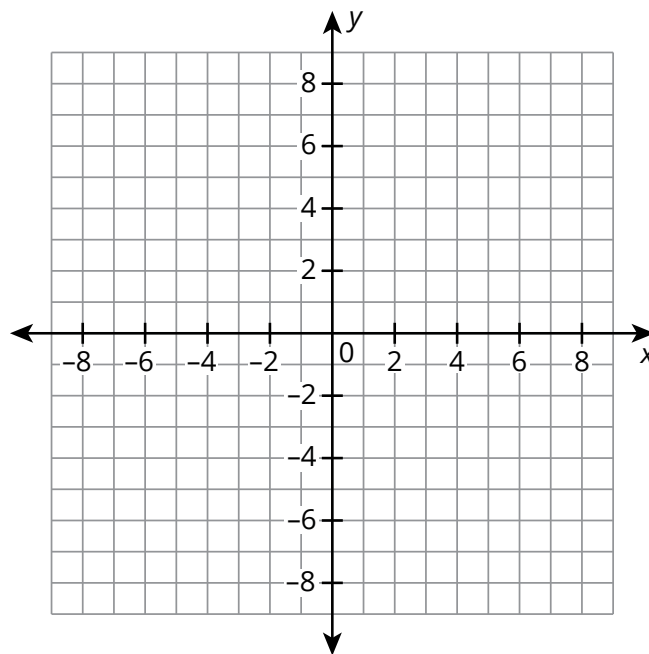


Lesson 3 Assignment

5. Consider $j(x) = -2 \cdot 10^{x-3} + 4$ which is a transformation of the function $g(x) = 10^x$.

a. Describe the transformation(s) on the graph of $g(x)$ to produce $j(x)$.

b. Graph and label $j(x)$.

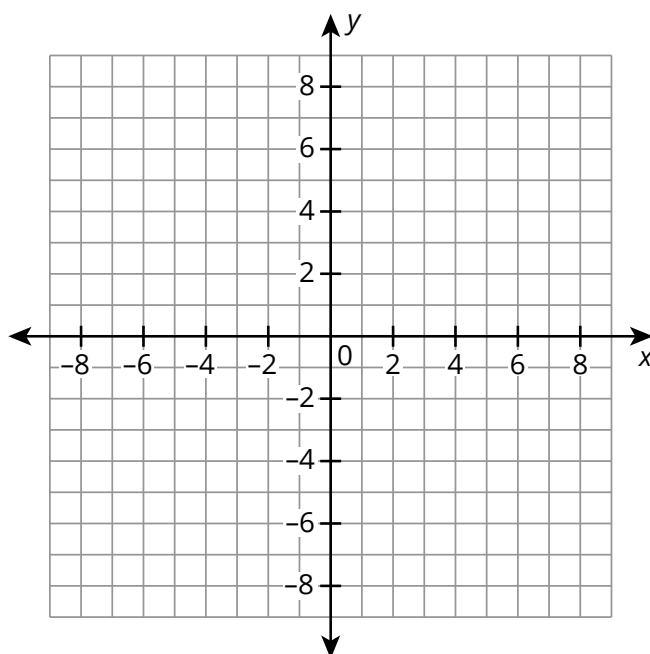


Lesson 3 Assignment

6. Consider $k(x) = 3 \cdot e^{x+2} - 5$ which is a transformation of the function $h(x) = e^x$.

a. Describe the transformation(s) on the graph of $h(x)$ to produce $k(x)$.

b. Graph and label $k(x)$.



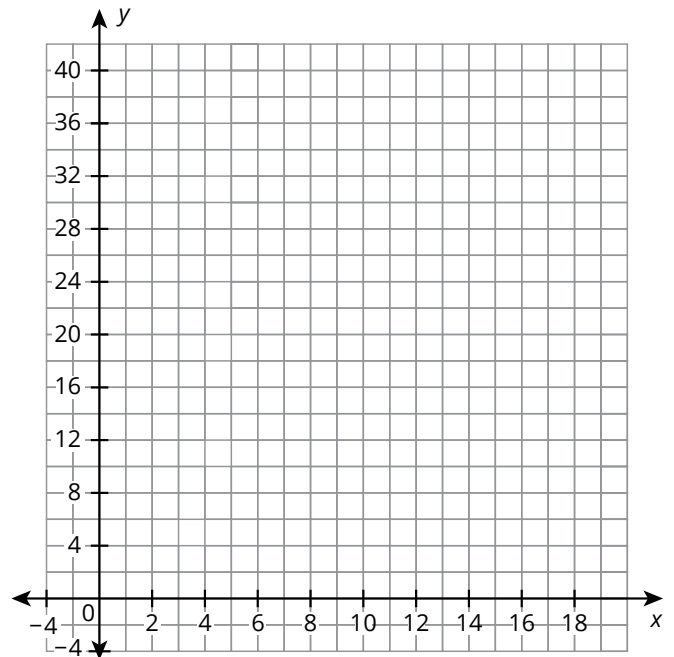
Lesson 3 Assignment

Prepare

While driving to their vacation spot, the Mitchell family kept a record of their gas purchases. Their information is recorded in the table.

Amount of Gas (gallons)	Total Cost (dollars)
9	21.33
16.6	40.00
11.8	26.08
10	24.70
13.2	28.91

1. Determine whether the data is best represented by a linear or quadratic function. Explain your reasoning.



Lesson 3 Assignment

2. Use technology to write the regression function.
3. What is the coefficient of determination, r^2 ? What does this value imply?

4

Modeling with Exponential Functions

OBJECTIVES

- Determine the family of functions that best represent a data set.
- Use graphing technology to write exponential regression functions.
- Use a regression function to make predictions and critical judgments.

.....

You have used technology to write linear and quadratic functions to represent a set of data.

How can you determine whether an exponential function best represents a set of data?

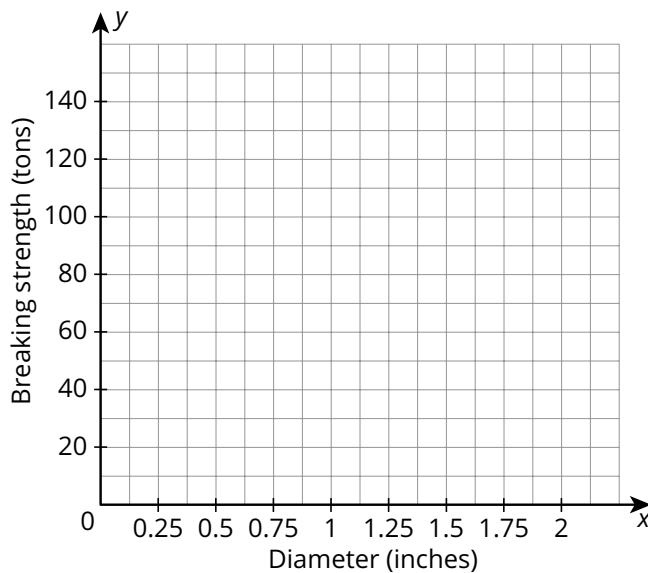
Getting Started

Breaking Down

The breaking strength in tons, $b(x)$, of steel cable with diameter in inches, x , is given in the table.

Diameter (inches)	0.25	0.50	0.75	1.00	1.25	1.50	1.75
Breaking Strength (tons)	5.2	9.8	18.6	29.3	54.2	84.4	122.7

1. Create a scatterplot of the data. Be sure to label your axes.



.....
Review the definition
of **represent** in the
Academic Glossary.
.....

2. Alyssa and Josh are analyzing the data. Alyssa says the data is best represented by a linear function because it is increasing. Josh says that the data is best represented by an exponential function. Who's correct? Explain your reasoning.

Exponential Regression

You have used graphing technology to determine a regression function to represent a data set. You have also used a regression function to make predictions based on a data set.

Recall that the coefficient of determination, r^2 , describes the strength of the relationship between the data and the regression function.

Consider the data from the Getting Started.

1. Use graphing technology to determine the exponential regression function for the data.
2. Is the exponential function a good fit for the data? Explain your reasoning.
3. The design of an elevator requires use of steel cable with a diameter of 1.2 inches. Predict the breaking strength of this steel cable.

4. The cable used for the towing rig of a commercial airplane requires a breaking strength of 120 tons. Use graphing technology to estimate the diameter of the steel cable to the nearest tenth. Explain your reasoning.

5. Make critical judgments as to the validity of Matthew's statement.

Matthew

An exponential regression best models the data. So, the domain of the problem situation is $(-\infty, \infty)$.

- a. According to the mathematical model, the diameter of a steel cable can extend to infinity. What do you think is a realistic upper bound for the diameter of steel cable? What information should you consider when determining the upper bound?
- b. According to the mathematical model, the breaking strength of steel cable can extend to infinity. What do you think is a realistic upper bound for the breaking strength of steel cable? What information should you consider when determining the upper bound?

c. Is Matthew correct? Explain your reasoning.

ACTIVITY
4.2**Comparing Linear, Quadratic and Exponential Data**

Remember you can use the first and second differences to determine if a data set represents a linear or a quadratic function.

Recall that exponential functions have a constant ratio. This means the ratio between successive y-values is constant.

1. Determine whether each table represents a linear, quadratic, or exponential function. Explain your reasoning.

a.

x	$f(x)$
-2	-9
-1	-7
0	-5
1	-3
2	-1

b.

x	$g(x)$
-1	2.5
0	5
1	10
2	20
3	40

c.

x	$h(x)$
-2	-3
-1	3
0	5
1	3
2	-3

You can use graphing technology to write a regression function for linear, quadratic, and exponential functions.

2. Use graphing technology to write a regression function for each table in Question 1.

a. $f(x)$

b. $g(x)$

c. $h(x)$



Talk the Talk

Home Sweet Data

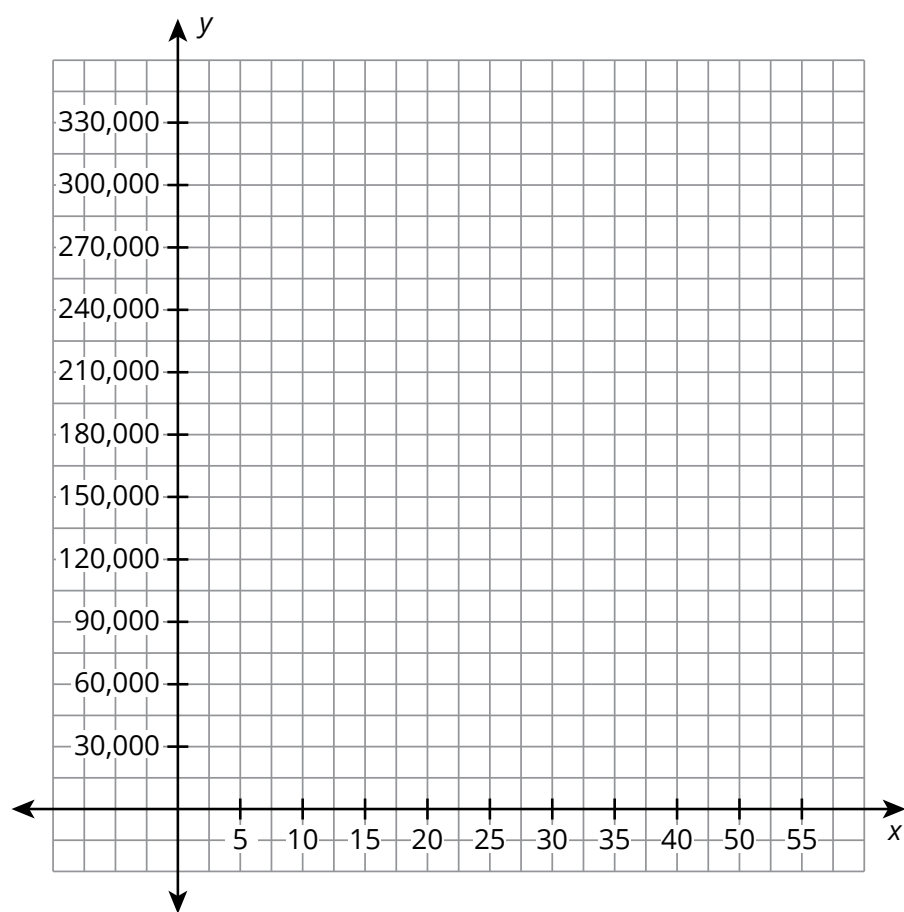
The U.S. average prices of a new home over five-year intervals since 1960 are shown.

Year	US Average Price of a New Home (dollars)
1960	16,500
1965	21,500
1970	26,500
1975	42,600
1980	76,400
1985	100,800
1990	149,800
2000	207,000
2005	297,000
2010	272,900
2015	329,000

Ask Yourself ...

How can you use regression functions in everyday life?

1. Create a scatterplot of the data on the coordinate plane.



2. Which family of functions do you think best fits the data? Explain your reasoning.

Lesson 4 Assignment

Write

Describe how to determine whether a data set is best represented with a linear, quadratic, or exponential function.

Remember

Determining and using a regression function is sometimes a step in the process of solving a more complex mathematical problem, rather than the final solution.

Practice

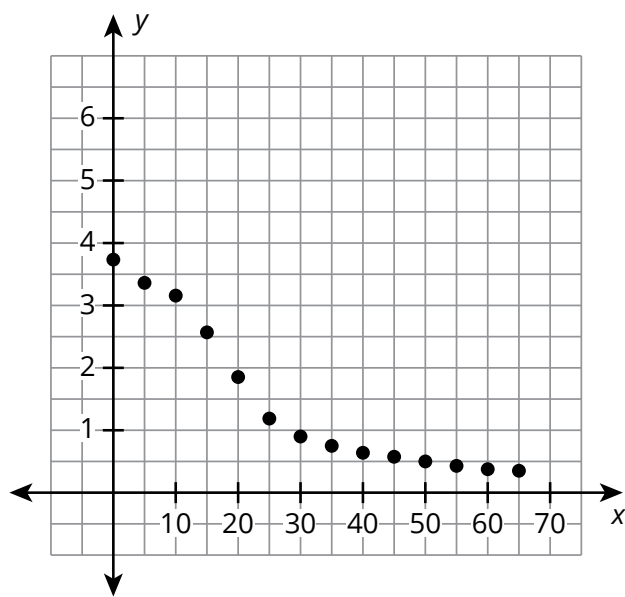
The table shows the purchasing value of the dollar, or the consumer price index, for consumers in the United States from 1955 to 2010. The table uses the year 1982 as a base period, so the consumer price index written in dollars and cents in 1982 is 1.00. For instance, in 1955, the consumer price index was 3.73. This means that a dollar in 1955 was worth 3.73 times what it was worth in 1982. Similarly, a dollar in 2010 was worth 0.46 times what it was worth in 1982.

Year	Consumer Price Index	Year	Consumer Price Index
1955	3.73	1990	0.77
1960	3.37	1995	0.66
1965	3.17	2000	0.58
1970	2.57	2005	0.51
1975	1.86	2010	0.46
1980	1.22	2015	0.42
1985	0.93	2020	0.38

Lesson 4 Assignment

The scatterplot shows the data in the table where x represents the number of years since 1955 and y represents the consumer price index.

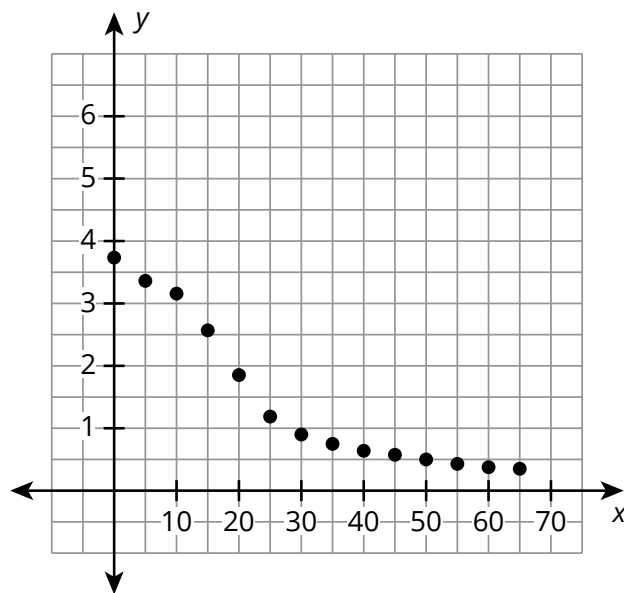
1. Describe how the consumer price index changes over time.
2. What function family best models this situation? Explain your reasoning.



Lesson 4 Assignment

3. Analyze the data and scatterplot.

- a. Determine the regression function for the model that best represents the data. Then, graph the model on the same grid as the scatterplot.



- b. Predict the consumer price index in 2025. Explain what your answer means in terms of the problem situation.

Lesson 4 Assignment

- c. Mr. Patel asks his students to calculate what the consumer price index was in 1950. Kaya says that you must evaluate the function at $x = 5$ to determine the consumer price index in 1950. Mario argues you must evaluate the function at $x = -5$ to determine the consumer price index in 1950. Who is correct? Explain your reasoning.
- d. Calculate the consumer price index for 1950. Show your work.
- e. The consumer price index in 1950 was actually 4.15. Compare this to the answer you calculated in part (d). Explain why these answers differ.

Lesson 4 Assignment

4. Determine whether each table represents a linear, quadratic, or exponential. Use graphing technology to write a regression function.

a.

x	$f(x)$
-3	-0.5
-2	-1.0
-1	-2.0
0	-4.0
1	-8.0

b.

x	$g(x)$
-1	-3.5
0	-2.0
1	-0.5
2	1.0
3	2.5

c.

x	$h(x)$
-2	4.5
-1	2.0
0	0.5
1	0
2	0.5

5

Logarithmic Functions

OBJECTIVES

- Graph the inverses of exponential functions with bases of 2, 10, and e .
- Recognize the inverse of an exponential function as a logarithm.
- Identify the domain and range of logarithmic functions.
- Investigate and verbally describe the graphs of logarithmic functions through intercepts, asymptotes, and intervals of increase and decrease.

KEY TERMS

- logarithm
- logarithmic function
- common logarithm
- natural logarithm

.....

You have investigated the inverses of linear, quadratic, and power functions.

What attributes can you identify from the inverse of exponential functions—logarithmic functions?

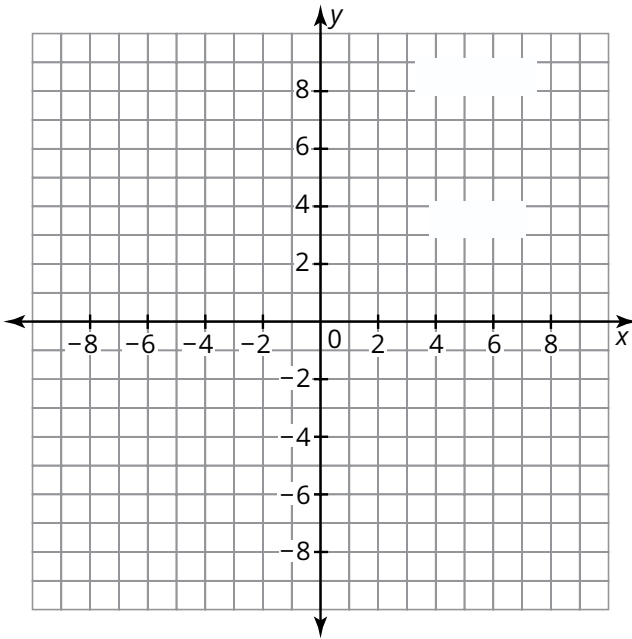
Getting Started

Across $y = x$

You have graphed exponential functions. Let's explore the inverse of an exponential function.

1. Complete the tables for $f(x) = 2^x$ and $f^{-1}(x)$. Then, graph each function.

x	$f(x) = 2^x$
-3	
-2	
-1	
0	
1	
2	
3	



x	$f^{-1}(x)$

.....
Remember...
For any function f with ordered pairs (x, y) , or $(x, f(x))$, the inverse of the function f is the set of all ordered pairs (y, x) , or $(f(x), x)$.
.....

2. Complete the table to identify the key attributes of each function.

	$f(x)$	$f^{-1}(x)$
Domain using inequalities		
Range using inequalities		
y-Intercept		
x-Intercept		
Asymptote		
Intervals of Increase and Decrease		

3. Compare the key attributes of the function and its inverse.

4. Write the equation for the inverse of $y = 2^x$. Explain your reasoning.

Logarithms Are Exponents

It is necessary to define a new function in order to write the equation for the inverse of an exponential function. The **logarithm** of a number is the exponent to which the given base must be raised in order to produce the number. If $y = b^x$, then x is the logarithm and can be written as $\log_b y = x$. The value of the base of a logarithm is the same as the base in the exponential expression b^x .

For example, you know $2^3 = 8$. So, the number 3 is the logarithm to which base 2 must be raised to produce the argument 8. The base is written as the subscript 2. The logarithm, or exponent, is the output 3. The argument of the logarithm is 8.

$$\begin{array}{c} \text{argument} \\ \swarrow \\ \log_2 8 = 3 \\ \nearrow \\ \text{base} \end{array} \quad \begin{array}{c} \text{logarithm,} \\ \text{or exponent} \\ \swarrow \end{array}$$

WORKED EXAMPLE

You can write any exponential equation as a logarithmic equation and vice versa.

Example	Exponential Form	$\Leftrightarrow$	Logarithmic Form
A	$y = b^x$	$\Leftrightarrow$	$\log_b y = x$
B	$16 = 4^2$	$\Leftrightarrow$	$\log_4 16 = 2$
C	$1000 = 10^3$	$\Leftrightarrow$	$\log_{10} 1000 = 3$
D	$32 = 16^{1.25}$	$\Leftrightarrow$	$\log_{16} 32 = 1.25$
E	$a = b^c$	$\Leftrightarrow$	$\log_b a = c$

1. Rewrite your equation from the Getting Started Question 4, in logarithmic form.

2. Analyze the exponential equation $y = b^x$ and its related logarithmic equation, $\log_b y = x$. State the restrictions, if any, on the variables. Explain your reasoning.

$y = b^x \Leftrightarrow \log_b y = x$		
Variable	Restrictions	Explanation
x		
b		
y		

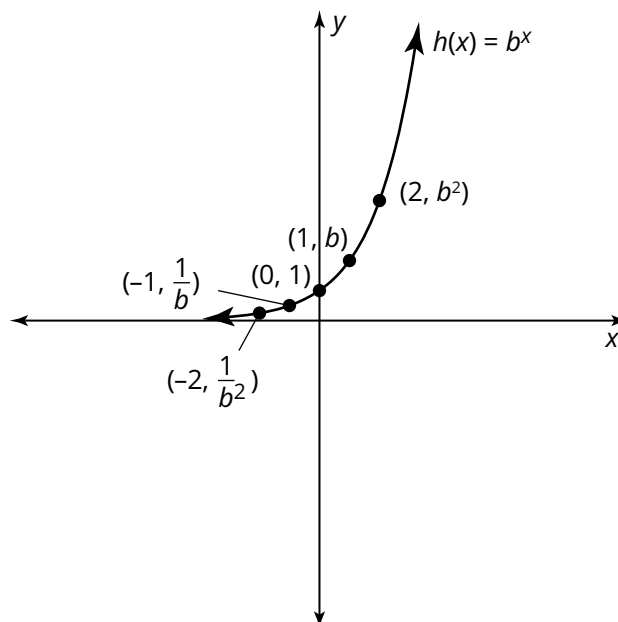
ACTIVITY 5.2

Graphing Logarithmic Functions

Recall that the logarithm of a number is the exponent to which the given base must be raised in order to produce the number. If $y = b^x$, then the logarithm is written as $\log_b y = x$, where $b > 0$, $b \neq 1$, and $y > 0$.

A **logarithmic function** is a function involving a logarithm. What do the graphs of logarithmic functions look like?

The graph of $h(x) = b^x$ is shown.



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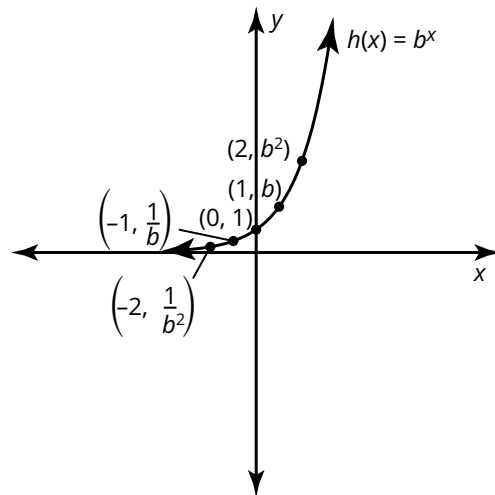
Think about ...

What are some methods you can use to sketch the graph of the inverse?

.....

1. Write the domain and range of $h(x)$ using interval notation.

2. Sketch the graph of the inverse of $h(x)$ on the same coordinate plane. Label coordinates of points on the inverse of $h(x)$.



3. Write the equation for the inverse of $h(x) = b^x$. Label the graph.

.....
 A function is
 invertible if its inverse
 is also a function.

4. Do you think all exponential functions are invertible? If so, explain your reasoning. If not, provide a counterexample.

5. Use the graph to analyze the domain and range of the inverse function. Write the domain and range of $h(x)$ using interval notation. Then, describe the relationship between the domain of an exponential function and the domain of its inverse.

The domain of the exponential function is the range of the inverse. The range of the exponential function is the domain of the inverse.

Logarithms were first conceived by a Swiss clockmaker and amateur mathematician Joost Bürgi, but they became more widely known and used after the publication of a book by Scottish mathematician John Napier in 1614. Tables of logarithms were originally used to make complex computations in astronomy, surveying, and other sciences easier and more accurate. With the invention of calculators and computers, the use of logarithm tables as a tool for calculation has decreased. However, many real-world situations can be modeled using logarithmic functions.

Two frequently used logarithms are logarithms with base 10 and base e . A **common logarithm** is a logarithm with base 10 and it is usually written without a specified base.

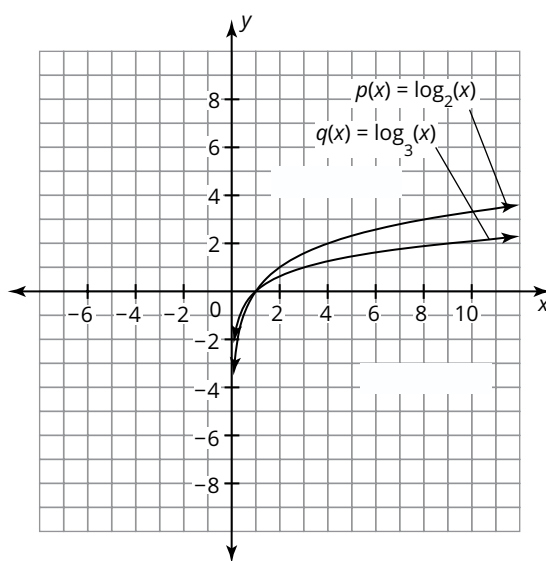
$$c(x) = \log_{10} x \Leftrightarrow c(x) = \log x$$

A **natural logarithm** is a logarithm with base e , and it is usually written as $\ln$.

$$n(x) = \log_e x \Leftrightarrow n(x) = \ln x$$

6. The functions $p(x) = \log_2 x$ and $q(x) = \log_3 x$ have been graphed for you.

a. Sketch and label the functions $c(x) = \log x$ and $n(x) = \ln x$.



- b. Explain how you determined the graphs of $c(x)$ and $n(x)$.
- c. Analyze the key attributes of $p(x)$, $q(x)$, $c(x)$, and $n(x)$. Describe the similarities and differences. Be sure to include the domain and range using set notation.
- d. What is the inverse of the logarithmic function $c(x) = \log x$?
- e. What is the inverse of the logarithmic function $n(x) = \ln x$?

Applying Logarithms

The Richter scale is used to rate the magnitude of an earthquake, or the amount of energy released. An earthquake's magnitude, M , is determined using the equation, $M = \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the earthquake being measured (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake), and I_0 is the intensity of a standard earthquake, or "threshold quake," whose seismograph amplitude is 10^{-4} cm.

1. Write an equation to model the magnitude of a standard earthquake.
2. An earthquake in California measured 6.8 on the Richter scale, while an earthquake in Japan measured 7.2. Use technology to determine how many times more intense the Japanese earthquake was than the Californian earthquake.
3. Early in the century, an earthquake in Indonesia registered 8.3 on the Richter scale. In the same year, another earthquake was recorded in Chile that was 4 times stronger. Use technology to determine the magnitude of the Chilean earthquake.



Talk the Talk

Getting Into a Logarithm

1. Identify the key attributes for exponential and logarithmic functions to complete the table.

	Parent Exponential Function	Parent Logarithmic Function
Function		
Graph		
Domain using set notation		
Range using set notation		
Intercepts		
Asymptotes		

Lesson 5 Assignment

Write

Write the term that best completes each sentence.

1. The _____ of a number for a given base is the exponent to which the base must be raised in order to produce that number.
2. A _____ is a logarithm with base e , and it is usually written as “ln.”
3. A _____ is a function involving a logarithm.
4. A _____ is a logarithm with a base 10, and it is usually written without a base specified.

Remember

All exponential functions are invertible.

A logarithm is an exponent. If $y = b^x$, then x is the logarithm and can be written as $x = \log_b y$.

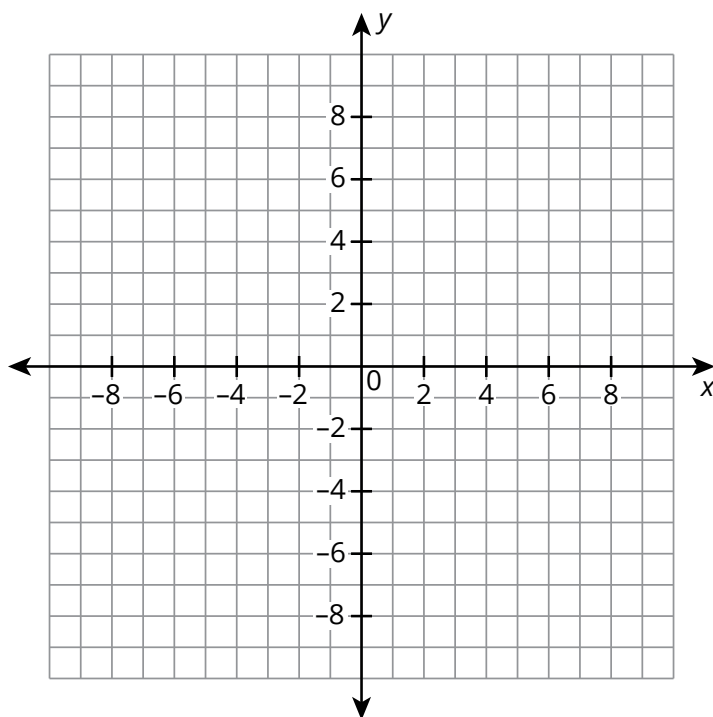
Key attributes of the basic logarithmic function include a domain of positive numbers, a range of all real numbers, and a vertical asymptote at $x = 0$.

Practice

1. Given $f(x) = 2^x$.
 - a. Write the function $f^{-1}(x)$, the inverse of $f(x) = 2^x$.

Lesson 5 Assignment

- b. Graph and label the functions $f(x)$ and $f^{-1}(x)$ on the same coordinate plane.



- c. Describe how to calculate $f^{-1}(3)$ without a calculator. Then, calculate $f^{-1}(2)$, $f^{-1}(4)$, and $f^{-1}(8)$.

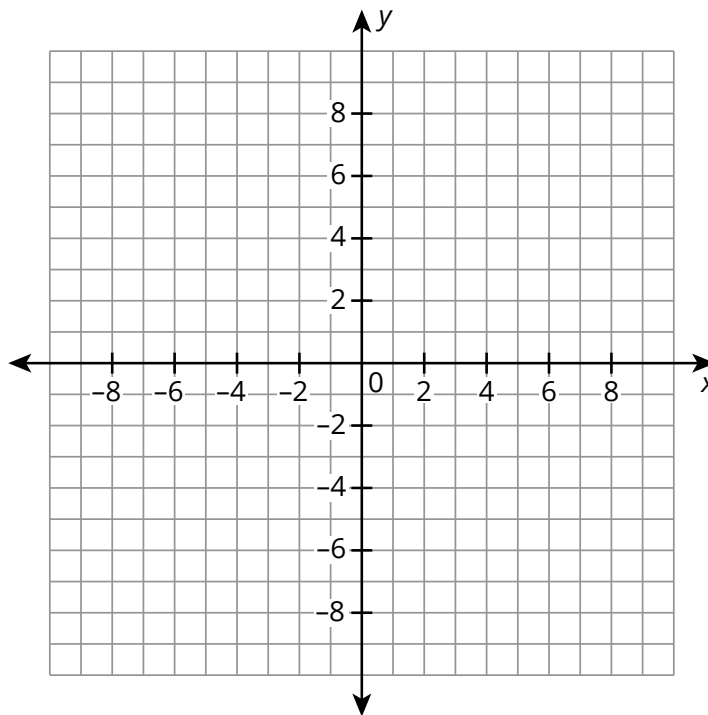
Lesson 5 Assignment

- d. Determine the domain, range, asymptotes, intercepts, and intervals of increase and decrease for $f^{-1}(x)$. Write the domain and range using inequalities.

2. Given $g(x) = 10^x$.

- a. Write the function $g^{-1}(x)$, the inverse of $g(x) = 10^x$.

- b. Graph and label the functions $g(x)$ and $g^{-1}(x)$ on the same coordinate plane.



Lesson 5 Assignment

c. Describe how to calculate $h^{-1}(10)$ without a calculator. Then, calculate $g^{-1}(10)$, $g^{-1}(100)$, and $g^{-1}(1000)$.

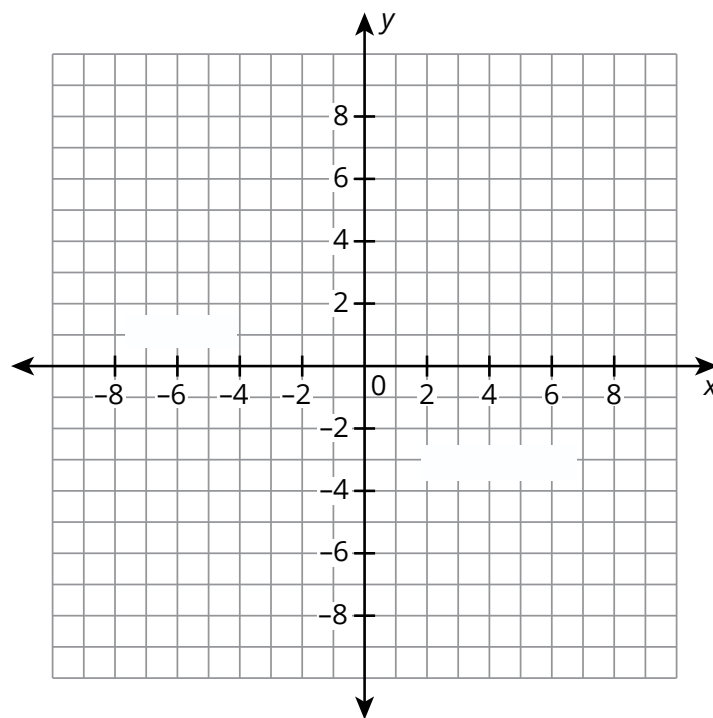
d. Determine the domain, range, asymptotes, intercepts, and intervals of increase and decrease for $g^{-1}(x)$. Write the domain and range using set notation.

3. Given $h(x) = e^x$.

a. Write the function $h^{-1}(x)$, the inverse of $h(x) = e^x$.

Lesson 5 Assignment

- b. Graph and label the functions $h(x)$ and $h^{-1}(x)$ on the same coordinate plane.



- c. Describe how to calculate $h^{-1}(e)$ without a calculator. Then, calculate $h^{-1}(e)$, $h^{-1}(e^2)$, and $h^{-1}(e^3)$.

Lesson 5 Assignment

d. Determine the domain, range, asymptotes, intercepts, and intervals of increase and decrease for $g^{-1}(x)$. Write the domain and range using interval notation.

4. Describe how the parent exponential function and the parent logarithm function are related. Be sure to include the relationship between the domain and range of the functions in your description.

Lesson 5 Assignment

Prepare

Determine the inverse of each function.

1. $f(x) = 2^x$

2. $g(x) = e^x$

3. $h(x) = \log x$

6

Transformations Logarithmic Functions

OBJECTIVES

- Dilate, reflect, and translate logarithmic functions using reference points and transformation function form.
- Investigate graphs logarithmic functions through intercepts, asymptotes, and intervals of increase and decrease.
- Verbally describe how transformations logarithmic functions affect their key attributes.
- Describe how transformations of a function affect the inverse of the function.

.....

You have explored transformations with functions in many function families.

How are the transformations of exponential and logarithmic functions related?

Getting Started

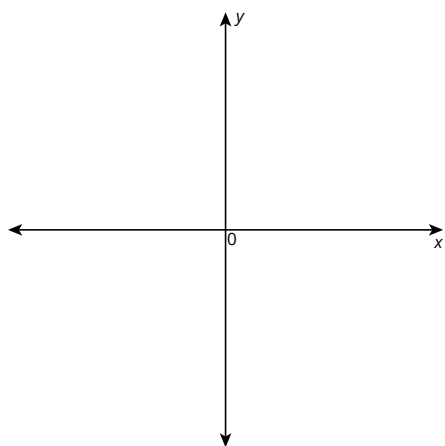
Don't Flip Out Again

Consider the two tables that show logarithmic functions represented as equations and as graphs.

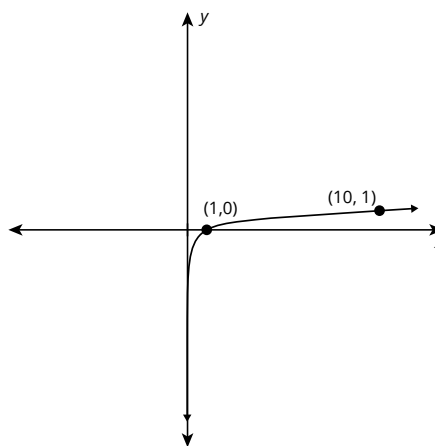
Logarithmic Equations	
$f(x) = \log_2 x$	$g(x) = -\log_2 x$
$h(x) = \log x$	$j(x) = -\log x$

Logarithmic Graphs

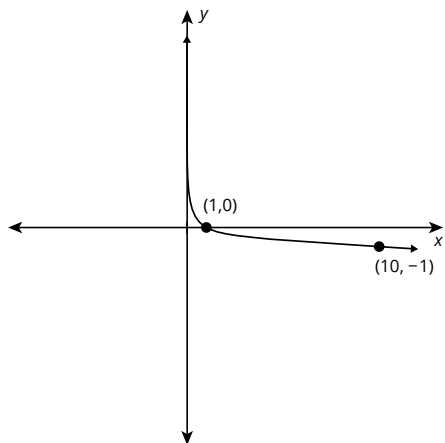
A.



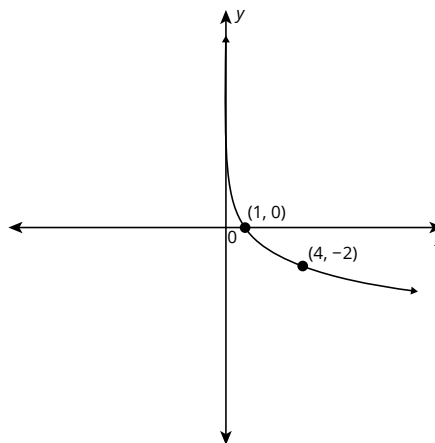
B.



C.



D.



1. Graph the function $f(x) = \log_2 x$ in the table for graph A. Then, analyze the graphs.
 - a. Match the remaining logarithmic equations to their corresponding graphs, and write the equation under the graph it represents.
 - b. Write an equation for transformation in terms of its parent function.

2. Complete the table to describe the key attributes of each function.

Function	Asymptote	Intervals of Increase and Decrease	x-intercept	Domain and Range in Interval Notation
$f(x) = \log_2 x$				
$g(x) = -\log_2 x$				
$h(x) = \log x$				
$j(x) = -\log x$				

- 1082 MODULE 5 • TOPIC 1 • LESSON 6

ACTIVITY
6.1

Graphing Transformations of Logarithmic Functions

In the Getting Started, you analyzed transformations of logarithmic functions that were a reflection across an axis.

You know that transformations performed on any function $f(x)$ to form a new function $g(x)$ can be described by the transformation function form.

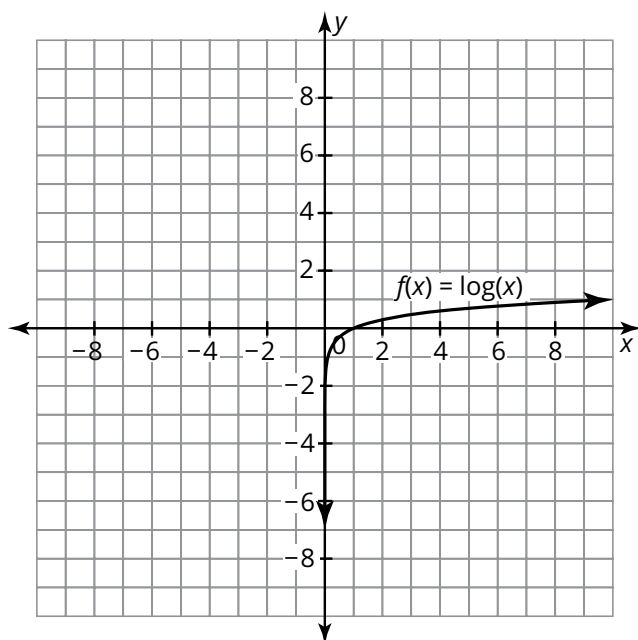
$$g(x) = af(x - c) + d$$

Let's consider single transformations of a logarithmic function.

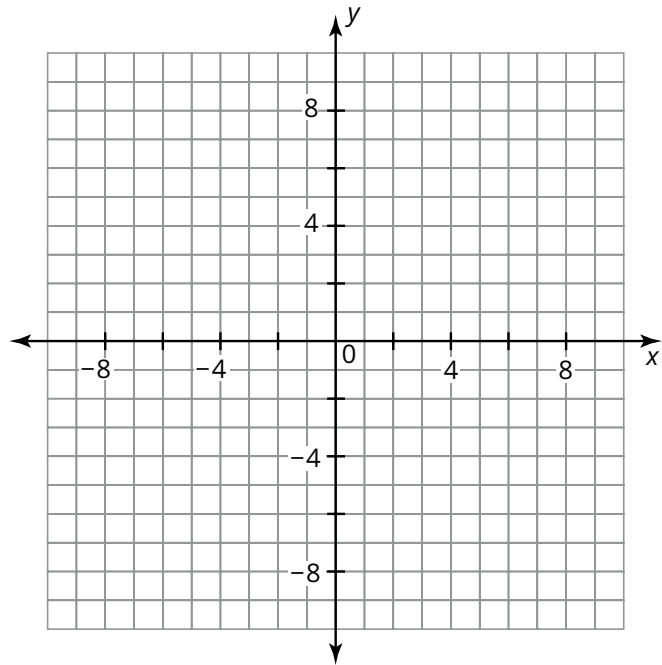
1. Consider each transformation to $f(x) = \log(x)$. Describe the effects of each transformation, including effects on the key attributes of the graph. Then, sketch the transformed graph.

Remember ...

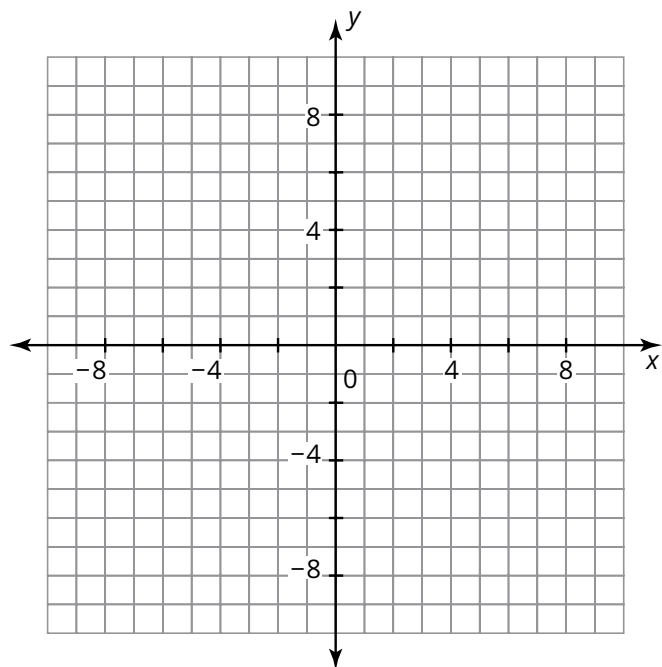
$$\log_{10}(x) = \log x$$



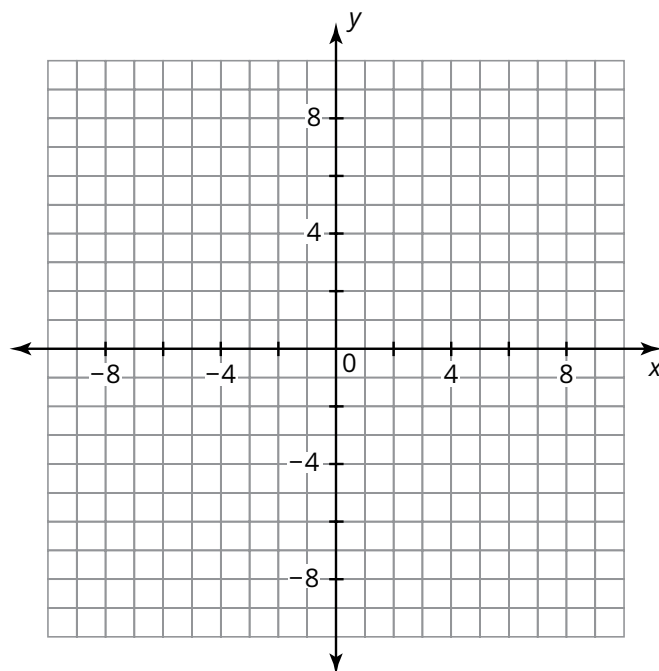
a. $g(x) = 3\log(x)$



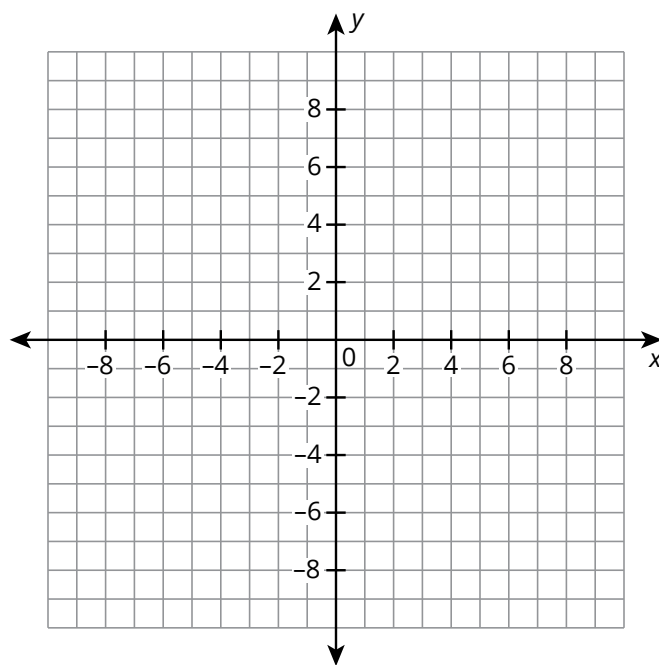
b. $p(x) = \log(x - 3)$



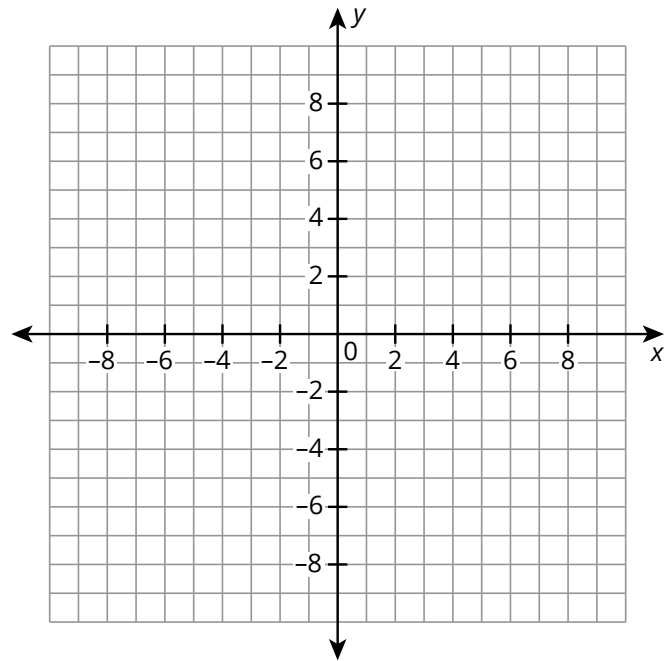
c. $k(x) = \log(x) - 3$



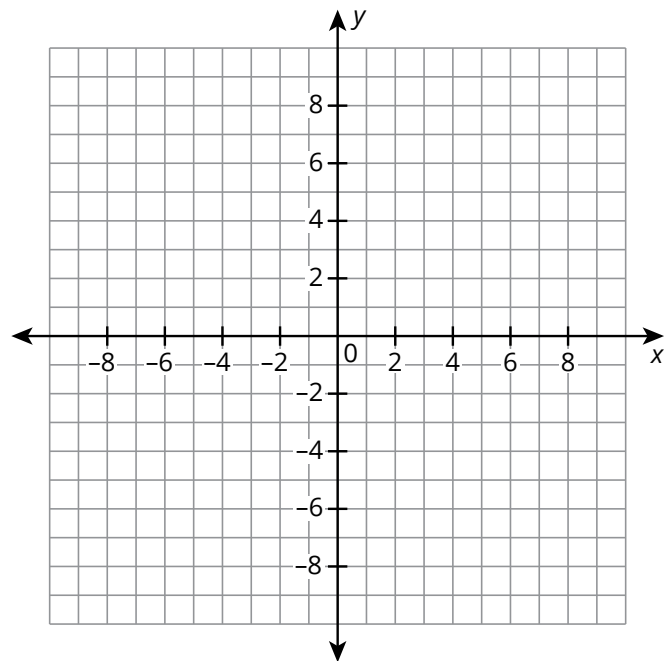
d. $h(x) = \log(x + 3)$



e. $q(x) = \log(x) + 3$

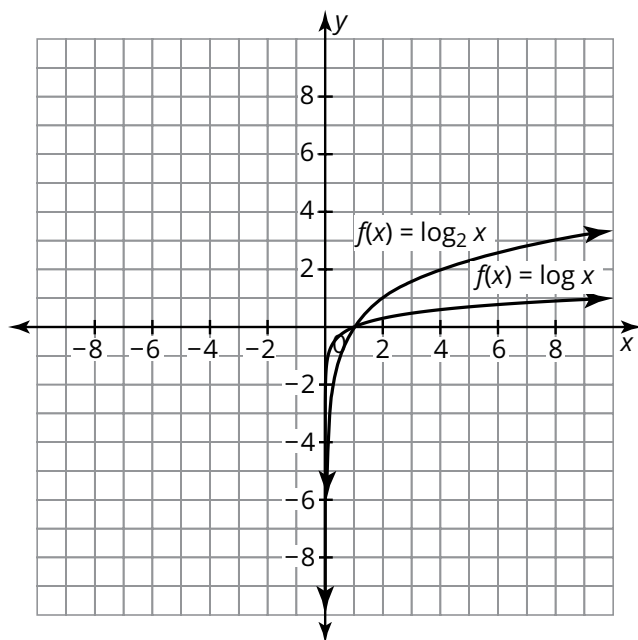


f. $m(x) = -3\log(x)$

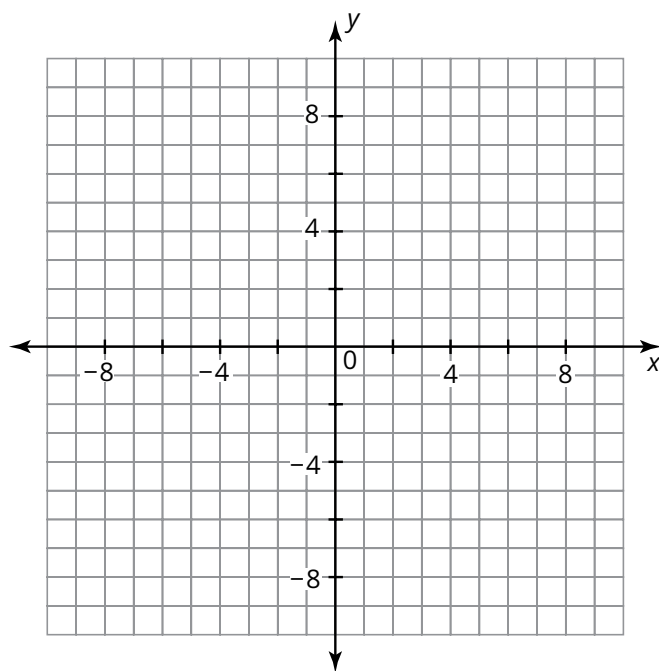


Now let's consider multiple transformations of a logarithmic function.

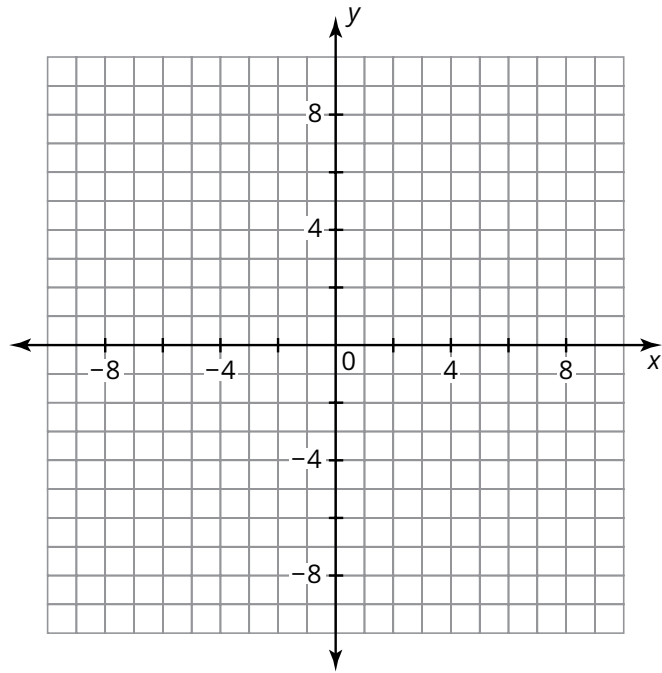
2. Consider the transformations to $f(x) = \log_2(x)$. Describe the effects of the multiple transformations, including effects on the key attributes of the graph. Then, sketch the transformed graph.



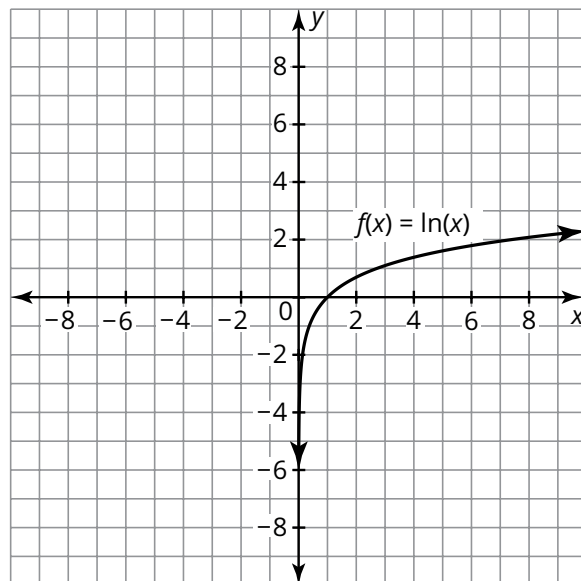
a. $s(x) = 4\log_2(x - 1) + 3$



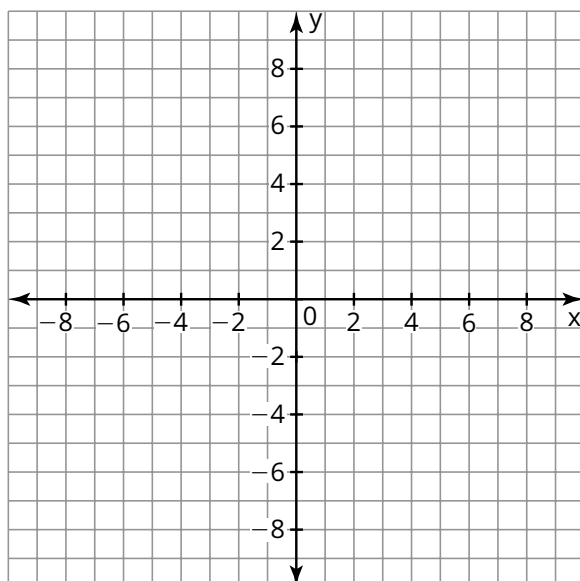
b. $t(x) = -\log_2(x + 3) - 4$



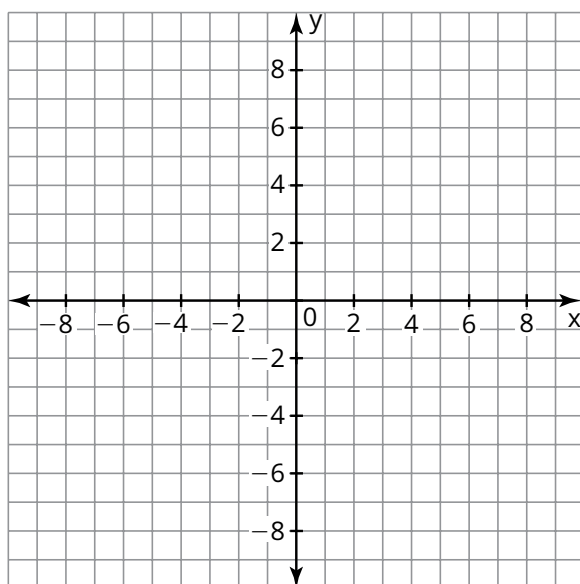
3. Consider the transformations to $f(x) = \ln(x)$. Describe the effects of the multiple transformations, including effects on the key attributes of the graph. Then, sketch the transformed graph.



a. $w(x) = 3 \ln(x - 2) - 4$



b. $z(x) = -4 \ln(x + 3) + 2$



4. Analyze the transformations performed on $f(x)$ in Questions 1 through 3.
- Which, if any, of these transformations affected the domain, range, and asymptotes?
 - What generalizations can you make about the effects of transformations on the domain, range, and asymptotes of logarithmic functions?

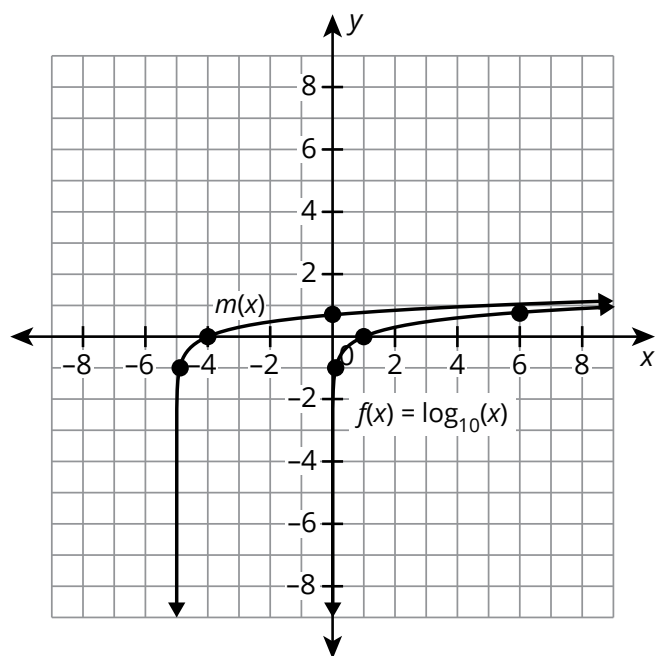
ACTIVITY
6.2

Describing Transformations of Logarithmic Functions

You can describe the transformation(s) performed on a function by comparing the graphs or equations of the original function and the transformed function.

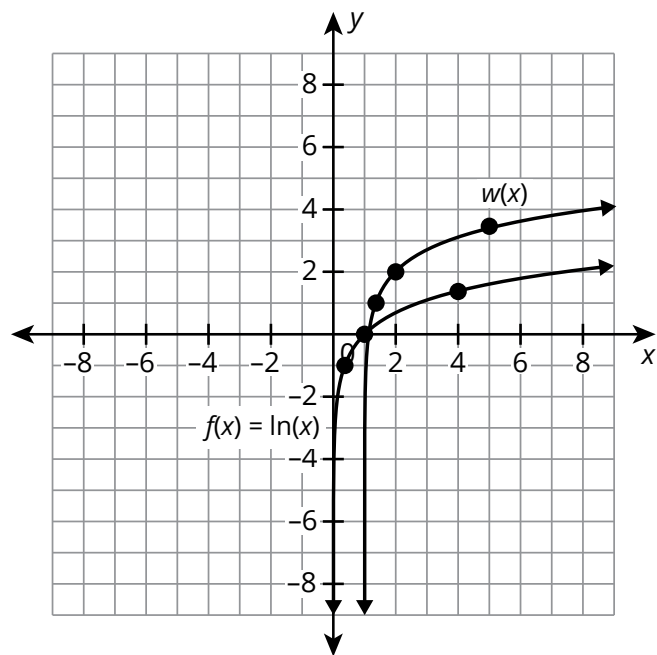
1. Analyze the graphs of $f(x)$ and the transformed function. Describe the transformations produced on $f(x)$ to create the transformed function. Then, write an equation for the transformed function in terms of $f(x)$. For each set of points shown on $f(x)$, the corresponding points are shown on the transformed function.

a. Describe the transformation:



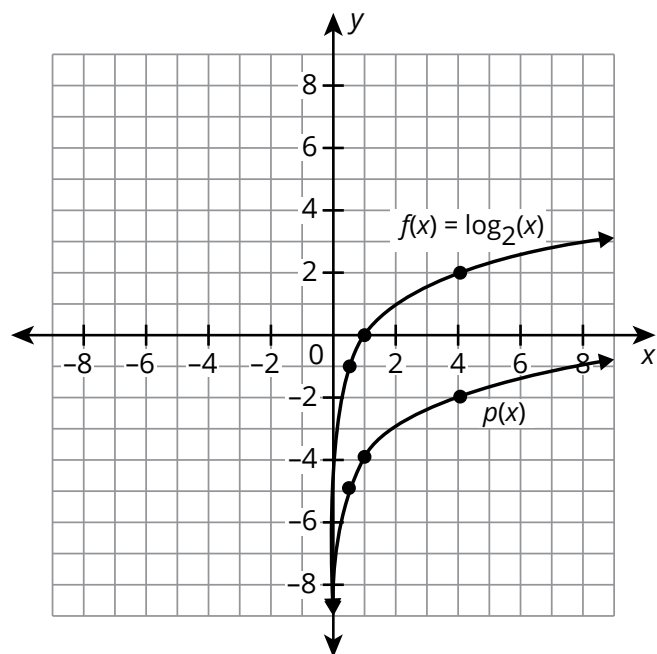
$m(x) =$ _____

b. Describe the transformation:



$w(x) =$ _____

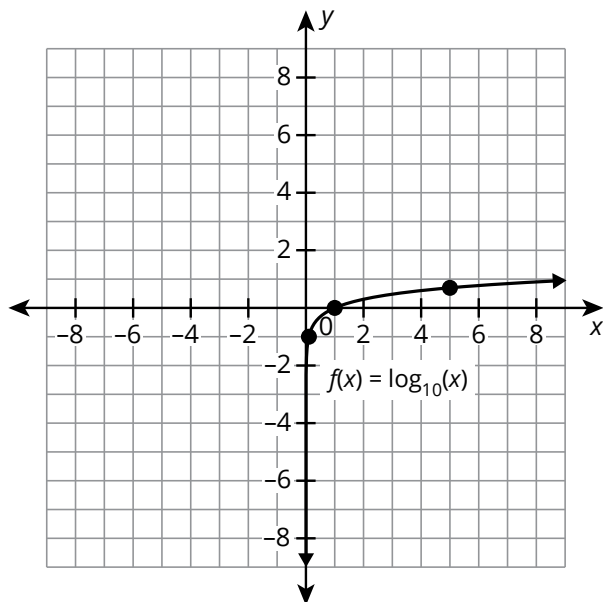
c. Describe the transformation:



$p(x) =$ _____

2. The graph of a parent logarithmic function $f(x)$ is shown. Graph each transformation of $f(x)$. State the domain, range, and asymptotes of your graph.

a. $c(x) = f(x - 2) + 6$

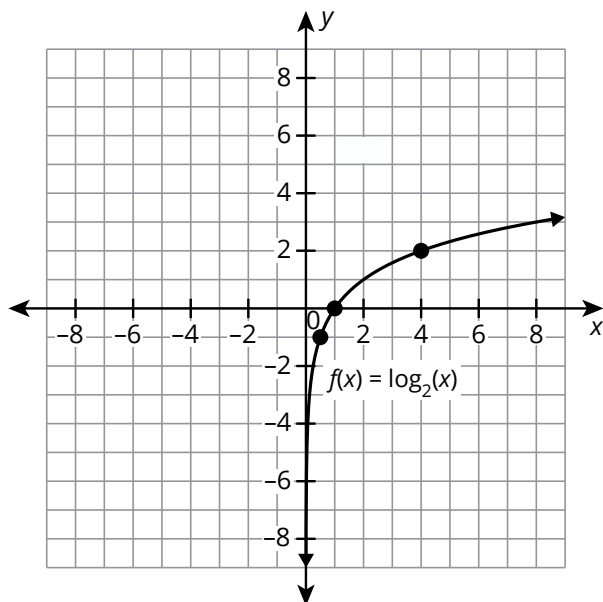


Domain of $c(x)$: (

Range of $c(x)$:

Asymptote of $c(x)$:

b. $n(x) = -f(x) + 3$

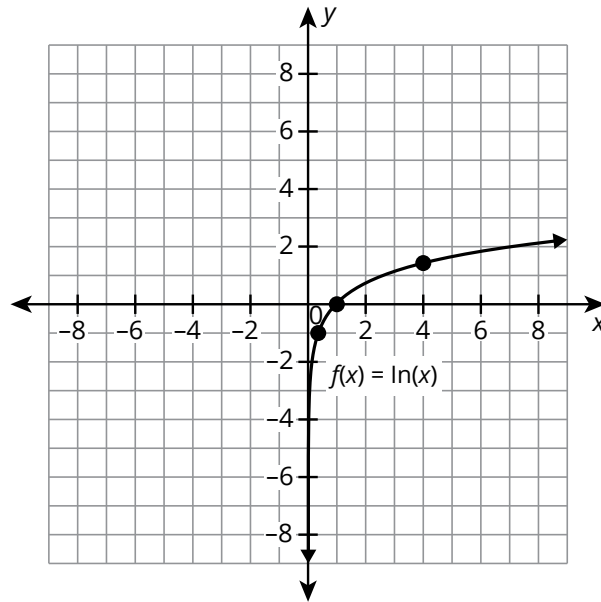


Domain of $n(x)$:

Range of $n(x)$:

Asymptote of $n(x)$:

c. $z(x) = 2f(x + 3)$



Domain of $z(x)$:

Range of $n(x)$:

Asymptote of $n(x)$:

3. The equation for a logarithmic function $m(x)$ is given. The equation for the transformed function $t(x)$ in terms of $m(x)$ is also given. Describe the graphical transformation(s) on $m(x)$ that produce(s) $t(x)$. Then, write a logarithmic equation for $t(x)$.

a. $m(x) = \log x$
 $t(x) = 3m(x) - 4$

b. $m(x) = \ln x$
 $t(x) = -m(x) - 1$

c. $m(x) = \log_2 x$
 $t(x) = 0.2m(x - 3)$

d. $m(x) = \log_2 x$
 $t(x) = m(x) + 5$

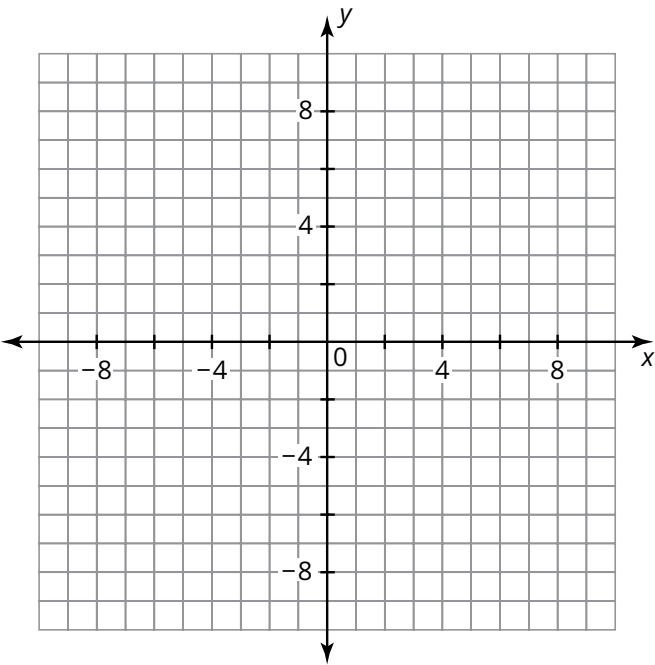
e. $m(x) = \log x$
 $t(x) = -4m(x + 2)$

f. $m(x) = \ln x$
 $t(x) = -\frac{1}{2}m(x + 5)$

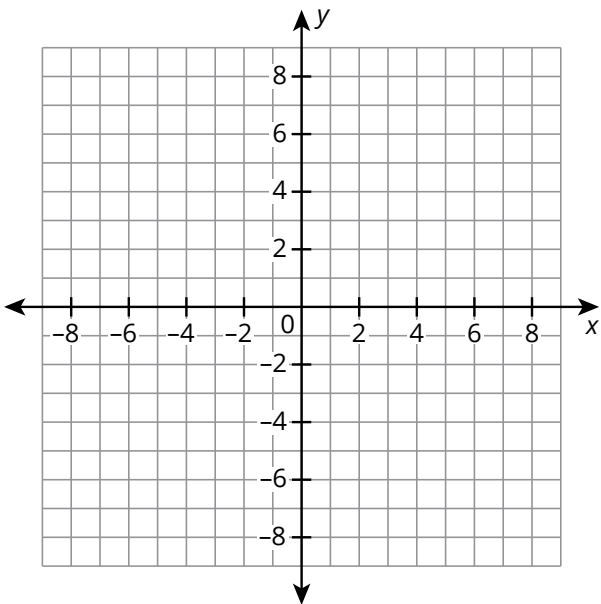
You can describe the transformation of a function given the original function and a key attribute, or attributes, of the transformed function.

4. Write a transformed logarithmic function in terms of $f(x)$ with the attribute(s) given. Then, graph the transformed function.

- a. $f(x) = \log_2 x$
 $m(x)$ vertical asymptote at $x = -3$



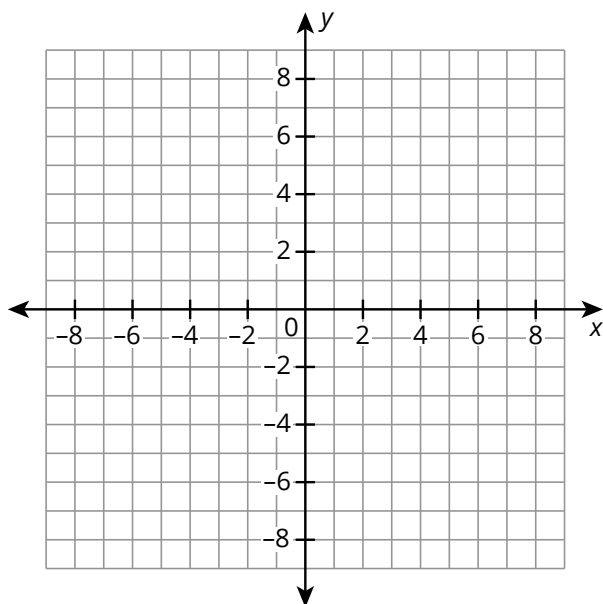
- b. $f(x) = \log x$



Reference Points on $f(x)$	→	Corresponding Points on $h(x)$
$\left(\frac{1}{10}, -1\right)$	→	$\left(\frac{1}{10}, -\frac{1}{2}\right)$
$(1, 0)$	→	$(1, 0)$
$(10, 1)$	→	$\left(10, \frac{1}{2}\right)$

c. $f(x) = \ln x$

$k(x)$ has an x-intercept at $(3, 0)$



Relating Inverses of Transformations

Recall that the inverse of an exponential function is a logarithmic function. Let's explore how the transformation of an exponential function and its inverse are related.

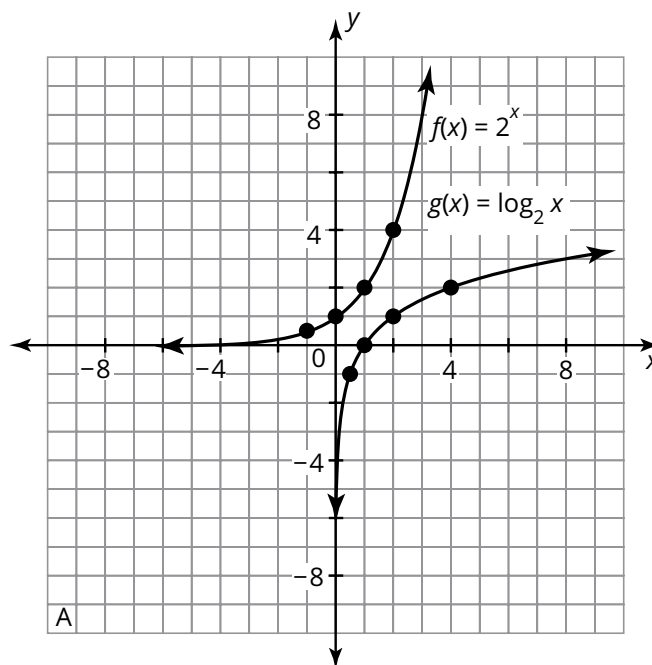
1. Consider the functions $f(x) = 2^x$ and $g(x) = f^{-1}(x)$, or $\log_2 x$. The graphs of $f(x)$ and $g(x)$ are provided.

- Graph the transformations on $f(x)$.
- Describe the transformation(s) performed on each.
- Then, graph the inverse of the transformed function.
- Finally, describe and label the graph of the inverse of the transformed function as a transformation on $g(x)$.

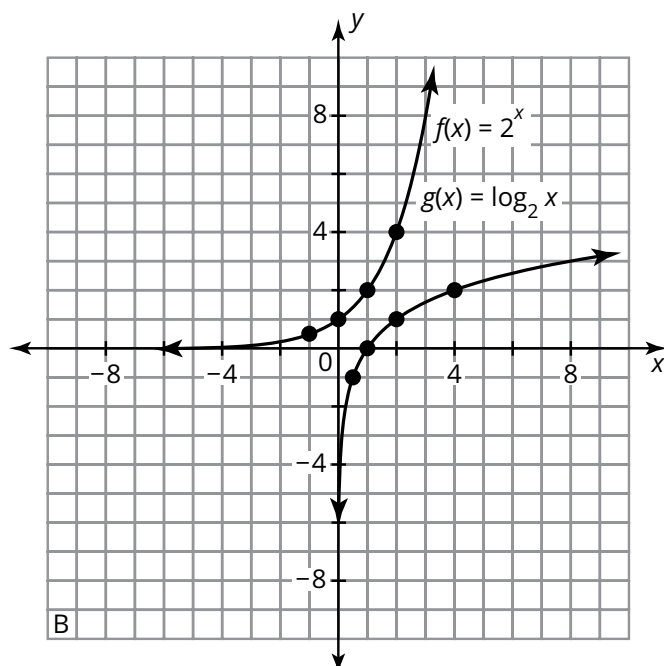
a. $f(x + 4)$

Think About . . .

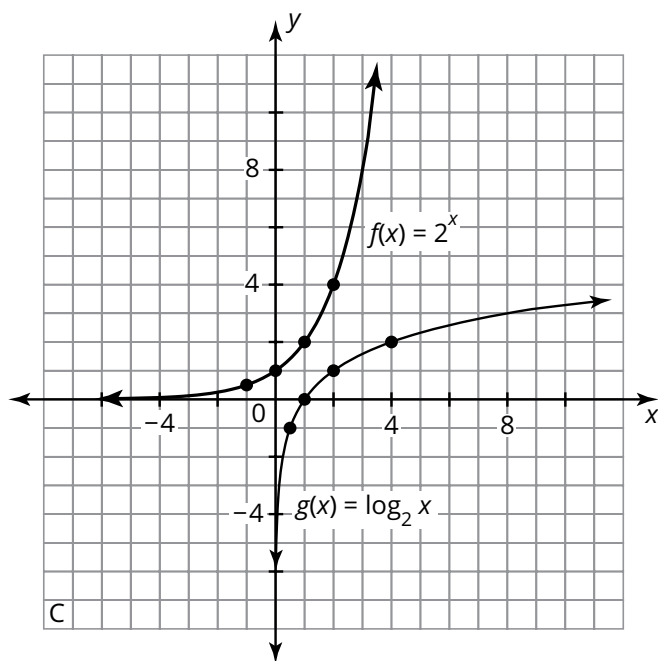
When graphing functions, which key attributes should be labeled?



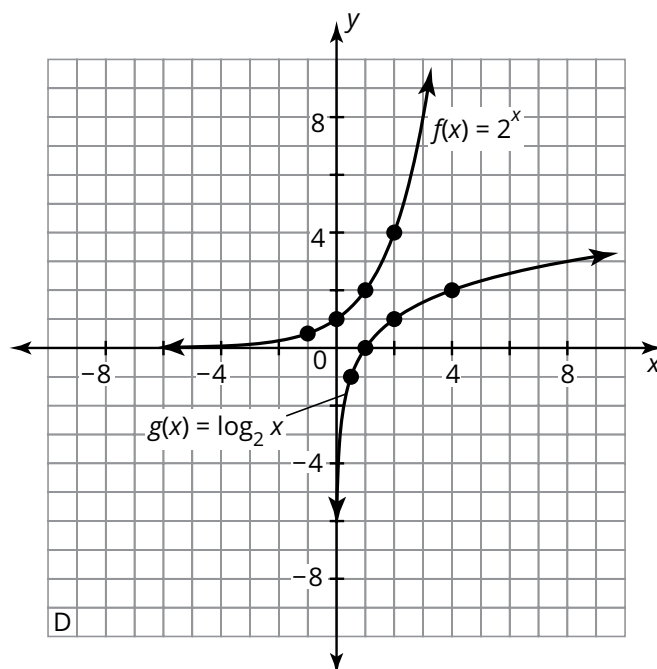
b. $f(x) + 5$



c. $f(x - 3) + 6$

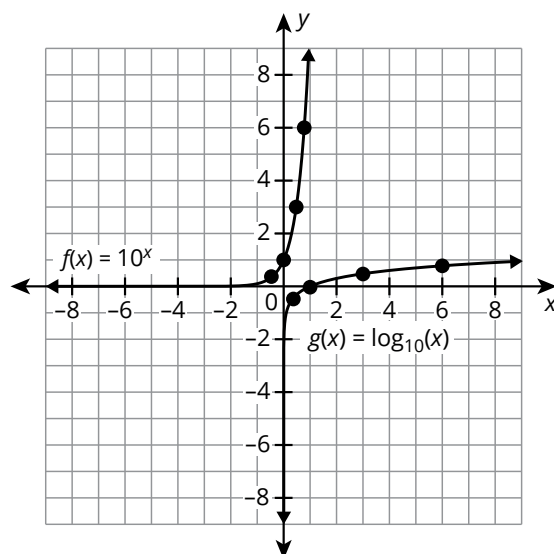


d. $f(x + 1) - 1$

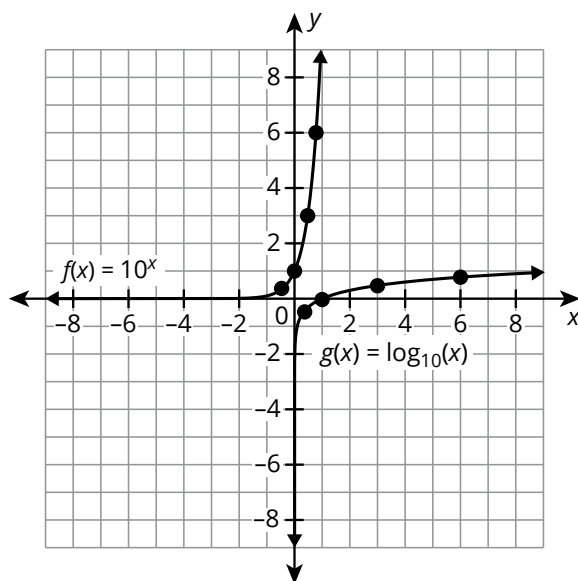


2. Consider the functions $f(x) = 10^x$ and $g(x) = f^{-1}(x)$, or $\log x$. The graphs of $f(x)$ and $g(x)$ are provided. Follow the same instructions from Question 1.

a. $f(x + 1) - 2$

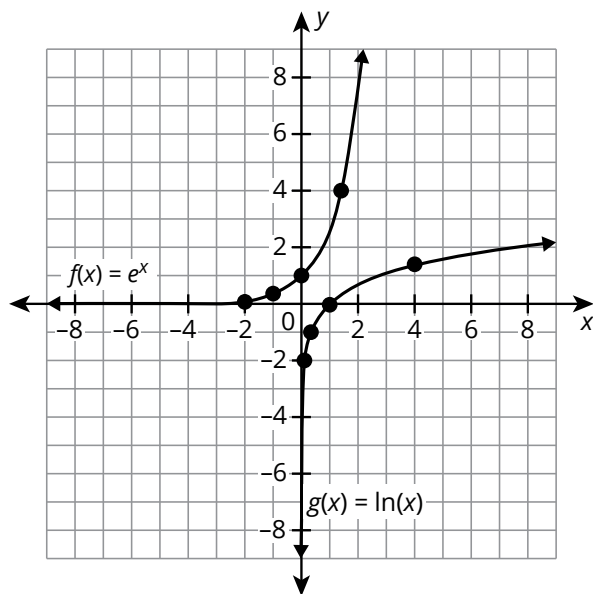


b. $f(x - 2) + 3$

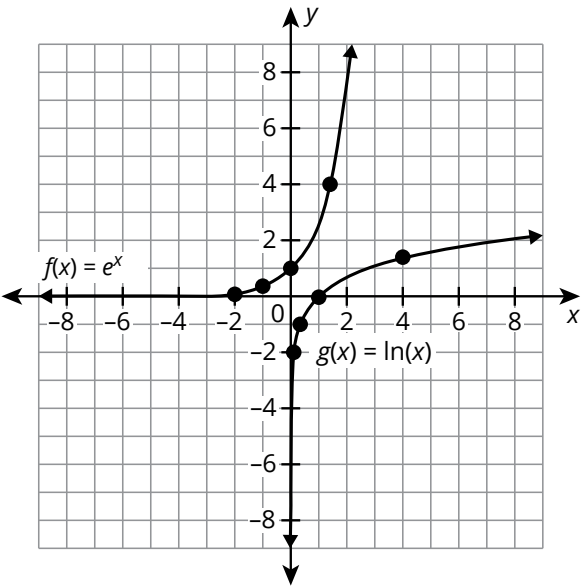


3. Consider the functions $f(x) = e^x$ and $g(x) = f^{-1}(x)$, or $\ln x$. The graphs of $f(x)$ and $g(x)$ are provided. Follow the same instructions from Question 1.

a. $f(x - 3) + 4$



b. $f(x + 2) - 3$



4. Generalize the effects of transformations on $f(x)$ and its inverse function, $f^{-1}(x)$. Complete the table to organize your results.

	Transformation on $f(x)$	Effect of Transformation on $f^{-1}(x)$
$f(x + c)$		
$f(x - c)$		
$f(x) + d$		
$f(x) - d$		

5. Consider the function $y = f(x)$ and the transformed function $g(x)$.
Write an equation for $g^{-1}(x)$ in terms of $f^{-1}(x)$.

a. $g(x) = f(x - 1)$ $g^{-1}(x) = \underline{\hspace{2cm}}$

b. $g(x) = f(x) - 2$ $g^{-1}(x) = \underline{\hspace{2cm}}$

c. $g(x) = f(x + 5)$ $g^{-1}(x) = \underline{\hspace{2cm}}$

d. $g(x) = f(x - 4) + 3$ $g^{-1}(x) = \underline{\hspace{2cm}}$



Talk the Talk

Moving Right Along Again

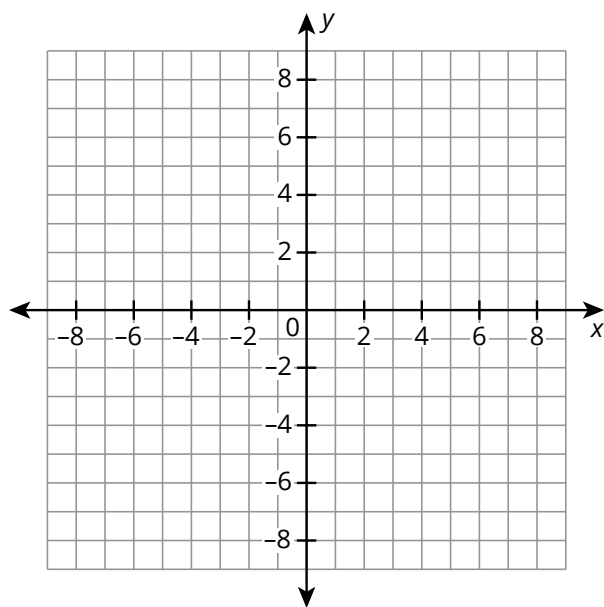
In this lesson, you graphed and analyzed transformations of logarithmic functions. You also explored how the transformation of an exponential function is related to the transformation of its inverse function, a logarithmic function.

1. Determine whether each statement is true or false. If the statement is false, rewrite the statement so that it is true.
 - a. The range of logarithmic functions is not affected by translations or dilations.
 - b. Translating a function horizontally left c units results in the inverse function translating down c units.
 - c. Horizontal translations do not affect the domain and vertical asymptote of logarithmic functions.
 - d. Translating a function up d units results in a translation down d units for the inverse function.
 - e. Vertical dilations affect the range and the vertical asymptote of logarithmic functions.

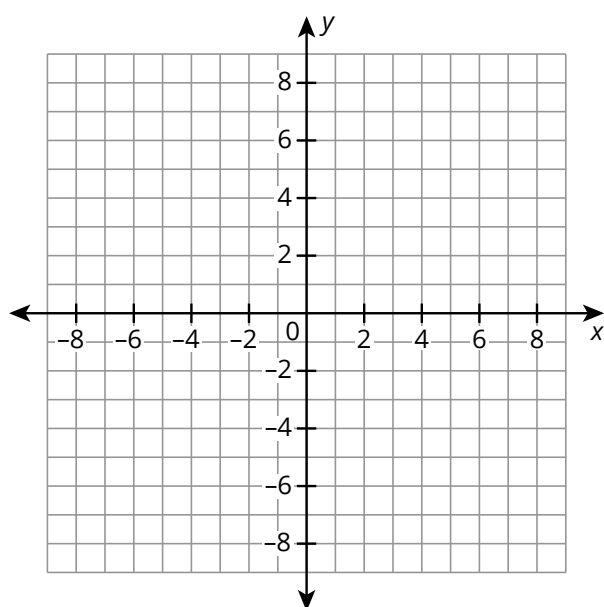
2. Given the function $f(x)$:

- Graph the function.
- Write the equation for the inverse function, $f^{-1}(x)$.
- Graph the inverse function.
- Identify the asymptotes of the function and inverse function.

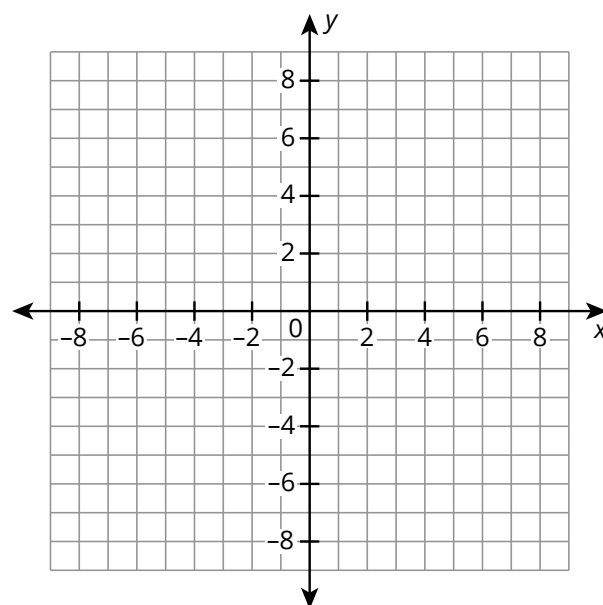
a. $f(x) = -2^x$



b. $f(x) = \log(x - 3)$



c. $f(x) = \ln x + 4$



Lesson 6 Assignment

Write

Match the transformation of $f(x)$ with its effect on $f^{-1}(x)$.

- | | |
|--------------------------------------|---|
| 1. $f(x)$ translates up d units | a. $f^{-1}(x)$ translates up c units |
| 2. $f(x)$ translates right c units | b. $f^{-1}(x)$ translates left d units |
| 3. $f(x)$ translates down d units | c. $f^{-1}(x)$ translates right d units |
| 4. $f(x)$ translates left c units | d. $f^{-1}(x)$ translates down c units |

Remember

The transformation function form $g(x) = af(x - c) + d$ can be applied to logarithmic functions.

A horizontal translation on a function produces a vertical translation on its inverse, while a vertical translation on a function produces a horizontal translation on its inverse. A vertical dilation on a function produces a horizontal dilation by the same factor on its inverse, while a horizontal dilation on a function produces a vertical dilation by the same factor on its inverse.

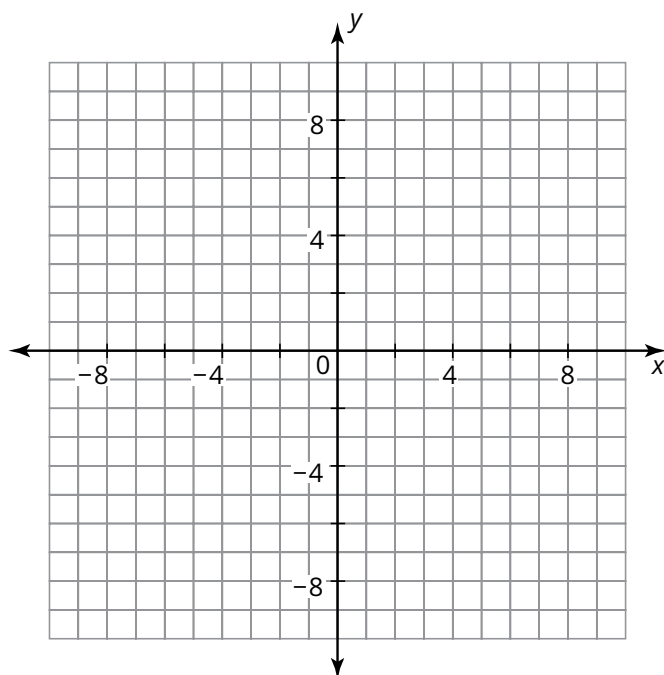
Practice

- Consider the function $g(x)$, which is formed by translating the function $f(x) = \log_2 x$ left 3 units and up 4 units.
 - Write $g(x)$ in terms of $f(x)$.
 - Complete the table by determining the corresponding point on $g(x)$ for each reference point on $f(x)$.

Reference Point on $f(x)$	Corresponding Point on $g(x)$
(0.5, -1)	
(1, 0)	
(2, 1)	
(4, 2)	

Lesson 6 Assignment

- c. Graph and label $f(x)$ and $g(x)$ on the same coordinate plane.



- d. Write $g(x)$ as a logarithmic function.
- e. List the domain, range, and any asymptotes of the logarithmic function $g(x)$. Write the domain and range using interval notation.

Lesson 6 Assignment

2. Describe the transformation of $f(x)$ that produces $g(x)$. Then, write $g(x)$ as a logarithmic function.

a. $f(x) = \ln x$, $g(x) = 2f(x - 1) - 3$

b. $f(x) = \log_2 x$, $g(x) = -3f(x) + 2$

c. $f(x) = \log x$, $g(x) = -f(x + 5) - 7$

d. $f(x) = \log_2 x$, $g(x) = -\frac{1}{2}f(x - 6)$

Lesson 6 Assignment

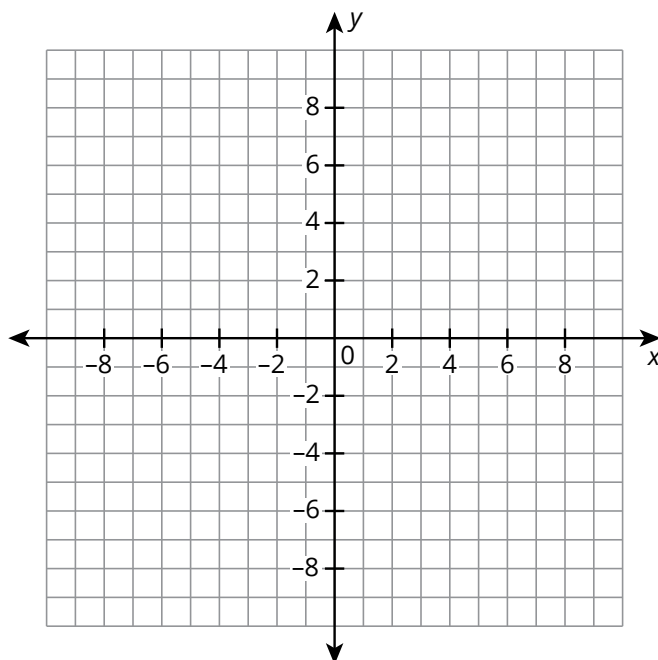
e. $f(x) = \ln x$, $g(x) = -0.3f(x + 9) + 4$

f. $f(x) = \log x$, $g(x) = \frac{1}{3}f(x) + 8$

3. Given the function $f(x)$:

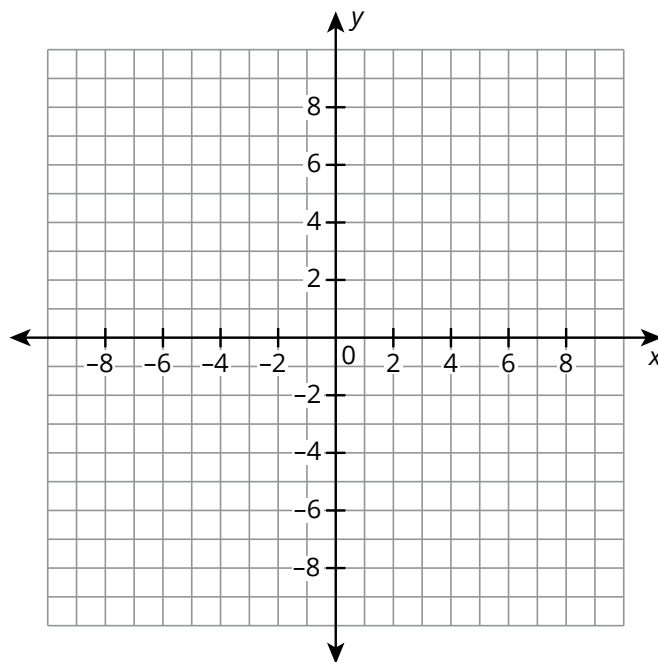
- Graph the function.
- Write the equation of the inverse function $f^{-1}(x)$.
- Graph the inverse function.

a. $f(x) = \log_2(x + 2) - 1$

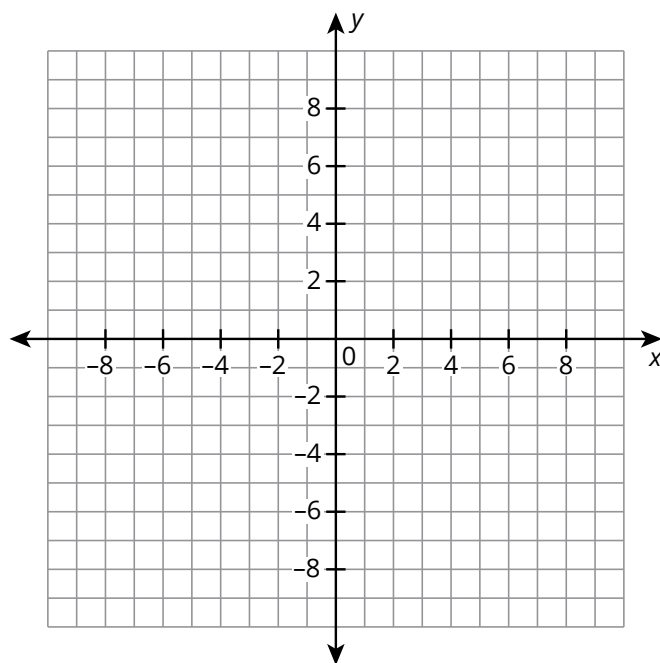


Lesson 6 Assignment

b. $f(x) = \log(x - 3) + 4$



c. $f(x) = \ln(x + 4) + 2$



Lesson 6 Assignment

Prepare

Rewrite each logarithmic equation as an exponential equation.

1. $\log_{10} 0.1 = x$

2. $\log_2 16 = x$

3. $\log_{10} 10^9 = x$

4. $\log_{10} \frac{1}{4} = x$

Exponential and Logarithmic Functions

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

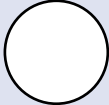
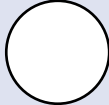
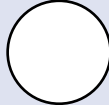
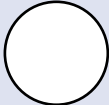
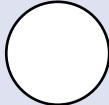
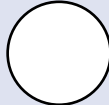
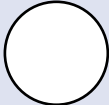
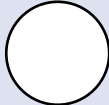
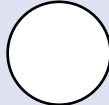
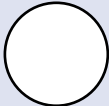
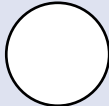
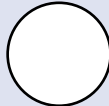
Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Exponential and Logarithmic Functions* topic by:

TOPIC 1: <i>Exponential and Logarithmic Functions</i>	Beginning of Topic	Middle of Topic	End of Topic
formulating an exponential function from a geometric sequence.			
identifying the key attributes of an exponential function from a table or graph.			
relating the key attributes of an exponential function to characteristics of a real-world scenario.			
classifying real-world problems as exponential growth or decay.			
explaining the importance of the irrational number e .			
using the irrational number e to solve problems involving continuously compounding interest and population growth.			
defining the inverse of an exponential function as a logarithmic function.			

continued on the next page

TOPIC 1 SELF-REFLECTION
continued

TOPIC 1: <i>Exponential and Logarithmic Functions</i>	Beginning of Topic	Middle of Topic	End of Topic
determining the values of exponential and logarithmic functions from a graph or table.			
identifying and describing the domain of an exponential graph and its inverse.			
graphing exponential functions and logarithmic functions with bases 2, 10, and e .			
describing key attributes of exponential and logarithmic functions through intercepts, asymptotes, and intervals of increase and decrease.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

- Describe a new strategy you learned in the *Exponential and Logarithmic Functions* topic.

- What mathematical understandings from the topic do you feel you are making the most progress with?

continued on the next page

TOPIC 1 SELF-REFLECTION *continued*

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 1 SUMMARY

Exponential and Logarithmic Functions Summary

LESSON

1

Comparing Linear and Exponential Functions

KEY TERMS

- half-life
- natural base e
[base natural e]
- logarithm
[logaritmo]
- logarithmic function
[función logarítmica]
- common logarithm
[logaritmo común/decimal]
- natural logarithm
[logaritmo natural]

The recursive formula for a geometric sequence is given by $a_n = a_{n-1} \cdot r$, where a_n represents the n th term of the sequence, a_{n-1} represents the previous term, and r represents the common ratio.

The explicit formula for a geometric sequence is given by $a_n = a_1 \cdot r^{(n-1)}$, where a_n represents the n th term of the sequence, a_1 represents the first term of the sequence, and r represents the common ratio.

Allison's income is shown. Her income represents a geometric sequence.

The recursive formula that represents Allison's income is $g_n = g_{n-1} \cdot 2$.

The explicit formula that represents Allison's income, $g_n = 0.01 \cdot 2^{(n-1)}$, can be written in function notation using the properties of powers.

Statement	Reason
$g_n = 0.01 \cdot 2^{(n-1)}$	Explicit formula for a geometric sequence
$g(n) = 0.01 \cdot 2^{(n-1)}$	Rewrite in function notation.
$g(n) = 0.01 \cdot 2^n \cdot 2^{-1}$	Product rule
$g(n) = 0.01 \cdot 2^n \cdot \frac{1}{2}$	Definition of negative exponents
$g(n) = 0.01 \cdot \frac{1}{2} \cdot 2^n$	Commutative property of multiplication
$g(n) = 0.005 \cdot 2^n$	Associative property of multiplication

So $g_n = 0.01 \cdot 2^{(n-1)}$ written in function notation is $f(n) = 0.005 \cdot 2^n$.

A geometric sequence, when written in function notation, is called an exponential function. The function gets its name from the variable in the exponent.

Week	Allison's Income (\$)
1	0.01
2	0.02
3	0.04
4	0.08
5	0.16
6	0.32
7	0.64
8	1.28
9	2.56
10	5.12
11	10.24

A **half-life** is the amount of time it takes a substance to decay to half of its original amount.

The table shows the amount of caffeine in a person's system over time, starting with 80 milligrams of caffeine. The caffeine has a half-life of 5 hours.

Time Elapsed (hours)	0	5	10	15	20	25
Caffeine in System (mg)	80	40	20	10	5	2.5
Number of Half-Life Cycles	0	1	2	3	4	5

An equation to represent the half-life of the caffeine would be,

$$g(n) = 80\left(\frac{1}{2}\right)^{\left(\frac{n}{5}\right)}, \text{ where } n \text{ is the number of hours.}$$

LESSON

2

Properties of Exponential Graphs

Parent exponential functions, whether increasing or decreasing, have the same domain, range, asymptote, and intercepts.

	Parent Exponential Functions	
	Exponential Growth $f(x) = b^x, b > 1$	Exponential Decay $f(x) = b^x, 0 < b < 1$
Domain in interval notation	$(-\infty, \infty)$	$(-\infty, \infty)$
Range in interval notation	$(0, \infty)$	$(0, \infty)$
Asymptote	$y = 0$	$y = 0$
Intercept	$(0, 1)$	$(0, 1)$
Intervals of Increase or Decrease	Increasing over $(-\infty, \infty)$	Decreasing over $(-\infty, \infty)$

For compounded interest, the amount in an account, A , can be represented with the equation $A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$, where P represents the principal, or starting amount, r represents the interest rate, and n represents the number of times the interest is compounded in a given amount of time.

The table below shows the amount of compound interest earned with an initial investment of \$1 over a year at various compounding intervals.

Period of Compounding	n	Formula	Amount
Yearly	1	$1\left(1 + \frac{1}{1}\right)^{1 \cdot 1}$	2.00
Semi-Annually	2	$1\left(1 + \frac{1}{2}\right)^{2 \cdot 1}$	2.25
Quarterly	4	$1\left(1 + \frac{1}{4}\right)^{4 \cdot 1}$	2.44141
Monthly	12	$1\left(1 + \frac{1}{12}\right)^{12 \cdot 1}$	2.61303
Weekly	52	$1\left(1 + \frac{1}{52}\right)^{52 \cdot 1}$	2.692597
Daily	365	$1\left(1 + \frac{1}{365}\right)^{365 \cdot 1}$	2.714567
Hourly	8760	$1\left(1 + \frac{1}{8760}\right)^{8760 \cdot 1}$	2.718127
Every Minute	525,600	$1\left(1 + \frac{1}{525,600}\right)^{525,600 \cdot 1}$	2.718279
Every Second	31,536,000	$1\left(1 + \frac{1}{31,536,000}\right)^{31,536,000 \cdot 1}$	2.718282

The amount that the interest earnings approaches as the compounding frequency increases is an irrational number called e .

$$e \approx 2.718281828459045 \dots$$

It is often referred to as the **natural base e** .

As the frequency of compounding, n , increases, the value $\left(1 + \frac{1}{n}\right)^n$

approaches e . Thus, the formula for compound interest, with continuous compounding, can be written as:

$$A = Pe^{rt}$$

LESSON

3

Transformations of Exponential Functions

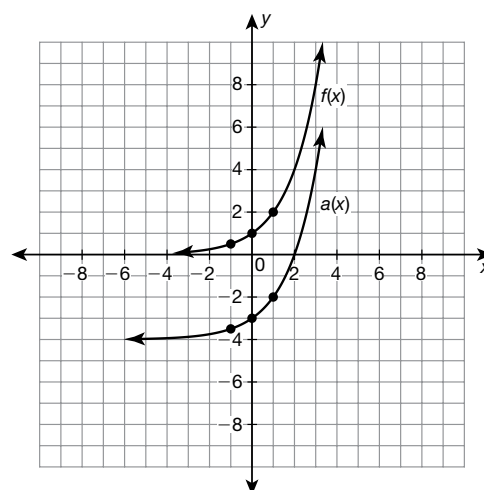
Consider the functions $y = f(x)$ and $g(x) = af(x + c) - d$. Recall that the d -value translates $f(x)$ vertically, the c -value translates $f(x)$ horizontally, and the a -value vertically stretches or compresses $f(x)$. Exponential functions are transformed in the same manner.

You can use reference points to graph transformations of exponential functions.

$$f(x) = 2^x$$

$$a(x) = f(x) + 4$$

Reference Points on $f(x)$	Corresponding Points on $a(x)$
$(21, \frac{1}{2})$	$(21, -\frac{7}{2})$
$(0, 1)$	$(0, -3)$
$(1, 2)$	$(1, -2)$



Domain of $a(x)$: $-\infty < x < \infty$

Range of $a(x)$: $y > -4$

Horizontal asymptote of $a(x)$: $y = -4$

For an exponential function with a base of b , the transformational function can be written as:

$$t(x) = a \cdot b^{(x-c)} + d$$

Effects of the a -value:

- When $a > 1$, the graph of $f(x) = b^x$ is stretched vertically by a factor of a .
- When $0 < a < 1$, the graph of $f(x) = b^x$ is compressed vertically by a factor of a .
- When $a < 0$, the graph of $f(x) = b^x$ is reflected across the x -axis and dilated by a factor of $|a|$.

Effects of the d -value:

- When $d < 0$, the graph of $f(x) = b^x$ is translated up d units.
- When $d > 0$, the graph of $f(x) = b^x$ is translated down d units.

Effects of the c -value:

- When $c > 0$, the graph of $f(x) = b^x$ is translated to the right c units.
- When $c < 0$, the graph of $f(x) = b^x$ is translated to the left c units.

Consider the functions $m(x)$ and $t(x)$.

$$m(x) = 10^x$$

$$t(x) = -m(x) - 3$$

The graph of the function $m(x)$ is translated vertically up 3 units and is reflected across the x -axis to produce $t(x)$.

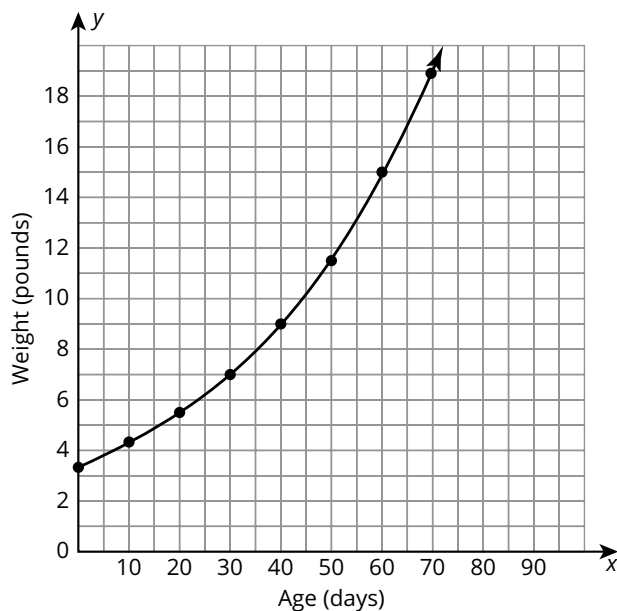
$$t(x) = -10^x - 3$$

Technology can be used to determine which exponential regression function best models real-world data. This regression function can be used to predict future values.

For example, the table shows the weight of a golden retriever puppy as recorded during its growth. A scatterplot and regression function of the data are shown.

The exponential regression function for this scenario is $f(x) = 3.31(1.025)^x$.

Age (days)	Weight (pounds)
0	3.25
10	4.25
20	5.5
30	7
40	9
50	11.5
60	15
70	19



The equation can be used to predict the puppy's weight at 80 days.

$$f(80) = 3.3(1.03)^{80}$$

$$f(80) = 35.1$$

The puppy's weight will be approximately 35.1 pounds on day 80.

The number of cell phone users increases with the increase of each year. Increasing functions can be modeled by a linear or exponential function. However, because the number of cell phone users does not increase at a constant rate, the data cannot be modeled by a linear function. The exponential regression function for the data is $f(x) = 285(1.75)^x$. This function, when graphed, closely models the data.

It is necessary to define a new function in order to write the equation for the inverse of an exponential function. The **logarithm** of a number for a given base is the exponent to which the base must be raised in order to produce the number. If $y = b^x$, then x is the logarithm and can be written as $x = \log_b y$. The value of the base of a logarithm is the same as the base in the exponential expression b^x .

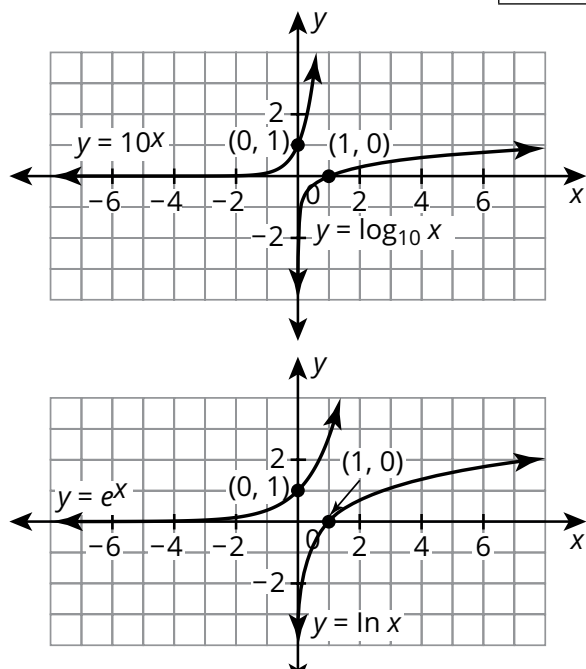
For example, the number 3 is the logarithm to which base 2 must be raised to produce the argument 8. The base is written as the subscript 2. The logarithm, or exponent, is the output 3. The argument of the logarithm is 8.

$$3 = \log_2 8$$

exponent
argument
↙
↘
3
2
8
↘
↙
base

You can write any exponential equation as a logarithmic equation and vice versa.

Exponential Form	Logarithmic Form
$y = b^x$	$x = \log_b y$
$16 = 4^2$	$2 = \log_4 16$
$1000 = 10^3$	$3 = \log_{10} 1000$
$a = b^c$	$c = \log_b a$



Recall that the logarithm of a number for a given base is the exponent to which the base must be raised in order to produce the number. If $y = b^x$, then the logarithm is written as $x = \log_b y$, where $b > 0$, $b \neq 1$, and $y > 0$.

A **logarithmic function** is a function involving a logarithm.

Two frequently used logarithms are logarithms with base 10 and base e . A **common logarithm** is a logarithm with base 10, and it is usually written as $\log$ without a base specified.

$$c(x) = \log_{10} x \Leftrightarrow c(x) = \log x$$

A **natural logarithm** is a logarithm with base e , and it is usually written as $\ln$.

$$n(x) = \log_e x \Leftrightarrow n(x) = \ln x$$

One example of the use of logarithms in real-world applications is the Richter scale. The Richter scale is used to rate the magnitude of an earthquake, or the amount of energy released. An earthquake's magnitude, M , is determined using the equation, $M = \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the earthquake being measured (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake), and I_0 is the intensity of a standard earthquake, or "threshold quake," whose seismograph amplitude is 10^{-24} cm.

Since the Richter Scale is a logarithmic scale, it is based on powers of ten. This can be used to determine the difference in intensity of earthquakes. For example, consider an earthquake in California that measured 6.8 on the Richter scale and an earthquake in Japan that measured 7.2. The difference between the two earthquakes is $7.2 - 6.8 = 0.4$, thus the earthquake in Japan was $10_{0.4} \approx 2.511$ times more intense than the one in California.

LESSON

6

Transformations of Logarithmic Functions

The transformation function form of a logarithmic function is

$$g(x) = a \cdot \log(x - c) + d.$$

The range of a logarithmic function is not affected by translations or dilations. Horizontal translations affect the domain and vertical asymptote of logarithmic functions.

You can describe a transformation in more than one way—in terms of the original function, as an equation in transformation function form, using key attributes, or with words.

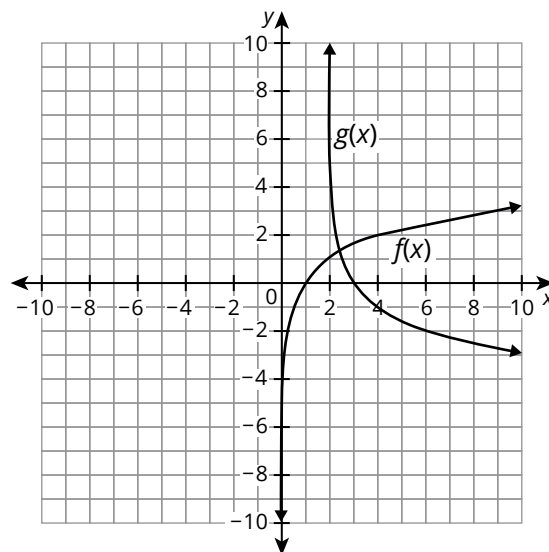
You can describe the transformation(s) performed on a function by comparing the graphs or equations of the original function and the transformed function.

To create $g(x)$, the graph of $f(x)$ is horizontally translated right 2 units and reflected across the x -axis.

$$g(x) = -f(x - 2)$$

You can describe the transformation of a function given the original function and a characteristic, or characteristics, of the transformed function. Consider the following function, $f(x) = \log_2(x)$. Create a transformed function that has a vertical asymptote at $x = -3$.

$$m(x) = f(x + 3)$$



By comparing the graphs of g and m , you can see that the following holds true for logarithmic functions as they have for other functions you have studied.

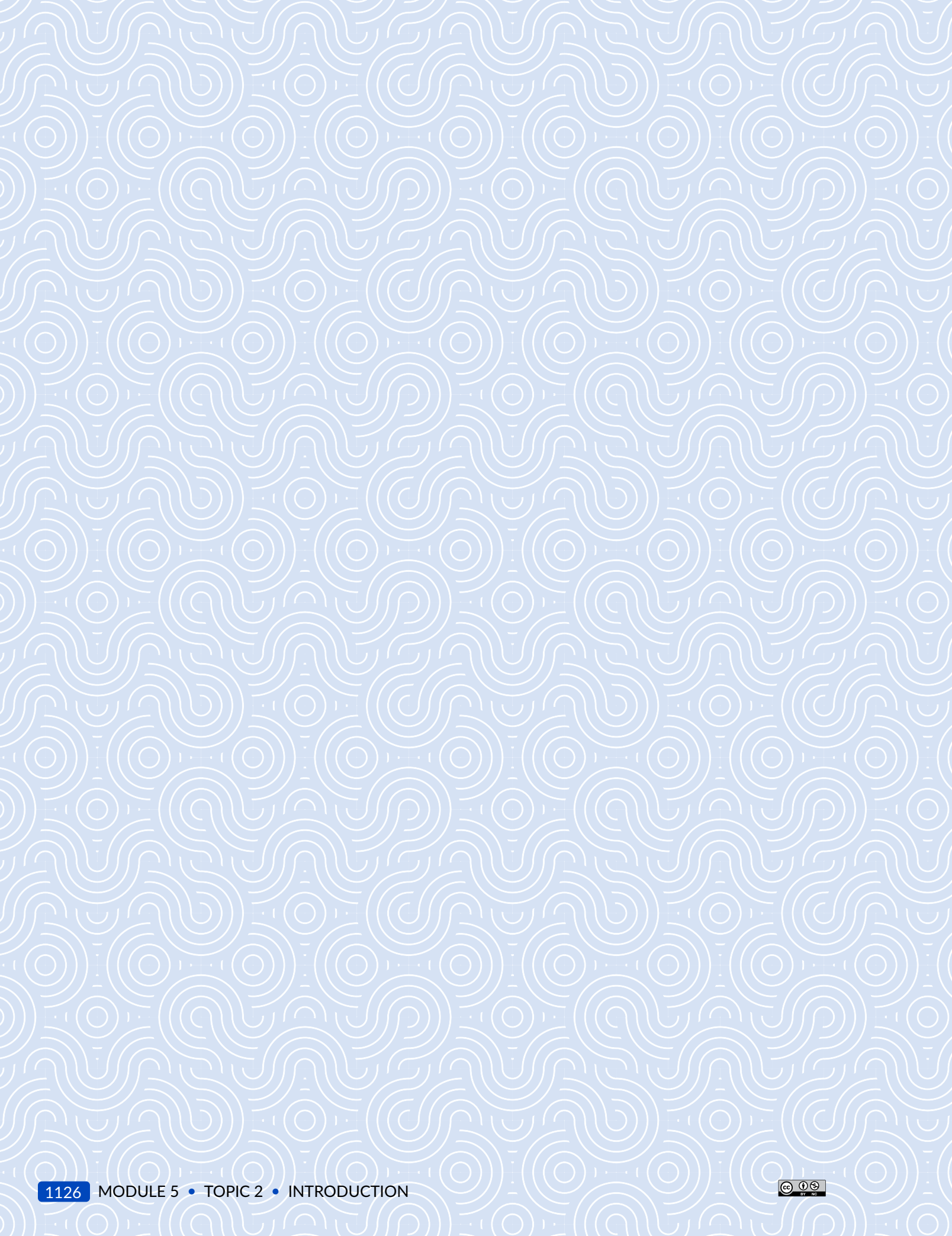
	Transformation on $f(x)$	Effect of transformation on $f^{-1}(x)$
$f(x + c)$	Translate horizontally left c units	Translate vertically down c units
$f(x - c)$	Translate horizontally right c units	Translate vertically up c units
$f(x) + d$	Translate vertically up d units	Translate horizontally right d units
$f(x) - d$	Translate vertically down d units	Translate horizontally left d units

A grayscale photograph showing a hand holding a petri dish. Inside the dish, there are numerous small, dark, circular bacterial colonies growing on a light-colored agar surface. The colonies are scattered across the dish, illustrating exponential growth.

Since bacteria typically grow by splitting, a bacterial population in a Petri dish is an excellent example of exponential growth.

Exponential and Logarithmic Equations

LESSON 1	Logarithmic Expressions	1127
LESSON 2	Solving Exponential Equations	1145
LESSON 3	Solving Logarithmic Equations	1161
LESSON 4	Applications of Exponential and Logarithmic Equations...	1175
LESSON 5	Modeling with Linear, Quadratic, and Exponential Functions	1195



1

Logarithmic Expressions

OBJECTIVES

- Convert exponential equations into logarithmic equations.
- Convert logarithmic equations into exponential equations.
- Solve exponential and simple logarithmic equations.
- Estimate the values of logarithms on a number line.
- Evaluate and verbally describe logarithmic expressions.

KEY TERM

- logarithmic expression

.....

You know that logarithmic functions are inverses of exponential functions.

How can you use inverses to solve exponential and logarithmic equations?

Getting Started

Two-Way Street

Recall that a logarithmic function is the inverse of an exponential function.

1. Write the equivalent form of the given exponential or logarithmic equation.

Exponential Form $y = b^x$	$\Leftrightarrow$	Logarithmic Form $x = \log_b y$
$12^2 = 144$	$\Leftrightarrow$	
	$\Leftrightarrow$	$\log_{16} 4 = \frac{1}{2}$
$10^5 = 100,000$	$\Leftrightarrow$	
	$\Leftrightarrow$	$\ln 20.086 \approx 3$
$\left(\frac{2}{3}\right)^3 = \frac{8}{27}$	$\Leftrightarrow$	
	$\Leftrightarrow$	$\log_9 27 = \frac{3}{2}$
	$\Leftrightarrow$	$\log_2 x = 8$
$6^x = 36$	$\Leftrightarrow$	
$n^5 = 243$	$\Leftrightarrow$	$\log_n 243 = 5$

Ask Yourself . . .

How does representing mathematics in multiple ways help to communicate reasoning?

ACTIVITY 1.1

Writing and Evaluating Logarithmic Equations

When you evaluate a logarithmic expression (logarithm), you are determining the value of the exponent in the corresponding exponential expression.

$$\text{base}^{\text{exponent}} = \text{argument} \Leftrightarrow \log_{\text{base}}(\text{argument}) = \text{exponent}$$

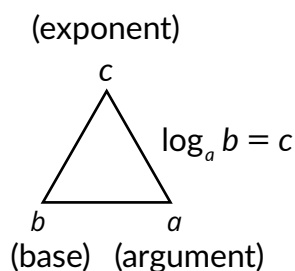
The variables of the logarithmic equation have the same restrictions as the corresponding variables of the exponential equation. The base, b , must be greater than 0, but not equal to 1; the argument must be greater than 0; and the value of the exponent has no restrictions.

It is important to become familiar with how the base, argument, and exponent fit into a *logarithmic equation*. A **logarithmic equation** is an equation that contains a logarithm.

.....
To write a logarithmic equation, it can be helpful to consider the exponential form first, and then convert it to logarithmic form.
.....

WORKED EXAMPLE

Triangle of Power



1. Use the triangle of power to arrange the given terms to create a true logarithmic equation.

a. 49, 2, 7

b. $-3, 6, \frac{1}{216}$

Ask Yourself . . .

How can you organize and record your mathematical ideas?

c. $4, 4, 1$

d. $256, 4, 4$

2. Write three logarithmic expressions that are equivalent to each given expression. Explain your strategy.

a. $\log_5 625$

b. $\log_7 \frac{1}{7}$

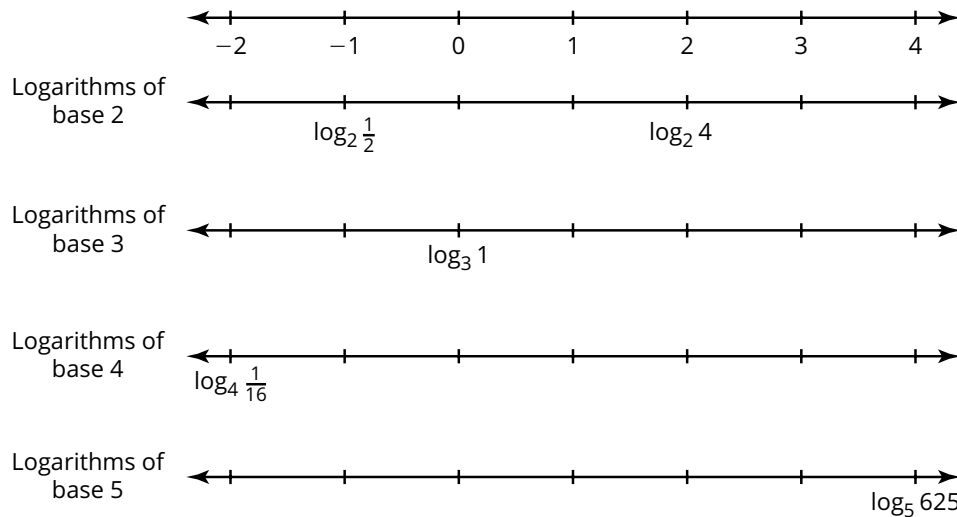
c. $\log_{64} 8$

d. $\log_2 -2$

Estimating with Logarithms

A logarithm is an exponent, so it can be any real number, even an irrational number.

1. Label each number line using logarithmic expressions with the indicated base.



2. Compare the logarithms on the number lines.

- a. Analyze all the logarithms that are equivalent to 0. Write a general statement using the base b to represent this relationship.
- b. Analyze all the logarithms that are equivalent to 1. Write a general statement using the base b to represent this relationship.

.....
Describe the
restrictions on
the variables
when appropriate.
.....

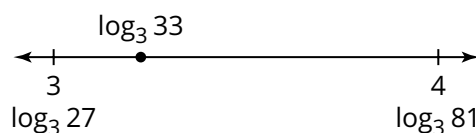
- c. Rewrite the general statements from parts (a) and (b) in exponential form. Use exponent rules to verify that each statement is true.

You can estimate the value of a logarithm that is not an integer using a number line as a guide.

WORKED EXAMPLE

Estimate the value of $\log_3 33$.

To estimate $\log_3 33$ to the nearest tenth, identify the closest integer logarithm whose argument is less than 33 and the closest integer logarithm whose argument is greater than 33 on a number line that represents base 3.



Closest integer logarithm
whose argument is
less than 33:

$$\log_3 27$$

Logarithm
you are
estimating:

$$\log_3 33$$

Closest integer logarithm
whose argument is
greater than 33:

$$\log_3 81$$

You know that $\log_3 27 = 3$ and $\log_3 81 = 4$. This means the estimate of $\log_3 33$ is between 3 and 4.

$$\log_3 27 < \log_3 33 < \log_3 81$$

$$3 < x < 4$$

Next, estimate the decimal digit.

Because 33 is closer to 27 than to 81, the value of $\log_3 33$ is closer to 3 than it is to 4.

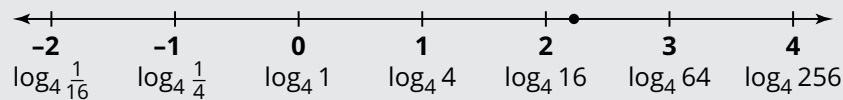
In this case, 3.2 is a good estimate for $\log_3 33$.

3. Andrew and Alejandro were each asked to estimate $\log_4 28$.

Andrew



I estimated $\log_4 28$ using the number line.



$$\begin{aligned} \log_4 16 &< \log_4 28 < \log_4 64 \\ 2 &< x < 3 \\ \log_4 28 &\approx 2.3 \end{aligned}$$

Alejandro



I estimated $\log_4 28$ by converting the log into exponential form and estimating based on powers of 4.

$$\begin{aligned} \log_4 28 &= x \\ 4^x &= 28 \end{aligned}$$

I know that $4^2 = 16$ and $4^3 = 64$, so the estimate of $\log_4 28$ must be between 2 and 3.

$$\log_4 28 \approx 2.4$$

Alejandro did not use the number line, but his estimate was about the same as Andrew's. Will Alejandro's method always work?

4. Estimate each logarithm to the nearest tenth and explain your reasoning.

a. $\log_2 10$

b. $\log_5 4$

.....
Review the definitions of **select** and **estimate** in the Academic Glossary.
.....



5. Trung and Eduardo were each asked to determine which base was used to estimate the value of $\log_b 58 = 2.9$.

Trung

The log of 58 falls between 2 and 3 when the base is 4.

$$\log_4 16 < \log_b 58 < \log_4 64$$
$$2 < 2.9 < 3$$

So, $b \approx 4$.

Eduardo

The log of 58 falls between 2 and 3 when the base is 5.

$$\log_5 25 < \log_b 58 < \log_5 125$$
$$2 < 2.9 < 3$$

So, $b \approx 5$.

Who is correct? Explain your reasoning.

6. Use the number lines from Question 1 to determine the appropriate base of each logarithm.

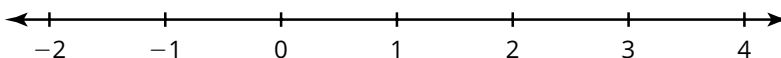
a. $\log_b 108 = 2.9$

b. $\log_b 0.4 = -1.3$

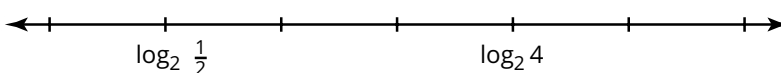
Estimating with Natural Logarithms

You have estimated with logarithms that have rational number bases. Let's consider how you can estimate the values of natural logarithms—those with a base of e .

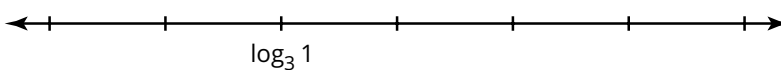
1. For a fixed base greater than 1, as the value of the argument gets larger, what happens to the value of the logarithm? Provide an example to illustrate your statement.
2. Plot $\log_2 18$, $\log_3 18$, $\log_4 18$, and $\log_5 18$ on the appropriate number lines in Question 1 in the previous activity. Then, use the number lines to estimate the numeric value of each logarithm to the tenths place. Verify your answers in exponential form.



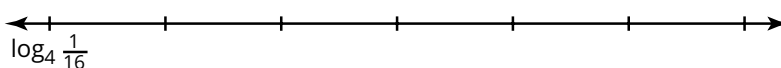
Logarithms of
base 2



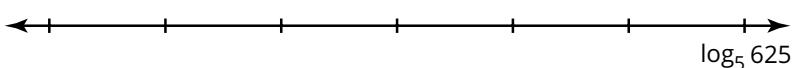
Logarithms of
base 3



Logarithms of
base 4



Logarithms of
base 5



3. For a fixed argument, when the value of the base is greater than 1 and increasing, what happens to the value of the logarithm?

4. How could you use the number lines to predict the value of $\ln 18$?

5. Make a prediction for the value of $\ln 18$.



Talk the Talk

Always, Sometimes, Never

1. Complete each sentence with *always*, *sometimes*, or *never* to make it true. Explain your reasoning.
 - a. The value of a logarithm is _____ equal to the exponent of the corresponding exponential equation.
 - b. The argument of a logarithmic expression is _____ a negative number.
 - c. The value of a logarithm is _____ equal to a negative number.
 - d. The base of a logarithm is _____ a negative number.
 - e. A logarithm is _____ a value that is not an integer.
 - f. For a base greater than 1, if $b > c$, then the value of $\log_a b$ is _____ greater than $\log_a c$.
 - g. If $a > b$, then the value of $\log_a 1$ is _____ less than $\log_b 1$.
 - h. The base of a logarithm is _____ equal to 1.

2. Use the triangle of power to arrange the given terms to create a true logarithmic equation.

a. 64, 2, 6

b. 27, 3, 3

c. $\frac{1}{16}$, 4, -2

d. 1000, 10, 3

Lesson 1 Assignment

Write

Describe how to estimate the value of a logarithm.

Remember

The value of a logarithmic expression is equal to the value of the exponent in the corresponding exponential expression.

For a fixed base greater than 1, as the value of the argument increases, the value of the logarithm increases as well.

For a fixed argument, when the value of the base is greater than 1 and increasing, the value of the logarithm is decreasing.

Practice

1. Write the equivalent form of the given exponential to logarithmic equation and vice versa.

a. $2^5 = 32$

b. $\log_3 81 = 4$

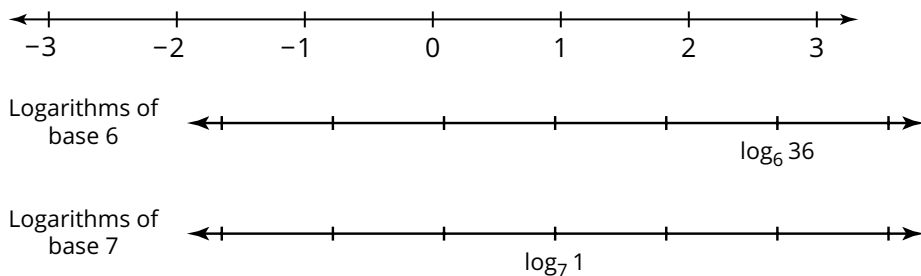
c. $7^{-1} = \frac{1}{7}$

d. $\log_8 \frac{1}{64} = -2$

Lesson 1 Assignment

2. Consider base 6 and base 7 logarithms.

- a. Label each number line using logarithmic expressions with the indicated base to match the given number line.



b. Estimate $\log_6 40$.

c. Estimate $\log_7 40$.

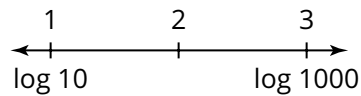
d. Estimate $\log_6 200$.

e. Estimate $\log_7 100$.

3. Estimate $\log_2 15$ to the nearest tenth. Explain your reasoning.

Lesson 1 Assignment

4. Two students are trying to finish labeling the number line with the base 10 logarithmic expression that equals 2. Chris says the missing logarithm should be $\log 505$ because 505 is halfway between 10 and 1000, just like 2 is halfway between 1 and 3. Jorge disagrees. He says the missing logarithm should be $\log 100$.



- a. Which student is correct? Explain your reasoning.
- b. For a logarithm with a base greater than 1, how does the argument change for every increase of 1 in the value of the logarithm?
- c. Estimate $\log 55$. Explain your reasoning.

Lesson 1 Assignment

Prepare

Identify the property of powers associated with each example.

1. $2^5 \cdot 2^2 = 2^7$

2. $2^1 = 2$

3. $2^0 = 1$

4. $\frac{2^{14}}{2^5} = 2^9$

2

Solving Exponential Equations

OBJECTIVES

- Solve exponential equations using common bases.
- Solve exponential equations by taking the log of both sides.
- Analyze different solution strategies to solve exponential equations and explain verbally and in writing when to use each.

KEY TERM

- None

.....

You have used the properties of equality to solve equations.

What strategies can you use to solve exponential equations?

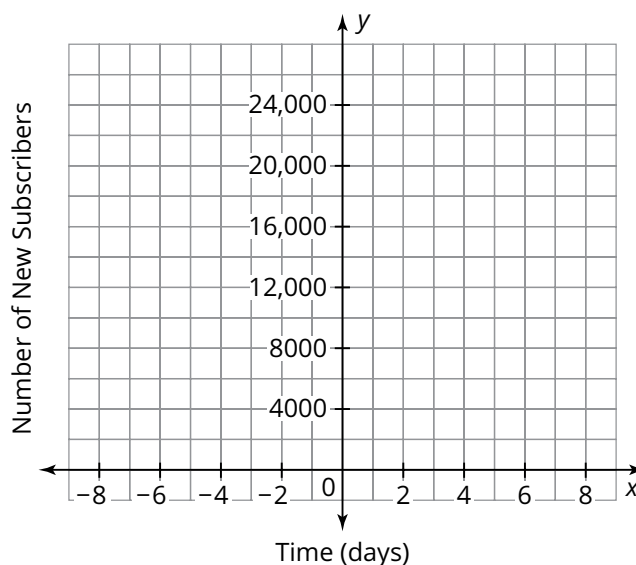
Getting Started

.....
A person is
considered a new
subscriber for one day
only, the actual day
they first subscribe.
.....

Don't Burst My Bubble

A company created a new game called Bubblez Burst. Before the game was even released, there were 50 early subscribers. The creators estimate that everyone who subscribes will then send 3 more people invitations to subscribe the next day. So, on the first day of its actual release, there were 150 new subscribers.

1. Write a function using the creator's estimate to model the number of new subscribers, P , who will be playing Bubblez Burst on day t . Use technology to graph your function, and then sketch it on the coordinate plane.

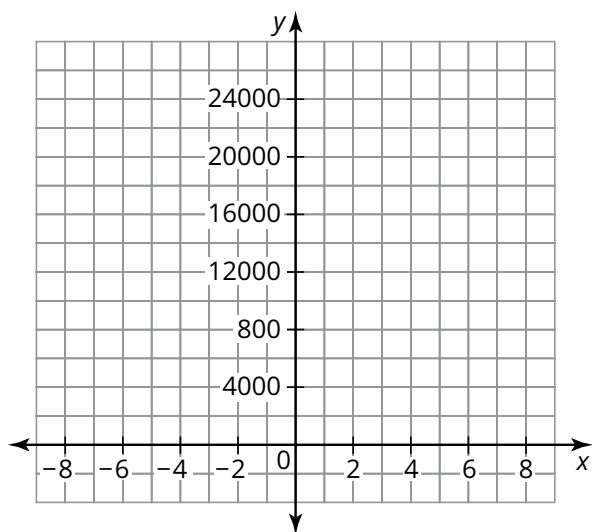


Ask Yourself ...

How can I check that my function makes sense for Day 1?

2. Use your graph to approximate each answer.
 - a. On what day will Bubblez Burst reach 4050 new subscribers?

b. On what day will Bubblez Burst reach 20,000 new subscribers?



Solving Exponential Equations Using Common Bases

An exponential function is a function written in the form $f(x) = a \cdot b^x$. To solve an exponential function, you can use what you know about the properties of exponents and common bases.

WORKED EXAMPLE

To solve the exponential equation $\frac{1}{32} = 2^{x-1}$, first determine the power of 2 that gives the result of 32:

$$(2)(2)(2)(2)(2) = 32$$

$$2^5 = 32$$

Then rewrite the equation to show common bases:

$$\frac{1}{32} = 2^{x-1}$$

$$\frac{1}{2^5} = 2^{x-1}$$

$$2^{-5} = 2^{x-1}$$

Because the expressions on both sides of the equals sign have the same base, you can set up and solve an equation using the exponents:

$$-5 = x - 1$$

$$-5 + 1 = x - 1 + 1$$

$$-4 = x$$

Therefore, $x = -4$.

Solve each equation for x .

1. $3^x = 81$

2. $2^{4x} = 1$

$$3. 4^{8-x} = \frac{1}{64}$$

$$4. 5^{9x} = 5$$

$$5. \frac{1}{3^{x+5}} = 243$$

$$6. 2^{-x} = \frac{1}{2}$$

In the Getting Started you used a graph to determine the number of days when Bubblez Burst would reach 4050 and 20,000 new subscribers.

7. Can you use the properties of exponents to solve each of the equations from Question 2 of the Getting Started? If so, solve the equation. If not, explain why not.

Taking the Logarithm of Both Sides of an Equation

Previously, you solved exponential equations with common bases, as well as exponential equations with common exponents.

1. Solve each exponential equation, and explain the strategy you used.

a. $2^x = 64$

b. $y^3 = 125$

2. Evaluate each expression.

a. $\log_2 2^3$

b. $\log_5 5^2$

c. $\log 10^2$

3. What conjecture can you make based on your answers to Question 2?

4. Alejandro solved the exponential equation $4^{x-1} = 50$.

Alejandro



$$\begin{aligned}4^{x-1} &= 50 \\x - 1 &= \log_4 (50) \\x &= \log_4 (50) + 1 \\x &\approx 3.822\end{aligned}$$

.....
Taking the logarithm
of both sides of an
equation keeps the
equation balanced.
.....

Describe Alejandro's strategy.

When solving equations, it can be helpful to isolate the term with the variable first before solving.

- There are multiple strategies for solving exponential equations. Let's consider some different methods used by students to solve the same exponential equation.

- Samantha



$$\begin{aligned} 9^{x+4} &= 27 \\ x+4 &= \log_9 27 \\ x &= \log_9 27 - 4 \\ x &= -2.5 \end{aligned}$$

Andrew



$$\begin{aligned} 9^{x+4} &= 27 \\ (3^2)^{x+4} &= 3^3 \\ 3^{2(x+4)} &= 3^3 \\ 2(x+4) &= 3 \\ 2x+8 &= 3 \\ 2x &= -5 \\ x &= -2.5 \end{aligned}$$

a. Describe each method used.

Ask Yourself . . .

What patterns do you notice?

b. Will each method work for every logarithmic equation? Describe any limitations of each method.



Talk the Talk

What's Your Strategy?

1. Solve each exponential equation. Explain why you chose the method that you used.

a. $2^{x-5} + 6 = 30$

b. $7 \cdot (2)^{3x} = 840$

c. $4^{x-3} = 16$

d. $10 \cdot \left(\frac{3^{2x}}{2}\right) = 40.5$

e. $2 \cdot 3^{5x} + 1 = 55$

Lesson 2 Assignment

Write

Explain when to use the common bases strategy to solve an exponential equation.

Remember

You can solve an exponential equation using common bases or by taking the logarithm of both sides.

Practice

1. Ten volunteers begin recruiting people to be volunteers for a large fundraising event. After one week, the total number of volunteers has doubled to 20. Each subsequent week, the total number of volunteers doubles.
 - a. Write a function to model the total number of volunteers, V , in the group after t weeks.
 - b. How many weeks will it take for the total number of volunteers to reach 1280?

Lesson 2 Assignment

- c. How many weeks will it take for the volunteers to reach their goal of 15,000 total volunteers?

2. Solve each exponential equation. Round each answer to the nearest hundredth.

a. $9^{x+2} + 3 = 6564$

b. $\frac{7^{3x-2}}{5} = 2$

Lesson 2 Assignment

Prepare

1. Rewrite each logarithmic expression as an exponential equation.

a. $\log_2 \frac{1}{8}$

b. $\log_8 512$

c. $\log 0.01$

3

Solving Logarithmic Equations

OBJECTIVES

- Solve for the base, argument, and exponent of logarithmic equations.
- Solve logarithmic equations arising from real-world situations in written form.

.....

You have used the properties of equality to solve many types of equations.

How can these properties be applied when solving logarithmic equations?

Sound Off

$$dB = 10 \log \left(\frac{I}{I_0} \right),$$

1. The sound in a quiet library is about 1000 times as intense as the threshold sound, or $I = 1000I_0$. Use a calculator to determine the decibel level of a quiet library.
2. The sound of traffic on a city street is calculated to be 500 million times as intense as the threshold sound. Use a calculator to determine the decibel level for city traffic.

ACTIVITY
3.1

Solving Logarithmic Equations

Remember, you can use a “Triangle of Power” to help you think about logarithms.

WORKED EXAMPLE

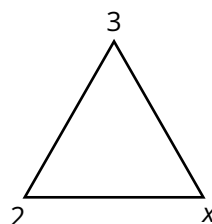
Consider the equation $2^3 = 8$.

Logarithmic Equation

Triangle of Power

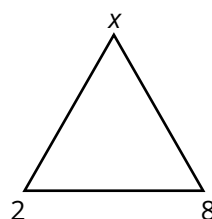
$$\log_2 x = 3$$

$$x = 8$$



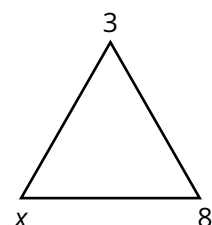
$$\log_2 8 = x$$

$$x = 3$$



$$\log_x 8 = 3$$

$$x = 2$$



1. Draw Triangles of Power to illustrate each equation.
Then rewrite in exponential form and solve the equation.

a. $\log_{10} 1000 = x$

⋮

b. $\log_5 x = 5$

⋮

c. $\log_x 9 = 1$

Let's consider the relationship between the base, argument, and exponent. You can use that relationship to solve for any unknown in a logarithmic equation.

WORKED EXAMPLE

To solve for any unknown in a simple logarithmic equation, begin by converting it to an exponential equation.

Argument Is Unknown Exponent Is Unknown Base Is Unknown

$$\log_4 y = 3$$

$$4^3 = y$$

$$64 = y$$

$$\log_4 64 = x$$

$$4^x = 64$$

$$4^x = 4^3$$

$$x = 3$$

$$\log_b 64 = 3$$

$$b^3 = 64$$

$$b^3 = 4^3$$

$$b = 4$$

2. Justify the last step of each case in the Worked Example.

a. If $4^3 = y$, why does $y = 64$?

b. If $4^x = 4^3$, why does $x = 3$?

c. If $b^3 = 4^3$, why does $b = 4$?

.....
It is important to note that you can convert a logarithmic equation to an exponential equation regardless of which term is unknown.
.....

3. Solve for the unknown in each logarithmic equation.

a. $\log_8 64 = n$

b. $\log_n \frac{1}{16} = -2$

c. $\log_{\frac{1}{2}} 64 = n$

d. $\log n = -3$

e. $\log_n \sqrt[3]{49} = \frac{2}{3}$

f. $\log_9 27 = n$

g. $\log_2 8 = 4x - 9$

h. $\log_6 (x^2 - x) = 1$

Recall from the Getting Started that a decibel is a unit used to measure the loudness of sound. The formula for the loudness of a sound is given by

$$dB = 10 \log \left(\frac{I}{I_0} \right),$$

where dB is the decibel level. The quantity I_0 is the intensity of the threshold sound—a sound that can barely be perceived. The quantity I is the number of times more intense a sound is than the threshold sound.

4. The sound of a crying baby registers at 115 decibels. How many times more intense is this sound than the threshold sound?

Ask Yourself . . .

How can you use solving logarithmic equations in everyday life?

PROBLEM SOLVING



ACTIVITY

3.2

Solving Problems with Logarithmic Equations

While most medications have guidelines for dosage amounts, doctors must also determine the amount of time a medication will remain in a patient's body before it is metabolized when they write prescriptions. The amount of medicine remaining in a patient's body can be predicted by the formula

$$t = \frac{\log \left(\frac{C}{A} \right)}{\log (1 - r)},$$

where t is the time in hours since the medicine was administered, C is the current amount of medicine remaining in the patient's body in milligrams, A is the original dose of the medicine in milligrams, and r is the rate at which the medicine is metabolized.

1. A patient is given 10 milligrams of medicine which is metabolized at the rate of 20% per hour. How long will it take for 2 milligrams of the medicine to metabolize?

Suppose you know that 4 hours after administering a 12-milligram dose of medicine, 9 milligrams remain in the patient's body. How can you determine the rate at which the medicine metabolizes?

You can solve for an unknown in a logarithmic equation by using the properties of equality and by rewriting the logarithmic equation as an exponential equation to isolate the variable.

WORKED EXAMPLE

$$4 = \frac{\log\left(\frac{9}{12}\right)}{\log(1-r)}$$

Substitute the values for t , C , and A into the formula.

$$4 \log(1-r) = \log(0.75)$$

Multiply both sides of the equation by $\log(1-r)$.

$$4 \log(1-r) < -0.125$$

Evaluate $\log(0.75)$.

$$\log(1-r) < -0.03125$$

Divide both sides of the equation by 4.

$$10^{-0.03125} < 1-r$$

Rewrite as an exponential equation.

$$r < 1 - 10^{-0.03125}$$

Isolate the variable r .

$$r < 0.0694$$

The medicine is metabolized at an approximate rate of 6.94% per hour.

2. Six hours after administering a 20-milligram dose of medicine, 5 milligrams remain in a patient's body. At what rate is the medicine metabolized?

Ask Yourself ...
Did you share your solution with others?

Did you share
your solution with
others?

3. A patient is undergoing an 8-hour surgery. When 4 milligrams of medicine must remain in the patient's body at the end of surgery, and the medicine is metabolized at the rate of 15% per hour, how much medicine must be administered at the start of surgery?

4. Consider your answers to Questions 1-3. How can you determine whether your answers are reasonable?



Talk the Talk

Mission Depreciation

Some items such as automobiles are worth less over time. The age of an item can be predicted using the formula $t = \frac{\log\left(\frac{V}{C}\right)}{\log(1-r)}$, where t is the age of the item in year, V is the value of the item after t years, C is the original value of the item, and r is the yearly rate of depreciation expressed as a decimal.

1. A car was originally purchased for \$32,000 and is currently valued at \$20,000. The average rate of depreciation for this car is 12% per year.

How old is the car to the nearest tenth of a year?

2. What is the current value of a car purchased for \$32,000 when the average rate of depreciation is 12% per year and the car is 5 years old?

3. How can you determine the reasonableness of your answers to Questions 1 and 2?

Lesson 3 Assignment

Write

Describe how to solve a logarithmic equation.

Remember

To solve for an unknown in a simple logarithmic equation, consider the relationship between the base, argument, and exponent.

Practice

1. In music, the cent, c , is a unit of measure that is used to measure the differences in frequencies between musical notes. The formula $c = 1200 \log_2 \left(\frac{f_1}{f_2} \right)$ can be used to determine the number of cents, c , between a musical note of a higher frequency, f_1 , and a musical note of a lower frequency, f_2 .
 - a. Determine the number of cents between the notes F4 and C4 if F4 has a frequency of 349.23 Hz (hertz) and C4 has a frequency of 261.63 Hz.
Round your answer to the nearest cent.
 - b. Determine the frequency (in hertz) of the note G4 when the notes A4 and G4 are separated by 200 cents, and A4 has a frequency of 440 Hz. The note A4 has a higher frequency than the note G4. Round your answer to the nearest hertz.

Lesson 3 Assignment

2. The magnitude limit, L , of a telescope refers to the magnitude of the faintest star that is observable with the telescope. Stars with lower magnitudes are brighter than stars with higher magnitudes. The equation $L = \frac{\log_6 d}{0.257} + 2$ can be used to determine the magnitude limit of a telescope with an objective lens diameter, d , in millimeters.
- a. Lizzie purchases a telescope that has an objective lens that is 80 mm in diameter. Determine the magnitude limit of her telescope to the nearest tenth.
- b. After using her telescope, Lizzie realizes she should have purchased a larger telescope. She wants to purchase a telescope with a magnitude limit of 13.5. Determine the minimum diameter of the objective lens to the nearest tenth for such a telescope.

Lesson 3 Assignment

3. Solve each logarithmic equation. Check your work.

a. $\log_x 216 = 1.5$

b. $\log_5 125 = y$

c. $\log_a \frac{1}{216} = -3$

d. $\log t = -1$

e. $\log_n \sqrt[4]{8} = \frac{3}{4}$

f. $\log_3 81 = 3x - 2$

g. $\log_4 64 = 2x + 1$

h. $\log_2 x^2 - 3x = 2$

Lesson 3 Assignment

Prepare

Solve each exponential equation.

1. $\left(\frac{1}{2}\right)^{-3} = y$

2. $5^a = 625$

3. $7^x = 30$

4

Applications of Exponential and Logarithmic Equations

OBJECTIVES

- Use exponential models to read and analyze problem situations and then respond in written form.
- Use logarithmic models to analyze problem situations.

KEY TERM

- None

.....

You have explored multiple methods for solving exponential and logarithmic equations.

How can you apply these methods to solve real-world problems modeled by exponential or logarithmic equations?

A mole is a counting unit—just a word that stands for a number. Just as *dozen* means 12, a *mole* means 6×10^{23} .

pHantastic Fun

The pH scale is a scale for measuring the acidity or alkalinity of a substance, which is determined by the concentration of hydrogen ions. The formula for pH is $\text{pH} = -\log \text{H}^+$, where H^+ is the concentration of hydrogen ions measured in moles per cubic liter. Solutions with a pH value less than 7 are acidic. Solutions with a pH value greater than 7 are alkaline. Solutions with a pH of 7 are neutral. For example, plain water has a pH of 7.

1. The H^+ concentration in orange juice is 0.000199 mole per cubic liter. Determine the pH level of orange juice, and then state whether it is acidic or alkaline.
2. The concentration of hydrogen ions in baking soda is 5.012×10^{-9} mole per cubic liter. Determine the pH level of baking soda, and then state whether it is acidic or alkaline.

3. Vinegar has a pH of 2.2. Determine the concentration of hydrogen ions in vinegar.
4. Lime water has a pH of 12. Determine the concentration of hydrogen ions in lime water.

Solving Problems with a Logarithmic Model

The pH level of soil is used to determine which plants grow best in an area. Different types of plants and vegetables require varying degrees of soil acidity. Generally, the soil tends to be acidic in moist climates and alkaline in dry climates.

The chart shows the optimal pH soil levels for a variety of fruits and vegetables.

Asparagus 6.5 – 7.5	Avocados 6.0 – 7.0	Beets 5.6 – 6.6
Carrots 5.0 – 6.0	Garlic 5.0 – 6.0	Lettuce 6.5 – 7.0
Mushrooms 7.0 – 8.0	Onions 6.2 – 6.8	Peanuts 5.0 – 6.0
Peppers 6.0 – 8.0	Potatoes 5.8 – 6.5	Raspberries 6.0 – 6.5
Spinach 5.0 – 7.0	Sweet Corn 6.0 – 7.0	Yams 6.0 – 8.0

1. A farmer measures the hydrogen ion concentration of the soil to be 6×10^{-7} mole per cubic liter. List the crops that the farmer can grow in this type of soil.

-
- Determine the range of the soil's optimal hydrogen ion concentration for growing onions.
- The farmer wants to plant spinach, carrots, and beets. She measures the hydrogen ion concentration of the soil to be 3.16×10^{-6} mole per cubic liter. Is her plan feasible? Explain your reasoning.

ACTIVITY
4.2

Solving Problems with an Exponential Model

Lizzie, a social media manager, uses the model for continuous population growth to project the monthly increase in the number of followers on her clients' social media sites.

1. Recall that the formula for continuous population growth is $N(t) = N_0 e^{rt}$. State what each quantity represents in terms of this problem situation.

2. Lizzie claims that when she started working with an up-and-coming boy band, they had 18,450 online followers, and she was able to increase their followers by 26% per month.
 - a. Use Lizzie's claim to write a function to represent the number of online followers that the boy band had after t months.

 - b. What is the growth factor in your equation?

.....
Don't round your
decimals until the
very end.
.....

- c. If Lizzie started working with the band on May 1st, how many online followers did they have by September 1st?
 - d. How long did it take for the band to surpass 100,000 online followers?
3. Lizzie's top client is a pro football quarterback. On her website, Lizzie claims that since she began managing his account three years ago, his online followers have grown to 105,326.
- a. If he had 4125 online followers when he hired Lizzie, at what monthly rate did his number of online followers grow?

- b. Write an exponential function to represent the number of online followers the quarterback will have in any given month, assuming the number of followers continues to grow at the same rate.
- c. How long would it take the quarterback to double his current online following of 105,326?
4. Lizzie also manages a running back on the same team who currently has 62,100 online followers. If he has seen a 14% increase in his followers, how many people were following him when he hired Lizzie 15 months ago?
5. Will the running back ever have the same number of online followers as the quarterback? If so, when? If not, explain your reasoning.

6. Lizzie acquires a talk show host as a new client. The host currently has 5200 online followers.
- a. Lizzie claims that she can more than triple the host's number of online followers in 6 months. Determine the rate per month of increased online following.

- b. Jaylen decides to check Lizzie's work.

Jaylen

$N(t) = 5200 e^{0.18t}$

$N(6) = 5200 e^{0.18(6)}$

$N(6) \approx 15312.33367$

$15312 \neq 3(5200)$

Lizzie did not triple the host's followers.



What is the error in Jaylen's method? Check Lizzie's work correctly.

- c. Assuming that Lizzie's promised rate of growth is true, project how many followers the talk show host will have in a year.

PROBLEM SOLVING



ACTIVITY

4.3

Newton's Law of Cooling

Newton's Law of Cooling explains that the rate at which an object cools is directly proportional to the difference between its temperature and that of its surroundings. In simpler terms, if an object is much hotter than its environment, it cools down faster than if it were only slightly hotter. This behavior is modeled by the equation $T(t) = T_{\text{env}} + (T_o - T_{\text{env}})e^{-kt}$ where

- $T(t)$ is the temperature at time t minutes
- T_{env} is the environment temperature
- T_o is the initial temperature of the object
- k is the colling constant, a positive constant that depends on the properties of the object and its surroundings
- e^{-kt} shows how the temperature difference decays over time.

1. A baked cake is removed from a 350°F degree oven and placed on a cooling rack in a room that is 70°F. After 5 minutes, the cake has cooled to 300°F.

- a. Use Newton's Law of Cooling to write an equation to represent the problem situation.

- b. Determine the value of the cooling constant k to the nearest ten thousandth.

.....
Notice that the formula assumes a constant room temperature. If the surrounding temperature is not constant, then there is a more involved formula.
.....

- c. Write an equation that can be used to determine when the cake will be cool to the temperature $T(t)$.
- d. The baker cannot frost the cake until it reaches 100°F . When is the soonest the baker can frost the cake?
2. You remove a cooked casserole from a 400°F oven and place it a cooling rack in a room that is 72°F . After 10 minutes, the casserole cooled down to 260°F . The ideal serving temperature for the casserole is 120°F . How long should you wait before you serve it?



Talk the Talk

Fruit Flies Like a Banana

A population of fruit flies increases exponentially. The initial population of fruit flies is 25. After 4 days, the population of fruit flies is 400.

1. Determine a function that models the growth of the fruit fly population.
2. What is the growth factor in your equation?
3. After how many days will the fruit fly population be 5000?

Some items such as laptop computers are worth less over time.

The depreciation constant, k , represents the continuously compounded rate at which the asset's value decreases over time.

It is determined by the formula $k = \frac{\ln\left(\frac{V_0}{V(t)}\right)}{t}$ where V_0 is the initial value of the asset at time $t = 0$, $V(t)$ is the value of the asset after time t , and t represents the amount of time that has elapsed in years.

4. You purchased a laptop 4 years ago that is now worth \$500. Write an equation for the depreciation constant of the laptop.

5. Use the equation you wrote in Question 4 to determine the initial cost of the laptop when the depreciation constant is 0.15.

6. Determine whether your answer to Question 5 is reasonable using substitution.

Lesson 4 Assignment

Write

What are the attributes of a data set that indicate that it can be modeled by an exponential function?

Remember

Exponential and logarithmic equations can be used to model situations in the real world and solve for unknowns.

Practice

1. Tiara found an equation to calculate the depreciated value of a vehicle. The equation is $t = \frac{\log\left(\frac{V}{C}\right)}{\log(1 - r)}$, where V represents the value of the vehicle after t years, C represents the original value of the vehicle, and r represents the average rate of depreciation as a decimal.
 - a. Tiara originally purchased her car for \$45,000. It is currently valued at \$36,520. The average rate of depreciation for the car is 16%. Determine the age of Tiara's car using the logarithmic equation she found.
 - b. Three years ago, Elena purchased her truck for \$52,000. The truck has an average rate of depreciation of 18.5%. Determine the current value of Elena's truck using the logarithmic equation Tiara found.

Lesson 4 Assignment

- c. Four years ago, Kaya purchased a car for \$36,000. The current value of the car is \$17,400. Determine the average rate of depreciation for the car.
- d. How do you determine if your answer to parts (a), (b), and (c) are reasonable? Explain your reasoning.

Lesson 4 Assignment

2. The equation used to determine the amount, A , in an account after t years of continuous compounding is $A = Pe^{rt}$, where P represents the principal (or original) amount in the account and r represents the annual interest rate as a decimal.
- a. Six years ago, Jamal invested \$5000 in an account with continually compounded interest at an annual interest rate of 5.2%. Determine the current amount of money in the account.
 - b. Chris invests \$2500 in an account in which the interest is compounded continuously at an annual interest rate of 6%. Determine the amount of money that will be in the account after 10 years.
 - c. Five years ago, Matthew invested money into an account in which the interest is compounded continuously at an annual interest rate of 4%. The account is currently valued at \$1099.26. Determine the amount of money Matthew invested to the nearest dollar.

Lesson 4 Assignment

- d. Determine the number of years it will take for Paola's initial investment of \$10,000 to reach a value of \$13,000 if she invests the money in an account in which the interest is compounded continuously at an annual interest rate of 6.5%.
- e. Determine the annual interest rate needed for an initial investment of \$4000 to reach an amount of \$6000 in 8 years if the money is invested in an account in which the interest is compounded continuously.
- f. How many years would it take for the amount in a continuously compounded account to triple in value if the annual interest rate was 6.3%?

Lesson 4 Assignment

Prepare

Consider the table.

x	1.6	3.4	6.1	8.7	11.5
y	415	342	231	140	34.1

1. Would a linear or exponential model best represent the function?
Explain your reasoning.
2. Using graphing technology to write a regression function to fit the data.

5

Modeling with Linear, Quadratic, and Exponential Functions

OBJECTIVES

- Read, discuss, and analyze data to select the appropriate model from among linear, quadratic, and exponential models.
- Use regression methods available through technology to write a linear, quadratic, and an exponential function from a given set of data.
- Predict and make decisions and critical judgments from a given set of data using linear, quadratic, and exponential models.

KEY TERM

None

You have used regression functions to model linear, quadratic, and exponential data.

How can you decide which family of functions best represents a set of data for a given problem situation?

Name that Pattern

You have used regression functions to model many different types of data.

Think about the types of real-world problems that you have modeled with regression functions.

1. For each function family, give three examples of real-world problems that are best represented by that function.

- a. Linear

- b. Quadratic

- c. Exponential

2. Compare your answers with your classmates. List at least 1 more example of each function from your classmates list.

a. Linear

b. Quadratic

c. Exponential

3. How is modeling data with a regression function important beyond the mathematics classroom?

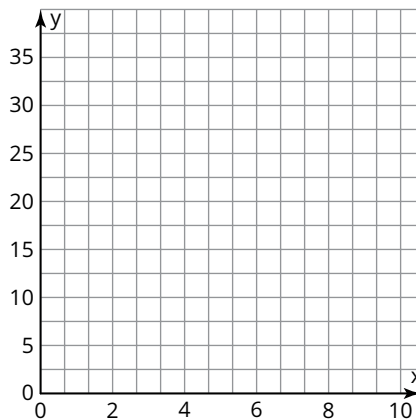
Choosing a Function Family

When analyzing data, it is a good idea to observe the general behavior of the data in order to determine the appropriate model using regression.

A biologist is studying the relationship between a tree's diameter in inches, x , and its height in feet, $h(x)$. She records the following data for 7 different trees.

Diameter (inches)	Height (feet)
2	8.2
3	11.4
4	15.8
5	17.1
6	21.7
7	25.3
8	28.6

1. Create a scatterplot to represent the data. Be sure to label the axes.



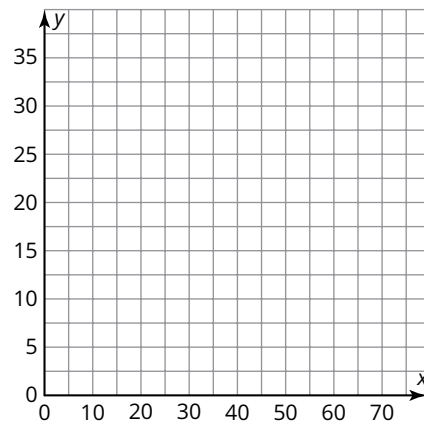
2. Use the scatterplot to determine whether a linear, exponential, or quadratic function best fits the data. Explain your reasoning.

3. Use graphing technology to determine the regression function for the data set.

A study compared the speed in miles per hour, x , and the average fuel economy in miles per gallon, $f(x)$, for cars. The results are recorded in a table.

Speed (miles per hour)	15	20	25	30	35	40	45	50	55	60	65	70
Fuel Economy (miles per gallon)	22.3	25.5	27.5	29.7	29.9	30.5	30.8	30.2	29.4	28.7	27.1	25.3

4. Create a scatterplot to represent the data. Be sure to label your axes.



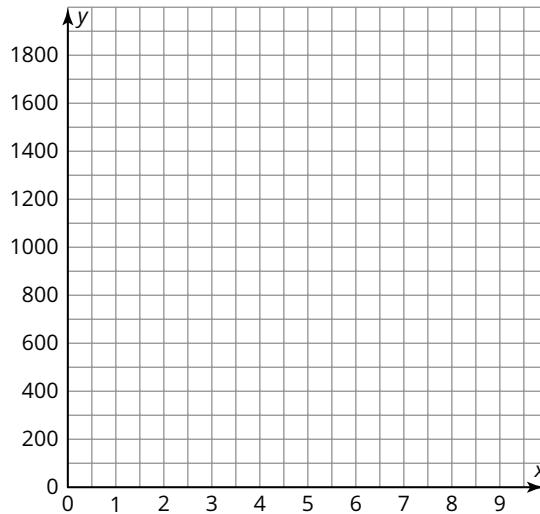
5. Use the scatterplot to determine whether a linear, exponential, or quadratic function best fits the data. Explain your reasoning.

6. Use graphing technology to determine the regression function for the data set.

The table shows the average salary of a professional sports team, $s(x)$, each year, x , from 2015 to 2024.

Number of Years Since 2015	0	1	2	3	4	5	6	7	8	9
Average Salary (thousands of dollars)	290	320	400	495	600	700	820	1000	1250	1580

7. Create a scatterplot to represent the data. Be sure to label your axes.



8. Use the scatterplot to determine whether a linear, exponential, or quadratic function best fits the data. Explain your reasoning.

9. Use graphing technology to determine the regression function for the data set.

Previously, you have practiced evaluating and solving linear, quadratic, and exponential equations. You can use a regression function to make predictions about data values that are not known. Using the regression function, substitute a value for one variable and evaluate or solve for the other variable, just as you did with any other equation.

10. Use your regression functions from Questions 3, 6, and 9 to make each prediction.
 - a. What is the diameter of a tree with a height of 30 feet?
 - b. What is the average fuel economy for a car traveling at 75 miles per hour?
 - c. What will the average salary of the team be in 2026?

You can also use the regression function to identify key attributes of the function that models the data, such as domain, range, intercepts, intervals of increase and decrease, maximum or minimum values, and asymptotes. This includes determining the limitations of the mathematical model with respect to the problem situation. For example, you may need to restrict the domain or range of the function to values that make sense in the problem.

11. What restrictions do you think each regression function from Questions 3, 6, and 9 have in terms of the problem situation?

a. Linear regression function

b. Quadratic regression function

c. Exponential regression function

Analyzing data is one form of critical judgment. Being critical involves asking yourself questions about what is being investigated and making judgments about truth, value, or usefulness. When you analyze and judge something, there is usually not one right answer, so you have to convince others that your judgment is credible and based on sound evidence.

To make any decision, whether in math class or in life, you should consider as much information as possible to make an informed decision. Tables, graphs, and equations, and the analysis of these representations are some types of information that may be useful. However, you may need to gather additional information from a variety of sources such as the library, the internet, or someone with expertise. Always be prepared to justify your decisions by clearly stating any assumptions and showing all necessary calculations.



Choosing Among Models

As the newly elected mayor of the city, you have called a meeting of all of your advisers to establish goals for your upcoming administration and to prepare a speech to the citizens of the city outlining those goals.

Each adviser has assembled data on significant trends within the past 10 years in the city. Four of these data sets are shown.

Year	Population (hundreds)
0	321.6
1	295.1
2	271.5
3	249.8
4	229.8
5	211.4
6	194.5
7	179.3
8	165.9
9	152.6

Year	Crimes (per 1000 people)
0	89.75
1	82.19
2	75.10
3	68.65
4	61.30
5	55.97
6	58.18
7	60.67
8	61.24
9	62.55

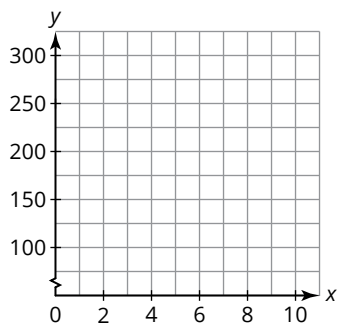
Year	Unemployment (% of population)
0	4.6
1	4.7
2	5.0
3	5.0
4	5.2
5	5.5
6	5.7
7	6.4
8	6.4
9	6.6

Year	Property Tax Rate (% of home value)
0	0.52
1	0.52
2	0.51
3	0.50
4	0.50
5	0.50
6	0.49
7	0.49
8	0.49
9	0.47

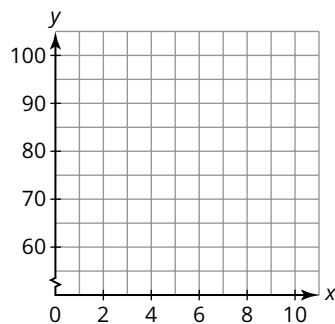
1. Analyze the data in each table. Which function family do you think would best model each data set? Justify your choices.

2. Create a scatterplot for each data set.

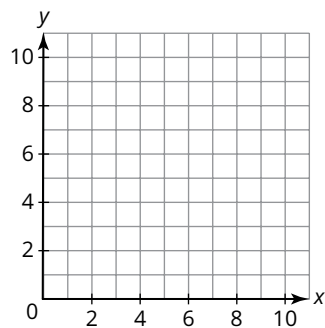
a. Population



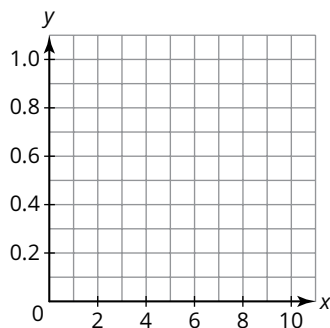
b. Crimes



c. Unemployment



d. Property Tax Rate



3. Four exponential models are shown. Explain which model best represents the population data.

$$f(x) = 320 \cdot 0.9^x$$

$$f(x) = -320 \cdot 0.9^x$$

$$f(x) = 320 \cdot 0.1^x$$

$$f(x) = -320 \cdot 0.9^x$$

4. Four quadratic function models are shown. Explain which model best represents the crime data.

$$f(x) = -10x^2 + 14x + 5$$

$$f(x) = 0.8x^2 - 10x + 100$$

$$f(x) = 8x^2 - x + 99$$

$$f(x) = -20x^2 + 150x + 400$$

5. Four linear function models are shown. Explain which model best represents the unemployment and property tax rate data.

$$f(x) = -20x$$

$$f(x) = -0.005x + 0.5$$

$$f(x) = 0.5x + 100$$

$$f(x) = 0.25x + 5$$

6. Use graphing technology to determine a quadratic regression function for the crime data. Write the function under each graph.

7. Use graphing technology to determine a linear regression function for the unemployment and property tax rate data. Write the function under each graph.

8. Suppose that one of your goals as mayor is to “bring people and businesses back to the city.” You want to include this goal in your speech by pointing to the trend in population.



- a. Your adviser, Mario, suggests making the claim shown.

Mario
Currently, the city's
population is declining by
92% every year.

Is Mario's claim valid? Explain your reasoning.

9. During your campaign, you promised voters to keep unemployment below the national average, which was 8% at the time.
- a. Insert a statement in your speech describing when an unemployment rate of 8% is currently predicted.

- b. You want to include in your speech a promise to reverse the trend in unemployment and return to an unemployment rate of 5% or lower in just 3 years. Describe the linear rate of unemployment decrease you will need each year to achieve this goal. Show your work.
10. Time to make some critical judgments! Continue your analysis of the data sets and functions. Can you draw a correlation between any of the four measures? Explain your reasoning.



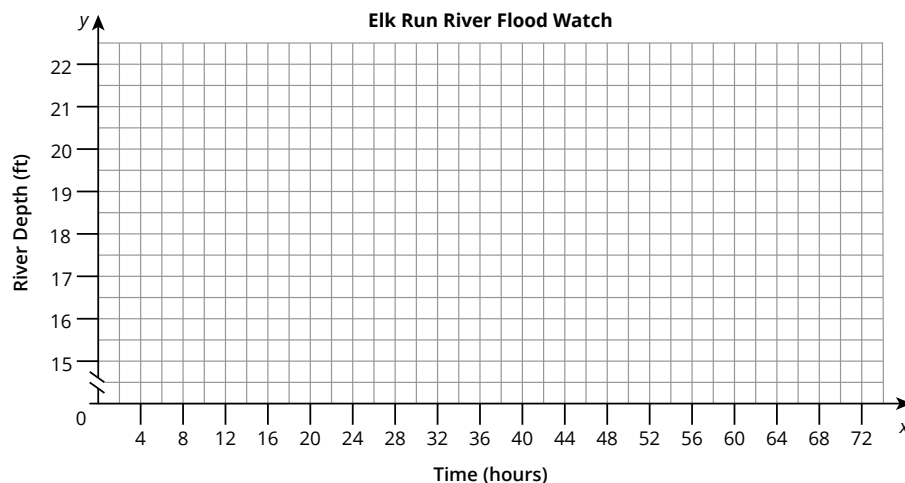
Talk the Talk

Can You Predict the Future?

A park ranger monitors the depth of the Elk Run River after a heavy rainstorm. The table shows the depth of the river at different times since the storm began.

Time (hours)	River Depth (ft)
3	16.2
8	17.4
15	18.3
22	19.5
27	20.1
33	20.6
39	21.1
48	21.5
55	21.2
60	20.7
64	20.1

1. Create a scatterplot of the data on the coordinate plane.



2. Describe how the level of the river changes as a function of time after the rainstorm.
3. Is the data best represented by a linear, quadratic, or exponential function? Explain your reasoning.
4. Use technology to determine the regression function that best models the data set. Sketch the graph of the regression model on the coordinate plane on which you created your scatterplot.
5. Is your regression model a good fit for the data set? Explain your reasoning.

6. Use the regression function to predict the depth of the Elk Run River 3 days after the storm began.

7. Over what time period do you predict the regression function will be a good model of the situation? Explain your reasoning.

Lesson 5 Assignment

Write

Describe how you can determine whether a data set is best modeled by a linear, quadratic, or exponential function.

Remember

You can use a regression function to model a set of real-world data and make predictions about the data set.

Practice

Determine whether the data set is linear, quadratic, or exponential. Then, determine the best regression function for the data set. Explain your reasoning.

1. The table shows the elevation above sea level (in feet) of a hot-air balloon at different times during a test flight.

Time (min)	0	2	4	6	8	10	12	14	16
Elevation (ft)	320	355	380	395	400	395	380	355	320

2. The table shows the total cost of purchasing different numbers of concert tickets.

Number of Tickets	2	4	6	8	10	12	14
Total Cost (\$)	85	150	215	280	345	410	475

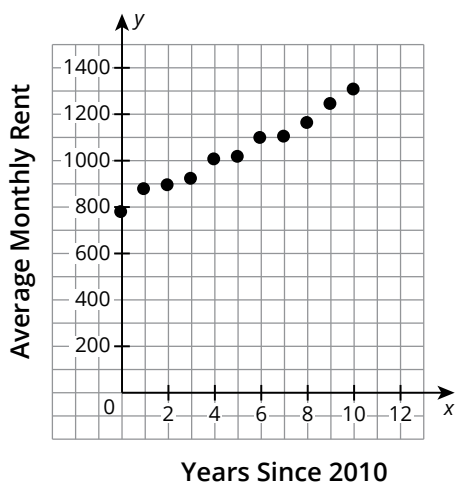
Lesson 5 Assignment

3. The table shows the value of a new car at different times since it was purchased.

Time (yr)	0	1	2	3	4	5	6	7
Value ($\times \$1000$)	28.5	24.9	21.8	19.1	16.7	14.6	12.8	11.2

4. Four linear function models are shown. Explain which model best represents the average monthly rent for a one-bedroom apartment data.

Average Monthly Rent for a One-Bedroom Apartment



$$f(x) = 49x + 801$$

$$g(x) = -49x + 801$$

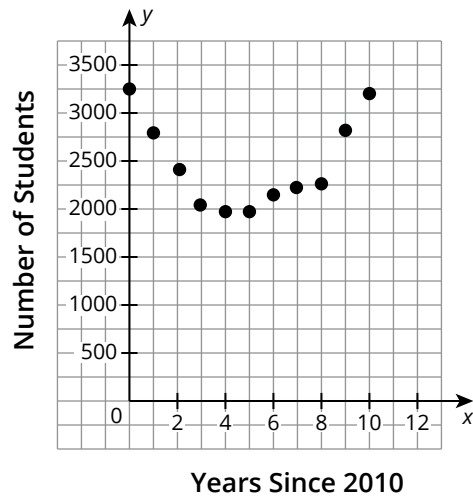
$$h(x) = 49x - 801$$

$$j(x) = -49x - 801$$

Lesson 5 Assignment

5. Four quadratic function models are shown. Explain which model best represents the high school enrollment data.

High School Enrollment (2010–2020)



$$f(x) = 5.1x^2 + 51x - 32$$

$$h(x) = 51x^2 - 510x + 3234$$

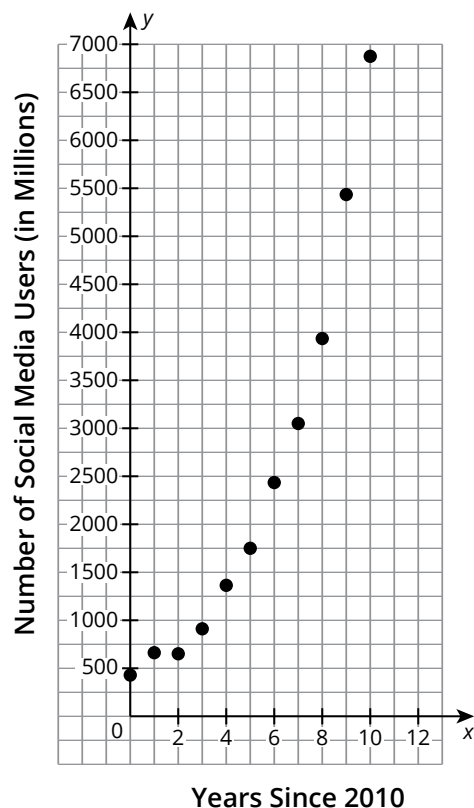
$$g(x) = -5.1x^2 + 51x - 32$$

$$j(x) = -51x^2 - 510x + 3234$$

Lesson 5 Assignment

6. Four exponential function models are shown. Explain which model best represents the growth of social media users data.

Growth of Social Media Users (2010–2020)



$$f(x) = -452 \cdot 1.3^x$$

$$g(x) = 452 \cdot 1.3^x$$

$$h(x) = -452 \cdot 0.3^x$$

$$j(x) = 452 \cdot 0.3^x$$

Exponential and Logarithmic Equations

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

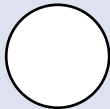
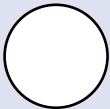
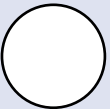
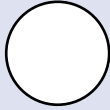
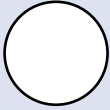
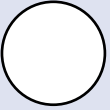
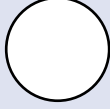
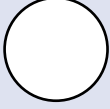
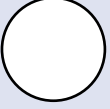
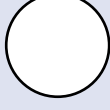
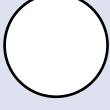
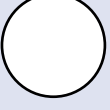
Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Exponential and Logarithmic Equations* topic by:

TOPIC 2: <i>Exponential and Logarithmic Equations</i>	Beginning of Topic	Middle of Topic	End of Topic
using the properties of exponents to rewrite exponential expressions in equivalent forms.			
understanding the inverse relationship between exponents and logarithms.			
converting exponential equations into logarithmic equations and vice versa.			
evaluating logarithmic expressions.			
solving exponential equations by using common bases and by taking the log of both sides.			
solving logarithmic equations when the base, exponent, or argument is unknown.			
formulating exponential and logarithmic functions to model real-world scenarios.			

continued on the next page

TOPIC 2 SELF-REFLECTION
continued

TOPIC 2: <i>Exponential and Logarithmic Equations</i>	Beginning of Topic	Middle of Topic	End of Topic
Using exponential and logarithmic equations to answer questions about real-world scenarios.			
determining whether a data set is best represented by a linear, quadratic, or exponential function.			
using graphing technology to determine a linear, quadratic, or exponential regression function for a set of data.			
using a regression function to make predictions and critical decisions.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Exponential and Logarithmic Equations* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

continued on the next page



TOPIC 2 SELF-REFLECTION *continued*

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

Exponential and Logarithmic Equations Summary

KEY TERM

- logarithmic expression [expresión logarítmica]

LESSON

1

Logarithmic Expressions

When you evaluate a logarithmic expression (logarithm), you are determining the value of the exponent in the corresponding exponential expression.

$$\text{base}^{\text{exponent}} = \text{argument} \Leftrightarrow \log_{\text{base}}(\text{argument}) = \text{exponent}$$

The variables of a logarithmic equation have the same restrictions as the corresponding variables of the exponential equation. The base, b , must be greater than 0, but not equal to 1; the argument must be greater than 0; and the value of the exponent has no restrictions.

A **logarithmic equation** is an equation that contains a logarithm. To write a logarithmic equation, it can be helpful to consider the exponential form first, and then convert it to logarithmic form.

You can convert a logarithmic equation to an exponential equation regardless of which term is unknown.

A logarithm can be any real number, even an irrational number. You can estimate the value of a logarithm that is not an integer by using a number line as a guide.

To estimate $\log_3 33$ to the tenth place, identify the closest logarithm whose argument is less than 33 and the closest logarithm whose argument is greater than 33 on the number line that represents base 3.

The closest logarithm whose argument is less than 33:
 $\log_3 27$

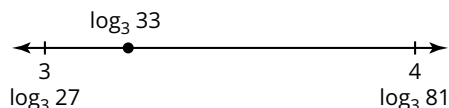
The logarithm you are estimating:
 $\log_3 33$

The closest logarithm whose argument is greater than 33:
 $\log_3 81$

You know that $\log_3 27 = 3$ because $3^3 = 27$ and $\log_3 81 = 4$ because $3^4 = 81$. This means the estimate of $\log_3 33$ is between 3 and 4.

$$\begin{array}{ccccccc} \log_3 27 & < & \log_3 33 & < & \log_3 81 \\ 3 & < & x & < & 4 \end{array}$$

Next, estimate the decimal digit. Because 33 is closer to 27 than it is to 81, the value of $\log_3 33$ is closer to 3 than it is to 4.



In this case, 3.2 is a good estimate for $\log_3 33$.

LESSON

2

Solving Exponential Equations

An exponential function is a function written in the form $f(x) = a \cdot b^x$. To solve an exponential equation, you can use what you know about the properties of exponents. To solve an exponential equation, first determine the power of the base that gives the result. Rewrite the equation in terms of the exponent. Then determine what x must be to equal that exponent.

For example solve the equation $5^{x-3} = 625$.

$$5^{x-3} = 625$$

$$5 \cdot 5 \cdot 5 \cdot 5 = 5^4 = 625$$

$$5^{x-3} = 5^4$$

$$\text{So, } x - 3 = 4$$

$$\text{Therefore, } x = 7.$$

Another way to solve equations that have a variable in the exponent is to apply a logarithm to both sides of the equation.

$$24^x = 5$$

$$\log 24^x = \log 5$$

$$x \cdot \log 24 = \log 5$$

$$x = \frac{\log 5}{\log 24}$$

$$x \approx 0.506$$

Solving Logarithmic Equations

You can use the relationship between the base, argument, and exponent to solve for any unknown in a logarithmic equation. To solve for any unknown in a simple logarithmic equation, begin by converting it to an exponential equation.

Argument Is Unknown	Exponent Is Unknown	Base Is Unknown
$\log_4 y = 3$	$\log_4 64 = x$	$\log_b 64 = 3$
$4^3 = y$	$4^x = 64$	$b^3 = 64$
$64 = y$	$4^x = 4^3$	$b^3 = 4^3$
	$x = 3$	$b = 4$

Applications of Exponential and Logarithmic Equations

Some problem situations can be modeled with either exponential or logarithmic equations. To solve these problems, identify what the question is asking and use the given model to determine the answer.

For example, consider the problem situation below.

The pH scale is a scale for measuring the acidity or alkalinity of a substance, which is determined by the concentration of hydrogen ions. The formula for pH is $\text{pH} = -\log H^+$ where H^+ is the concentration of hydrogen ions in moles per cubic liter.

Vinegar has a pH of 2.2. Determine the concentration of hydrogen ions in vinegar.

$$2.2 = -\log H^+$$

$$-2.2 = \log H^+$$

$$10^{-2.2} = H^+$$

$$H^+ \approx 0.006$$

The concentration of hydrogen ions in vinegar is 0.006 moles per cubic liter.

To determine an appropriate regression model, graph the data set. Then, use technology to write the function based on the given data. This function helps predict outcomes and make critical judgments and decisions about the data.

Consider the situation and table of data.

A cup of tea is poured into a teacup. The table shows the temperature of the tea, $t(x)$, in $^{\circ}\text{F}$, x minutes after the tea is poured.

Time (Minutes)	0	5	7	9	11	15	18	22
Temperature ($^{\circ}\text{F}$)	179.9	164.3	156.1	148.3	141.2	135.1	128.7	125.1

Write a regression function that best models the data and use the equation to predict the number of minutes that it takes the tea to cool to 137.5° .

The data decreases over the given domain, but not at a constant rate, so an exponential regression best models the data. The function that models the data best is $t(x) = 176.1738 \cdot 0.9831^x$.

The predicted number of minutes it will take the tea to cool to 137.5°F is approximately 14.5397 minutes.

$$y = 176.1738 \cdot 0.9831^x$$

$$137.5 = 176.1738 \cdot 0.9831^x$$

$$0.7805 \approx 0.9831^x$$

$$\log(0.7805) \approx \log(0.9831)^x$$

$$\log(0.7805) \approx x \log(0.9831)$$

$$\frac{\log(0.7805)}{\log(0.9831)} \approx x$$

$$14.5397 \approx x$$

Consider the table and the three linear functions shown.

x	y
1	4.8
2	7.2
3	9.5
4	11.9

- $f(x) = -2.5x + 2$
- $g(x) = 2.5x + 2$
- $h(x) = 2.5x + 8$

The linear function $g(x) = 2.5x + 2$ best represents the data as it increases steadily with a positive slope. When $x = 1$, the function $g(x)$ gives an output of 4.5 which is close to the corresponding value of 4.8 in the table.

Consider the table and the three exponential functions shown.

x	y
1	4
2	6
3	10
4	19

- $f(x) = 2^x + 7$
- $g(x) = \left(\frac{1}{2}\right)^x + 4$
- $h(x) = 2^x + 3$

The exponential function $h(x) = 2^x + 3$ best represents the data as it increases with a growth factor greater than 1. When $x = 1$, the function $h(x)$ gives an output of 5 which is close to the corresponding value of 4 in the table.

Consider the table and the three quadratic functions shown.

x	y
1	8
2	11
3	12
4	11

- $f(x) = -x^2 + 6x + 2$
- $g(x) = x^2 - 6x + 13$
- $h(x) = -2x^2 + 12x - 3$

The quadratic function $f(x) = -x^2 + 6x + 2$ best represents the data as it increases to a maximum and then decreases. When $x = 1$, the function $f(x)$ gives an output of 7 which is close to the corresponding value of 8 in the table.

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