ENGLISH

Bluebonnet Learning™
Secondary Mathematics
EDITION 1

# Geometry

## Family Guides

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Secondary Mathematics

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# Geometry

Family Guides

**Acknowledgment**

Thank you to all the Texas educators and stakeholders who supported the review process and provided feedback. These materials are the result of the work of numerous individuals, and we are deeply grateful for their contributions.

**Notice**

These learning resources have been built for Texas students, aligned to the Texas Essential Knowledge and Skills, and are made available pursuant to Chapter 31, Subchapter B-1 of the Texas Education Code.

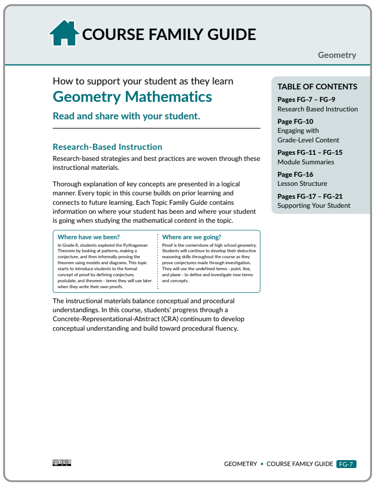
If you have further product questions or to report an error, please email openeducationresources@tea.texas.gov.

## Dear Family,

We recognize that learning outside of the classroom is crucial to your student's success at school. This letter serves as an introduction to the resources designed to assist you as you talk to your student about what they are learning. Resources available to you include:

- Course Family Guide
- Topic Family Guides
- Topic Summaries
- Math Glossary

## Course Family Guide


COURSE FAMILY GUIDE

Geometry

How to support your student as they learn

Geometry Mathematics

Read and share with your student.

Research-Based Instruction

Research-based strategies and best practices are woven through these instructional materials.

Thorough explanation of key concepts are presented in a logical manner. Every topic in this course builds on prior learning and connects to future learning. Each Topic Family Guide contains information on where your student has been and where your student is going when studying the mathematical content in the topic.

Where have we been?

In Grade 8, students explored the Pythagorean Theorem by looking at patterns, making a conjecture, and then informally proving the theorem using models and diagrams. This topic starts to introduce students to the formal concept of proof by defining conjecture, postulate, and theorem - terms they will use later when they write their own proofs.

Where are we going?

Proof is the cornerstone of high school geometry. Students will continue to develop their deductive reasoning skills throughout the course as they prove conjectures made through investigation. They will use the undefined terms - point, line, and plane - to define and investigate new terms and concepts.

The instructional materials balance conceptual and procedural understandings. In this course, students' progress through a Concrete-Representational-Abstract (CRA) continuum to develop conceptual understanding and build toward procedural fluency.

TABLE OF CONTENTS



GEOMETRY • COURSE FAMILY GUIDE FG-7

Following this letter, there is the Course Family Guide that will walk you through the research-based instructional approach, how the course is structured, how the mathematics applies to the real-world, using Talking Points from the Topic Family Guide, and using the TEKS mathematical process standards to initiate discussions.

Research and classroom experience guided course development, with the foundation being a scientific understanding of how people learn and a real-world understanding of how to apply that science to mathematics instructional materials. The instructional design elements presented in the Course Family Guide incorporate research-based strategies to develop conceptual understanding and creative problem solvers.

The Course Family Guide provides an overview of the structure of the course. The course consists of both a Learning Together component and a Learning Individually component. The teacher facilitates a collaborative learning experience during the Learning Together Days and uses data to target specific skills on the Learning Individually Days.

Next, the Course Family Guide includes Module Overviews of each module in the course, which include a detailed summary of what your student will be learning in each topic within the module. Below the topic summaries are facts and information that connect the concepts to the real world. Read and discuss the information below the topic summaries with your student and continue to come back to these pages as your student moves from one topic to the next within each module.

The Course Family Guide also highlights the lesson structure. Each lesson is structured the same way and includes four parts: Objectives & Essential Question, Getting Started, Activities, and the Talk the Talk.

# Topic Family Guide

Each course is organized into modules. Each module consists of topics with corresponding Topic Family Guides. These guides all have the same structure. This consistency will allow you and your student to understand how to reference the content of each topic.

The Topic Family Guide begins with an overview of the content in the topic. This introduction includes a brief explanation of what your student will learn in the topic, the prior knowledge they will use to help them understand this topic, and a connection to future learning.

The next section of the Topic Family Guide is the Talking Points section. The Talking Points section provides skills you can discuss with your students and a sample question based on the math in the topic that you can talk through with them.

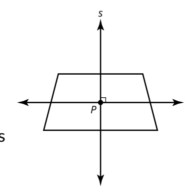
### *TALKING POINTS*

**DISCUSS WITH YOUR STUDENT**

Sequences of rigid motion transformations, including translations, rotations, and reflections, maintain the size and shape of objects while analyzing movements. This is useful in applications such as designing machinery, creating animations, and mapping out paths in navigation systems.

**HERE IS A SAMPLE QUESTION**

The figure shows two perpendicular lines, *s* and *r*, intersecting at point *P* in the interior of a trapezoid. Line *r* is parallel to the bases of the trapezoid and bisects both legs of the trapezoid. Line *s* bisects both bases of the trapezoid.




Which transformation will always carry the figure onto itself?

A reflection across line *s* will carry the figure onto itself.

Next, the Topic Family Guide lists all the key terms of the topic and details some of the math strategies students will learn in the topic. Finally, each Topic Family Guide contains a Math in the Real-World section. Understanding how the math content is used in the real-world boosts student engagement, bridges theory and practice, and helps students to think about connections to other courses and careers they may be interested in.

### *Math in the Real World*

Ever wonder how video games look so amazing or how GPS knows exactly where you are? It's all thanks to the magic of the coordinate plane! Think of it as a giant grid that helps us map out the world—or even imaginary worlds!

Game developers use coordinates to place characters, obstacles, and treasure chests in just the right spots. Architects use it to design skyscrapers, making sure everything lines up perfectly. Even athletes are in on the action—when a coach analyzes plays in basketball or soccer, they're essentially plotting moves on a giant invisible coordinate plane. So next time you're cruising through your favorite game or finding the quickest route to your friend's house, thank those *x*- and *y*-axes for keeping everything in check!

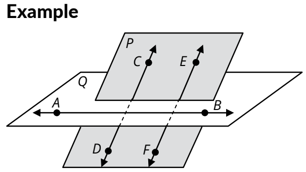
## Topic Summary

A Topic Summary is provided for students at the end of each topic. The Topic Summary lists all key terms of the topic and provides a summary of each lesson. Each lesson summary defines key terms and reviews key concepts, strategies, and/or Worked Examples. The Topic Summary provides an opportunity for you and your student to discuss the key concepts from each lesson, review the examples, and do the math together.

LESSON 1 **Points, Lines, Planes, Rays, and Line Segments**

There are three essential building blocks of geometry- the *point*, the *line*, and the *plane*. These three terms are called undefined terms; we can only describe and create mathematical models to represent them. A **point** is a location in space that has no size or shape. A **line** is a straight continuous arrangement of an infinite number of points. A **plane** is a flat surface that has an infinite length and width, but no depth. **Collinear points** are points that are located on the same line. **Coplanar** lines are two or more lines that are located in the same plane. **Non-collinear points** are points are points that do not lie on the same line. **Skew lines** are two or more lines that are not in the same plane.

**Example**




Points $A$ and $B$ lie on $\overline{AB}$, points $C$ and $D$ lie on $\overline{CD}$, and points $E$ and $F$ lie on $\overline{EF}$.

Line $AB$ lies in plane $Q$. Lines $CD$ and $EF$ lie in plane $P$.

Points $A$ and $B$ are collinear. Points $C$ and $D$ are collinear. Points $E$ and $F$ are collinear.

Lines $CD$ and $EF$ are coplanar.

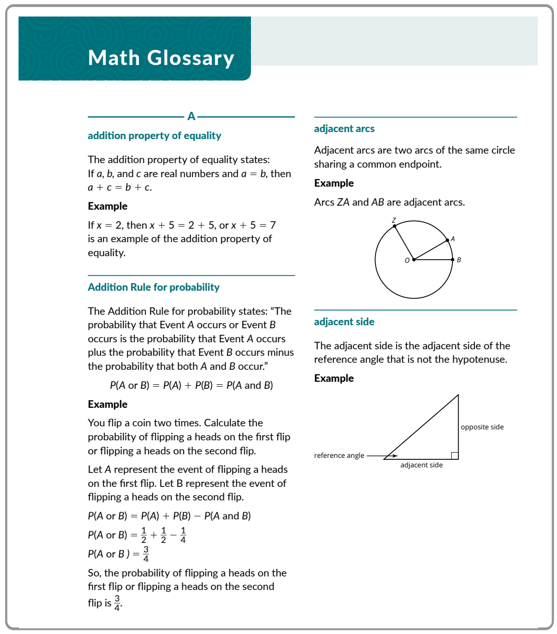
Lines $AB$ and $CD$ are skew. Lines $AB$ and $EF$ are skew.

Planes $P$ and $Q$ intersect.

Evidence of the TEKS mathematical process standards are present in the Topic Summaries. Each lesson within the topic highlights one or more of the TEKS mathematical process standards. These processes will help your student develop effective communication and collaboration skills that are essential for becoming a successful learner. Discuss with your student the “I can” statements associated with each of the TEKS mathematical process standards to help them develop their mathematical learning and understanding. The "I can" statements for each of the TEKS mathematical process standards are included in the Course Family Guide. With your help, your student can develop the habits of a productive mathematical thinker.

# Math Glossary

The Math Glossary for each course is a tool for your student to utilize and reference during their learning. Along with the definition of a term, the glossary provides examples to help further their understanding.



## Math Glossary

A

**addition property of equality**

The addition property of equality states:
If $a$, $b$, and $c$ are real numbers and $a = b$, then $a + c = b + c$.

**Example**

If $x = 2$, then $x + 5 = 2 + 5$, or $x + 5 = 7$ is an example of the addition property of equality.

**Addition Rule for probability**

The Addition Rule for probability states: "The probability that Event *A* occurs or Event *B* occurs is the probability that Event *A* occurs plus the probability that Event *B* occurs minus the probability that both *A* and *B* occur."

$P(A \text{ or } B) = P(A) + P(B) = P(A \text{ and } B)$

**Example**

You flip a coin two times. Calculate the probability of flipping a heads on the first flip or flipping a heads on the second flip.

Let *A* represent the event of flipping a heads on the first flip. Let B represent the event of flipping a heads on the second flip.

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$P(A \text{ or } B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$

$P(A \text{ or } B) = \frac{3}{4}$

So, the probability of flipping a heads on the first flip or flipping a heads on the second flip is $\frac{3}{4}$.

**adjacent arcs**

Adjacent arcs are two arcs of the same circle sharing a common endpoint.

**Example**

Arcs *ZA* and *AB* are adjacent arcs.



**adjacent side**

The adjacent side is the adjacent side of the reference angle that is not the hypotenuse.

**Example**



We all have the same goal for your student: to become a successful problem solver and use mathematics efficiently and effectively in daily life. Encourage them to use the mathematics they already know when seeing new concepts and communicate their thinking while providing a critical ear to the thinking of others.

Thank you for supporting your student.

# COURSE FAMILY GUIDE



How to support your student as they learn

# Geometry Mathematics

### Read and share with your student.

## TABLE OF CONTENTS



## Research-Based Instruction

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Proof is the cornerstone of high school geometry. Students will continue to develop their deductive reasoning skills throughout the course as they prove conjectures made through investigation. They will use the undefined terms - point, line, and plane - to define and investigate new terms and concepts.

The instructional materials balance conceptual and procedural understandings. In this course, students' progress through a Concrete-Representational-Abstract (CRA) continuum to develop conceptual understanding and build toward procedural fluency.

| Concrete | Representational | Abstract |
|---|---|---|
| Students explore transformations on and off the coordinate plane. They start by tracing a figure onto patty paper and physically translating the figure on the plane. | Students translate figures using a directed line segment on and off the plane. | Students represent translations using coordinate notation. |

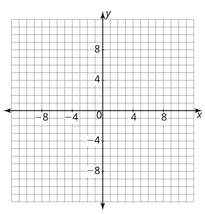
**Concrete:**

Getting Started

Translating Geometric Figures on the Coordinate Plane

Trace the figure shown onto patty paper and then cut out the figure.

1. Graph trapezoid *ABCD* by plotting the points *A* (3, 9), *B* (3, 4), *C* (11, 4), and *D* (11, 10).




**Representational:**

A **translation** moves a set of points a specified distance in a specified direction along parallel lines. When an image is horizontally translated *c* units on the coordinate plane, the value of the *x*-coordinates change by *c* units. When an image is vertically translated *c* units on the coordinate plane, the value of the *y*-coordinate changes by *c* units. **Coordinate notation** describes how the original coordinates of a point are changed into new coordinates after one of more transformations. For example, the mapping $(x, y) \rightarrow (x - 3, y - 2)$ represents a translation left 3 units and down 2 units. The coordinates of an image after a translation are summarized in the table.

| Original Point | Horizontal Translation to the Left | Horizontal Translation to the Right | Vertical Translation Up | Vertical Translation Down |
|---|---|---|---|---|
| $(x, y)$ | $(x - c, y)$ | $(x + c, y)$ | $(x, y + c)$ | $(x, y - c)$ |

**Abstract:**

A **translation** moves a set of points a specified distance in a specified direction along parallel lines. When an image is horizontally translated *c* units on the coordinate plane, the value of the *x*-coordinates change by *c* units. When an image is vertically translated *c* units on the coordinate plane, the value of the *y*-coordinate changes by *c* units. **Coordinate notation** describes how the original coordinates of a point are changed into new coordinates after one of more transformations. For example, the mapping $(x, y) \rightarrow (x - 3, y - 2)$ represents a translation left 3 units and down 2 units. The coordinates of an image after a translation are summarized in the table.

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|---|---|---|---|---|
| $(x, y)$ | $(x - c, y)$ | $(x + c, y)$ | $(x, y + c)$ | $(x, y - c)$ |


Support is provided to students as they persevere in problem solving. These instructional materials include a problem-solving model, which includes questions your student can ask when productively engaging in real-world and mathematical problems. Prompts will encourage your student to use the problem-solving model throughout the course.

These instructional materials include features that support learners. Worked Examples throughout the product provide explicit instruction and provide a model your student can continually reference.

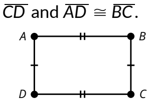
**When you see a Worked Example:**

- Take your time to read through it.
- Question your own understanding.
- Think about the connection between steps.

WORKED EXAMPLE

Markers are used to indicate congruent segments in geometric figures. If a diagram has more than one set of congruent segments then you use sets of markers.

The figure shows $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$.



Thumbs Up, Thumbs Down, and Who's Correct questions address your student's common misconceptions and provide opportunities for peer work analysis.

**Thumbs Up Thumbs Down**

**When you see a Thumbs Up icon:**

- Take your time to read through the correct solution.
- Think about the connection between steps.

**Ask Yourself**

- Why is this method correct?
- Have I used this method before?

**When you see a Thumbs Down icon:**

- Take your time to read through the incorrect solution.
- Think about what error was made.

**Ask Yourself**

- Where is the error?
- Why is it an error?
- How can I correct it?

Mason
I marked 4 outcomes, and there are 4 possible outcomes. But, I marked 1 of the outcomes twice. So, I have to subtract one of the outcomes and the probability of a heads up result for the first coin or a tails up result for the second coin is $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-\frac{1}{4}=\frac{3}{4}$.

Sebastian
I circled 2 outcomes and drew a rectangle around 2 outcomes, and there are 4 possible outcomes. So, the probability of flipping a heads on the first coin or a tails on the second is $\frac{2+2}{4}=\frac{4}{4}=1$.

**Who's Correct**

**When you see a Who's Correct icon:**

- Take your time to read through the situation.
- Question the strategy or reason given.
- Determine if correct or incorrect.

**Ask Yourself**

- Does the reasoning make sense?
- If the reasoning makes sense, what is the justification?
- If the reasoning does not make sense, what error was made?

7. Mr. Davis wrote the statement shown on the board.

If $\overline{AC} \cong \overline{BC}$, then point $C$ is the midpoint of $\overline{AB}$.

He asked his students to discuss the truth of this conditional statement.

Aaliyah said she believed the statement to be true in all situations. Mateo disagreed with Aaliyah and said that the statement was not true all of the time.

What is Mateo thinking and who is correct?

Targeted Skills Practice provides support to your student as they work to gain proficiency of the course material. Spaced Practice provides a spaced retrieval of key concepts to your student. Extension opportunities provide challenges to accelerate your student's learning.

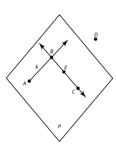
**Skills Practice** TOPIC 1 Geometric Reasoning

Name ______ Date ______

I. Points, Lines, Planes, Rays, and Line Segments

**Topic Practice**

A. Use the diagram to determine what each name represents and tell whether it represents an undefined or defined term.




1. $P$
2. $\overrightarrow{AB}$
3. $\overleftrightarrow{BC}$
4. $\ell$
5. $D$
6. $\overleftrightarrow{BEC}$
7. $\overline{EC}$

TOPIC 1 Geometric Reasoning

**Extension**

1. Name a line and plane in multiple ways.

**Spaced Practice**

Determine the unknown side length of each right triangle.

1. 6, $b$, 10
2. 8, 15, $c$
3. $a$, 9, 12

Each lesson features one or more ELPS (English Language Proficiency Standards) and provides the teacher with implementation strategies incorporating best practices for supporting language acquisition. In addition, students are provided with cognates for Key Terms in the Topic Summaries and Topic Family Guides.

## KEY TERMS

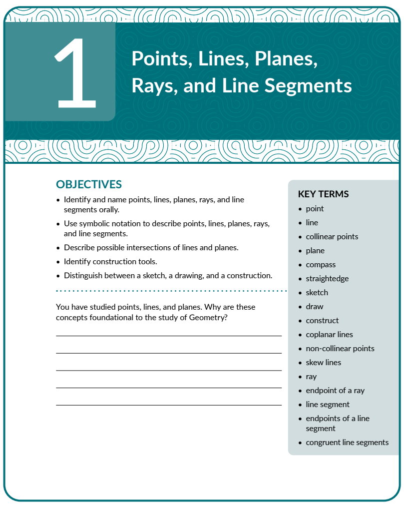
- arc [arco]
- midpoint [punto medio]
- segment bisector [bisectriz de segmento]
- perpendicular bisector [bisectriz perpendicular]
- transformation [transformación]
- rigid motion [movimiento rígido]
- distance formula [fórmula de la distancia]
- extracting perfect squares
- radical [radical]
- radicand [radicando]
- midpoint formula [fórmula del punto medio]
- composite figure [figura compuesta]
- regular polygon [polígono regular]
- apothem [apotema]

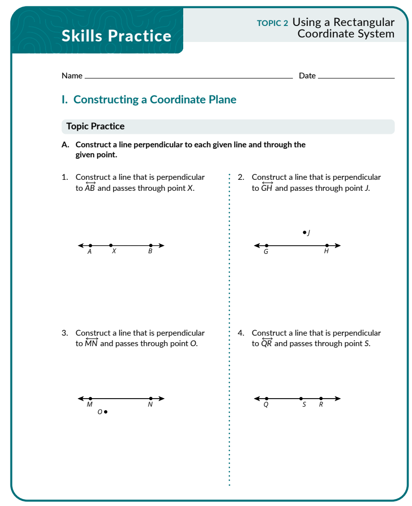
Refer to the Math Glossary for definitions of Key Terms.

## Engaging with Grade Level Content

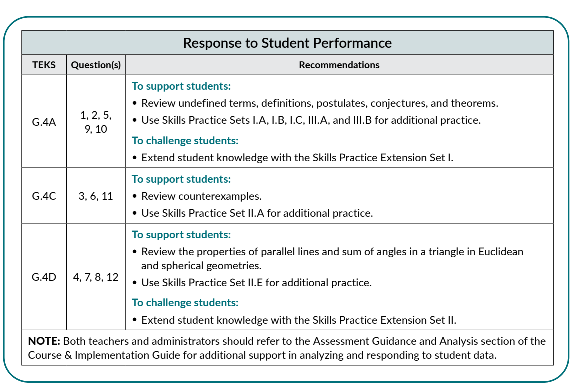
Your student will engage with grade-level content in multiple ways with the support of the teacher.

| Learning Together | Learning Individually |
|---|---|
| The teacher facilitates active learning of lessons so that students feel confident in sharing ideas, listening to each other, and learning together. Students become creators of their mathematical knowledge. | Skills Practice provides students the opportunity to engage in additional skill building that aligns to each Learning Together lesson. The Learning Individually Days target discrete skills that may require additional practice to achieve proficiency. |




At the end of each topic, your student will take an assessment aligned to the standards covered in the topic. This assessment consists of multiple-choice, multiselect, and open-ended questions designed for your student to demonstrate learning. Each assessment also includes a scoring guide for teachers to ensure consistent scoring. The scoring guide includes ways to support or challenge your student based on their responses to the questions on the assessment. The purpose of the assessment is for the teacher and student to reflect on the learning. Teachers will use your student's assessment results to target individual skills your student needs for proficiency or to accelerate and challenge your student.

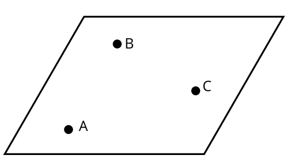
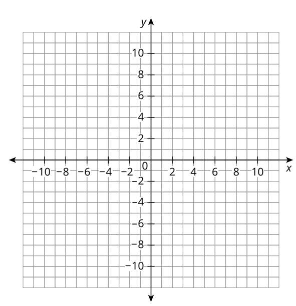
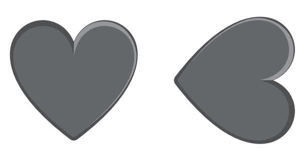
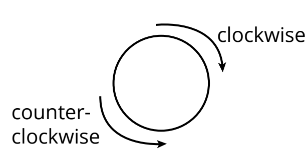
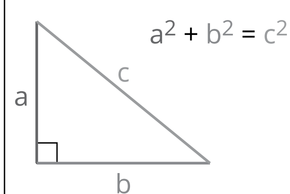

Response to Student Performance

| TEKS | Question(s) | Recommendations |
|---|---|---|
| G.4A | 1, 2, 5, 9, 10 | To support students:<br>• Review undefined terms, definitions, postulates, conjectures, and theorems.<br>• Use Skills Practice Sets I.A, I.B, I.C, III.A, and III.B for additional practice.<br>To challenge students:<br>• Extend student knowledge with the Skills Practice Extension Set I. |
| G.4C | 3, 6, 11 | To support students:<br>• Review counterexamples.<br>• Use Skills Practice Set II.A for additional practice. |
| G.4D | 4, 7, 8, 12 | To support students:<br>• Review the properties of parallel lines and sum of angles in a triangle in Euclidean and spherical geometries.<br>• Use Skills Practice Set II.E for additional practice.<br>To challenge students:<br>• Extend student knowledge with the Skills Practice Extension Set II. |

**NOTE:** Both teachers and administrators should refer to the Assessment Guidance and Analysis section of the Course & Implementation Guide for additional support in analyzing and responding to student data.

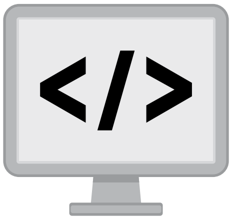
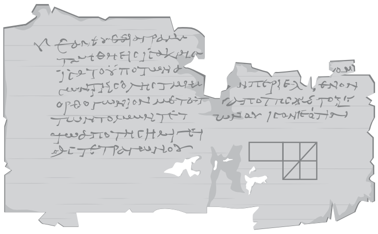
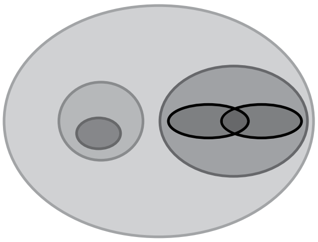
# MODULE 1 Reasoning with Shapes

In this module, your student will take the first steps to transition form informal reasoning to formal reasoning. They begin to reason about the defining characteristics of shapes they are familiar with: squares, circles, and triangles. There are four topics in this module: *Reasoning with Shapes*, *Using a Rectangular Coordinate System*, *Sequences of Rigid Motions*, and *Congruence Through Transformations*.

<table>
<tr><th>TOPIC 1<br>Geometry Reasoning</th><th>TOPIC 2<br>Using a Rectangular Coordinate System</th><th>TOPIC 3<br>Sequences of Rigid Motions</th><th>TOPIC 4<br>Congruence Through Transformations</th></tr>
<tr><td>Your student will be introduced to the building blocks of geometry: points, lines, and planes.</td><td>Your student will study the properties of squares and will learn strategies for determining the perimeters and areas of figures on the coordinate plane.</td><td>Your student will study rigid motions with a transformation machine, then consider each as a function.</td><td>Your student will use formal reasoning to prove geometric theorems.</td></tr>
<tr><td>Did you know?<br><br>Any three non-collinear points determine a unique plane. This means that if you take three points that don’t all lie on the same straight line, there is exactly one flat surface (plane) that contains all three of them.<br><br>
</td><td>Plane vs. Plane<br><br>A plane can mean an airplane.<br><br><br><br>A plane can also be a flat surface.<br><br>A coordinate plane is formed by the intersection of horizontal and vertical lines.<br><br></td><td>What in the world?<br><br>A transformation machine is like making a shake. You put in the ingredients (the input), you blend them together (the transformation), and the result is a delicious shake (the output).<br><br>How has this image of a heart been transformed?<br><br><br><br>[This heart shape has been rotated 90 degrees clockwise.]<br><br>
</td><td>What is a theorem?<br><br>A theorem is a math rule that has been proven to be true.<br><br>
<br><br>A well-known example is the Pythagorean Theorem.</td></tr>
</table>

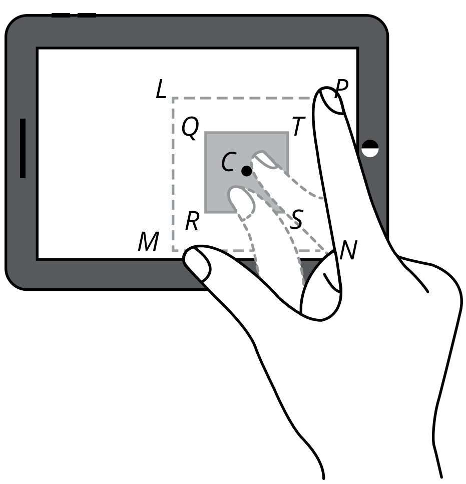
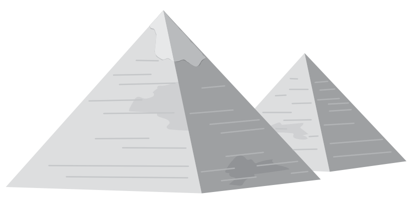
# MODULE 2 Justifying Mathematical Ideas and Arguments

In this module your student will deepen their understanding of congruence by constructing and deconstructing shapes, and defining the line and angle relationships that will ultimately justify congruence. There are three topics in this module: *Composing and Decomposing Shapes*, *Justifying Line* and *Angle Relationships, and Using Congruence Theorems.*

| TOPIC 1<br>Composing and Decomposing Shapes | TOPIC 2<br>Justifying Line and Angle Relationships | TOPIC 2<br>Using Congruence Theorems |
|---|---|---|
| Your student will learn and define many of the angles, lines, arcs, shapes, and parts of proofs necessary to understand and prove theorems in this topic. | Your student will learn to prove many of the angle, line, and shape theorems needed to later prove congruence in the next topic. | Your student will use the theorems that they have proven to prove new theorems about triangles, quadrilaterals, and angles formed in circles. |
| If… Then…<br><br>Understanding how conditional statements work and being able to use them effectively is a staple of programming and coding.<br><br> | Did you know?<br><br>Euclid of Alexandria, often known as the “founder of geometry”, formed the basis on which we still use for proofing today, something he started over 2000 years ago!<br><br> | So many quadrilaterals!<br><br>Though there are many names for quadrilaterals, they also share many properties. It may be helpful to categorize them into a Venn Diagram like the one below!<br><br> |

# MODULE 3 Investigating Proportionality

In this module, your student will deepen their understanding of similarity and proportionality. Using these new ideas of proportionality, students develop trigonometric ratios for right triangles. There are two topics in this module: *Similarity* and *Trigonometry*.

| TOPIC 1<br>Similarity | TOPIC 2<br>Trigonometry |
|---|---|
| Your student will use what they know about dilations to learn about similarity and proportionality. | Your student will be introduced to trigonometry and how to use trigonometric ratios with right triangles to find missing side length and angle measures. |
| Did you know?<br>Without realizing it we probably use dilations everyday. Anytime you pinch and zoom on a smartphone or other touch screen device, that is actually applying a dilation.<br><br> | Ahead of their time...<br>The Egyptians were the first ones to use trigonometric ratios during the construction of the pyramids. They used a variation of tangent to make sure that their buildings were accurately built.<br> |

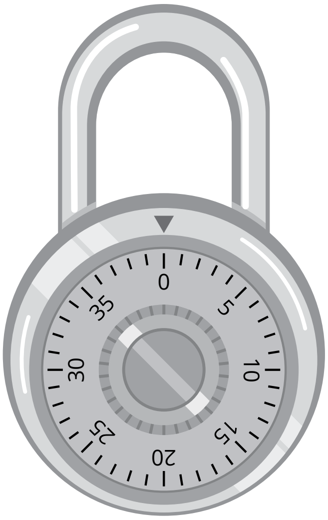
# MODULE 4 Connecting Geometric and Algebraic Descriptions

In this module, your student will deepen their understanding of circles and three dimensional shapes along with their attributes. There are two topics in this module: *Circles* and *Building Three-Dimensional Shapes*.

| TOPIC 1<br>Circles | TOPIC 2<br>Building Three-Dimensional Shapes |
|---|---|
| Your student will prove theorems about chords, tangents, and secants of circles. They will write equations for circles on the coordinate plane. | Your student will calculate various attributes for different three-dimensional objects. |
| Did you know?<br>All circles are similar to each other. This means that no matter the size of a circle—whether it's tiny like a coin or as big as a Ferris wheel—all circles have the same shape and can be resized to match any other circle perfectly. Their proportions (like the ratio of circumference to diameter, which is always $\pi$) stay constant! | Did you know?<br>Archimedes of Syracuse, Sicily, who lived from 287 BC to 212 BC, was an ancient Greek mathematician, physicist, and engineer.<br>Archimedes discovered formulas for computing volumes of spheres, cylinders, and cones. Archimedes has been honored in many ways for his contributions. He has appeared on postage stamps in East Germany, Greece, Italy, Nicaragua, San Marino, and Spain.<br> |

# MODULE 5 Making Informed Decisions

In this module your student will deepen their understanding of probability and how to determine the likelihood of certain outcomes, or end results. There are two topics in this module: *Independence and Conditional Probability* and *Computing Probabilities*.

| TOPIC 1<br>Independence and Conditional Probability | TOPIC 2<br>Computing Probabilities |
|---|---|
| Your student will learn about probability using situations involving coins, number cubes, bags of marbles, and playing cards. Different probabilities will be calculated differently if the situations use words like *and* or *or*, as well as if items are being replaced or not replaced. | Your student will organize events in frequency tables and two-way tables to help determine the probability of specific outcomes. Then they will learn the differences between, and make calculations for, permutations and combinations. |
| Let's Make A Deal!<br><br>There are three doors. Behind one door is a prize. Behind the other two doors are donkeys. You choose one door. The game show host opens one of the doors that you did not choose to reveal a donkey. Then, the host asks you if you would like to stay with the door you chose or switch to the other unopened door. Should you stay or switch? Or does it matter?<br><br> | Permutation lock, not a combination lock!<br><br>Normally we use a "combination lock" for lockers, safes, garages, gates, and even bikes! So as long as you know the 3 or 4 numbers needed for the lock, it should open. But really it should be called a "permutation lock" because the order that the numbers are used matters!<br><br> |

# Lesson Structure

Each lesson in the course is laid out in the same way to develop deep understanding. Read through the parts of the lesson to learn more about your student's learning in their math classroom.

## Objectives & Essential Question

Each lesson begins with objectives, listed to help students understand the objectives. Also included is an essential statement connecting students' learning with a question to ponder. The question is asked again at the end of each lesson to see how much your student understands.

## Getting Started

The Getting Started engages your student in the learning. In the Getting Started, your student uses what they already know about the world, what they've already learned, and their intuition to get them thinking mathematically and prepare them for what's to come in the lesson.

## Activities

In the Activities, students develop their mathematical knowledge and build a deep understanding of the math. These activities provide your student with the opportunity to communicate and work with others in their math classroom.

When your student is working through these activities, keep in mind:

- It's not just about answer-getting. Doing the math and talking about it is important.
- Making mistakes is an important part of learning, so take risks.
- There is often more than one way to solve a problem.

## Talk the Talk

The Talk the Talk gives your student an opportunity to reflect on the main ideas of the lesson and demonstrate their learning.

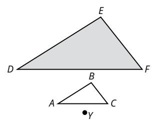
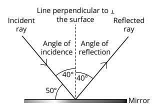
## Lesson Assignment

The lesson assignment provides your students with practice to develop fluency and build proficiency. The lesson assignment also includes a section to help prepare students for the next lesson.

## Key Concepts of the Lesson

At the end of each topic, the Topic Summary provides a summary of each lesson in the topic. Encourage your student to use these as a tool to review and retrieve the key concepts of a lesson.

**Where are we now?**

| | |
|---|---|
| A **dilation** is a transformation of the figure in which the figure stretches or shrinks with respect to a fixed point, or center of dilation.<br><br>Triangle *DEF* is a dilation of $\triangle ABC$. The center of dilation is point *Y*. |  |
| The **geometric mean** of two positive numbers *a* and *b* is the positive number *x* such that $\frac{a}{x} = \frac{x}{b}$.<br><br>The geometric mean of 3 and 12 is 6. | $\frac{3}{x} = \frac{x}{12}$<br>$x^2 = 36$<br>$x = 6$ |
| The **angle of reflection** is the angle formed by the reflected ray and a line perpendicular to the surface of a mirror.<br><br>In this example, the angle of reflection measures 40°. |  |

## The Topic Family Guide

The Topic Family Guide provides an overview of the mathematics of the topic, how that math is connected to what students already know, and how that knowledge will be used in future learning. It provides an example of a math model or strategy your student is learning in the topic, real-world connections to the mathematics, talking points to discuss and/or questions to ask your student, and the key terms your student will learn. You and your student can use the Math Glossary to look up the definitions to the key terms. Encourage your student to reference the key terms in the Topic Family Guide and/or Math Glossary when completing math tasks.

Learning outside of the classroom is crucial for your student's success. While we don't expect you to be a math teacher, the Topic Family Guide can assist you as you talk to your student about the mathematical content of the course. The hope is that both you and your student will read and benefit from the guides.

### Where have we been?

In Grade 8, students explored the Pythagorean Theorem by looking at patterns, making a conjecture, and then informally proving the theorem using models and diagrams. This topic starts to introduce students to the formal concept of proof by defining conjecture, postulate, and theorem - terms they will use later when they write their own proofs.

### Where are we going?

Proof is the cornerstone of high school geometry. Students will continue to develop their deductive reasoning skills throughout the course as they prove conjectures made through investigation. They will use the undefined terms - point, line, and plane - to define and investigate new terms and concepts.

### *Math in the Real World*

Ever wonder how video games look so amazing or how GPS knows exactly where you are? It's all thanks to the magic of the coordinate plane! Think of it as a giant grid that helps us map out the world—or even imaginary worlds!

Game developers use coordinates to place characters, obstacles, and treasure chests in just the right spots. Architects use it to design skyscrapers, making sure everything lines up perfectly. Even athletes are in on the action—when a coach analyzes plays in basketball or soccer, they're essentially plotting moves on a giant invisible coordinate plane. So next time you're cruising through your favorite game or finding the quickest route to your friend's house, thank those *x*- and *y*-axes for keeping everything in check!

## Math in the Real-World

Math is used in everyday activities like budgeting, cooking, and sports, making it a valuable life skill. It also plays a crucial role in financial literacy, logical reasoning, and decision-making. By connecting math to real-life situations, students gain confidence, see its relevance, and develop critical thinking skills that will benefit them in both academics and daily life. The Math in the Real-World section in the Topic Family Guide provides examples of where the math concepts of the topic are used beyond the classroom.

Examples of these real-world connections include:

**Math in the Real World:** Imagine you're at an amusement park, staring up at the tallest roller coaster in the world. How would you figure out how tall it is without climbing to the top? That's where similar triangles come to the rescue! By measuring the shadow of the coaster and using a smaller, similar triangle you create with a stick and its shadow, you can calculate the height of the coaster without leaving the ground.

# Supporting Your Student

Cartographers (mapmakers) and surveyors use similar triangles to measure things like rivers, mountains, and cityscapes, all while keeping their feet dry and their gear light. Next time you take a selfie, think about this: the way the background shrinks while your face stays big is all about perspective, which relies on—you guessed it—similar triangles! Artists and designers use this to create realistic drawings and 3D effects. So whether you're scaling a building, mapping uncharted lands, or snapping the perfect photo, similar triangles are involved.

**Math in the Real-World:** You might not realize it, but lines and angles are everywhere in your life, even if you're not actively thinking about them. From the moment you wake up, you're using them! The corners of your phone screen? Angles. The way the sun shines through your window? That's light following the rules of angles and reflection. Even when you take a selfie, you're positioning your phone at just the right angle to get the best lighting and show your best side—geometry in action!

Architects use line and angle relationships to design buildings that won't fall down. Engineers use them to create roads, bridges, and roller coasters with perfect curves and supports. A basketball court or soccer field relies on parallel lines (the sidelines) and transversals (passes and plays across the field) to create strategies and guide movement. Geometry is basically the VIP pass to understanding how the world is built and moves!

## The TEKS Mathematical Process Standards

Each module will focus on TEKS mathematical process standards that will help your student become a mathematical thinker. The TEKS mathematical process standards are listed below. Discuss with your student the "I can" statements below the standards to help them develop their mathematical learning and understanding. With your help, your student can become a productive mathematical thinker.

*Apply mathematics to problems arising in everyday life, society, and the workplace.*

I can:

- use the mathematics that I learn to solve real-world problems.
- interpret mathematical results in the contexts of a variety of problem situations.

# Supporting Your Student

*Use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution.*

I can:

- explain what a problem "means" in my own words.
- create a plan and change it when necessary.
- ask useful questions to understand the problem.
- explain my reasoning and defend my solution.
- reflect on whether my results make sense.

*Select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems.*

I can:

- use a variety of different tools that I have to solve problems.
- recognize when a tool that I have to solve problems might be helpful and when it has limitations.
- look for efficient methods to solve problems.
- estimate before I begin calculations to inform my reasoning.

*Communicate mathematical ideas, reasoning, and their implications using multiple representations including symbols, diagrams, graphs, and language as appropriate.*

I can:

- explain what a problem "means" in my own words.
- create a plan and change it if necessary.
- ask useful questions when trying to understand the problem.
- explain my reasoning and defend my solution.
- reflect on whether my results make sense.

*Create and use representations to organize, record, and communicate mathematical ideas.*

I can:

- consider the units of measure involved in a problem.
- label diagrams and figures appropriately to clarify the meaning of different representations.
- create an understandable representation of a problem situation.

# Supporting Your Student

*Analyze mathematical relationships to connect and communicate mathematical ideas.*

I can:

- identify important relationships in a problem situation.
- use what I know to solve new problems.
- analyze and organize information.
- look closely to identify patterns or structure.
- look for general methods and more efficient ways to solve problems.

*Display, explain, and justify mathematical ideas and arguments using precise mathematical language in written or oral communication.*

I can:

- work carefully and check my work.
- distinguish correct reasoning from reasoning that is flawed.
- use appropriate mathematical vocabulary when I talk with my classmates, my teacher, and others.
- specify the appropriate units of measure when I explain my reasoning.
- calculate accurately and communicate precisely to others.

# Supporting Your Student

## Reflecting on Learning and Progress

You can support your student by encouraging your student to reflect on the learning process. The instructional resources include a student Topic Self-Reflection for each topic. Encourage your student to accurately and frequently reflect on learning and progress throughout each topic. Talk about the specific concepts in the Topic Self-Reflection with your student and celebrate the progress from the beginning to the end of the topic. Remind your student to refer to the Topic Self-Reflection on Learning Individual Days after targeting specific skills and concepts. You can have your student explain concepts from the self-reflection using the topic summaries or lesson assignments to demonstrate understanding. In addition, encourage your student to reflect after taking an assessment. An Assessment Reflection is available to your student to assist with this process. Encourage your student to consider what went well and how to prepare for the next assessment. Ask your student how you can support them when preparing for the next assessment.

TOPIC 1 SELF-REFLECTION

Name: ______________________

### Geometry Reasoning

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me,** 2 represents **I am building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Geometry Reasoning* topic by:

| TOPIC 1: *Geometry Reasoning* | Beginning of Topic | Middle of Topic | End of Topic |
|---|---|---|---|
| distinguishing between undefined terms, definitions, conjectures, postulate, theorems, and proofs. | ○ | ○ | ○ |
| explaining the difference between the terms draw, sketch, and construct. | ○ | ○ | ○ |
| verifying a conjecture is false using a counterexample. | ○ | ○ | ○ |
| identifying the hypothesis and conclusion given a conditional statement. | ○ | ○ | ○ |
| determining the validity of a conditional statement. | ○ | ○ | ○ |
| recognizing a linear, exponential, or quadratic function by its equation or graph. | ○ | ○ | ○ |
| explaining the difference between Euclidean geometry and spherical geometry. | ○ | ○ | ○ |
| making a conjecture and using deductive reasoning to validate it. | ○ | ○ | ○ |

*continued on the next page*



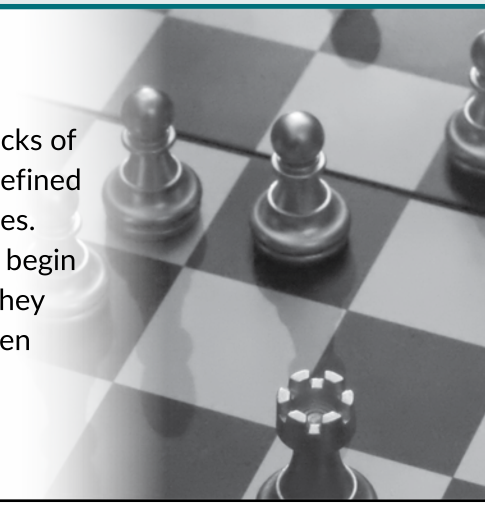
MODULE 1 • TOPIC 1 • SELF-REFLECTION 55

## Thanks!

Enjoy the fun mathematical adventure that is ahead for you and your student! Remember the supports available to you, and thank you for supporting your student's learning.

## TOPIC 1 Geometry Reasoning

In this topic, students are introduced to the three essential building blocks of geometry - the point, the line, and the plane. They use these three undefined terms to define additional terms such as line segment, ray, and skew lines. Students differentiate between inductive and deductive reasoning and begin to build the foundation of proof by exploring conditional statements. They then learn about making conjectures and explore the difference between postulates and theorems.



### Where have we been?

In Grade 8, students explored the Pythagorean Theorem by looking at patterns, making a conjecture, and then informally proving the theorem using models and diagrams. This topic starts to introduce students to the formal concept of proof by defining conjecture, postulate, and theorem - terms they will use later when they write their own proofs.

### Where are we going?

Proof is the cornerstone of high school geometry. Students will continue to develop their deductive reasoning skills throughout the course as they prove conjectures made through investigation. They will use the undefined terms - point, line, and plane - to define and investigate new terms and concepts.

### *TALKING POINTS*

#### DISCUSS WITH YOUR STUDENT

People use deductive reasoning every day to make logical decisions, solve problems, and draw conclusions based on facts in school, work, and everyday life.

#### HERE IS A SAMPLE QUESTION

If the statement “If a number is divisible by 6, then it is divisible by 3” is true, which of the following must also be true?

A. If a number is divisible by 3, then it is divisible by 6.
B. If a number is not divisible by 6, then it is not divisible by 3.
C. If a number is not divisible by 3, then it is not divisible by 6.
D. If a number is divisible by 6, then it is not divisible by 3.

Students can use a counterexample to show that A, B, and D are not true. For example, For choice A, 3 is divisible by 3 but is not divisible by 6.

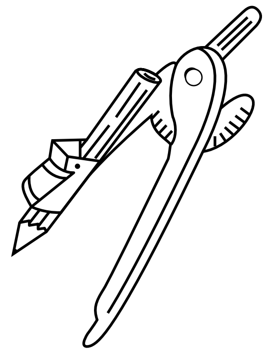
The statement “If a number is not divisible by 3, then it is not divisible by 6” is a true statement.

## KEY TERMS

- point [punto]
- line
- collinear points [puntos colineales]
- plane [plano]
- noncollinear points [puntos no colineales]
- compass [compás]
- straightedge
- sketch
- draw
- construct [construir]
- coplanar lines
- skew lines
- ray [rayo]
- endpoint of a ray
- line segment
- endpoints of a line segment
- congruent line segments
- conditional statement
- biconditional statement
- counterexample [contrajemplo]
- hypothesis [hipótesis]
- conclusion [conclusión]
- postulate [postulado]
- theorem [teorema]
- proof [prueba]
- validity [validez]
- Euclidean geometry [geometría euclidiana]
- great circle

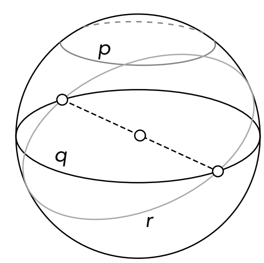
## Where are we now?

A **compass** is a tool used to create arcs and circles.



A **counterexample** is a single example that shows that a statement is not true.
For example, a friend claims you add fractions by adding the numerators and then adding the denominators.
A counterexample is $\frac{1}{2} + \frac{1}{2} = 1$. If you use your friend's method the result is $\frac{1+1}{2+2} = \frac{2}{4}$ or $\frac{1}{2}$. The counterexample shows that your friend's method is incorrect.

**Spherical geometry** is a geometry that substitutes a sphere for a plane, which makes it different from plane geometry in significant ways.
In spherical geometry, lines are defined as *great circles* of a sphere, which divide the sphere into two congruent hemispheres.




A **great circle** is the intersection of a sphere and a plane passing through the center of the sphere.

In **Lesson 1: Points, Lines, Rays, and Line Segments**, students are introduced the three undefined terms – *point*, *line*, and *plane* – that form the building blocks of geometry.

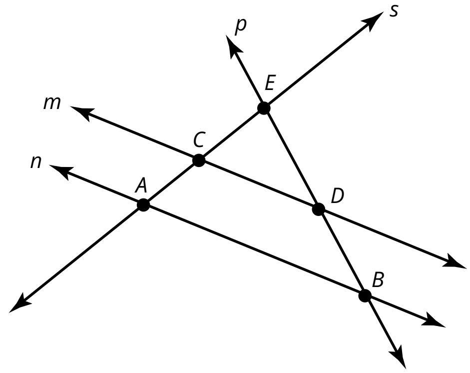
## Points, Lines, and Planes

Although the terms point, line, and plane are undefined, we can describe and use mathematical models to represent them.

A **point** is described simply as a location. A point in geometry has no size or shape, but it is often represented using a dot.

A **line** is described as a straight, continuous arrangement of an infinite number of points. A line has an infinite length but no width.

A mathematical model of several points and lines is shown.




A **plane** is described as a flat surface. A plane has an infinite length and width but no depth and extends infinitely in all directions. One real-world model of a plane is the surface of a still body of water.

**Lesson 2: Formal Reasoning in Euclidean Geometry**, introduces formal reasoning as a foundation for proving geometric theorems.

## Conditional Statements

A **conditional statement** is a statement that can be written in the form "if *p*, then *q*". You can write it using symbols as $p \rightarrow q$, which is read as "*p* implies *q*". In a conditional statement, the **hypothesis** is the part of the "if-then" statement that follows "if". The **conclusion** is the part of the "if-then" statement that follows the "then."

conditional statement

$$\underbrace{\text{If } x^2 = 36,}_{\text{Hypothesis}}\ \underbrace{\text{then } x = 6 \text{ or } x = -6}_{\text{Conclusion}}.$$

In **Lesson 3: Making Conjectures and Deductive Reasoning**, students are introduced to inductive and deductive reasoning and explore several fundamental postulates.

- spherical geometry [geometría esférica]
- conjecture [conjetura]
- inductive reasoning [razonamiento inductivo]
- deductive reasoning [razonamiento deductivo]
- diagonal [diagonal]
- angle [ángulo]
- vertex of an angle [vértice de un ángulo]
- linear pair postulate [postulado del par lineal]
- segment addition postulate
- angle addition postulate
- auxiliary line

Refer to the Math Glossary for definitions of Key Terms.

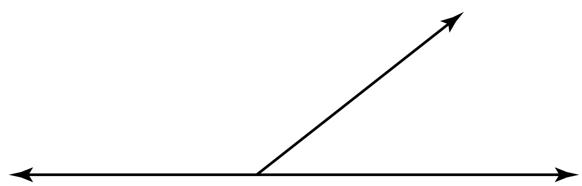
## Math in the Real World

Deductive reasoning isn't just for solving angle puzzles or proving triangles congruent—it's your secret weapon for life! It's the power of starting with facts and building logical steps to reach a conclusion, whether you're cracking a geometry proof or figuring out why your friend is always late to meet up. This skill fuels everything from designing skyscrapers to debating who gets the last slice of pizza. It helps scientists develop theories, lawyers build cases, and detectives solve mysteries. So next time you're working through a proof, remember—you're not just learning math; you're training to be a problem-solving superhero in whatever field you choose!

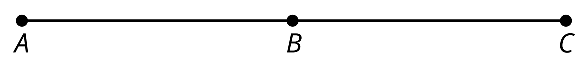
## The Linear Pair Postulate, Segment Addition Postulate and Segment Addition Postulate

Three fundamental postulates—the linear pair postulate, the segment addition postulate, and the angle addition postulate—can be used to make various conjectures. If the conjectures are proven, then they will become theorems.

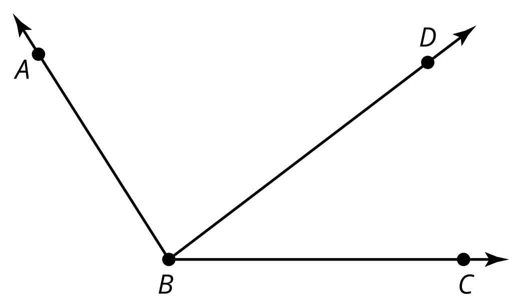
The **linear pair postulate** states, "If two angles form a linear pair, then the angles are supplementary."



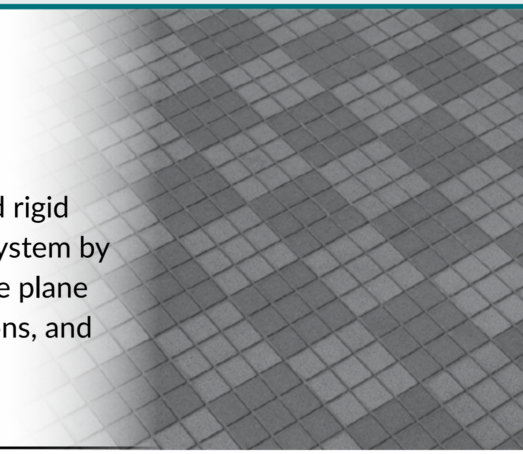
The **segment addition postulate** states, "If point *B* is on $\overline{AC}$ and between points *A* and *C*, then $AB + BC = AC$."




The **angle addition postulate** states, "If point *D* lies in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$."



# TOPIC 2 Using a Rectangular Coordinate System



Students begin this topic by reviewing the properties of squares and rigid motions. They use constructions to build a rectangular coordinate system by creating and transforming squares. Students then use the coordinate plane to study parallel and perpendicular line relationships, classify polygons, and determine the area and perimeter of shapes.

## Where have we been?

Students performed rigid motion transformations of geometric objects in previous grades and have explored the properties of triangles, and some quadrilaterals. They have studied informal demonstrations of geometric congruence using parallel lines and have a wealth of experience with the coordinate plane from elementary and middle school.

## Where are we going?

In this topic, students make conjectures—a theme that will continue into the early parts of the next topic. They use what they have learned in previous courses to ask formal questions about shapes and lines. These questions will be addressed formally with proofs as students move into later topics in this course.

### *TALKING POINTS*

#### DISCUSS WITH YOUR STUDENT

The coordinate plane is fundamental in mathematics and has numerous real-world applications. It provides a structured way to represent and analyze relationships between numbers, objects, or events in two dimensions, enabling precision and visualization in various fields including navigation, urban planning, and engineering.

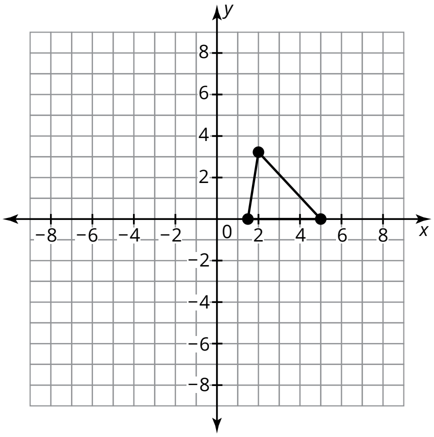
#### HERE IS A SAMPLE QUESTION

In the *xy*-plane, a triangle has vertices at $(5, 0)$, $(\sqrt{2}, 0)$, and $(2, \sqrt{10})$. What is the approximate area of the triangle?

You can think of the base as the horizontal line segment. Its base is $5 - \sqrt{2}$, and the height is $\sqrt{10}$.

So, the area is

$\frac{1}{2}(5 - \sqrt{2})(\sqrt{10}) < 5.67$




So, the area of the triangle is approximately 5.67 square units.

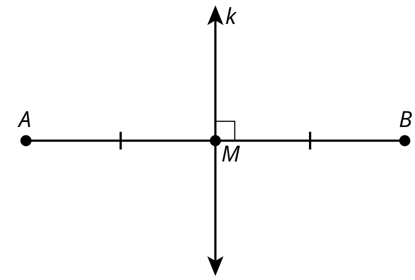
## KEY TERMS

- arc [arco]
- midpoint [punto medio]
- segment bisector [bisectriz de segmento]
- perpendicular bisector [bisectriz perpendicular]
- transformation [transformación]
- rigid motion [movimiento rígido]
- distance formula [fórmula de la distancia]
- extracting perfect squares
- radical [radical]
- radicand [radicando]
- midpoint formula [fórmula del punto medio]
- composite figure [figura compuesta]
- regular polygon [polígono regular]
- apothem [apotema]

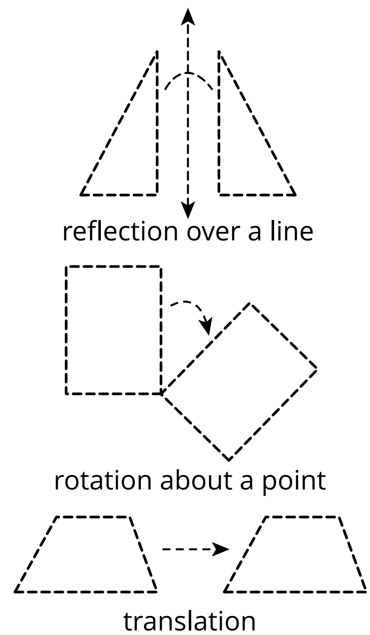
Refer to the Math Glossary for definitions of Key Terms.

## Where are we now?

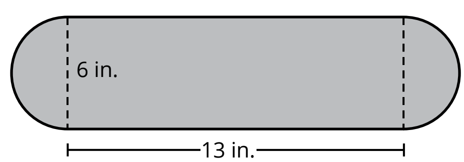
A **perpendicular bisector** is a line, line segment, or ray that intersects the midpoint of a line segment at a 90° angle. Line *k* is the perpendicular bisector of $\overline{AB}$.




A **transformation** is an operation that maps, or moves, a figure called the pre-image, to form a new figure called the image. Three types of transformations are reflections, rotations, and translations.



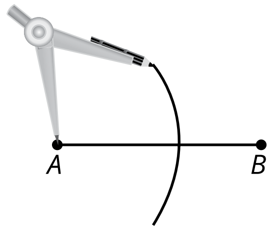

A **composite figure** is a figure that is formed by combining different shapes. For example, this composite figure combines a rectangle and two half circles.

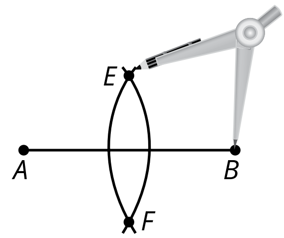


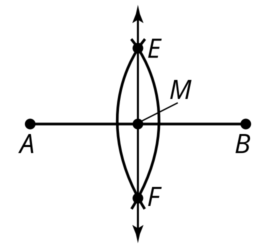

In **Lesson 1: Constructing a Coordinate Plane**, students learn some basic constructions. They construct a square and use transformations to use the square to construct a coordinate plane.

## Constructing a Segment Bisector

To construct a segment bisector using only a compass and straightedge, you make use of the fact that all the radii of a circle have an equal length.




**Construct an Arc**

Open the radius of the compass to more than half the length of $\overline{AB}$. Use endpoint *A* as the center and construct an arc.

→

**Construct Another Arc**

Keep the compass radius and use point *B* as the center as you construct another arc. Label the points formed by the intersection of the arcs point *E* and point *F*.

→

**Construct a Line**

Connect points *E* and *F* with a straightedge. Line segment *EF* is the segment bisector of $\overline{AB}$.
The point *M* represents the midpoint of $\overline{AB}$.

Line *EF* bisects $\overline{AB}$. Point *M* is the midpoint of $\overline{AB}$.

In **Lesson 2: Parallel and Perpendicular Lines**, students extend their understanding of linear relationships to write equations of parallel and perpendicular lines.

## Writing an Equation Parallel to a Given Line

Parallel lines have the same slope. For example, to write an equation for a line parallel to the line $y = 2x + 1$ that passes through the point $(4, 11)$, first determine the slope of the line.

Using the given point and point-slope form, you can write an equation parallel to the given line.

$y - y_1 = m(x - x_1)$

$y - 11 = 2(x - 4)$

$y - 11 = 2x - 8$

$y = 2x + 3$

So, the line parallel to $y = 2x + 1$ that passes through the point $(4, 11)$ is $y = 2x + 3$.

## Math in the Real World

Ever wonder how video games look so amazing or how GPS knows exactly where you are? It's all thanks to the magic of the coordinate plane! Think of it as a giant grid that helps us map out the world—or even imaginary worlds!

Game developers use coordinates to place characters, obstacles, and treasure chests in just the right spots. Architects use it to design skyscrapers, making sure everything lines up perfectly. Even athletes are in on the action—when a coach analyzes plays in basketball or soccer, they're essentially plotting moves on a giant invisible coordinate plane. So next time you're cruising through your favorite game or finding the quickest route to your friend's house, thank those *x*- and *y*-axes for keeping everything in check!

In **Lesson 3: Classifying Quadrilaterals on the Coordinate Plane** and **Lesson 4: Classifying Triangles on the Coordinate Plane**, students derive and use the *slope formula*, *distance formula*, and *midpoint formula*. They use these formulas to solve problems on the coordinate plane.

# The Distance Formula

You can use the Pythagorean Theorem to derive the distance formula, which determines the distance between two points on the coordinate plane. The distance formula states that if $(x_1, y_1)$ and $(x_2, y_2)$ are two points on the coordinate plane, then the distance, *d*, between if $(x_1, y_1)$ and $(x_2, y_2)$ is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

For example, determine the distance between the points (3, 7) and (–1, 4).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(-1 - 3)^2 + (4 - 7)^2}$$

$$d = \sqrt{(-4)^2 + (-3)^2}$$

$$d = \sqrt{16 + 9}$$

$$d = \sqrt{25}$$

$$d = 5$$

The distance between the points (3, 7) and (–1, 4) is 5 units.

# The Midpoint Formula

The **midpoint formula** states that if $(x_1, x_2)$ and $(x_2, y_2)$ are two points on the coordinate plane, then the midpoint of the line segment that joins these points is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

To determine the midpoint of the segment with endpoints (2, 3) and (8, 5), substitute the *x* and *y* values of the ordered pairs into the midpoint formula and then rewrite the ordered pair.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\left(\frac{2 + 8}{2}, \frac{3 + 5}{2}\right)$$

$$\left(\frac{10}{2}, \frac{8}{2}\right)$$

$$(5, 4)$$

The midpoint of the segment with the end points (2, 3) and (8, 5) is located at (5, 4).

In **Lesson 5: Area and Perimeter on the Coordinate Plane**, students apply the distance and midpoint formulas to determine the area and perimeter of figures on the coordinate plane. They also determine the area of *regular polygons*.

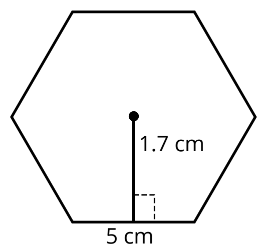
## Determining the Area of a Regular Polygon

A **regular polygon** is a polygon with all sides congruent and all angles congruent. The **apothem** of a regular polygon is a perpendicular segment from the center of a regular polygon to one of its sides. The formula for the area of a regular polygon is $A = \frac{1}{2}aP$ where $P$ represents the perimeter of the regular polygon, and $a$ represents the length of the apothem.

For example, determine the area of the regular hexagon.

The perimeter of the hexagon is $5 \cdot 6 = 30$ cm.

The length of the apothem is 1.7 cm.




$A = \frac{1}{2}aP$

$A = \frac{1}{2} \cdot 1.7 \cdot 30$

$A = 25.5$

The area of the regular hexagon is 25.5 $cm^2$.

# TOPIC 3 Sequences of Rigid Motions

In this topic, students build on their understand of rigid motions from middle school they consider each of the rigid motions - translations, reflections, and rotations - on a coordinate plane and off the coordinate plane. Then, students perform sequences of rigid motion transformations. They also use sequences of transformations to prove that two figures are congruent.

## Where have we been?

In middle school, students explored translations, reflections, and rotations. They described the effect of rigid motion on two-dimensional figures using coordinates. This topic builds on this knowledge, requiring students to rely on geometric reasoning to perform sequences of rigid motion transformations.

## Where are we going?

In this topic, students begin the formal study of congruence and lay the groundwork for their study of similarity and trigonometry. This topic is part of a long progression in understanding geometric and algebraic transformations. In the next topic, students explain how the conditions for triangle congruence (SSS, SAS, and ASA) follow from the definition of congruence in terms of rigid motion. They will use the triangle congruence theorems in future topics to prove a wide range of geometric theorems.

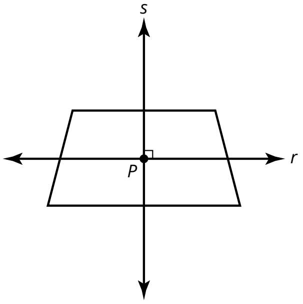
## *TALKING POINTS*

### DISCUSS WITH YOUR STUDENT

Sequences of rigid motion transformations, including translations, rotations, and reflections, maintain the size and shape of objects while analyzing movements. This is useful in applications such as designing machinery, creating animations, and mapping out paths in navigation systems.

### HERE IS A SAMPLE QUESTION

The figure shows two perpendicular lines, *s* and *r*, intersecting at point *P* in the interior of a trapezoid. Line *r* is parallel to the bases of the trapezoid and bisects both legs of the trapezoid. Line *s* bisects both bases of the trapezoid.




Which transformation will always carry the figure onto itself?

A reflection across line *s* will carry the figure onto itself.

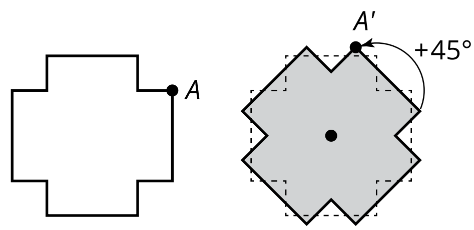
## KEY TERMS

- coordinate notation [notación de coordenadas]
- congruent angles [ángulos congruentes]
- congruent triangles [triángulos congruentes]
- translation [traslación]
- angle bisector [bisectriz de un ángulo]
- reflection [reflexión]
- perpendicular bisector theorem [teorema de la bisectriz perpendicular]
- rotation [rotación]
- angle of rotation [ángulo de rotación]
- central angle [ángulo central]
- concentric circle [círculo concéntrico]

Refer to the Math Glossary for definitions of Key Terms.

# Where are we now?

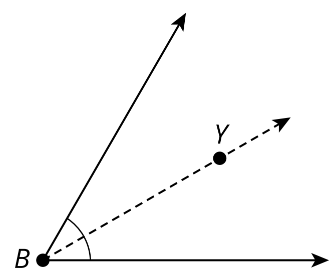
An **angle of rotation** is a directed angle based on a circle. Positive rotation angles turn counterclockwise, and negative rotation angles turn clockwise. The angle of rotation shown rotates point *A* 45° counterclockwise.




**Coordinate notation** describes how the original coordinates of a point are changed into new coordinates after one or more transformations. The coordinate notation of a mapping is written in the form $(x, y) \rightarrow$ (new expression in terms of $x$, new expression in terms of $y$).

For example, $(x, y) \rightarrow (x + 2, y - 1)$ describes that the point $(x, y)$ was translated right 2 units and down 1 unit.

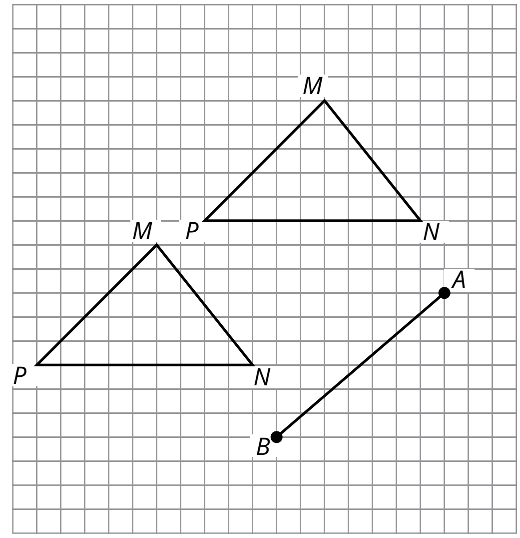
An **angle bisector** is a ray drawn through the vertex of an angle that divides the angle into two angles of equal measure, or two congruent angles.




In **Lesson 1: Translations On and Off the Plane**, students review what they learned about translations from middle school and use construction tools to complete translations off the coordinate plane. Then, in **Lesson 2: Sequences of Multiple Translations,** students use what they know about translations to describe a sequence of translations that map a pre-image onto its image.

## Translations

A translation can be measured as a directed line segment.




$\triangle MNP$ was translated to produce $\triangle M'N'P'$. The triangle was translated a distance equal to the distance between points *A* and *B*. It was translated in the direction from point *A* to point *B*.

So, $\overline{AB}$ is the directed line segment used to measure this translation.

In **Lesson 3: Reflections On and Off the Coordinate Plane**, students continue to explore rigid motion transformations. This time they explore reflections.

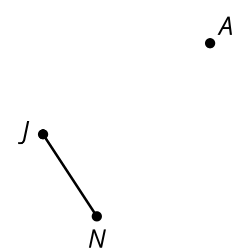
## Reflections

When an image on the coordinate plane is reflected across the *x*-axis, the value of the *y*-coordinate of the image is opposite the *y*-coordinate of the pre-image. The coordinates of an image after a reflection on the coordinate plane are summarized in the table.

| Original Point | Coordinates of Image After a Reflection Over the *x*-axis | Coordinates of Image After a Reflection Over the *y*-axis |
|---|---|---|
| $(x, y)$ | $(x, -y)$ | $(-x, y)$ |

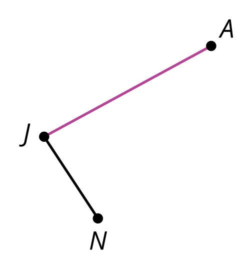
Next, students explore rotations in **Lesson 4: Rotations On and Off the Coordinate Plane**.

## Rotations Off the Coordinate Plane



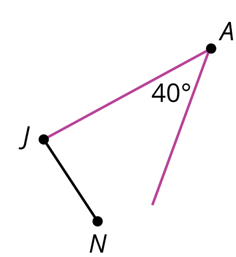

Rotate $\overline{JN}$ 40° using point *A* as the center of rotation.

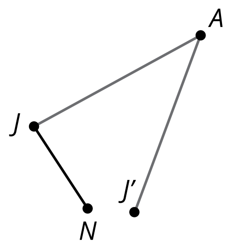
1. Draw a line segment from the center of rotation, *A*, to one endpoint of the line segment.



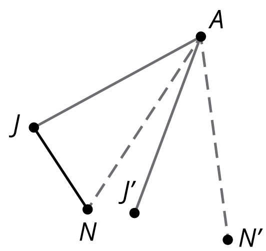

2. Using a protractor, draw a 40° angle. Use the center of rotation, *A*, as the vertex and the line segment drawn, $\overline{AJ}$, as one side of the angle.

   Since the angle measure is positive, place the angle in the counterclockwise direction of the line segment drawn.

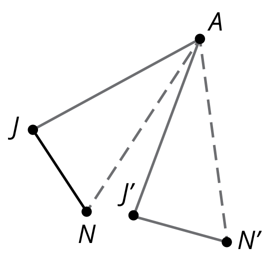





3. Use a ruler to extend the side of the angle so that is the same length as $\overline{AJ}$. Label the other endpoint $J'$.




4. Repeat steps 1, 2, and 3 using the other endpoint of the original line segment.




5. Connect endpoints $J'$ and $N'$.

   Segment $\overline{J'N'}$ is the result of a 40° rotation of $\overline{JN}$ about point $A$.

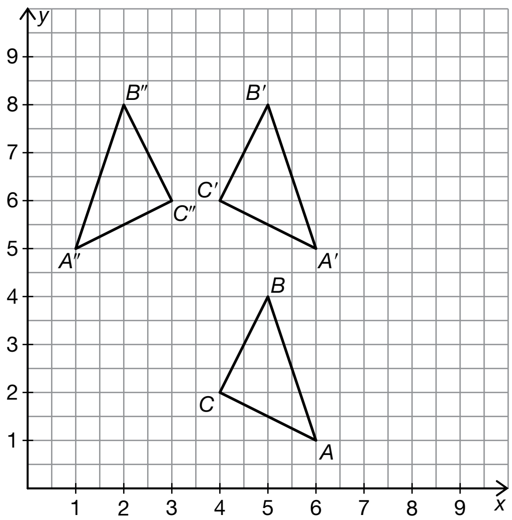
In **Lesson 5: Compositions of Functions**, students combine what they have learned throughout the topic to perform compositions of rigid motion transformations on the coordinate plane. They identify possible sequences of rigid motion transformations that map a rectangle on the coordinate plane to its image.

## Multiple Transformations

You can perform a composition of rigid motion transformations on the coordinate plane.

For example, $\triangle A''B''C''$ is the result of a composition of rigid motions on $\triangle ABC$.

- a translation up 4 units
- a reflection over the line $x = 3\frac{1}{2}$




The coordinates of $\triangle ABC$ are $A(6, 1)$, $B(5, 4)$, $C(4, 2)$.

- Translate $\triangle ABC$ up 4 units.
  The coordinates of $\triangle A'B'C'$ are $A'(6, 5)$, $B'(5, 8)$, $C'(4, 6)$.
- Reflect $\triangle A'B'C'$ over the line $x = 3\frac{1}{2}$.
  The coordinates of $A''B''C''$ are $A''(1, 5)$, $B''(2, 8)$, $C''(3, 6)$.

## Math in the Real World

Imagine trying to design a roller coaster or choreograph a dance routine—both are all about moving objects while keeping their shapes and sizes intact. They represent sequences of rigid motion transformations! These transformations include translations (sliding objects across a surface), rotations (spinning them around a point), and reflections (flipping them like looking in a mirror).

Rigid motion transformations help architects and engineers ensure structures are built correctly and consistently. They are responsible for creating lifelike animations in movies and programming robots that can move precisely to complete tasks like assembling cars. Video games use these concepts to create smooth character movements and realistic environments. So next time you see a flawless gymnastics flip, a rotating Ferris wheel, or a seamless animated scene, remember: sequences of rigid motions make it all happen!

## TOPIC 4 Congruence Through Transformations

In this topic, students use what they know about rigid motions to prove triangle congruence theorems by construction. They prove the Side-Side-Side, Side-Angle-Side, Angle-Side-Angle, Angle-Angle Side, and Hypotenuse Angle congruence theorems. To prove these congruence theorems, students use a sequence of transformations that maps one triangle onto another, given the congruence of three corresponding parts. Integrating their knowledge of geometry and algebra, students encounter triangles on the coordinate plane that require them to use the distance formula to apply the congruence theorems to triangles with given measurements on the plane.

### Where have we been?

In Grade 6, students constructed triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle. Through that hands-on exploration, they developed an intuition regarding the minimum criteria for determining whether triangles are congruent.

### Where are we going?

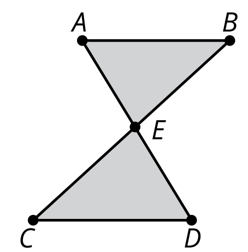
In this topic, students use proof by construction. In Module 2, they will learn to write two-column, flowchart, and paragraph proofs. They will continue to develop their deductive reasoning skills as they prove conjectures that they have made through investigation. The triangle congruence theorems proven in this topic will be used in many of these upcoming proofs.

### *TALKING POINTS*

**DISCUSS WITH YOUR STUDENT**

Congruent triangles are important for ensuring precision in construction and engineering and for applications like facial recognition and animation.

**HERE IS A SAMPLE QUESTION**




In the figure, $\overline{AB}$ and $\overline{CD}$ are 8 cm apart, congruent, and parallel. If $AB = 4$ cm, what is the shaded area?

$\angle AEB \cong \angle CED$ because they are vertical angles. Alternate interior angles are also congruent, so the two triangles are congruent by Angle-Side-Angle.

Since the triangles are congruent, they have congruent altitudes, so each has an altitude of 4 cm. Thus, each triangle's area is $\frac{1}{2}(4)(4)$ or 8 square centimeters. The total area shaded, then, is 16 square centimeters.

**KEY TERMS**

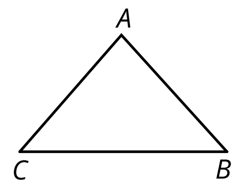
- reflectional symmetry [simetría de reflexión]
- rotational symmetry [simetría de rotación]
- Side-Side-Side congruence theorem (SSS)
- corresponding parts of congruent triangles are congruent (CPTTC) [las partes correspondientes de triángulos congruentes son congruentes]
- Side-Angle-Side congruence theorem (SAS)
- included angle [ángulo incluido]
- Angle-Side-Angle congruence theorem (ASA)
- included side
- Hypotenuse Angle congruence theorem (HA) [Teorema de congruencia del ángulo hipotenusa]
- Angle-Angle-Side congruence theorem (AAS)
- reflexive property of congruence [propiedad reflexiva de la congruencia]

Refer to the Math Glossary for definitions of Key Terms.

## Where are we now?

An **included side** is a line segment between two consecutive angles of a figure.

In $\Delta ABC$, $\overline{AB}$ is the included side formed by consecutive angles *A* and *B*.

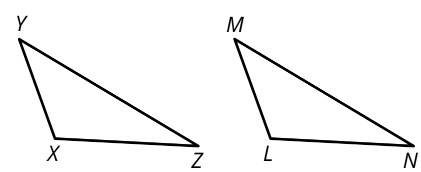



An **included angle** is an angle between two consecutive line segments of a figure. In $\triangle ABC$, $\angle C$ is the included angle formed by consecutive sides $\overline{AC}$ and $\overline{CB}$.

**Corresponding parts of congruent triangles are congruent (CPCTC)** indicates that if two triangles are congruent, then each part of one triangle is congruent to the corresponding part of the other triangle.

In the triangles shown, $\triangle XYZ \cong \triangle LMN$. Because corresponding parts of congruent triangles are congruent (CPCTC), the following corresponding parts are congruent.

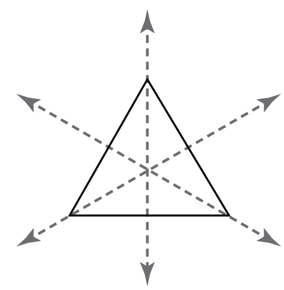
- $\angle X \cong \angle L$
- $\angle Y \cong \angle M$
- $\angle Z \cong \angle N$
- $\overline{XY} \cong \overline{LM}$
- $\overline{YZ} \cong \overline{MN}$
- $\overline{XZ} \cong \overline{LN}$




In **Lesson 1: Reflectional and Rotational Symmetry**, students identify reflectional and rotational symmetry in geometric figures, letters of the alphabet, and activity titles. They also identify the relationship between the rotational symmetries of a regular figure and the measure of its interior angle.

## Reflectional and Rotational Symmetry

A plane figure has **reflectional symmetry** if you can draw a line so that the figure to one side of the line is a reflection of the figure on the other side of the line. The line you can draw on each shape is called the line of symmetry. A figure can have more than one line of symmetry. For example, the triangle shown has three lines of reflectional symmetry.

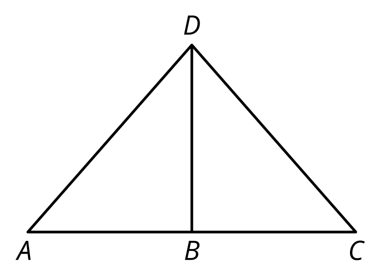


A plane figure can also have **rotational symmetry** if you can rotate the figure more than 0° but less than 360°, and the resulting figure is the same as the original figure in the original position. For example, a rectangle has rotational symmetry. A rotation of 180° carries a rectangle onto itself.

In **Lesson 2: Proving Triangle Congruence Theorems**, students verify proofs of the Side-Side-Side, Side-Angle-Side, and Angle-Side-Angle congruence theorems using transformations. They also prove and apply the Angle-Angle-Side congruence theorem.

The **Side-Side-Side congruence theorem (SSS)** states: “If three sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent.”

Given: $\overline{AD} \cong \overline{DC}$ and $\overline{DB}$ bisects $\overline{AC}$.




You can use the Side-Side-Side congruence theorem to demonstrate that $\Delta ADB$ is congruent to $\Delta CDB$.

- Using the definition of a bisector, $B$ is the midpoint of $\overline{AC}$ and therefore $\overline{AB} \cong \overline{BC}$.
- Since $\overline{DB}$ is the same side in each triangle, then $\overline{DB} \cong \overline{DB}$.
- Therefore, $\triangle ADB \cong \triangle CDB$ by SSS.

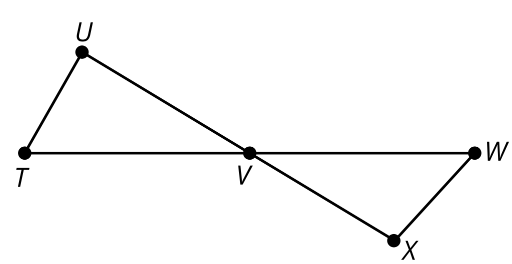
## Side-Angle-Side Congruence Theorem (SAS)

The **Side-Angle-Side congruence theorem** states: If two sides and the included angle of one triangle are congruent to the corresponding two sides and the included angle of a second triangle, then the triangles are congruent.

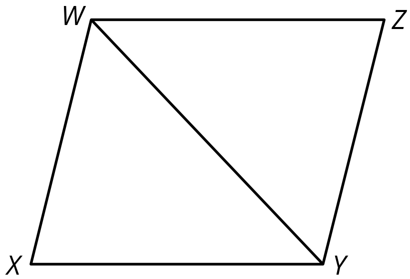
Given: $V$ is the midpoint of $\overline{UX}$ and $V$ is the midpoint of $\overline{TW}$.

You can use the Side-Angle-Side congruence theorem to demonstrate that $\triangle UVT$ is congruent to $\triangle XVW$.

- By definition of a midpoint, $\overline{UV} \cong \overline{XV}$ and $\overline{TV} \cong \overline{WV}$.
- Using the definition of vertical angles, $\angle UVT \cong \angle XVW$.
- Therefore, $\triangle UVT \cong \triangle XVW$ by SAS.



## Angle-Side-Angle Congruence Theorem (ASA)

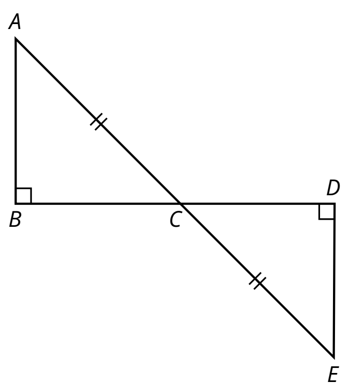



The **Angle-Side-Angle congruence theorem** states: "If two angles and the included side of one triangle are congruent to the corresponding two angles and the included side of another triangle, then the triangles are congruent."

Given: $\overline{WZ} \parallel \overline{XY}$ and $\overline{WX} \parallel \overline{ZY}$.

You can use the Angle-Side-Angle congruence theorem to demonstrate that $\triangle WXY$ is congruent to $\triangle YZW$.

- Using the definition of alternate interior angles, $\angle ZWY \cong \angle XYW$ and $\angle XWY \cong \angle ZYW$.
- Since $\overline{WY}$ is the same side in each triangle, then $\overline{WY} \cong \overline{WY}$.
- Therefore, $\triangle WXY \cong \triangle YZW$ by ASA.

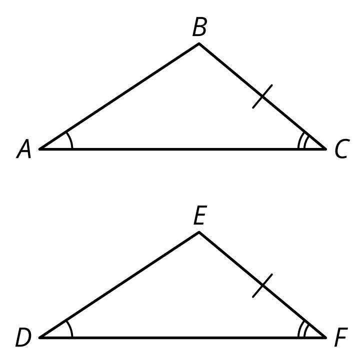



The **Hypotenuse-Angle (HA) congruence theorem** states: "If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and an acute angle of another right triangle, then the two triangles are congruent."

In right triangle *ABC*, $\overline{AC}$ is the hypotenuse and in right triangle *EDC*, $\overline{EC}$ is the hypotenuse. Since $\overline{AC} \cong \overline{EC}$ and $\angle ACB \cong \angle ECD$ by the Vertical Angle theorem, then $\triangle ABC \cong \triangle EDC$.

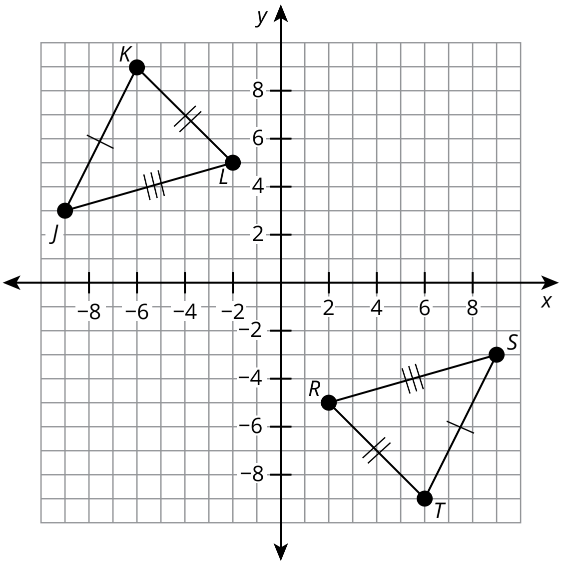
The **Angle-Angle-Side (AAS) congruence theorem** states: "If two angles and the non-included side of one triangle are congruent to two angles and the non-included side of another triangle, then the two triangles are congruent."

For example, $\overline{BC} \cong \overline{EF}$, $\angle A \cong \angle D$, and $\angle C \cong \angle F$. Therefore, $\triangle ABC \cong \triangle DEF$ by *AAS*.



In **Lesson 3: Applying Congruence Theorems**, students combine their knowledge of geometry to solve problems using the triangle congruence theorems. They use the distance formula to apply the congruence theorems to triangles with given measurements on the coordinate plane.

## Using the Distance Formula to Show Two Triangles are Congruent by SSS




Use the distance formula to determine the length of each side.

$JK = \sqrt{(-9 - (-6))^2 + (3 - 9)^2}$
$= \sqrt{9 + 36}$
$= \sqrt{45}$

$KL = \sqrt{(-6 - (-2))^2 + (9 - 5)^2}$
$= \sqrt{16 + 16}$
$= \sqrt{32}$

$JL = \sqrt{(-9 - (-2))^2 + (3 - 5)^2}$
$= \sqrt{49 + 4}$
$= \sqrt{53}$

$ST = \sqrt{(9 - 6)^2 + (-3 - (-9))^2}$
$= \sqrt{9 + 36}$
$= \sqrt{45}$

$RT = \sqrt{(6 - 2)^2 + (-5 - (-9))^2}$
$= \sqrt{16 + 16}$
$= \sqrt{32}$

$RS = \sqrt{(2 - 9)^2 + (-5 - (-3))^2}$
$= \sqrt{49 + 4}$
$= \sqrt{53}$

The lengths of the corresponding sides are equal. Therefore, the triangles are congruent by the SSS congruence theorem.

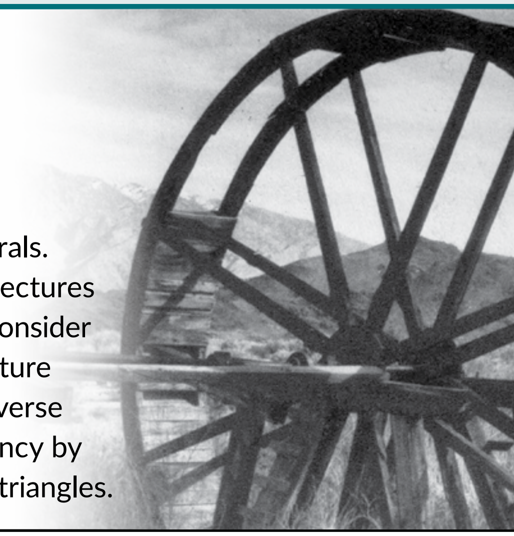
### *Math in the Real World*

How does chat software know where to apply the funny face filter, even as you move your face around on the screen? The software uses advanced technology called facial recognition. This technology identifies key points on your face, such as your eyes, nose, and mouth, and connects these points with imaginary lines to create a mesh of triangles. It uses triangles to understand the shape and position of your face. For example, if you raise an eyebrow or tilt your head, the software detects how the triangles change and adjusts the filter to stay aligned with your face.

To do this efficiently, the software relies on a congruent triangles. Congruent triangles are triangles that have the same shape and size, even if they're rotated, reflected, or translated. By identifying congruent triangles, the software can quickly and accurately track how your face moves without needing to process every single detail of your appearance. This makes the filter work in real time, even if you move your face quickly or make funny expressions.

# Family Guide

MODULE 2 Justifying Mathematical Ideas and Arguments Geometry



## TOPIC 1 Composing and Decomposing Shapes

In this topic, students use circles to conjecture about line and angle relationships. Students also use circles to conjecture about quadrilaterals. They then explore the properties of isosceles triangles and make conjectures about triangles. As students begin to formalize their conjectures and consider the validity of statements and their converses, they consider a conjecture about the base angles of isosceles triangles and test whether the converse is true. To conclude the topic, students investigate points of concurrency by constructing the circumcenter, incenter, centroid, and orthocenter of triangles.

### Where have we been?

Throughout elementary and middle school, students have informally investigated many of the relationships explored in this topic. For example, in Grade 8, they used informal arguments to establish facts about angle pairs created when parallel lines are cut by a transversal. Students have an intuitive understanding about the properties of many of the shapes considered in this topic.

### Where are we going?

In future topics, students will formally prove many of the conjectures they write in this topic. Through exploration, construction, and conjecture, students can develop an intuitive understanding of whether they believe a relationship to be actually true. Once they believe that it's true, they will be better equipped to prove that it is true in all cases.

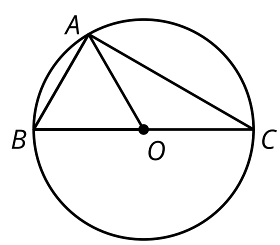
### *TALKING POINTS*

**DISCUSS WITH YOUR STUDENT**

Conjectures are an important topic to know for fostering critical thinking and innovation, as they encourage proposing and testing ideas that can lead to real-world advancements and solutions.

### HERE IS A SAMPLE QUESTION

$\overline{BC}$ is a diameter of circle $O$ and $\triangle ABC$ is inscribed in the circle. If $AB = AO$, what is the degree measure of $\angle ABO$?




To solve this problem, you should notice that $\overline{AO}$ is a radius of the circle. All radii of a circle are congruent, so $AB = AO = OB$, and $\triangle ABO$ is an equilateral triangle.

An equilateral triangle has three congruent angle measures which must sum to 180°, so the measure of $\angle ABO$ must be 60°.

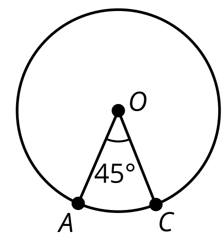
## KEY TERMS

- central angle [ángulo central]
- major arc [arco mayor]
- minor arc [arco menor]
- secant [secante]
- chord [cuerda]
- inscribed angle [ángulo inscrito]
- intercepted arc [arco interceptado]
- tangent [tangente]
- circumscribed angle [ángulo circunscrito]
- radius [radio]
- diameter [diámetro]
- coincident [coincidente]
- interior angle of a polygon [ángulo interior de un polígono]
- kite
- isosceles trapezoid [trapecio isósceles]
- midsegment [segmento medio]
- quadrilateral inscribed in a circle [cuadrilátero inscrito en un círculo]
- convex polygon [polígono convexo]
- concave polygon [polígono cóncavo]

# Where are we now?

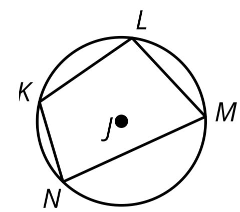
A **central angle** of a circle is an angle whose sides are radii. The measure of a central angle is equal to the measure of its intercepted arc.

In circle *O*, $\angle AOC$ is a central angle and $\overset{\frown}{AC}$ is its intercepted arc. If $m\angle AOC = 45°$, then $m\overset{\frown}{AC} = 45°$.



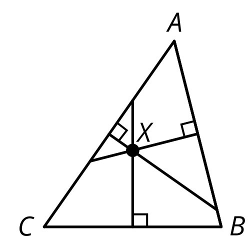

An **inscribed polygon** is a polygon drawn inside another polygon or circle in which all the vertices of the interior polygon lie on the outer figure.

Quadrilateral *KLMN* is inscribed in circle *J*.




The **circumcenter** of a triangle is the point at which the perpendicular bisectors intersect.

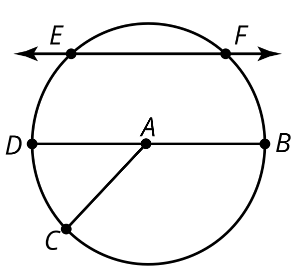
Point *X* is the circumcenter of $\triangle ABC$.

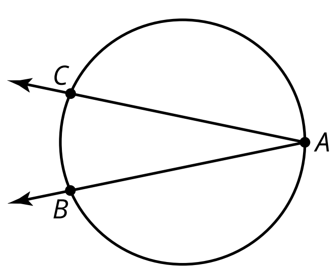


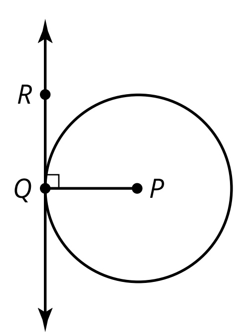

In **Lesson 1: Using Circles to Make Conjectures**, students explore and identify lines and angles associated with the interior and exterior of a circle.

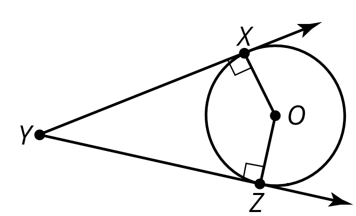
## Angles and Arcs

Students may already know some of the more basic names of geometric parts like **radius** and **diameter** as they relate to circles, but now they will use them to define the arcs and angles they form such as *major* and *minor arcs*, *intercepted arcs*, *inscribed angles*, *central angles*, and *circumscribed angles*. Each of these angles has their own specific properties and will have theorems and postulates that relate to them.




See what parts of these diagrams your student may already know and be able to identify!

In **Lesson 2: Conjectures About Quadrilaterals**, students investigate the properties of quadrilaterals and use them to make conjectures. They construct several quadrilaterals from the diameters of concentric circles.

## Properties of Quadrilaterals

Various quadrilaterals are defined by the properties that they hold and by triangles that are formed when dividing a quadrilateral along any of its diagonals.

| Property | Parallelogram | Rectangle | Rhombus | Square | Isosceles Trapezoid | Kite |
|---|---|---|---|---|---|---|
| Opposite sides are parallel. | X | X | X | X | | |
| Only one pair of opposite sides is parallel. | | | | | X | |

In **Lesson 3: Constructing an Inscribed Regular Polygon**, students learn how to construct three regular polygons: regular hexagons, squares, and equilateral triangles.

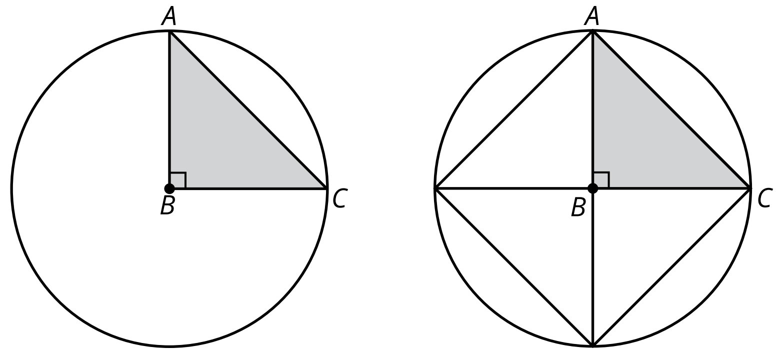
- inscribed polygon [polígono inscrito]
- base angles [ángulos de la base]
- exterior angle [ángulo exterior]
- remote interior angles [ángulos interiores remotos]
- Triangle Inequality theorem [Teorema de la desigualdad triangular]

Refer to the Math Glossary for definitions of the Key Terms.

## Constructing a Square

To construct a square, you can use a transformation of an inscribed polygon.

For example, circle *B* contains a right triangle, $\triangle ABC$, with points *A* and *C* on the circle and point *B* at the center. To create a square, rotate $\triangle ABC$ to create an inscribed square with all 4 vertices touching the circle.



In **Lesson 4: Conjectures About Triangles**, students decompose quadrilaterals to investigate triangles. They construct an equilateral triangle using circles and conjecture about the sum of the interior and exterior angles of a triangle.

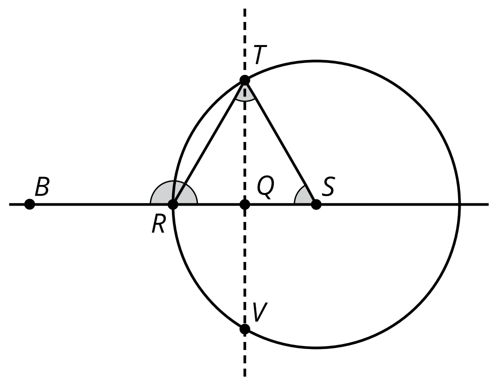
## Exploring Triangles

The sum of the measures of the interior angles of a triangle is 180°.

An **exterior angle** of a polygon is an angle that forms a linear pair with an interior angle of the polygon.

For example, $\angle TRB$ is an exterior angle of $\triangle RST$.

The **remote interior angles** of a triangle are the two angles that are not adjacent to the specified exterior angle.




For example, $\angle RST$ and $\angle STR$ are remote interior angles of $\triangle RST$.

The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

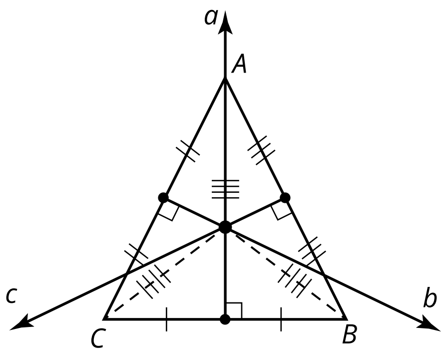
For example, $m\angle RST + m\angle STR = m\angle TRB$

In **Lesson 5: Points of Concurrency**, student explore *concurrent* lines, rays, and line segments.

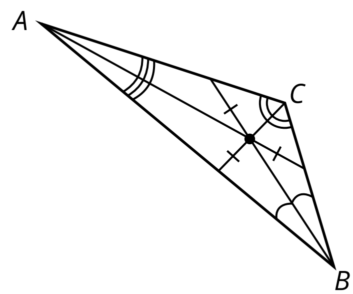
## Concurrency

**Concurrent** lines, rays, and line segments are three or more lines, rays, or line segments that intersect at a single point. The **point of concurrency** is the point at which concurrent lines, rays, or segments intersect.

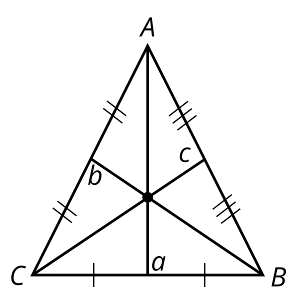
The **circumcenter** is the point of concurrency of the three perpendicular bisectors of a triangle. It is equidistant from the vertices.



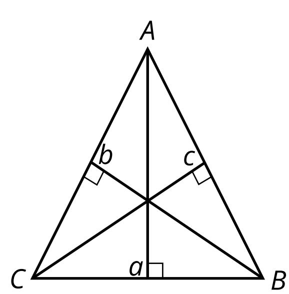

The **incenter** is the point of concurrency of the three angle bisectors of a triangle. It is equidistant from the sides.




A **median** of a triangle is a line segment that connects a vertex to the midpoint of the opposite side. The **centroid** is the point of concurrency of the three medians of a triangle.




An **altitude** of a triangle is a line segment that is perpendicular to a side of the triangle and has one endpoint at the opposite vertex. The **orthocenter** is the point of concurrency of the three altitudes of a triangle.




## Math in the Real World

Circles are everywhere in the real world, from the wheels of bicycles to the moon in the night sky, but perhaps one of the most iconic circular creations is the Ferris wheel.

The invention of the Ferris wheel is credited to George Washington Gale Ferris, Jr., who debuted his new ride at the World's Columbian Exposition in Chicago, Illinois, in 1893. It was 264 feet tall, had a capacity of 2160 people, took approximately 10 minutes to complete a revolution, and cost 50 cents to ride.

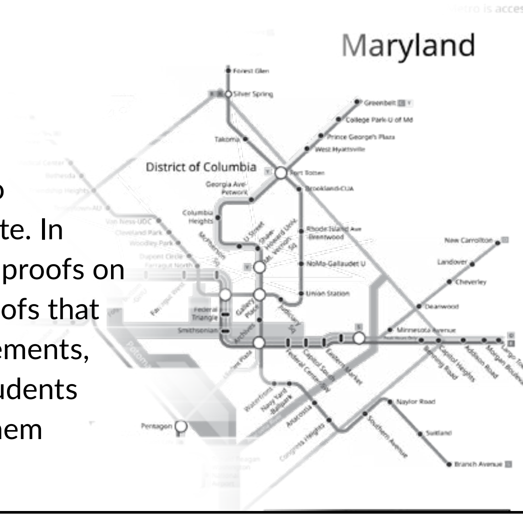
Engineers and architects rely on precise circle constructions when designing everything from bridges to buildings, ensuring structural integrity and balance. Urban planners use circular layouts in roundabouts to improve traffic flow, while astronomers map celestial objects using circular orbits. Even in everyday tasks like crafting gears or creating artistic designs, the geometry of circles proves essential for accuracy and efficiency.

# Family Guide

MODULE 2 Justifying Mathematical Ideas and Arguments

Geometry

## TOPIC 2 Justifying Line and Angle Relationships




This topic moves from the conjectures made in the previous topic to formal proof. The development of proof-writing is slow and deliberate. In preparation for writing proofs independently, students engage with proofs on several levels: reading and analyzing completed proofs, finishing proofs that have been partially completed, supplying the reasons for given statements, and mirroring a two-column proof from a given flow chart proof. Students have the opportunity to experience proofs before having to write them entirely on their own.

### Where have we been?

In elementary and middle school, students investigated lines, angles, triangles, and quadrilaterals. In Grade 8, they used informal arguments to establish facts about the angle sum and exterior angles of triangles, as well as the angles created when parallel lines are cut by a transversal. In the previous topic, students explored and conjectured about each of the concepts that they prove in this topic.

### Where are we going?

As the theorems proven in this topic are used to prove other theorems in future topics, students are building a system of geometric relationships and seeing how these geometric ideas are connected. Seeing a system of relationships allows students to reason and generalize beyond the specificity of any given figure and prepares them to solve more complicated problems.

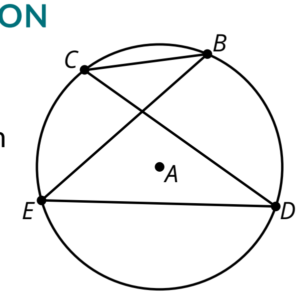
### *TALKING POINTS*

#### DISCUSS WITH YOUR STUDENT

Line and angle relationships are an important topic to know for designing and constructing structures, ensuring precision in engineering, and optimizing layouts in fields like architecture, transportation, and technology.

### HERE IS A SAMPLE QUESTION

In the figure shown, $\angle BED$ and $\angle BCD$ are inscribed in circle $A$. What is the relationship between $\angle BED$ and $\angle BCD$?




To solve this problem, you need to know the inscribed angle theorem. Both of the inscribed angles are half the measure of the central angle, $\angle BAD$, because they intercept the same arc, $\overset{\frown}{BD}$. Since they are both half the measure of the same angle, the two inscribed angles must be congruent.

## KEY TERMS

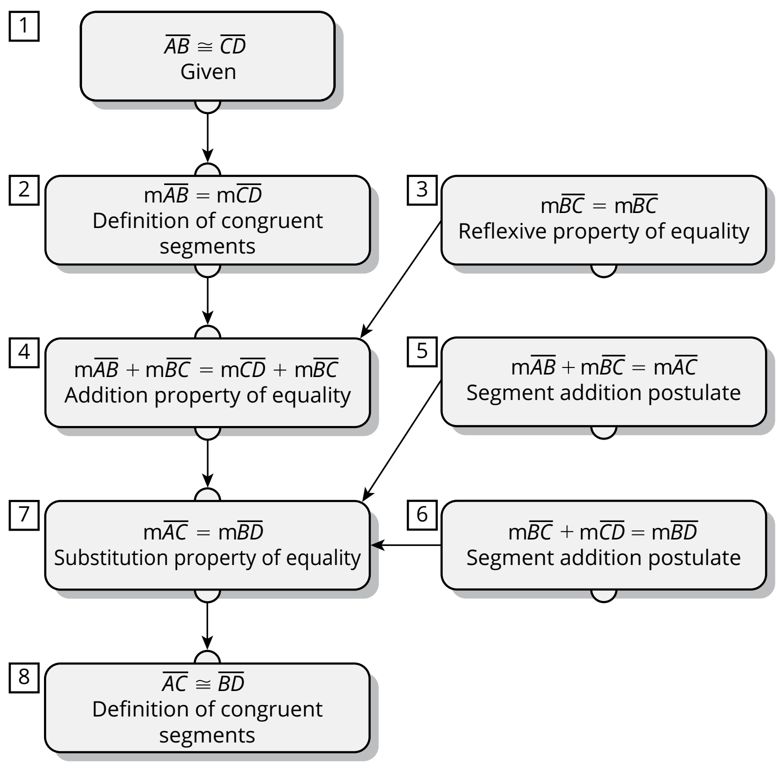
- addition property of equality
- subtraction property of equality
- substitution property of equality [propiedad de sustitución de la igualdad]
- transitive property of equality [propiedad transitiva de la igualdad]
- flow chart proof
- two-column proof [prueba de dos columnas]
- right angle congruence postulate
- congruent supplement theorem [teorema del suplemento congruente]
- vertical angle theorem
- paragraph proof [prueba en párrafo]
- corresponding angles theorem [teorema de los ángulos correspondientes]
- corresponding angles converse theorem
- same-side interior angles theorem
- alternate interior angles theorem [teorema de los ángulos interiores alternos]
- alternate exterior angles theorem [teorema de los ángulos exteriores alternos]
- same-side exterior angles theorem

# Where are we now?

A **flow chart** proof is a proof in which the steps and corresponding reasons are written in boxes.

Arrows connect the boxes and indicate how each step and reason is generated from one or more steps and reasons.

**Flow Chart Proof**



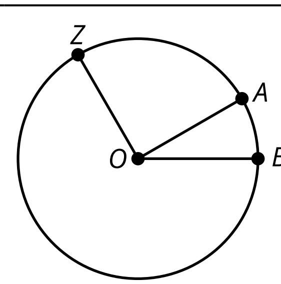

The process of eliminating a radical from the denominator of a fraction is called *rationalizing the denominator.*

To **rationalize the denominator**, multiply by a form of one so that the radicand of the radical in the denominator is a perfect square.

$$\begin{aligned}\frac{5}{\sqrt{3}} &= \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{5\sqrt{3}}{\sqrt{9}} \\ &= \frac{5\sqrt{3}}{3}\end{aligned}$$

**Adjacent arcs** are two arcs of the same circle sharing a common endpoint.

Arcs *ZA* and *AB* are adjacent arcs.



In **Lesson 1: Forms of Proof**, students apply real number properties to angles measures and distances. They are introduced to proof by construction, flow chart proofs, two-column proofs, and paragraph proofs.

## The Properties of Equality

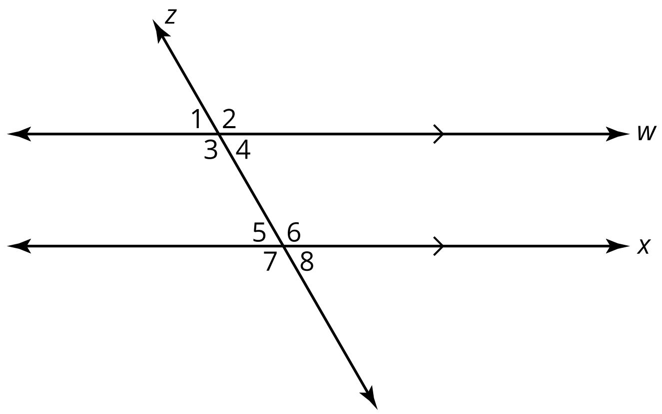
Many properties of real numbers can be applied in geometry. These properties are important when making conjectures and proving new theorems.

| Property | Statements |
|---|---|
| addition property of equality | If $a$, $b$, and $c$ are real numbers and $a = b$, then $a + c = b + c$. |
| subtraction property of equality | If $a$, $b$, and $c$ are real numbers and $a = b$, then $a - c = b - c$. |
| reflexive property of equality | If $a$ is a real number, then $a = a$. |
| substitution property of equality | If $a$ and $b$ are real numbers and $a = b$, then $a$ can be substituted for $b$. |
| transitive property of equality | If $a$, $b$, and $c$ are real numbers, $a = b$, and $b = c$, then $a = c$. |

In **Lesson 2: Proving Parallel Line Theorems**, students prove angle relationships when two parallel lines are cut by a transversal.

## Corresponding Angles Theorem

The **corresponding angles theorem** states, "If two parallel lines are intersected by a transversal, then corresponding angles are congruent." For example, $\angle 1 \cong \angle 5$, $\angle 2 \cong \angle 6$, $\angle 3 \cong \angle 7$, and $\angle 4 \cong \angle 8$.




- alternate interior angles converse theorem
- same-side interior angles converse theorem
- alternate exterior angles converse theorem
- same-side exterior angles converse theorem
- perpendicular/parallel line theorem [teorema de líneas perpendiculares y paralelas]
- triangle sum theorem [teorema de la suma del triángulo]
- exterior angle theorem [teorema del ángulo exterior]
- perpendicular bisector converse theorem
- isosceles triangle base angles theorem [teorema de los ángulos de la base del triángulo isósceles]
- isosceles triangle base angles converse theorem
- 30°–60°–90° triangle theorem [teorema del triángulo 30°-60°-90°]
- 45°–45°–90° triangle theorem [teorema del triángulo 45°–45°–90°]
- rationalize the denominator [racionalizar el denominador]
- inverse [inverso]

- contrapositive [contrapositivo]
- direct proof [prueba directa]
- indirect proof or proof by contradiction [prueba indirecta o prueba indirecta contradicción]
- hinge theorem
- hinge converse theorem
- degree measure of an arc
- arc addition postulate
- adjacent arcs [arcos adyacentes]
- inscribed angle theorem [teorema del ángulo inscrito]
- inscribed right triangle-diameter theorem
- inscribed quadrilateral-opposite angles theorem [teorema del cuadrilátero inscrito-ángulos opuestos]
- interior angles of a circle theorem [teorema de los ángulos interiores de un círculo]
- exterior angles of a circle theorem [teorema de los ángulos exteriores de un círculo]
- tangent to a circle theorem [teorema de la tangente a un círculo]

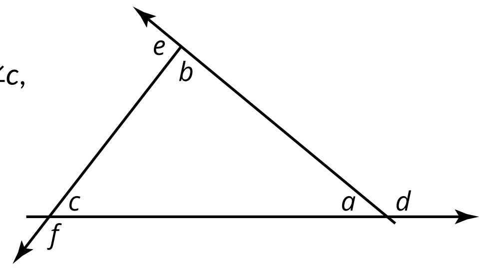
Refer to the Math Glossary for definitions of the Key Terms.

In **Lesson 3: Interior and Exterior Angles of Polygons**, students continue to investigate polygons and their angles. They explore the triangle sum theorem and exterior angle theorem.

## Exterior Angle Theorem

The **exterior angle theorem states**: "The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles."

For example, $m\angle d = m\angle b + m\angle c$, $m\angle e = m\angle a + m\angle c$, and $m\angle f = m\angle a + m\angle b$.



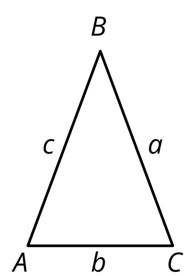

In **Lesson 4: Perpendicular Bisector and Isosceles Triangle Theorems**, students continue to prove the theorems they explored in the previous topic.

## The Isosceles Triangle Base Angles Theorem

The **isosceles triangle base angles theorem** states: "If two sides of a triangle are congruent, then the angles opposite the sides are congruent."

For example, in $\Delta ABC$, if side $a$ is congruent to side $c$,

Then $\angle A \cong \angle C$.




In **Lesson 5: Inverse, Contrapositive, Direct Proof, and Indirect Proof**, students analyze conditional statements and write the inverse and contrapositive.

## Inverse and Contrapositive of a Conditional Statement

To state the **inverse** of a conditional statement, negate both the hypothesis and the conclusion. To state the **contrapositive** of a conditional statement, negate both the hypothesis and the conclusion and then reverse them.

For example, consider the conditional statement, “If a triangle is equilateral, then it is isosceles.”

Inverse: If a triangle is not equilateral, then it is not isosceles.

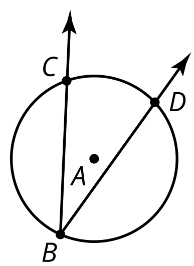
Contrapositive: If a triangle is not isosceles, then it is not equilateral.

In **Lesson 6: Angle Relationships Inside and Outside Circles**, students explore and prove theorems for determining the measures of angles inside and outside of a circle.

## Inscribed Angle Theorem

The **inscribed angle theorem** states: “The measure of an inscribed angle is half the measure of the intercepted arc.”

For example, $m\angle CBD = \frac{1}{2}m\overset{\frown}{CD}$.




### *Math in the Real World*

You might not realize it, but lines and angles are everywhere in your life, even if you’re not actively thinking about them. From the moment you wake up, you’re using them! The corners of your phone screen? Angles. The way the sun shines through your window? That’s light following the rules of angles and reflection. Even when you take a selfie, you’re positioning your phone at just the right angle to get the best lighting and show your best side—geometry in action!

Architects use line and angle relationships to design buildings that won’t fall down. Engineers use them to create roads, bridges, and roller coasters with perfect curves and supports. A basketball court or soccer field relies on parallel lines (the sidelines) and transversals (passes and plays across the field) to create strategies and guide movement. Geometry is basically the VIP pass to understanding how the world is built and moves!

# Family Guide

MODULE 2 Justifying Mathematical Ideas and Arguments

Geometry

## TOPIC 3 Using Congruence Theorems

In this topic, students use the theorems that they have proved to prove new theorems about triangles, quadrilaterals, and angles formed in circles. Students use triangle congruence theorems to verify properties of parallelograms, and they use the congruence theorems they have proved to prove theorems related to the chords of circles.

### Where have we been?

Students build from the fundamentals of proof they have learned in the previous topics. They have explained how the conditions for the SSS, SAS, and ASA theorems follow from the definition of congruence in terms of rigid motion. They have also proved the AAS congruence theorem and the HA congruence theorem for right triangles. They now use these theorems to prove additional congruence theorems for right triangles.

### Where are we going?

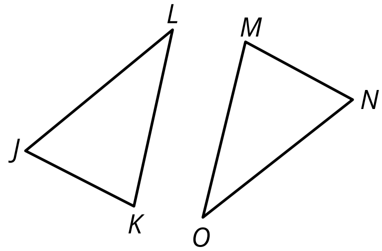
Students will use logical reasoning not just in geometry but as they progress through advanced mathematics. Mathematics is about understanding and providing valid reasons why numeric, algebraic, and geometric relationships exist and whether or not they exist in all cases.

### *TALKING POINTS*

**DISCUSS WITH YOUR STUDENT**

Triangle congruence theorems are an important topic to know for real-world applications like construction, where ensuring two structures or components are identical in shape and size is crucial for stability and precision.

**HERE IS A SAMPLE QUESTION**




In the diagram above, $\overline{LJ} \cong \overline{ON}$ and $\overline{LK} \cong \overline{OM}$. List a congruence relationship that would be sufficient to prove that the two triangles are congruent.

To prove the two triangles are congruent by *SSS*, you can show that $\overline{JK} \cong \overline{NM}$.

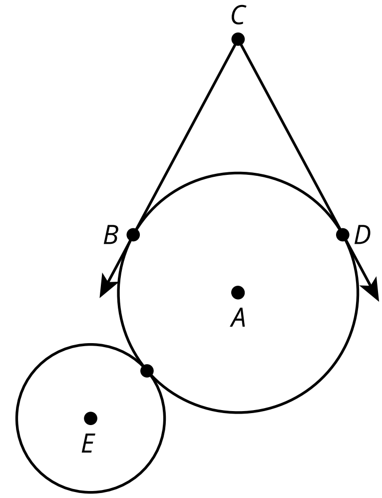
**KEY TERMS**

- Hypotenuse-Leg congruence theorem
- Leg-Leg congruence theorem
- Leg-Angle congruence theorem
- tangent segment theorem [teorema del segmento tangente]
- tangent circles [círculos tangentes]
- diameter chord theorem [teorema de diámetro y cuerda]
- equidistant chord theorem [teorema de las cuerdas equidistantes]
- equidistant chord converse theorem
- congruent chord-congruent arc theorem [teorema de cuerda congruente-arco congruente]
- congruent chord-congruent arc converse theorem
- parallelogram/congruent parallel sides theorem [teorema del paralelogramo/lados paralelos congruentes]
- base angles of a trapezoid [ángulos base de un trapecio]
- trapezoid midsegment theorem [teorema del segmento medio del trapecio]

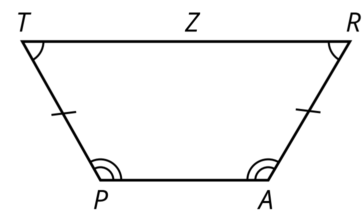
Refer to the Math Glossary for definitions of the Key Terms.

## Where are we now?

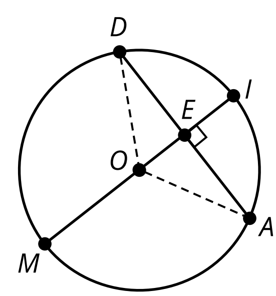
A **tangent segment** is a line segment formed by connecting a point outside of the circle to a point of tangency. **Tangent circles** are circles that lie in the same plane and intersect at exactly one point.




The **base angles of a trapezoid** are either pair of angles that share a base as a common side. An isosceles trapezoid is a trapezoid with congruent non-parallel sides. The base angles of an isosceles trapezoid are congruent.




The **diameter-chord theorem** states: "If a circle's diameter is perpendicular to a chord, then the diameter bisects the chord and bisects the arc determined by the chord."



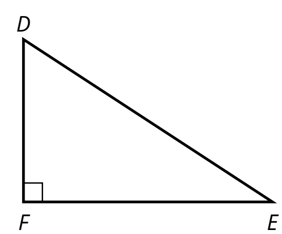
In **Lesson 1: Using Triangle Congruence to Determine Relationships Between Segments**, students prove the Hypotenuse Leg congruence theorem. They also compare the Leg-Leg congruence theorem and Leg-Angle congruence theorem to the triangle congruence theorems they have already proven.

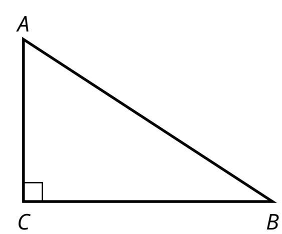
## Right Triangle Congruence Theorems

The **Hypotenuse-Leg (HL) congruence theorem** states: "If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent." For example, given $\overline{AC} \cong \overline{DF}$ and $\overline{AB} \cong \overline{DE}$, then $\triangle ABC \cong \triangle DEF$.

The **Leg-Leg (LL) congruence theorem** states: "If the two corresponding shorter legs of two right triangles are congruent, then the two triangles are congruent." For example, given $\overline{AC} \cong \overline{DF}$ and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

The **Leg-Angle (LA) congruence theorem** states: "If the leg and an acute angle of one right triangle are congruent to the corresponding leg and acute angle of another right triangle, then the triangles are congruent." For example, given $\overline{AC} \cong \overline{DF}$ and $\angle A \cong \angle D$, then $\triangle ABC \cong \triangle DEF$.




In **Lesson 2: Using Congruent Triangles to Determine Relationships Between Chords and Tangents**, students prove relationships between chords and tangents.

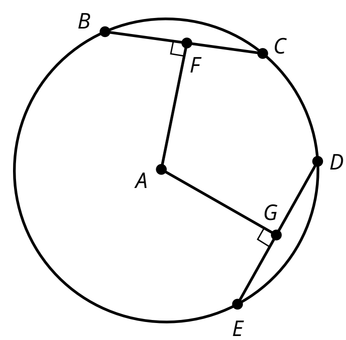
## Chord Theorems

The **equidistant chord theorem** states: "If two chords of the same circle or congruent circles are congruent, then they are equidistant from the center of the circle." For example, in circle $A$, chord $\overline{BC} \cong$ chord $\overline{DE}$. Therefore, $AF = AG$.

The **equidistant chord converse theorem** states: "If two chords of the same circle or congruent circles are equidistant from the center of the circle, then the chords are congruent."

The **congruent chord-congruent arc theorem** states: "If two chords of the same circle or congruent circles are congruent, then their corresponding arcs are congruent." For example, in circle *A*, $\overset{\frown}{BC} \cong \overset{\frown}{DE}$.

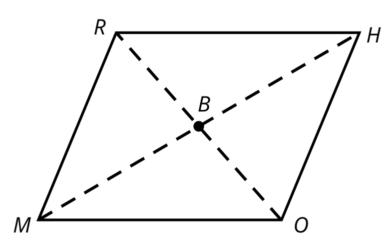
The **congruent chord-congruent converse theorem** states: "If two arcs of the same circle or congruent circles are congruent, then their corresponding chords are congruent."




In **Lesson 3: Properties of Quadrilaterals**, students prove properties of parallelograms, rectangles, squares, rhombi, and trapezoids.

## Parallelogram/Congruent-Parallel Side Theorem

A parallelogram is a quadrilateral with both pairs of opposite sides parallel. The **parallelogram/congruent–parallel side theorem** states: "If one pair of opposite sides of a quadrilateral is both congruent and parallel, then the quadrilateral is a parallelogram." For example, given quadrilateral *RHOM*, $\overline{RM} \cong \overline{OH}$, and $\overline{RM} \parallel \overline{OH}$, then quadrilateral *RHOM* is a parallelogram.



## *Math in the Real World*

Chords are related to many real-world situations, especially in engineering, design, and construction. A chord is a straight line connecting two points on a circle, and the theorems about chords help us figure out relationships between lengths, angles, and distances in circles. For example, architects use these principles when designing curved structures like domes, arches, or round windows. Knowing how chords and their properties work allows them to calculate precise measurements and ensure the design is stable and symmetrical.

In fields like robotics and automotive design, chord theorems are useful when engineers are working with gears, wheels, or anything circular. They can use chord theorems to figure out how different parts interact with one another. Even in everyday stuff like mapping or GPS, these ideas help calculate distances and positions when dealing with circular paths or areas.

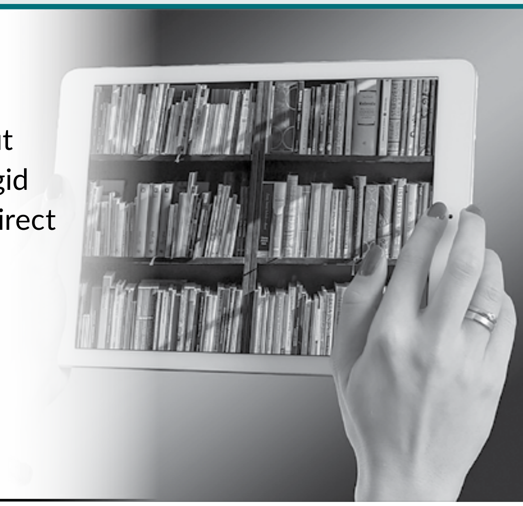
It's not just about big projects, though! Think about designing logos, creating art, or even cutting a pizza into equal slices—chord theorems help ensure everything looks balanced and proportional. Chords are really useful tools for solving all kinds of creative and practical problems in the real world.

MODULE 3

# Investigating Proportionality

## TOPIC 1 Similarity

This topic begins with a review of what students already know about dilations from Grade 8. They perform sequences of rigid and non-rigid transformations on and off the coordinate plane. Students solve indirect measurement problems using similarity and right triangles.



### Where have we been?

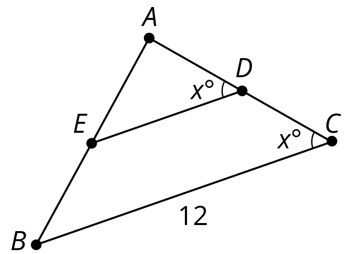
In middle school, students developed their understanding of proportional reasoning through explorations of multiplicative relationships. Students have used scale factors to solve problems. They have informally investigated dilations and described their effect on two-dimensional figures using coordinates.

### Where are we going?

Understanding similarity further develops proportional reasoning, which began in middle school and continues throughout high school mathematics. It provides the opportunity for students to connect spatial and numeric reasoning and lays the groundwork for understanding trigonometric ratios, which students will explore in the next topic.

### HERE IS A SAMPLE QUESTION

**In the figure, if $AE = EB$, what is the length of $\overline{ED}$?**




You can notice that corresponding angles are congruent, so $\overline{ED}$ and $\overline{BC}$ are parallel. Since $\overline{AE} = \overline{EB}$ and $\overline{ED}$ and $\overline{BC}$ are parallel, line segment $ED$ is a midsegment of $\Delta ABC$. Since it is a midsegment, the triangle midsegment theorem tells us that it is half the length of $\overline{BC}$. So, $m\overline{ED} = 6$.

### *TALKING POINTS*

#### DISCUSS WITH YOUR STUDENT

Similarity is an important topic to know for solving real-world problems such as estimating heights of tall objects, creating accurate maps, and designing scale models in engineering and architecture.

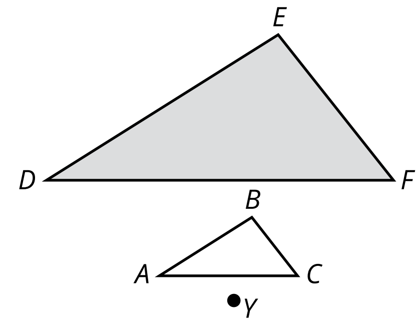
## KEY TERMS

- dilation [dilatación]
- similar figures [figuras similares]
- similar triangles [triángulos similares]
- Angle-Angle similarity theorem [teorema de similitud ángulo-ángulo (AA)]
- Side-Side-Side similarity theorem
- Side-Angle-Side similarity theorem
- angle bisector/ proportional side theorem [teorema de la bisectriz del ángulo/ lado proporcional]
- Triangle Proportionality theorem [teorema de proporcionalidad del triángulo]
- converse of the Triangle Proportionality theorem
- proportional segments theorem [teorema de segmentos proporcionales]
- triangle midsegment theorem [Teorema del segmento medio del triángulo]
- right triangle/altitude similarity theorem
- geometric mean
- right triangle altitude/ hypotenuse theorem
- right triangle altitude/ leg theorem
- indirect measurement

# Where are we now?

A **dilation** is a transformation of the figure in which the figure stretches or shrinks with respect to a fixed point, or center of dilation.

Triangle *DEF* is a dilation of $\triangle ABC$. The center of dilation is point *Y*.




The **geometric mean** of two positive numbers *a* and *b* is the positive number *x* such that $\frac{a}{x} = \frac{x}{b}$.

The geometric mean of 3 and 12 is 6.

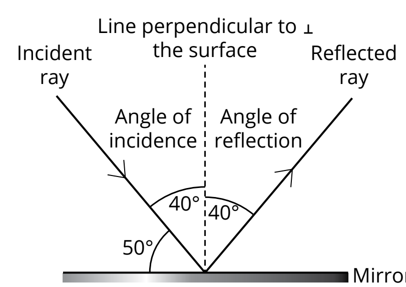
$$\frac{3}{x} = \frac{x}{12}$$

$$x^2 = 36$$

$$x = 6$$

The **angle of reflection** is the angle formed by the reflected ray and a line perpendicular to the surface of a mirror.

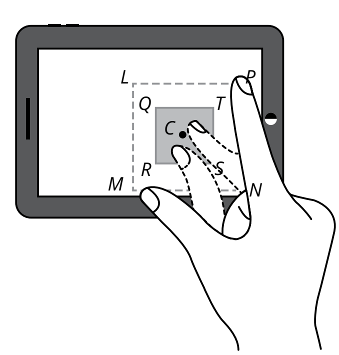
In this example, the angle of reflection measures 40°.




In **Lesson 1: Dilating Figures to Create Similar Figures**, students dilate figures on and off the coordinate plane. They investigate relationships between corresponding sides and corresponding angles in similar triangles.

## Dilating to Create Similar Figures

A *dilation* can produce an enlargement, a reduction, or a congruent figure. For example, when you pinch and zoom on a tablet computer, you can create dilations.

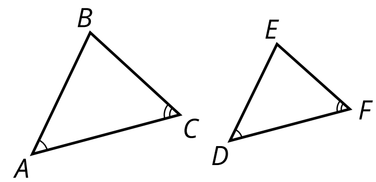


In **Lesson 2: Establishing Triangle Similarity Criteria** students prove the Angle-Angle, Side-Side-Side, and Side-Angle-Side similarity theorems.

## Angle-Angle Similarity Theorem

The **Angle-Angle similarity theorem** states: "If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar."

For example, if $m\angle A = m\angle D$ and $m\angle C = m\angle F$, then $\angle ABC \sim \angle DEF$.

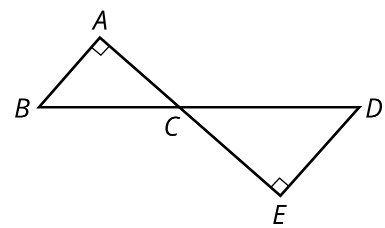



You can apply the Angle-Angle similarity theorem to verify that two triangles are similar.

For triangles *ABC* and *EDC*, you know that:

$\angle A \cong \angle E$ because the two angles are right angles.

$\angle ACB \cong \angle ECD$ because the two angles are vertical angles.




The Angle-Angle similarity theorem tells you that triangles *ABC* and *EDC* are similar because they have two pairs of congruent angles.

In **Lesson 3: Theorems About Proportionality**, students prove theorems about proportionality including the the Triangle Proportionality theorem, the converse of the Triangle Proportionality theorem, and the angle bisector/proportional side theorem.

## Angle Bisector/Proportional Side Theorem

Consider $\triangle ABC$. If $\overline{CD}$ bisects $\angle C$, we can use the angle bisector/proportional side theorem to determine $\overline{BD}$.

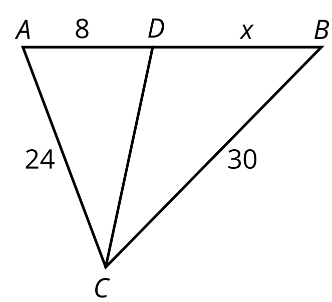
$\frac{AC}{BC} = \frac{AD}{BD}$

$\frac{24}{30} = \frac{8}{x}$

$24x = 240$

$x = 10$

$\overline{BD} = 10$




Students continue to explore theorems related to similar triangles in **Lesson 4: More Similar Triangles**. Then, in **Lesson 5: Applications of Similar Triangles**, students apply what they have learned about similar triangles to solve real-world problems.

- angle of incidence [ángulo de incidencia]
- angle of reflection [ángulo de reflexión]
- law of reflection [ley de la reflexión]
- directed line segment

Refer to the Math Glossary for definitions of the Key Terms.

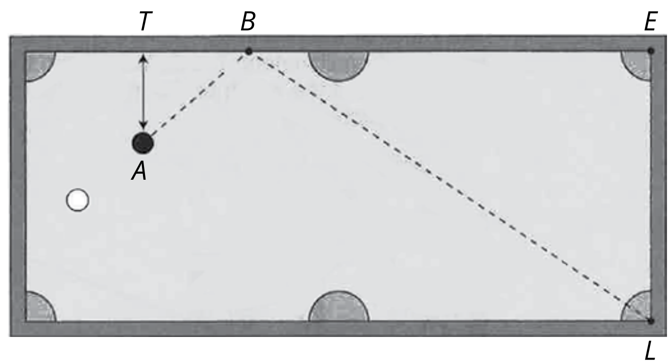
## Math in the Real World

Imagine you're at an amusement park, staring up at the tallest roller coaster in the world. How would you figure out how tall it is without climbing to the top? That's where similar triangles come to the rescue! By measuring the shadow of the coaster and using a smaller, similar triangle you create with a stick and its shadow, you can calculate the height of the coaster without leaving the ground.

Cartographers (mapmakers) and surveyors use similar triangles to measure things like rivers, mountains, and cityscapes, all while keeping their feet dry and their gear light. Next time you take a selfie, think about this: the way the background shrinks while your face stays big is all about perspective, which relies on—you guessed it—similar triangles! Artists and designers use this to create realistic drawings and 3D effects. So whether you're scaling a building, mapping uncharted lands, or snapping the perfect photo, similar triangles are involved.

## Applications of Similarity and Proportionality

Students will use the theorems they have learned to solve problems with real-world application: calculating heights of objects based on shadows, the widths of creeks and canyons, or even just making a tricky shot in billiards.

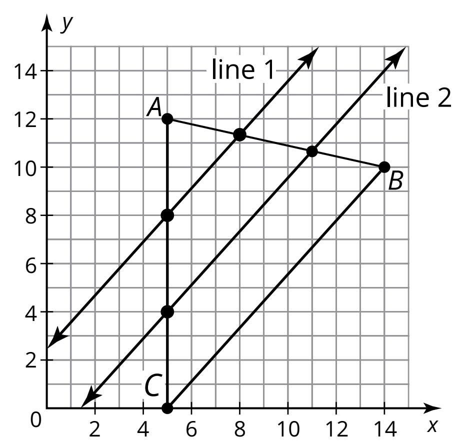



In **Lesson 6: Partitioning Segments in Given Ratios**, students use the distance and midpoint formulas to determine the midpoints of segments on the coordinate plane. They also divide line segments and number lines into given ratios.

## Partitioning a Direct Line Segment

Divide the directed segment *AB* into a 1 : 2 ratio, draw $\overline{AC}$ such that point *C* is at (5, 0) and draw $\overline{BC}$ to form $\Delta ABC$.

Plot points on $\overline{AC}$ to divide the segment into thirds. Draw lines that are parallel to $\overline{BC}$ and pass though each point you plotted. Line 1 divides directed segment $\overline{AB}$ into a 1 : 2 ratio. Determine the equation of the line.




line 1: $y = \frac{10}{9}x + \frac{22}{9}$

Write and solve a system of equations to determine the coordinates of the point on directed segment *AB* that divides it into a 1 : 2 ratio.

$$\begin{cases} y = \frac{10}{9}x + \frac{22}{9} & \longleftarrow \text{equation for line 1} \\ y = -\frac{2}{9}x + \frac{118}{9} & \longleftarrow \text{equation for the line containing } \overline{AB} \end{cases}$$

The point $\left(8, \frac{102}{9}\right)$ divides directed segment *AB* into a 1 : 2 ratio.

# Family Guide

MODULE 3 Investigating Proportionality

Geometry

## TOPIC 2 Trigonometry

This topic introduces students to trigonometric ratios through an investigation of right triangles. Students to discover and analyze these ratios and use them to solve application problems. They also explore the inverses of the basic trigonometric ratios to determine unknown angle measures. Students explore complement angle relationships in right triangles and then solve real-world problems.

### Where have we been?

In middle school, students learned that slope is the steepness and direction of a line. They used similar triangles to explain why the slope between any two points on a line is the same. This understanding lays the groundwork for the development of tangent as the ratio of the side opposite (the vertical distance of a slope triangle) to the side adjacent (the horizontal distance of a slope triangle). For this reason, the tangent ratio is developed first, followed by sine and cosine.

### Where are we going?

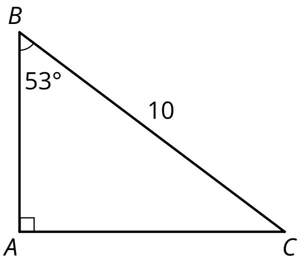
Trigonometry provides an important bridge between geometry and algebra. Understanding the trigonometric ratios in terms of side length ratios prepares students to study trigonometric functions in future courses. Students experience a concrete representation of the trigonometric ratios using triangles in this course.

### *TALKING POINTS*

#### DISCUSS WITH YOUR STUDENT

Trigonometry an important topic to know for understanding the relationships between angles and sides in triangles, which has applications in fields like construction, architecture, and engineering.

#### HERE IS A SAMPLE QUESTION




**Based on the figure above, what is the approximate length of side $\overline{AB}$?**

Use $\angle B$ as the reference angle and $x$ for the unknown side. The cosine of the reference angle is the ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$, or $\frac{x}{10}$. Use a calculator to determine that $\cos(53°) \approx 0.602$. So, $0.602 \approx \frac{x}{10}$, which means that $x$, the unknown side, has a length of approximately 6.

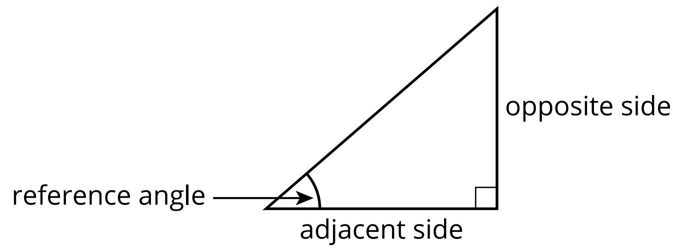
## KEY TERMS

- reference angle [ángulo de referencia]
- opposite side
- adjacent side
- tangent (tan) [tangente]
- inverse tangent (arctangent) [tangente inversa / arco tangente]
- sine (sin) [seno]
- inverse sine (arcsine) [seno inverso / arco seno]
- cosine (cos) [coseno]
- inverse cosine (arccosine) [coseno inverso / arco coseno]

Refer to the Math Glossary for definitions of the Key Terms.

# Where are we now?

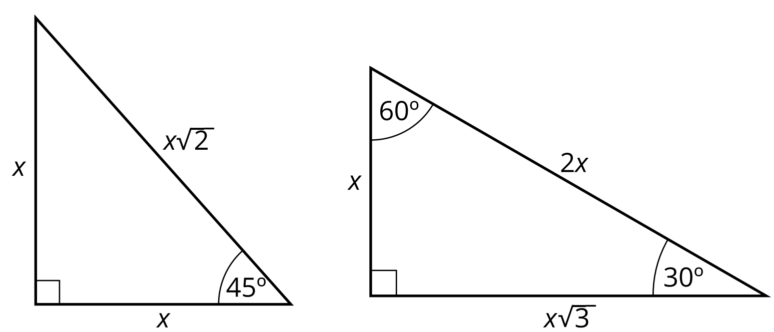
The legs of a right triangle are often referred to as the *opposite side* and the *adjacent side*. These references are based on the angle of the triangle that you are considering, which is called the **reference angle**. The **opposite side** is the opposite side of the reference angle. The **adjacent side** is the adjacent side of the reference angle that is *not* the hypotenuse.




In **Lesson 1: Introduction to Trigonometry**, students learn that there are common ratios that exist between the side lengths within a triangle. They gain understanding of the ratios first through measurement and then generalize, or make conclusions, based on investigations of multiple triangles.

**30°-60°-90° and 45°-45°-90° Triangles**

Students use what they know about 30°-60°-90° and 45°-45°-90° Triangles to begin to informally explore the trigonometric ratios. They gain understanding of the ratios first through measurement and then generalize, or make conclusions, based on investigations of multiple triangles.




In **Lesson 2: Tangent Ratio and Inverse Tangent,** students formalize what they discovered about the tangent ratio in Lesson 1.

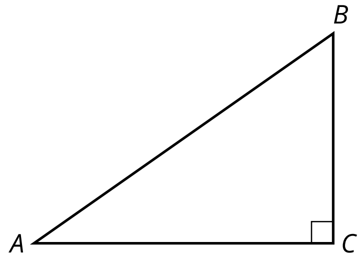
# The Tangent Ratio

The **tangent** (**tan**) of an acute angle in a right triangle is the ratio of the length of the opposite side of the reference angle to the length of the adjacent side of the reference angle. The expression "tan *A*" means "the tangent of ∠$A$." The tangent of an angle can be used to describe the slope of a line.

Consider ∠$A$ in the right triangle.

The tangent ratio describes the relationship between ∠$A$, the opposite side of ∠$A$, and the adjacent side of ∠$A$.

$$\tan A = \frac{\text{length of opposite side of } \angle A}{\text{length of opposite side to } \angle A} = \frac{BC}{AC}$$

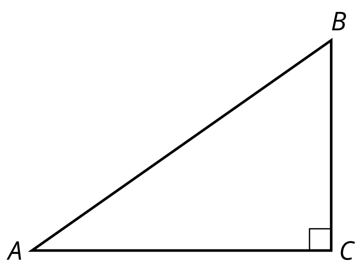



Remember, the ratios of corresponding side lengths between two similar triangles are congruent. The tangent values of congruent angles in similar triangles are always the same.

In **Lesson 3: Exploring Sine and Cosine Ratios**, students formalize their understanding of the sine and cosine ratios they explored in Lesson 1.

## The Sine and Cosine Ratios

The **sine** (**sin**) of an acute angle in a right triangle is the ratio of the length of the opposite side of the angle to the length of the hypotenuse. The expression "sin $A$" means "the sine of ∠$A$."




In right triangle $ABC$, $\sin A = \frac{\text{length of opposite side of } \angle A}{\text{length of hypotenuse}} = \frac{BC}{AB}$.

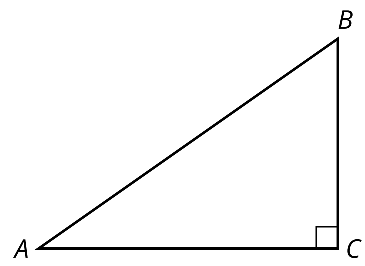
The **cosine** (**cos**) of an acute angle in a right triangle is the ratio of the length of the adjacent side of the angle to the length of the hypotenuse. The expression "cos $A$" means "the cosine of ∠$A$."

### *Math in the Real World*

Right triangle trigonometry might seem like just another math topic, but it has a lot of practical applications in the real world and many careers. Take architecture and engineering, for example. When designing a skyscraper, engineers use trigonometry to calculate the correct slope and length of support beams to ensure the structure is both functional and safe. Or think about a surveyor working to map out land for a new highway. They use trigonometry to measure distances and angles that would be impossible to measure directly, such as the width of a river or the slope of a hill. Without trigonometry, projects like highways, bridges, and buildings couldn't be planned with such precision.

The applications go far beyond construction. Pilots use trigonometry to calculate descent angles and determine how far out they need to start descending for a safe landing, especially on shorter runways. In medicine, ultrasound technicians rely on trigonometry to interpret sound waves bouncing off tissues, helping doctors diagnose and treat patients. Whether you dream of designing skyscrapers, mapping landscapes, flying planes, or working in healthcare, trigonometry plays a role in making those goals a reality.

In right triangle $ABC$, $\cos A = \frac{\text{length of adjacent side of } \angle A}{\text{hypotenuse}} = \frac{AC}{AB}$.




In **Lesson 4: Complement Angle Relationships**, students explore the complementary relationships involved with trigonometric ratios and use them to solve real-world problem situations.

The table displays the trigonometric ratios for a 30°, 60°, and 45° angle based on the Pythagorean Theorem and complementary relationships.

| Reference Angle | sin | cos |
|---|---|---|
| 30° | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ |
| 60° | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ |
| 45° | $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ | $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ |

# Family Guide

## MODULE 4 Connecting Geometric and Algebraic Descriptions

Geometry

# TOPIC 1 Circles

In this topic, students apply proportional reasoning to solve problems involving circles. Students use translation and dilation to establish the similarity of all circles, which plays a critical role in an informal argument for the circumference formula. Students use the distance formula to derive equations for a circle centered at the origin and a circle centered at point $(h, k)$.

## Where have we been?

In previous modules, students studied different forms of proof and criteria for triangle similarity. They developed an understanding of equivalent ratios in Grade 6 and began identifying the constant of proportionality in Grade 7. Students learned the Pythagorean Theorem in Grade 8 and have used it to solve for distances on the coordinate plane.

## Where are we going?

In pre-calculus, students will use radian measures and the trigonometric ratios to develop their understanding of the unit circle. They will "unroll" the unit circle along the $x$-axis of the coordinate plane to represent the sine, cosine, and tangent functions graphically. Understanding trigonometric functions on a coordinate plane allows students to model periodic behavior and to solve more complex real-world problems.

### *TALKING POINTS*

**DISCUSS WITH YOUR STUDENT**

Functions are an important topic to know for shaping everything from wheels and gears for transportation to clocks, pipes, and even planetary orbits, ensuring efficiency, symmetry, and functionality in daily life.

### HERE IS A SAMPLE QUESTION

$$\mathbf{x^2 + (y + 1)^2 = 4}$$

**The graph of the equation above on the coordinate plane is a circle. When the center of the circle is translated 2 units down and the radius is increased by 1, what is the equation of the resulting circle?**

A circle with its center at $(h, k)$ is described by the equation $(x - h)^2 + (y - k)^2 = r^2$, where $r$ is the radius. So, the center of the original circle is at $(0, -1)$ and it has a radius of 2. The new center will be at $(0, -3)$, and its radius will be 3.

So, the equation for the resulting circle will be $x^2 + (y + 3)^2 = 9$.

## KEY TERMS

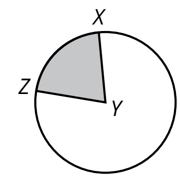
- segments of a chord [segmentos de una cuerda]
- segment chord theorem [teorema del segmento-cuerda]
- external secant segment [segmento secante externo]
- secant segment theorem [teorema del segmento secante]
- secant tangent theorem [teorema de la sacante y la tangente]
- arc length
- radian [radián]
- sector of a circle [sector de un círculo]
- segment of a circle [segmento de un círculo]

Refer to the Math Glossary for definitions of the Key Terms.

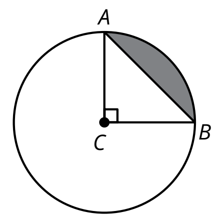
# Where are we now?

A **sector of a circle** is a region of the circle bounded by two radi and the included arc.

In circle $Y$, $\widehat{XZ}$, radius $\overline{XY}$, and radius $\overline{YZ}$ form a sector.




A **segment of a circle** is a region of the circle bounded by a chord and the included arc.



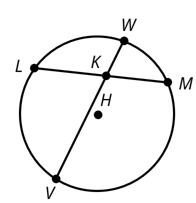

In **Lesson 1: Tangents, Segments, and More Chords,** students explore and prove theorems related to segments of chords, secants, and tangents of a circle.

# Segment Chord Theorem

**Segments of a chord** are the segments formed on a chord when two chords of a circle intersect.

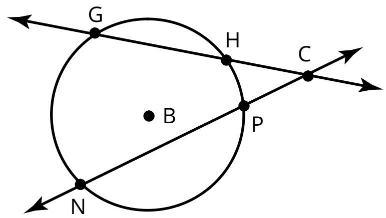
The **segment chord theorem** states: "If two chords in a circle intersect, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the second chord."

For example, in circle $H$, chords $\overline{LM}$ and $\overline{VW}$ intersect to form $\overline{LK}$ and $\overline{MK}$ of chord $\overline{LM}$ and $\overline{WK}$ and $\overline{VK}$ of chord $\overline{VW}$. So $LK \cdot MK = WK \cdot VK$.




The **secant segment theorem** states: "If two secant segments intersect in the exterior of a circle, then the product of the lengths of one secant segment and its external secant segment is equal to the product of the lengths of the second secant segment and its external secant segment."

For example, in circle $B$, secant segments $\overline{GH}$ and $\overline{NP}$ intersect at point $C$ outside the circle. So, $GC \cdot HC = NC \cdot PC$.



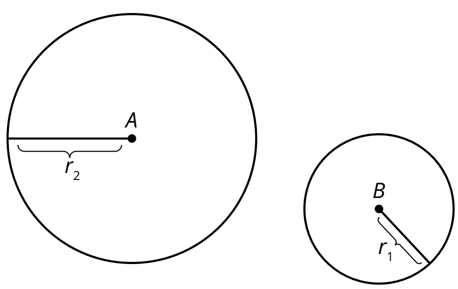

In **Lesson 2: Similarity Relationships in Circles**, student use dilations to show all circles are similar.

## Similarity in Circles

To show that any two circles are similar, only a translation and a dilation are necessary. To determine their scale factor, find the ratio of the radii.

For example, a dilation is needed to increase circle $B$ to the size of circle $A$.

The scale factor, $k$, is $k = \frac{r_2}{r_1}$




In **Lesson 3: Sectors and Segments of a Circle**, students explore and describe methods for determining the area of a sector and the area of a segment of a circle.

## Area of a Sector

To determine the area of a sector, $A$, multiply the area of the circle by a fraction that represents the portion of the area determined by the central angle measure, $m$. There is a proportional relationship between the measure of the area of a circle sector, $A$, and the area of the circle.

$$A = \frac{m}{360°} \cdot \text{area of a circle}$$

$$\frac{A}{\text{area of a circle}} = \frac{m}{360°}$$

The formula for sector area can also be written as follows:

$$A = \frac{m}{360°} \cdot \pi r^2$$

## Math in the Real World

Imagine you're designing a water fountain with perfectly round arcs of water shooting into the air. The equations of circles help engineers and designers determine the exact path of the water, ensuring that each stream lands precisely where it should. Using equations for circles, they can calculate the perfect curvature for pipes and nozzles to create mesmerizing displays. Whether it's the layout of circular garden paths or the curved edges of a stadium roof, these equations ensure beauty, balance, and functionality in design.

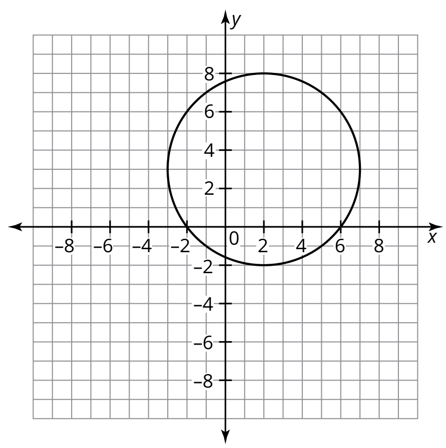
Beyond fountains and architecture, circles play a huge role in technology and navigation. When astronomers track the movement of planets or scientists design the lenses in cameras and telescopes, they rely on equations of circles to ensure accuracy. Even engineers working on roundabouts in city planning or designing the perfect spinning gears for machines use these formulas to maintain precision. So, the next time you see a massive telescope capturing images of distant galaxies or a round clock keeping time perfectly, just remember—somewhere in the process, an equation of a circle made it all possible!

In **Lesson 4: Deriving the Equation of a Circle** and **Lesson 5: Determining Points on a Circle**, students explore the equation of a circle on the coordinate plane.

## Equation of a Circle

The equation of a circle centered at the origin is $x^2 + y^2 = r^2$, where $r$ is the radius of the circle.

The standard form of the equation of a circle centered at $(h, k)$ with radius $r$ can be expressed as $(x - h)^2 + (y - k)^2 = r^2$.




For example, a circle with a center at (2, 3) and a radius of 5 has the equation $(x - 2)^2 + (y - 3)^2 = 25$.

# Family Guide

MODULE 4 Connecting Geometric and Algebraic Descriptions

Geometry

## TOPIC 2 Building Three-Dimensional Shapes

In this topic, students build three-dimensional figures. The also determine the formulas for volume, surface area, and lateral surface area of each three-dimensional figure. Students apply these concepts to mathematical and real-world problems.

### Where have we been?

Students are first introduced to the formula for the volume of a rectangular prism in Grade 5. From Grade 6 to Grade 8, they learned and used the formulas for the volume and surface area of numerous solids to solve problems. In *Building Three-Dimensional Shapes*, students develop informal arguments for each formula, understanding how each arises from the structure of the figure.

### Where are we going?

Students develop an understanding of the structure of three-dimensional solids allowing them to represent solids as geometric shapes and apply volume or surface area formulas to approximate measurements.

### *TALKING POINTS*

**DISCUSS WITH YOUR STUDENT**

Volume and surface area are an important topics to know for designing everything from packaging and architecture to medicine and engineering, ensuring efficiency, proper capacity, and material usage.

### HERE IS A SAMPLE QUESTION

**The water level in a 4 ft long by 3 ft wide by 2 ft tall fish tank is 1 foot. All of this water is poured into a 3 ft long by 2 ft wide by 4 ft tall fish tank. What is the height of the water in the second tank?**

The volume of the water in the first tank is $4 \cdot 3 \cdot 1$, or $12\ ft^3$. This is poured into a fish tank with a base whose area is $6\ ft^2$. To get a volume of $12\ ft^3$, you have to multiply $6\ ft^2$ by 2 ft.

So, the height of the water in the second tank must be 2 ft.

## KEY TERMS

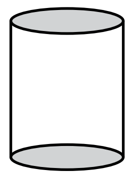
- disc [disco]
- lateral face
- isometric paper [papel isométrico]
- right triangular prism
- oblique triangular prism [prisma triangular oblicuo]
- right rectangular prism
- oblique rectangular prism [prisma rectangular oblicuo]
- right cylinder
- oblique cylinder [cilindro oblicuo]
- cross-section
- sphere [esfera]
- radius of a sphere [radio de una esfera]
- diameter of a sphere [diámetro de una esfera]
- hemisphere [hemisferio]
- lateral surface area [área de superficie lateral]
- total surface area [área de superficie total]

Refer to the Math Glossary for definitions of the Key Terms.

## Where are we now?

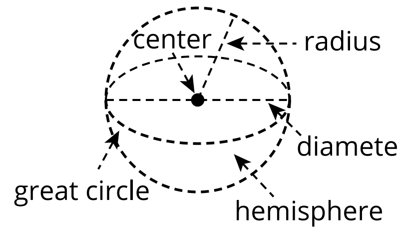
A **disc** is the set of all points on the circle and in the interior of the circle.

A disc translated through space in a direction perpendicular to the plane containing the disc forms a **right cylinder**.



A **sphere** is the set of all points in space that are a given distance from a fixed point called the center of the sphere.

A sphere is shown.




In **Lesson 1: Building Three-Dimensional Figures**, students use transformations to build three-dimensional figures from two-dimensional figures.

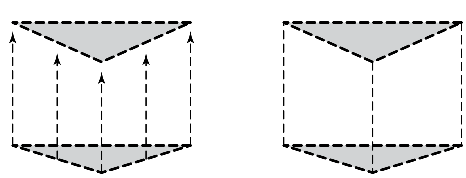
## Building Three-Dimensional Figures

You can perform rotations in three-dimensional space to create a model of a three-dimensional figure.

For example, a cylinder is formed by rotating a rectangle about one of its sides, or about an axis of symmetry. A sphere is formed by rotating a disc about an axis of symmetry. A cone is formed by rotating any triangle about an axis of symmetry, or by rotating a right triangle about one of its legs.

You can translate a two-dimensional figure through space to create a model of a three-dimensional figure. You can use **isometric paper**, or dot paper, to create a two-dimensional representation of a three-dimensional figure.

When you translate a triangle through space in a direction that is perpendicular to the plane containing the triangle, the solid formed is a **right triangular prism**, as shown. Each lateral face of a right triangular prism is a rectangle. A **lateral face** of a three-dimensional object is a face that is not a base.

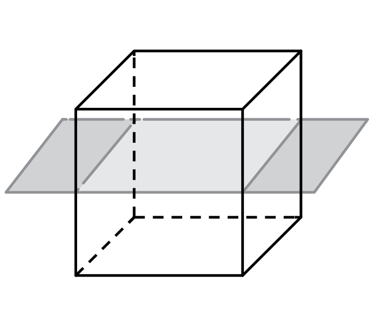


In **Lesson 2: Cross-Sections**, students reason about when the intersection of a plane and a geometric solid creates a cross-section that is a single point or a line segment.

## Cross-Sections

A **cross-section** of a three-dimensional solid results from the intersection of a plane with the solid. When the plane passes through the solid, a two-dimensional cross-section is formed. A point or a line segment may be the cross-section of a plane and a solid when the plane intersects just a vertex or an edge of the solid.

For example, consider some of the cross-sections formed by the intersection of a plane and a cube.



In **Lesson 3: Building Volume Formulas for Pyramids, Cones, and Spheres**, students build and apply the volume formulas of three-dimensional shapes.

## Volume

The table below shows the formulas for determining the volume of some three-dimensional objects. Students will come up with a set of formulas for finding surface area, both lateral and total, for different figures.

| Prisms | Pyramids | Cylinders | Cones | Spheres | Hemisphere |
|---|---|---|---|---|---|
| $V =$ (area of base)(height)<br>$V = Bh$ | $V = \frac{1}{3}Bh$ | $V = \pi r^2 h$ | $V = \frac{1}{3}\pi r^2 h$ | $V = \frac{4}{3}\pi r^3$ | $V = \frac{2}{3}\pi r^2$ |

In **Lesson 4: Building Lateral and Surface Area Formulas for Pyramids, Cones, and Spheres**, students build and apply the lateral surface area and total surface area formulas for three-dimensional figures.

## Surface Area Formulas

The formulas for some of the figures students explore are shown.

| Figure | Lateral Surface Area | Total Surface Area |
|---|---|---|
| Right Rectangular Prism | $2\ell h + 2wh$<br><br>$\ell$ = length<br>$w$ = width<br>$h$ = height | $2\ell w + 2\ell h + 2wh$, or $2B + L$<br><br>$\ell$ = length<br>$b$ = width<br>$h$ = height<br>$B$ = area of base<br>$L$ = lateral surface area |
| Square Pyramid | $2\ell s$<br><br>$\ell$ = length of square<br>$s$ = slant height | $2\ell s + \ell^2$, or $B + L$<br><br>$\ell$ = length of square<br>$s$ = slant height<br>$B$ = area of base<br>$L$ = lateral surface area |
| Right Cylinder | $2\pi rh$<br><br>$r$ = radius of cylinder<br>$h$ = height of cylinder | $2\pi r^2 + 2\pi rh$, or $2B + L$<br><br>$r$ = radius of cylinder<br>$h$ = height of cylinder<br>$B$ = area of base<br>$L$ = lateral surface area |

## *Math in the Real World*

Imagine you're designing the perfect ice cream cone. You want the cone to hold as much ice cream as possible without overflowing and making a mess. That's where the volume formula of a cone comes in handy—it helps you figure out exactly how much ice cream fits inside! Meanwhile, the surface area formula is just as important because it determines how much waffle cone material is needed to wrap around the ice cream. Whether it's designing food packaging, measuring how much soda fits in a can, or even calculating how much cake batter fills a baking pan, understanding volume and surface area makes sure everything is the correct size.

Beyond food, these math concepts are everywhere! Architects use surface area to figure out how much glass is needed for skyscraper windows, while engineers designing fuel tanks for rockets must calculate volume to ensure there's enough space for fuel without adding unnecessary weight. Even medical professionals rely on these formulas when creating medicine bottles to ensure the right dosage and storage capacity. So, the next time you sip from a water bottle or ride in a car with a perfectly shaped fuel tank, just know—surface area and volume played a major role in making it work!

# Family Guide

MODULE 5 Making Informed Decisions

Geometry

## TOPIC 1 Independence and Conditional Probability

In this topic, students investigate compound probability with an emphasis toward modeling and analyzing sample spaces to determine rules for calculating probabilities in different situations. Students explore various probability models and calculate compound probabilities with independent and dependent events in a variety of problem situations.

### Where have we been?

In Grade 7, students learned about simple probability involving simple events and described the probabilities of those events both informally and formally. Students have also explored both experimental and theoretical probability and sample spaces for compound events.

### Where are we going?

Students formalize notions about probability. In the next topic they broaden their understanding of both probability and sample spaces to include combinations, permutations, and expected value. Probabilistic reasoning is an important component of statistical reasoning as they work with randomness in advanced courses.

### *TALKING POINTS*

#### DISCUSS WITH YOUR STUDENT

Your student is learning about probability. You can further support your student's learning by discussing probability in everyday life such as in the weather forecast, probability of running into traffic when driving at different times of the day, or in games.

#### HERE IS A SAMPLE QUESTION

**A teacher hands out gumballs from a jar randomly to her class. If the jar has 50 gumballs in all—15 licorice, 10 banana, 20 watermelon, and 5 grapefruit— what is the probability that the first three gumballs picked out are licorice flavored?**

The probability of choosing a licorice-flavored gumball on the first draw is $\frac{15}{50}$. On the next draw, there are 49 gumballs and 14 licorice-flavored ones, so the probability on the second draw is $\frac{14}{49}$. The probability on the third draw is $\frac{13}{48}$.
We can see that this is "and" probability, so multiply all the probabilities:

$$\frac{15}{50} \cdot \frac{14}{49} \cdot \frac{13}{48} = \frac{2730}{117{,}600} = \frac{13}{560}$$

## KEY TERMS

- outcome
- sample space
- event [evento]
- probability [probabilidad]
- probability model [modelo de probabilidad]
- uniform probability model [modelo de probabilidad uniforme]
- non-uniform probability model [modelo de probabilidad no uniforme]
- complement of an event [complemento de un evento]
- tree diagram
- organized list [lista organizada]
- set
- element [elemento]
- disjoint sets
- intersecting sets
- union of sets
- independent events [eventos independientes]
- dependent events [eventos dependientes]
- Fundamental Counting Principle [Principio fundamental de conteo]
- compound event [evento compuesto]
- compound probability rule involving and [regla de probabilidad compuesta con “y”]

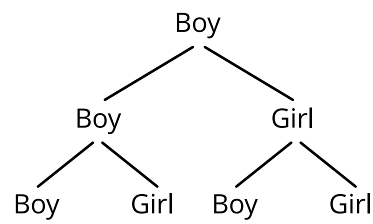
# Where are we now?

A **probability model** lists the possible outcomes and the probability for each outcome. In a probability model, the sum of the probabilities must equal 1.

The table shows a probability model for flipping a fair coin once.

| **Outcomes** | Head (H) | Tails (T) |
|---|---|---|

A **tree diagram** is a diagram that illustrates sequentially the possible outcomes of a given situation.




An **organized list** is a visual model for determining the sample space of events.
The sample space for flipping a coin 3 times can be represented as an organized list.

| | |
|---|---|
| HHH | THH |
| HHT | THT |
| HTH | TTH |
| HTT | TTT |

In **Lesson 1: Compound Sample Spaces,** students learn strategies for determining the sample space of compound events.

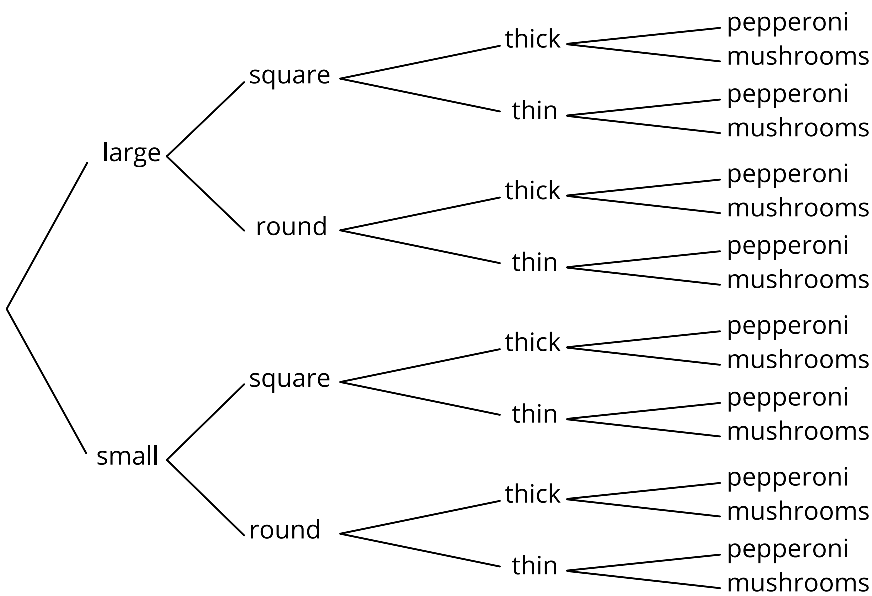
## Sample Spaces

An **outcome** is a result of an experiment. The **sample space** is the set of all the possible outcomes of an experiment. An **event** is an outcome or set of outcomes in a sample space.

The **probability** of an event is the ratio of the number of desired outcomes to the total number of possible outcomes. The probability of event $A$ is $P(A) = \frac{\text{desired outcomes}}{\text{possible outcomes}}$.

A **tree diagram** is a visual model for determining the sample space of multiple events.

For example, the sample space for all of the possible types of pizza specials that can be made at a pizzeria is be modeled by the tree diagram shown.




An **organized list** is a visual model for determining the sample space of events. You can abbreviate the names of the outcomes as long as a key is provided.

For example, an organized list for the pizza tree diagram could use the following abbreviation key: lg-large, sm-small, sq-square, ro-round, thk-thick, thn-thin, pepp-pepperoni, mush-mushroom. The resulting list would be as follows:

| | | | |
|---|---|---|---|
| Lg-sq-thk-pepp | Lg-ro-thk-pepp | Sm-sq-thk-pepp | Sm-ro-thk-pepp |
| Lg-sq-thk-mush | Lg-ro-thk-mush | Sm-sq-thk-mush | Sm-ro-thk-mush |
| Lg-sq-thn-pepp | Lg-ro-thn-pepp | Sm-sq-thn-pepp | Sm-ro-thn-pepp |
| Lg-sq-thn-mush | Lg-ro-thn-mush | Sm-sq-thn-mush | Sm-ro-thn-mush |

The *Fundamental Counting Principle* is used to determine the number of outcomes in the sample space. The **Fundamental Counting Principle** states that if an action *A* can occur in *m* ways, and for each of these *m* ways an action *B* can occur in *n* ways, then actions *A* and *B* can occur in $m \cdot n$ ways. The Fundamental Counting Principle can be generalized to more than two actions that happen in succession. If for each of the *m* and *n* ways *A* and *B* occur, there is also an action *C* that can occur in *s* ways, then Actions *A*, *B*, and *C* can occur in $m \cdot n \cdot s$ ways.

- conditional probability [probabilidad condicional]
- Addition Rule for probability

Refer to the Math Glossary for definitions of the Key Terms.

For example, if you were to visit an ice cream store that sells either chocolate, vanilla, or strawberry ice cream in a cone or cup with or without sprinkles, you would have $3 \cdot 2 \cdot 2 = 12$ possible outcomes.

In **Lesson 2: Compound Probability with *And*,** students determine the probability of two or more independent events and two or more dependent events.

## Compound probability rule involving *and*

A **compound event** is an event that consists of two or more events.

The **compound probability rule involving *and*** states: "If Event *A* and Event *B* are independent events, then the probability that Event *A* happens and Event *B* happens is the product of the probability that Event *A* happens and the probability that Event *B* happens, given that Event *A* has happened."

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

If Event *A* and Event *B* are dependent, then the probability that Event *A* happens and Event *B* happens is the product of the probability that Event *A* happens and the probability that Event *B* happens, given that Event *A* has already happened.

For example, suppose you have 2 red, 1 blue, and 3 green socks in a drawer. You reach into the drawer without looking and choose a sock, replace it, and then choose another sock. You choose a total of 2 socks.

These are independent events, so use the compound probability rule to calculate. $P(\text{green and red}) = P(\text{green}) \cdot P(\text{red}) = \frac{3}{6} \cdot \frac{2}{6} = \frac{6}{36} = \frac{1}{6}$

Now suppose you reach into the drawer without looking and choose a sock, do not replace it, and then choose another sock. You choose a total of 2 socks.

These are dependent events, so $P(A \text{ and } B) = P(A) \cdot P(B, \text{ given } A)$. Therefore, $P(\text{green and red}) = \frac{3}{6} \cdot \frac{2}{5} = \frac{6}{30} = \frac{1}{5}$.

A **conditional probability** is the probability of event *B*, given that event *A* has already occurred. The notation for conditional probability is $P(B|A)$, which reads, "the probability of event *B*, given event *A*."

In **Lesson 3: Compound Probability with *Or*,** students determine the probability of two or more independent events and two or more dependent events.

## Addition Rule for probability

In a compound event that is related by the word *or*, there can be possible outcomes that are in the sample space for each event. These outcomes should only be counted once when determining the compound probability.

The **Addition Rule for probability** states: "The probability that Event $A$ occurs or Event $B$ occurs is the probability that Event $A$ occurs plus the probability that Event $B$ occurs minus the probability that both $A$ and $B$ occur."

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

For example, consider determining the probability of drawing a jack or a club from a deck of cards. Let $A$ = a jack and $B$ = a club.

$$P(\text{jack or club}) = P(\text{jack}) + P(\text{club}) - P(\text{jack and club})$$
$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

In **Lesson 4: Calculating Compound Probability,** students determine the probability of independent events $P(A \text{ and } B)$ with replacement, independent events $P(A \text{ or } B)$ with replacement, dependent events $P(A \text{ and } B)$ without replacement, and dependent events $P(A \text{ or } B)$ without replacement.

## Compound Probability

You can use the same methods to calculate the probabilities for more than two compound events.

You will often see the phrases "with replacement" and "without replacement" in probability problems. These phrases refer to whether or not the total sample space changes from the first to the last event.

For example, the probability for drawing 3 of a kind of a given number or face card from a deck of cards with replacement is

$P(\text{3 of a kind}) = \frac{4}{52} \cdot \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{2197}.$

The probability of drawing 3 of a kind without replacement is

$P(\text{3 of a kind}) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = \frac{1}{5525}.$

### *Math in the Real World*

People use probability when making decisions to determine risks and predict outcomes. You might check the weather forecast to see what to wear the next day. The weather forecast is based on probability. Meteorologists use satellite data and weather stations to predict the chance of rain, snow, or sunshine. A sports analyst uses probability to talk about a team's chances of winning a game. Financial analysts use probability to make predictions about stock prices. Insurance companies rely on probability to figure out how likely it is that you'll file a claim, so they know how much to charge.

Probability is a powerful tool. Understanding how probability works can help you make better decisions and understand the world around you in a whole new way. Whether you're deciding whether to bring an umbrella or predicting the best time to purchase a plane ticket, probability helps you make choices.

# TOPIC 2 Computing Probabilities

In this topic, students apply and extend the probability concepts they learned in the previous topic to explore expected value and conditional probability. They also use combinatorial techniques to construct and reason with large sample spaces. Students consider simple and complex event interactions in this topic and organize them to derive information from them about probabilities.

## Where have we been?

In the previous topic, students revisited simple probability and deepened their understanding of compound probability in preparation for this topic. Students were introduced to the Fundamental Counting Principle in the previous topic as a precursor to combinations and permutations introduced in this topic.

## Where are we going?

Combinations and permutations used to construct large sample spaces are widely used concepts in a variety of mathematical subfields, including number theory and computer science. Conditional probability and expected value are important concepts in fields such as economics and statistics.

## *TALKING POINTS*

### DISCUSS WITH YOUR STUDENT

Conditional probability is important because it helps us make more informed decisions. Understanding conditional probability helps us understand how one event influences another and is used in many fields including statistics, science, finance, and artificial intelligence.

### QUESTIONS TO ASK

**A committee of 10 people will elect three representatives. How many different groups of three representatives can they choose?**

It is possible to choose two groups of the same three people, which only differ by the order in which they were chosen. Those groups would be permutations of each other. But, what we want are combinations—where no two groups have the same people.

There is a formula for choosing *r* combinations from a set of *n* elements: $\frac{n!}{(n-r)!r!}$. So, the number of different groups of three representatives that can be chosen is:

$$\frac{10!}{(10-3)! \cdot 3!} = \frac{3,628,800}{30,240} = 120$$

## KEY TERMS

- two-way table
- frequency table [tabla de frecuencia(s)]
- two-way frequency table
- categorical data [datos categóricos]
- relative frequency [frecuencia relativa]
- two-way relative frequency table
- factorial [factorial]
- permutation [permutación]
- circular permutation [permutación circular]
- combination [combinación]
- geometric probability [probabilidad geométrica]
- expected value [valor esperado]

Refer to the Math Glossary for definitions of the Key Terms.

# Where are we now?

A **frequency table** shows the frequency of an item, number, or event appearing in a sample space.

The frequency table shows the number of times a sum of two number cubes occurred.

| Sum of Two Number Cubes | Frequency |
|---|---|
| 2 | 1 |
| 3 | 2 |
| 4 | 3 |
| 5 | 4 |
| 6 | 5 |
| 7 | 6 |
| 8 | 5 |
| 9 | 4 |
| 10 | 3 |
| 11 | 2 |
| 12 | 1 |

**Qualitative data**, are data for which each piece of data fits exactly one of several different groups or categories.

Animals: lions, tigers, bears, etc.
Colors: blue, green, red, etc.

The **factorial** of *n*, written as *n*!, is the product of all non-negative integers less than or equal to *n*.
$3! = 3 \cdot 2 \cdot 1 = 6$

In **Lesson 1: Compound Probability for Data Displayed in Two-Way Tables**, analyze two-way tables and determine compound probabilities and compound events.

## Compound Probabilities

A **relative frequency** is the ratio of occurrences within a category to the total number of occurrences. To determine the ratio for each category, determine the part-to-whole ratio for each category. A **two-way relative frequency table** displays the relative frequencies for two categories of data.

Sports Participation

| | Individual | Team | Does Not Play | Total |
|---|---|---|---|---|
| **Left** | $\frac{3}{63} \approx 4.8\%$ | $\frac{13}{63} \approx 25.4\%$ | $\frac{8}{63} \approx 12.7\%$ | $\frac{24}{63} \approx 38.1\%$ |
| **Right** | $\frac{6}{63} \approx 9.5\%$ | $\frac{23}{63} \approx 36.5\%$ | $\frac{4}{63} \approx 6.3\%$ | $\frac{33}{63} \approx 52.4\%$ |
| **Mixed** | $\frac{1}{63} \approx 1.6\%$ | $\frac{3}{63} \approx 4.8\%$ | $\frac{2}{63} \approx 3.2\%$ | $\frac{6}{63} \approx 9.5\%$ |
| **Total** | $\frac{10}{63} \approx 15.9\%$ | $\frac{39}{63} \approx 61.9\%$ | $\frac{14}{63} \approx 22.2\%$ | $\frac{63}{63} = 100\%$ |

You can use a two-way relative frequency table to calculate the probability of events occuring.

For example, $P(\text{left handed}) = \frac{24}{63} \approx 38.1\%$, $P(\text{right handed in a team sport}) = \frac{23}{63} \approx 36.5\%$, or $P(\text{mixed handed or does not play}) = P(\text{mixed handed}) + P(\text{does not play}) - P(\text{mixed handed and does not play}) = \frac{6}{63} + \frac{14}{63} - \frac{2}{63} = \frac{18}{63} \approx 28.6\%$.

In **Lesson 2: Conditional Probability**, students derive a formula for computing conditional probability and apply the formula in several different situations.

## Calculating Conditional Probabilities

A conditional probability is the probability of event *B*, given that event *A* has already occurred.

The notation for conditional probability is $P(B|A)$, which is read, "the probability of event *B*, given event *A*."

The conditional probability, $P(B|A)$, for independent events can be represented as $\frac{\text{desired outcomes}}{\text{total outcomes}}$.

$$\begin{aligned} P(B|A) &= \frac{\text{desired outcomes}}{\text{total outcomes}} \\ &= \frac{A \text{ and } B}{A} \\ &= \frac{A \text{ and } B}{A} \cdot \frac{\frac{1}{\text{total}}}{\frac{1}{\text{total}}} \\ &= \frac{\frac{A \text{ and } B}{\text{total}}}{\frac{A}{\text{total}}} \\ &= \frac{P(A \text{ and } B)}{P(A)} \end{aligned}$$

For example, consider the possible sums when rolling two number cubes.

Let $A$ = rolling a 5

Let $B$ = rolling a sum of 10

$P(B|A) = \frac{\frac{1}{72}}{\frac{1}{6}} = \frac{1}{12}$

You can check if these two events are independent by also calculating $P(B) = \frac{3}{36} = \frac{1}{12}$, thus they are independent events.

In **Lesson 3: Conditional Probability**, students derive the formulas to calculate permutations and combinations and apply them in different situations.

## Permutations and Combinations

An ordered arrangement of items without repetition is called a **permutation**. Being able to determine the different arrangement of three letters from the first four letters of the alphabet is an example of a permutation. There are different notations that are used for the permutations of $r$ elements taken from a collection of $n$ items.

$$_nP_r = P(n, r) = P_r^n$$

You can calculate permutations using the formula $_nP_r = \frac{n!}{(n - r)!}$.

For example, consider the number of permutations of two letters from the first four letters of the alphabet.

$$_4P_2 = \frac{4!}{(4 - 2)!} = \frac{4!}{2!} = 4 \cdot 3 = 12$$

There are 12 different combinations of two letters using only the first four letters of the alphabet.

A **combination** is an unordered collection of items. Different notations can be used for the combinations of *r* elements taken from a collection of *n* elements.

$$_{n}C_{r} = C(n, r) = C_{r}^{n}$$

You can calculate combinations using the formula $_{n}C_{r} = \frac{n!}{(n-r)!r!}$.

For example, the possible combinations of three-letter strings from the letters DEFG is

$$_{4}C_{3} = \frac{4!}{(4-3)!3!} = \frac{4!}{1!3!} = 4.$$

In **Lesson 4: Independent Trials**, students apply a formula using combinations to calculate probabilities for two independent events over multiple trials.

## Multiple Trials

Independent events can happen in more than one way when performing multiple trials. The probability of the events happening is multiplied by the number of different ways the events can happen.

For example, the table below summarizes the probability of a basketball player making each number of free throws, using combinations.

| Number of Free Throws | Number of Free Throws | Probability of Making | Probability of Missing | Probability of Making *r* |
|---|---|---|---|---|
| Attempts | Makes | Each Free Throw | Each Free Throw | out of *n* Attempts |
| $n$ | $r$ | $p$ | $(1-p)$ | $_{n}C_{r}(p)^{r}(1-p)^{n-r}$ |
| 1 | 0 | $\frac{2}{3}$ | $\frac{1}{3}$ | $_{1}C_{0}\left(\frac{2}{3}\right)^{0}\left(\frac{1}{3}\right)^{1}$ |
| | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ | $_{1}C_{1}\left(\frac{2}{3}\right)^{1}\left(\frac{1}{3}\right)^{0}$ |
| 2 | 0 | $\frac{2}{3}$ | $\frac{1}{3}$ | $_{2}C_{0}\left(\frac{2}{3}\right)^{0}\left(\frac{1}{3}\right)^{2}$ |
| | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ | $_{2}C_{1}\left(\frac{2}{3}\right)^{1}\left(\frac{1}{3}\right)^{1}$ |
| | 2 | $\frac{2}{3}$ | $\frac{1}{3}$ | $_{2}C_{2}\left(\frac{2}{3}\right)^{2}\left(\frac{1}{3}\right)^{0}$ |
| 3 | 0 | $\frac{2}{3}$ | $\frac{1}{3}$ | $_{3}C_{0}\left(\frac{2}{3}\right)^{0}\left(\frac{1}{3}\right)^{3}$ |
| | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ | $_{3}C_{1}\left(\frac{2}{3}\right)^{1}\left(\frac{1}{3}\right)^{2}$ |
| | 2 | $\frac{2}{3}$ | $\frac{1}{3}$ | $_{3}C_{2}\left(\frac{2}{3}\right)^{2}\left(\frac{1}{3}\right)^{1}$ |
| | 3 | $\frac{2}{3}$ | $\frac{1}{3}$ | $_{3}C_{3}\left(\frac{2}{3}\right)^{3}\left(\frac{1}{3}\right)^{0}$ |

## Math in the Real World

Conditional probability plays a critical role in many fields like insurance, finance, and artificial intelligence. Conditional probability helps professionals make informed decisions based on given conditions and available data.

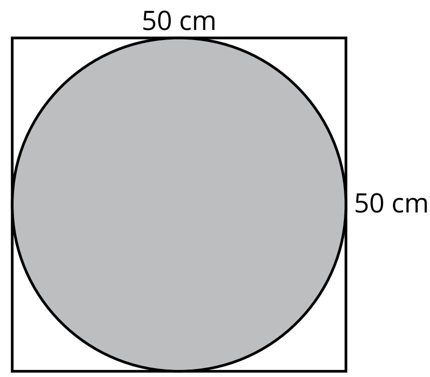
In insurance, conditional probability helps calculate the likelihood of a claim based on certain conditions, like the probability of a claim given the driver's history or the weather. In finance, it's used to evaluate whether an investment will perform well based on certain market conditions. Conditional probability is also used to predict whether an individual will default on a loan based on their financial history. In artificial intelligence, conditional probability is part of many algorithms, like in spam filters. For example, the probability that an email is spam given certain features (like specific words or phrases) is calculated using conditional probability, helping systems make more decisions based on the available information.

In **Lesson 5: Expected Value**, students apply geometric probability to dartboards containing geometric shapes and use expected value to make decisions.

## Geometric Probability

To determine the probability of hitting a shaded section on a dartboard, you can use *geometric probability*. **Geometric probability** is probability that involves a geometric measure, such as length, area, volume, and so on.

For example, consider the dartboard shown, which is a square that measures 50 cm by 50 cm.




The probability of a dart hitting the shaded region of the dartboard is $\frac{\text{area of shaded region}}{\text{area of dartboard}}$.

$$\frac{\text{area of shaded region}}{\text{area of dartboard}} = \frac{\pi(25)^2}{50 \cdot 50} \approx \frac{1962.5}{2500}$$

The probability that the dart hits the shaded region is $\frac{1962.5}{2500}$, or about 78.5%.