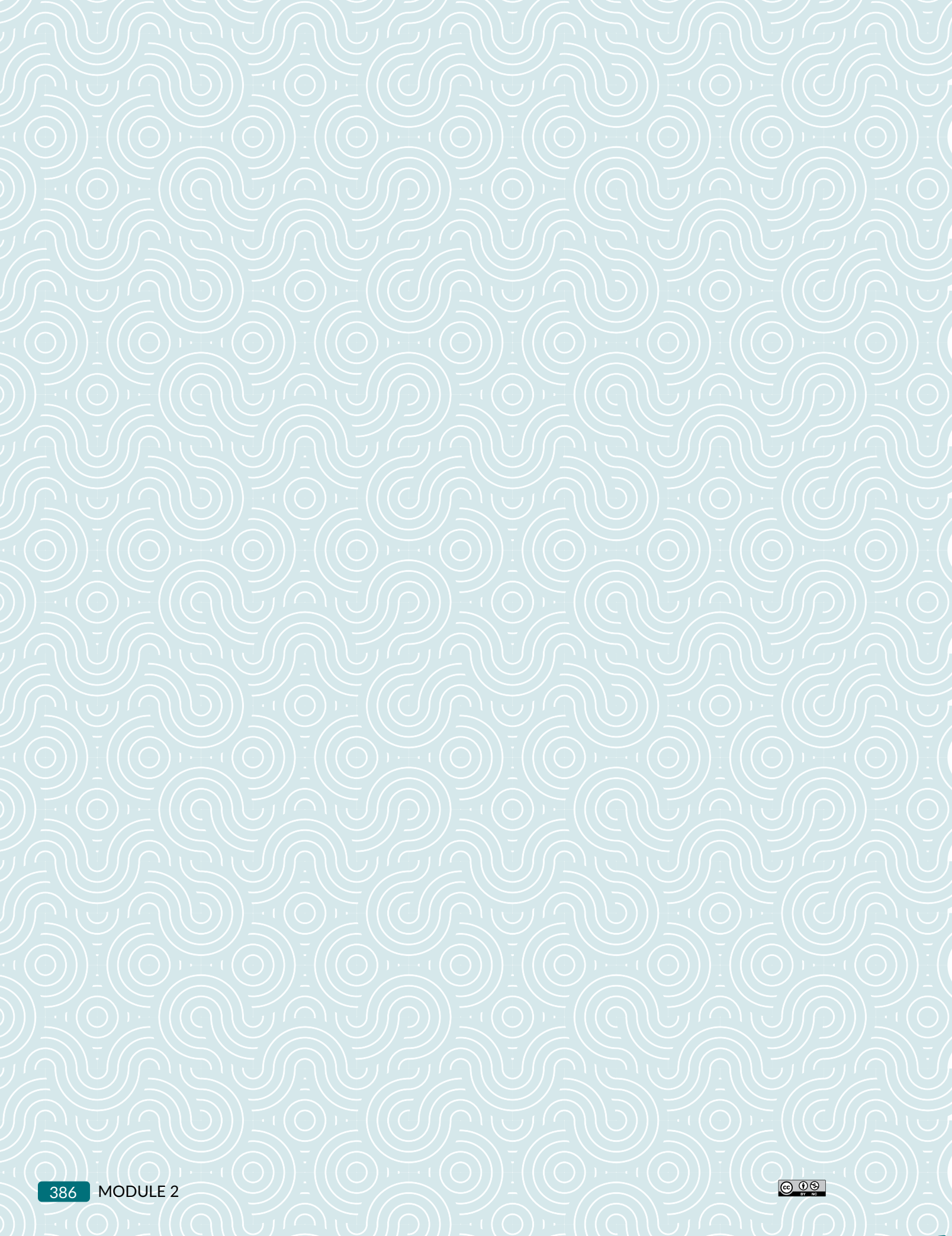
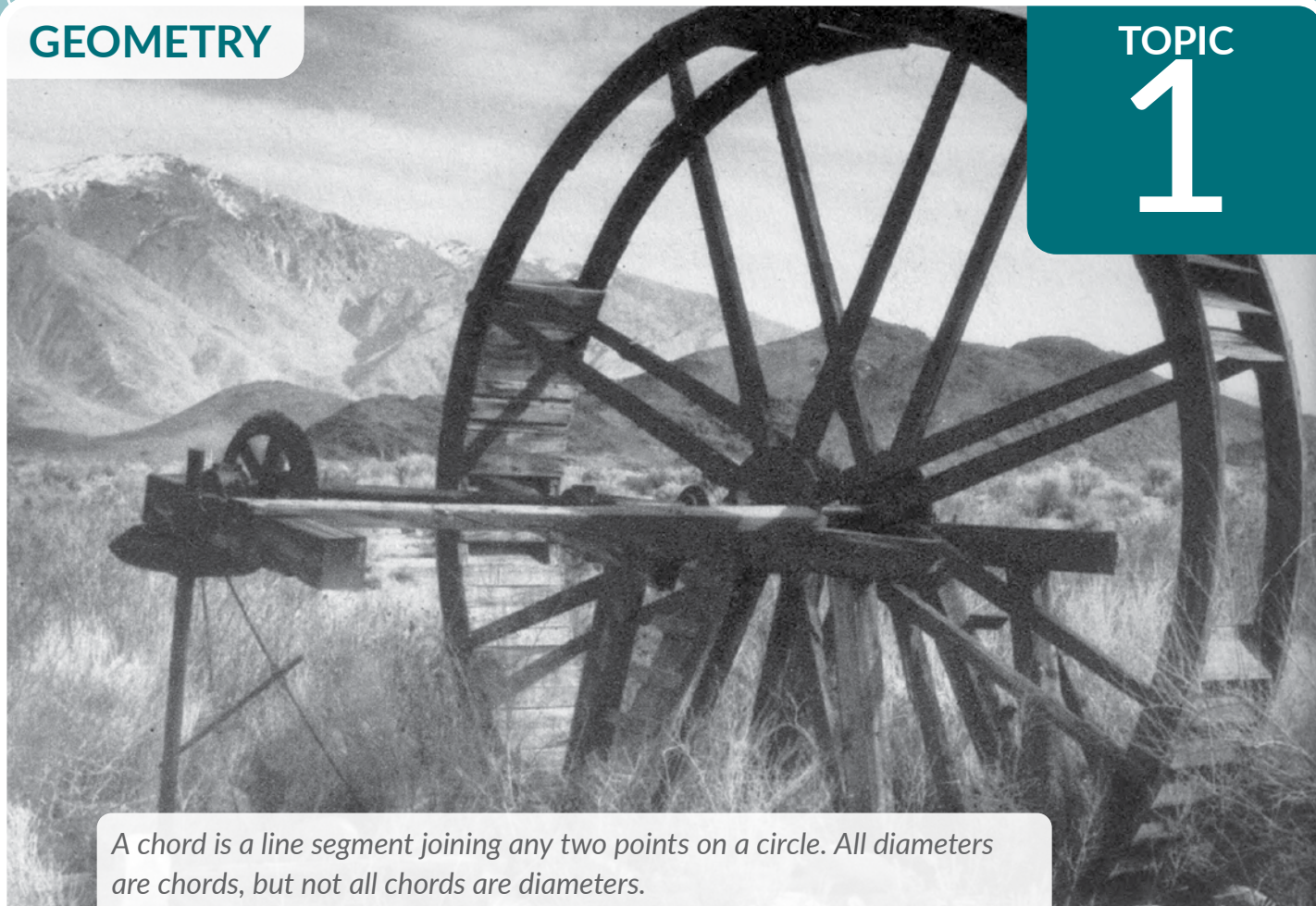


Justifying Mathematical Ideas and Arguments

TOPIC 1	Composing and Decomposing Shapes	387
TOPIC 2	Justifying Line and Angle Relationships	493
TOPIC 3	Using Congruence Theorems	665





A chord is a line segment joining any two points on a circle. All diameters are chords, but not all chords are diameters.

Composing and Decomposing Shapes

LESSON 1	Using Circles to Make Conjectures	389
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1

Using Circles to Make Conjectures

OBJECTIVES

- Construct circles, parallel lines, and perpendicular bisectors.
- Identify line segments, lines, and angles associated with the interior and exterior of circles.
- Make conjectures about vertical angles, alternate interior angles, corresponding angles, and points on the perpendicular bisector of a line segment in order to write theorems.
- Make conjectures about inscribed angles on a diameter and the angles formed where tangent lines intersect radii of a circle.

.....

You know a lot about geometry and can reason with geometric objects. How can you use what you know about circles to make conjectures about line and angle relationships?

KEY TERMS

- radius
- diameter
- central angle
- major arc
- minor arc
- secant
- chord
- inscribed angle
- intercepted arc
- tangent
- circumscribed angle

Getting Started

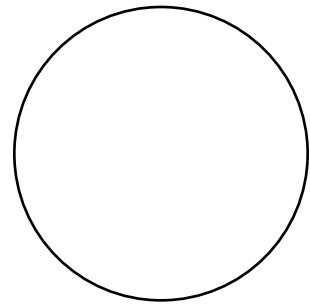
Freehand Circle Drawing Championship

A teacher, as a little joke, told his math classes for over a decade that he was a “World Freehand Circle Drawing Champion” because he could draw perfect circles on the board.

But the championship didn’t really exist, until a video of the teacher’s circle drawing ability went viral, launching the first World Freehand Circle Drawing Championship in 2007.

Can you draw a perfect circle freehand?

1. Try to draw a perfect circle, like the one shown, without tracing or using tools.



2. Explain how you could decide whether one circle is closer to “perfect” than another. Use your criteria to judge your and your classmates’ best circles.
3. What is the measure of an angle that forms a perfect circle? What is the arc measure of that angle?

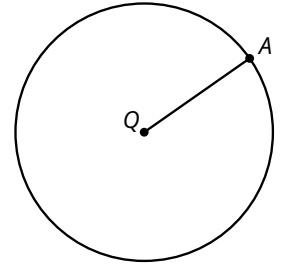
1.1

Circle Parts and Bisectors

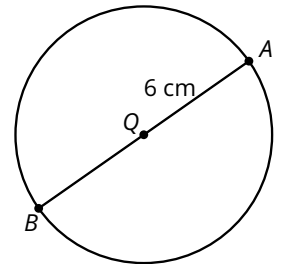
Let's investigate what you know about circles and their parts.

1. Use a compass to construct a circle in the space provided.
Label the center point, O .
2. Use the definitions to complete the construction and answer each question.
 - a. Construct a diameter of the circle. Label the endpoints D and M . Point O is the midpoint of the diameter.
 - b. How many central angles and arcs are produced by drawing a diameter and what are their measures? Explain how you know.
 - c. What name is given to each half of the circle created by the diameter?

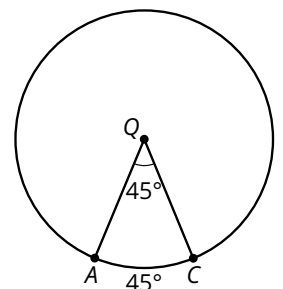
A **radius** is a line segment from the center of a circle to a point on the circle.



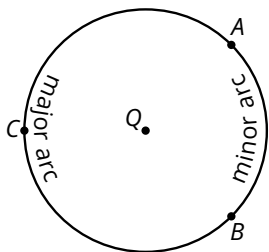
A **diameter** of a circle is a line segment passing through the center of the circle with endpoints on the circle.



A **central angle** is an angle with its vertex at the center of a circle, like $\angle AQC$. An arc is a part of a circle, like $\widehat{AC}$. The measure of $\widehat{AC}$ is the measure of the central angle formed by the endpoints.



Two points on a circle determine a major arc and a minor arc. The arc with the greater measure, $\widehat{ACB}$, is the **major arc**. The other arc, $\widehat{AB}$, is the **minor arc**.



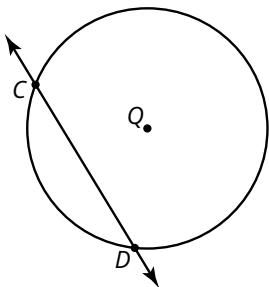
3. Construct a perpendicular bisector of your circle's diameter. Label the points of intersection of the perpendicular bisector and the circle as points P and B .

4. Use the definitions to identify the circle parts and answer each question.

a. Identify all the drawn radii of the circle created by the diameter and its perpendicular bisector.

b. What central angles and arcs are produced by the diameter and its perpendicular bisector and what are their measures? Explain how you know.

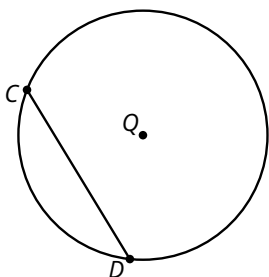
A **secant** of a circle is a line that intersects the circle at two points. In this circle, $\overleftrightarrow{CD}$ is a secant.



5. Draw a line through two of your labeled points on the circle to form a secant and a chord that is not a diameter.

a. Use a protractor to measure the angles of the triangle and a ruler to measure the side lengths.

A **chord** is a segment whose endpoints are points on a circle. In this circle, $\overline{CD}$ is a chord.



b. Classify the triangle you have created.

6. Compare your circle diagram with your classmates' diagrams.

- a. Did everyone create the same triangle? The same type of triangle?

- b. Are all circles similar? Congruent?

7. What conjecture can you make about this type of triangle?

.....

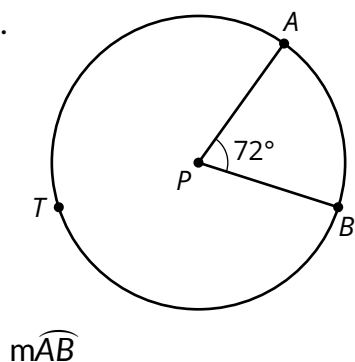
Think About...

Are all circles similar?

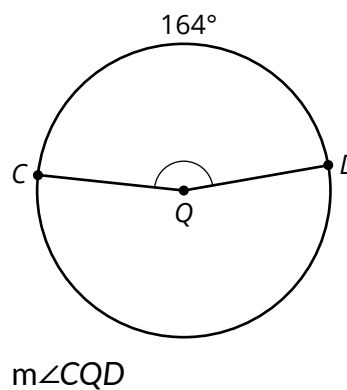
.....

8. Determine the measure of each central angle or arc. If a measure of an arc is determined, identify whether the arc is a major arc or a minor arc.

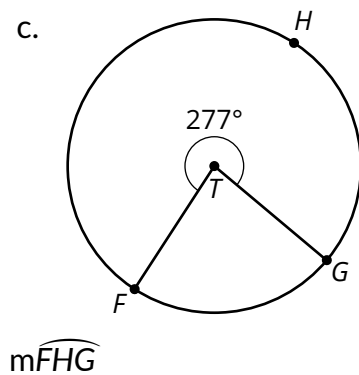
a.



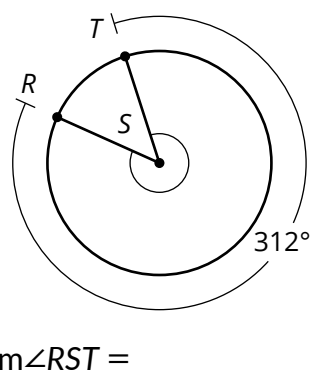
b.



c.



d.



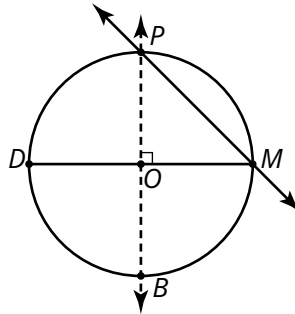
9. Determine the measure of each arc.

a. $m\widehat{ATB}$ from Question 8, part a.

b. $m\widehat{RT}$ from Question 8, part d.

Angle and Arc Relationships

Let's use your geometric knowledge to make a few more conjectures. Consider the circle and triangle Malik constructed.



Angles POM and DOB are vertical angles. Angles BOM and POD are vertical angles.



1. Abby and Luis each made a conjecture about the measures of vertical angles. Consider their conjectures and determine who's correct. Explain why the other conjecture is incorrect.

Abby

All vertical angles are congruent and measure 90° .

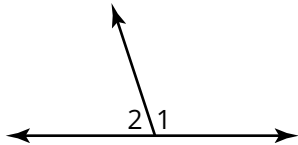
Luis

All vertical angles are congruent.

.....

Remember ...

A *linear pair* of angles are two adjacent angles that have noncommon sides that are opposite rays.



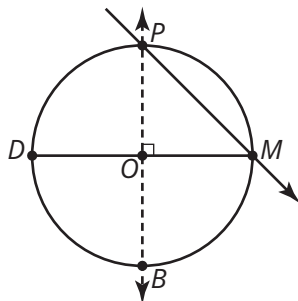
The angles of a linear pair have measures whose sum is 180° .

.....

2. Consider the central angles MOP and POD and their arcs MP and PD .

- What do you know about the sum of $m\angle MOP + m\angle POD$?
- What can you conjecture about the sum of $m\widehat{MP} + m\widehat{PD}$?

3. Construct $\overleftrightarrow{EL}$ through the center of the circle parallel to $\overleftrightarrow{PM}$, so that $\angle EOP$ and $\angle LOM$ are both acute angles. Then, make conjectures about each angle pair.



- $\angle PMO$ and $\angle EOD$
- $\angle MPO$ and $\angle POE$
- Make a conjecture about another angle pair that you know something about.



3. Hannah conjectured that the measure of an inscribed angle is equal to the measure of a central angle when both angles intercept the same arc.

Do you agree or disagree with Hannah's conjecture? Draw examples to justify your answer.

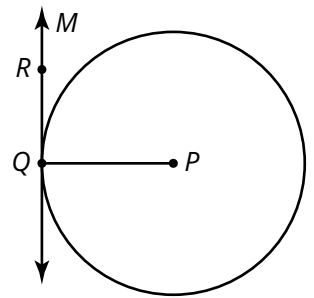
4. Consider the measures of inscribed angles that intercept the same arc. What conjecture can you make about this relationship? Use examples to explain your reasoning.



5. Nicky measured each of the inscribed angles that intercept a semicircle. He conjectured that the measure of any inscribed angle that intercepts a semicircle arc is equal to 90° .

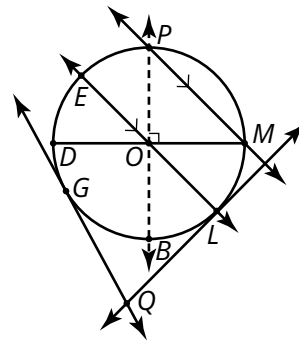
Do you agree or disagree with Nicky's conjecture? Draw examples to justify your answer.

A **tangent** to a circle is a line that intersects a circle in exactly one point, called the *point of tangency*.

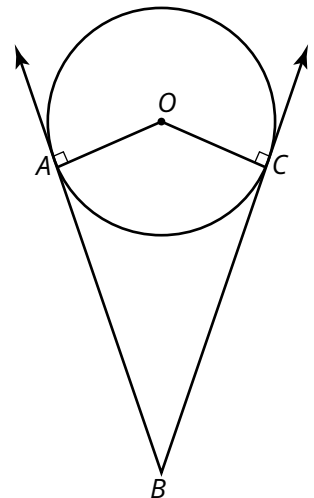


Tad drew two *tangent* lines to the circle, one through point L and one through point G to form a *circumscribed angle*, $\angle GQL$.

6. What conjecture can you make about the measures of angles formed by tangent lines and the radii of a circle? Draw examples to explain your reasoning.



A **circumscribed angle** has its two sides tangent to the circle.





Talk the Talk

Draw Whatcha Know

You have explored and measured different relationships to make and analyze several conjectures throughout this lesson. In mathematics, you often have to prove a solution is correct. Remember that in geometry, theorems are used to verify statements. A theorem is a statement that can be proven true using definitions, postulates, or other theorems. A postulate is a mathematical statement that is not proven but is considered true.

The table shows the conjectures that were made in this lesson. You will prove these as theorems in upcoming lessons.

1. Based on your explorations in this lesson, draw an example of each conjecture.

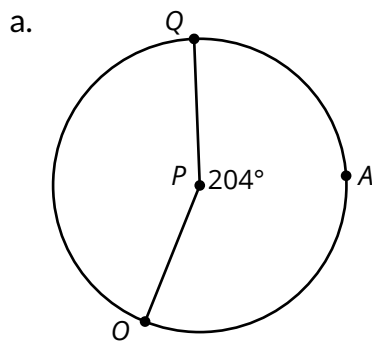
a. All circles are similar.	
b. Vertical angles are congruent.	
c. When two adjacent angles have noncommon sides that are opposite rays, the sum of the angles is 180°	

d. When the measures of two arcs x and y are added, the sum is the measure of another arc, z .	
e. When two parallel lines are crossed by a transversal, alternate interior angles are congruent.	
f. When two parallel lines are crossed by a transversal, corresponding angles are congruent.	
g. When an inscribed angle intercepts the same arc as a central angle, the inscribed angle has half the measure of the central angle.	
h. Two inscribed angles that intercept the same arc have the same measure.	

- i. When an inscribed angle intercepts a semicircle arc, the inscribed angle measures 90° .

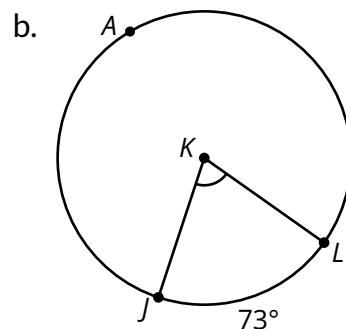
- j. The angle formed by a tangent line and a radius of a circle is 90° .

2. Determine the measure of each central angle or arc. Identify whether the arc is a major arc or a minor arc.



$$m\widehat{OAQ} =$$

$$m\widehat{OQ} =$$



$$m\angle JKL =$$

$$m\widehat{JAL} =$$

Lesson 1 Assignment

Write

Match each term with the best description.

- | | |
|------------------------|--|
| 1. secant | a. a line segment from the center of a circle to a point on the circle |
| 2. major arc | b. a line that intersects a circle at exactly one point |
| 3. minor arc | c. an angle that has two sides tangent to a circle |
| 4. radius | d. an angle with a vertex that is on a circle and sides that contain chords and/or secants of the circle |
| 5. inscribed angle | e. the arc with the greater measure |
| 6. intercepted arc | f. an angle with its vertex at the center of a circle |
| 7. tangent | g. a line segment that passes through the center of a circle with endpoints on the circle |
| 8. circumscribed angle | h. a line that passes through two points on a circle |
| 9. diameter | i. the arc of a circle with endpoints that are intersected by two rays of an angle |
| 10. central angle | j. the arc with the lesser measure |

Remember

Circles can be helpful in constructing geometric figures in order to make conjectures about line and angle relationships.

When you conjecture, you use what you know through experience and reasoning to presume that something is true. The proven statement of a conjecture is called a theorem.

Lesson 1 Assignment

Practice

1. Write a conjecture about each geometric object described. Draw examples to test your conjecture.

a. Vertical angles

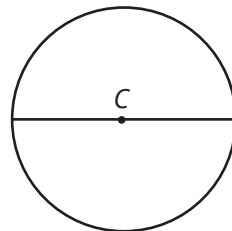
b. Points on the perpendicular bisector of a line segment

c. Inscribed angles that intercept the same arc of a circle

d. Tangent

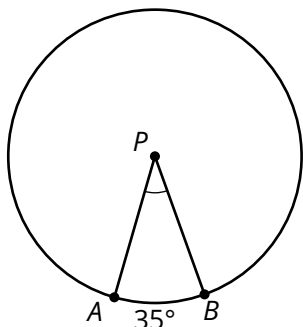
Lesson 1 Assignment

2. Draw examples of inscribed angles that intercept the diameter of the circle. What conjecture can you make about the measure of the inscribed angle?



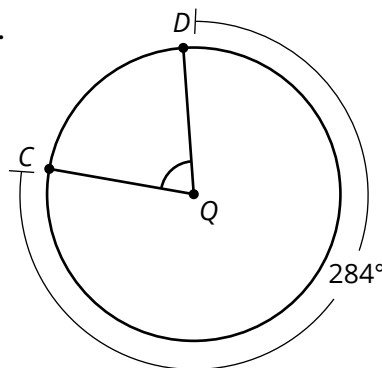
3. Determine the measure of each central angle.

a.



$$m\angle APB =$$

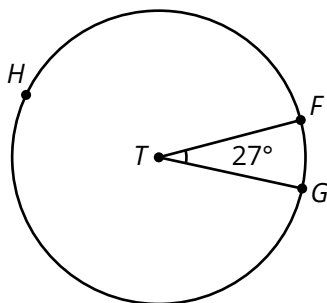
b.



$$m\angle CQD =$$

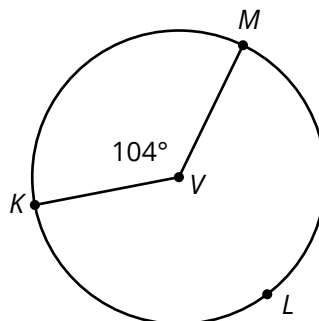
4. Determine each arc measure. Determine whether the arc is a minor arc or major arc.

a.



$$m\widehat{FG} =$$

b.



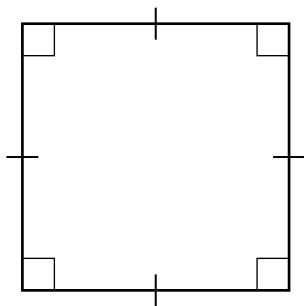
$$m\widehat{KLM} =$$

Lesson 1 Assignment

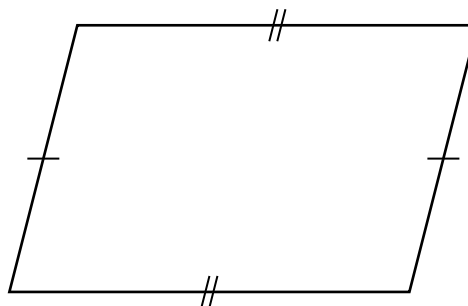
Prepare

Classify each figure using as many names as possible.

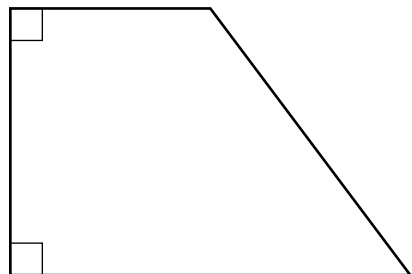
1.



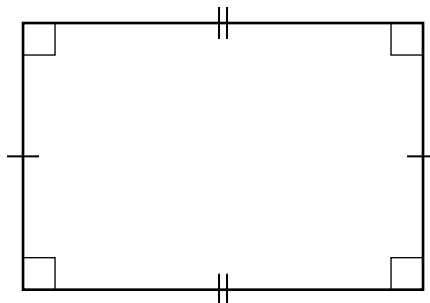
2.



3.



4.



2

Conjectures About Quadrilaterals

OBJECTIVES

- Use diagonals to draw quadrilaterals.
- Make conjectures about the diagonals of special quadrilaterals.
- Make conjectures about the angle relationships of special quadrilaterals.
- Categorize quadrilaterals based upon their properties.
- Make conjectures about the midsegments of quadrilaterals.
- Understand through listening and reading that the vertices of inscribed quadrilaterals lie on the same circle.

.....

You have classified quadrilaterals by their side measurements and side relationships.

What conjectures can you make about different properties of quadrilaterals?

KEY TERMS

- coincident
- interior angle of a polygon
- kite
- isosceles trapezoid
- midsegment
- quadrilateral inscribed in a circle
- convex polygon
- concave polygon

Getting Started

Cattywampus

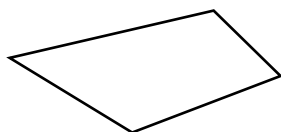
Ask yourself . . .

Why can a concave quadrilateral have only one angle greater than 180° ?

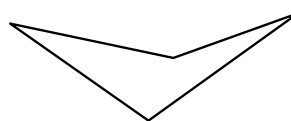
A quadrilateral may be convex or concave. The quadrilaterals you are most familiar with—trapezoids, parallelograms, rectangles, rhombi, and squares—are convex. A **convex polygon** contains all of the line segments connecting any pair of points. It is a **concave polygon** if and only if at least one of its interior angles is greater than 180° .

Consider the two quadrilaterals shown. A quadrilateral has exactly two diagonals.

Convex



Concave



1. Draw the diagonals in the two quadrilaterals shown. What do you notice?
2. Make a conjecture about the diagonals of a convex quadrilateral and about the diagonals of a concave quadrilateral.

The diagonals of any convex quadrilateral create two pairs of vertical angles and four linear pairs of angles.

3. Label the vertices of the convex quadrilateral as well as the point of intersection of the diagonals. Identify each pair of vertical angles and each linear pair of angles. State the relationship for each pair of angles you identified.

ACTIVITY 2.1

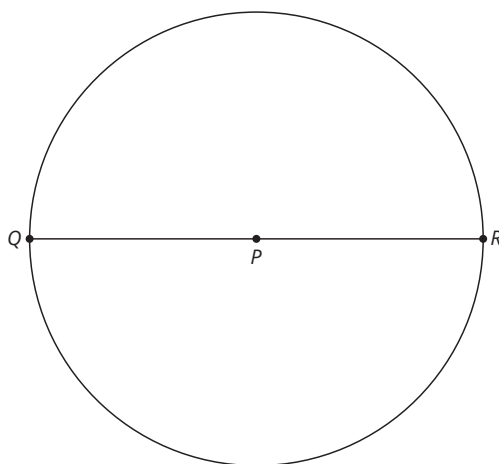
Quadrilaterals Formed Using a Circle

Let's explore the diagonals of different convex quadrilaterals. Circle P with diameter $\overline{QR}$ is shown.

Use a new piece of patty paper for each quadrilateral. For precision, it is important to use a straightedge when tracing or drawing line segments.

.....
Two line segments are **coincident** when they lie exactly on top of each other.
.....

1. Draw quadrilateral $QSRT$ by following these steps.
 - Construct the perpendicular bisector to $\overline{QR}$ through point P .
 - Use patty paper to trace $\overline{QR}$.
 - Without moving the patty paper, draw diameter $\overline{ST}$ of circle P in such a way that it is not perpendicular or *coincident* to $\overline{QR}$.
 - Connect the endpoints of the diameters to draw quadrilateral $QSRT$.



- a. What do you know about $\overline{QR}$ and $\overline{ST}$?

Ask Yourself ...

Are both pairs of opposite sides of the quadrilateral parallel?

- b. Use a ruler to measure the lengths of each side of the quadrilateral. Make a conjecture about the side lengths of the quadrilateral.

- c. Use a protractor to determine the measure of each interior angle of the quadrilateral. Which conjecture that you made in the previous lesson does this measurement support?

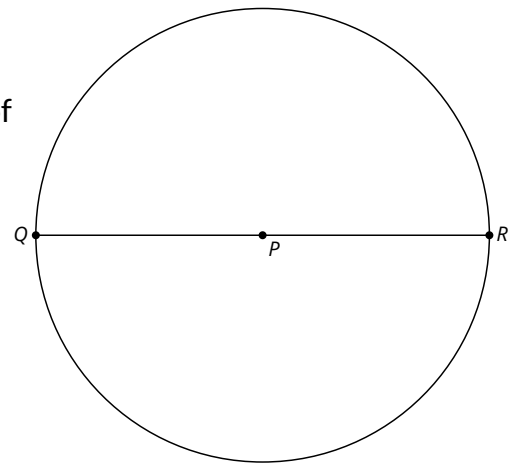
.....
An **interior angle of a polygon** is an angle inside the polygon between two adjacent sides.
.....

- d. What names can be used to describe quadrilateral $QSRT$? Explain your reasoning. Write the most specific name for the quadrilateral on the patty paper.

.....
Keep this quadrilateral to compare with other quadrilaterals that you will create in this lesson.
.....

2. Draw quadrilateral $QVRW$ by following these steps.

- Use patty paper to trace $\overline{QR}$.
- Without moving the patty paper, draw diameter $\overline{VW}$ of circle P perpendicular to $\overline{QR}$.
- Connect the endpoints of the diameters to draw quadrilateral $QVRW$.



- a. What do you know about $\overline{QR}$ and $\overline{VW}$?

- b. Use a ruler to measure the lengths of each side of the quadrilateral. Make a conjecture about the side lengths of the quadrilateral?

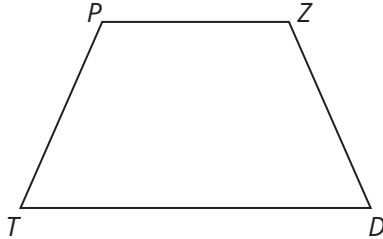
- c. Use a protractor to determine the measure of each interior angle of the quadrilateral. Does this measurement support the same conjecture from Question 1? Explain your reasoning.

- d. What names can be used to describe quadrilateral QVRW? Explain your reasoning. Write the most specific name for the quadrilateral on the patty paper.

A trapezoid is a quadrilateral with only one pair of parallel sides, known as the bases. The non-parallel sides are known as the legs. A special type of trapezoid is an *isosceles trapezoid*. An **isosceles trapezoid** is a trapezoid with congruent legs.

3. Consider the isosceles trapezoid shown.

- a. Use a piece of patty paper to trace the trapezoid and draw diagonals $\overline{TZ}$ and $\overline{PD}$.



- b. Use a ruler to measure $\overline{TZ}$ and $\overline{PD}$. What do you notice?
- c. Do $\overline{TZ}$ and $\overline{PD}$ bisect each other? Explain how you determined your answer.

.....
Save the quadrilaterals you drew on your pieces of patty paper. You'll need them later.
.....

- d. Use a protractor to determine the measure of each interior angle of the quadrilateral. Make a conjecture about the interior angles of the quadrilateral.

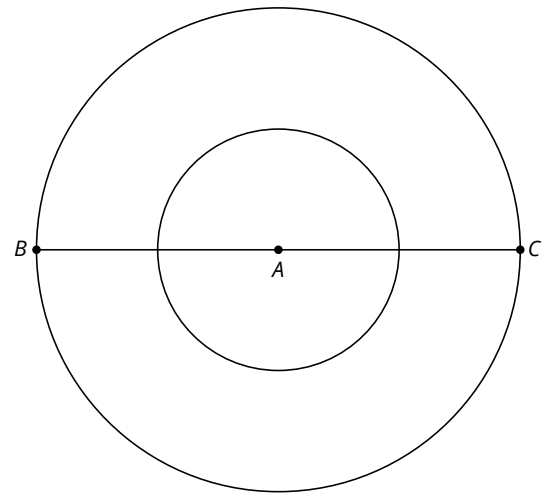
ACTIVITY
2.2**Quadrilaterals Formed Using Concentric Circles**

Now let's draw quadrilaterals using concentric circles. Consider a pair of concentric circles with center A . Diameter $\overline{BC}$ is shown.

Use a new piece of patty paper for each quadrilateral. For precision, it is important to use a straightedge when tracing or drawing line segments.

1. Draw quadrilateral $BDCE$ by following these steps.

- Construct the perpendicular bisector to $\overline{BC}$ through point A of the concentric circles.
- Use patty paper to trace $\overline{BC}$.
- Without moving the patty paper, draw diameter $\overline{DE}$ of the inner circle in such a way that it is not perpendicular or *coincident* to $\overline{BC}$.
- Connect the endpoints of the diameters to draw quadrilateral $BDCE$.

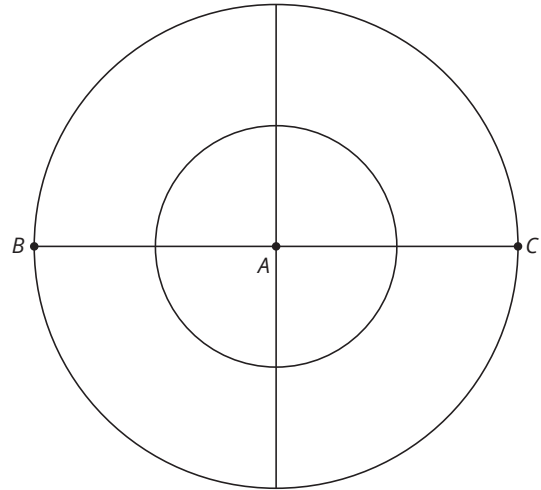


- a. What do you know about $\overline{BC}$ and $\overline{DE}$?
- b. Use a ruler to measure the side lengths of each side of the quadrilateral. Make a conjecture about the side lengths of a quadrilateral.
- c. Use a protractor to determine the measure of each interior angle of the quadrilateral. Make a conjecture about the interior angles of the quadrilateral.

- d. What do you know about $\overline{BE}$ and $\overline{DC}$?
- e. What names can be used to describe quadrilateral $BDCE$?
Explain your reasoning. Write the name of the quadrilateral on the patty paper.

2. Draw quadrilateral $BFCG$ by following these steps.

- Use patty paper to trace $\overline{BC}$.
- Without moving the patty paper, draw diameter $\overline{FG}$ of the inner circle perpendicular to $\overline{BC}$.
- Connect the endpoints of the diameters to draw quadrilateral $BFCG$.



- a. What do you know about $\overline{BC}$ and $\overline{FG}$?
- b. Use a ruler to measure the side lengths of each side of the quadrilateral. Make a conjecture about the side lengths of a quadrilateral
- c. Use a protractor to determine the measure of each interior angle of the quadrilateral. Make a conjecture about the interior angles of the quadrilateral.

- d. What names can be used to describe quadrilateral $BFCG$?
Explain your reasoning. Write the name of the quadrilateral on the patty paper.

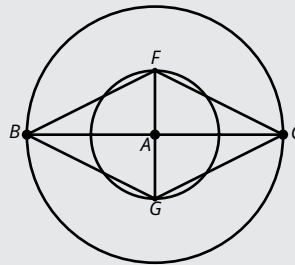
.....
Think About...

How can you use same-side interior angles to help determine the name of the quadrilateral?
.....

3. Consider William's statement. What error did William make?

William

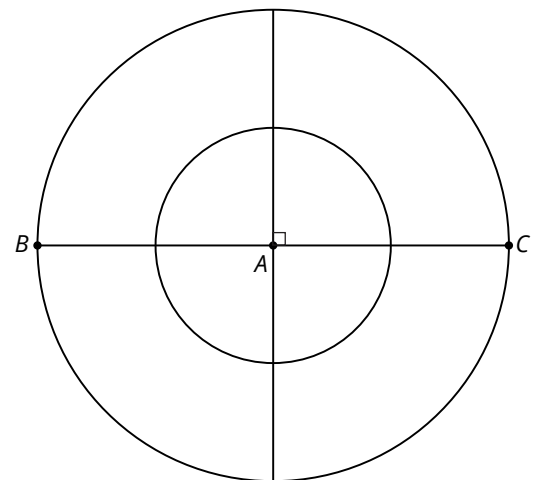
In the quadrilateral $BFCG$ that I drew, $\angle BFC$ and $\angle CGB$ are circumscribed angles.



4. Draw quadrilateral $BHCJ$ by following these steps.

- Use patty paper to trace $\overline{BC}$.
- Without moving the patty paper, draw point H where the perpendicular bisector intersects the inner circle and point J where it intersects the outer circle, such that $\overline{HJ}$ contains point A .
- Connect the points to draw $\overline{HJ}$.
- Connect the endpoints of the line segments to create quadrilateral $BHCJ$.

- a. What do you know about $\overline{BC}$ and $\overline{HJ}$?

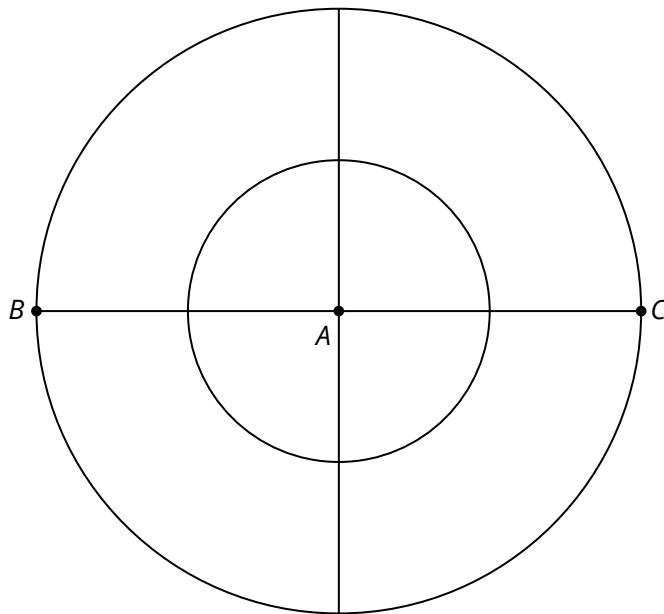


- b. Use a ruler to measure the side lengths of each side of the quadrilateral. Make a conjecture about the side lengths of a quadrilateral.

- c. Use a protractor to determine the measure of each interior angle of the quadrilateral. Make a conjecture about the interior angles of the quadrilateral.

The quadrilateral you drew is a *kite*. A **kite** is a quadrilateral with two pairs of equal adjacent sides. If the diagonals of a quadrilateral are perpendicular, non-congruent, and only one bisects the other, it can only be classified as a kite.

- 5. Write the name of the quadrilateral on the patty paper.



ACTIVITY
2.3

Making Conjectures About Quadrilaterals

In the previous two activities, you used the properties of the diagonals to discover each member of the quadrilateral family. You investigated the relationships between the diagonals of quadrilaterals.

1. Make a conjecture about the diagonals of the described quadrilaterals. Explain your reasoning using examples.
 - a. parallelograms
 - b. rectangles
 - c. kites

You also investigated the relationships between the interior angles of quadrilaterals.

2. Make a conjecture about the interior angles of rectangles. Explain your reasoning using examples.



3. Huyen observed the relationship between the interior angles of the isosceles trapezoid. She conjectured that trapezoids have two pairs of congruent angles that are adjacent to each other.

Do you think Huyen's conjecture is correct? Draw examples to justify your answer.

4. Determine whether each statement is true or false. If it is false, explain why.
- a. A square is also a rectangle.
 - b. A rectangle is also a square.
 - c. A parallelogram is also a trapezoid.
 - d. The diagonals of a trapezoid are congruent.
 - e. A kite is also a parallelogram.
 - f. The diagonals of a rhombus bisect each other.

Knowing certain properties of each quadrilateral makes it possible to construct the quadrilateral given only a single diagonal.

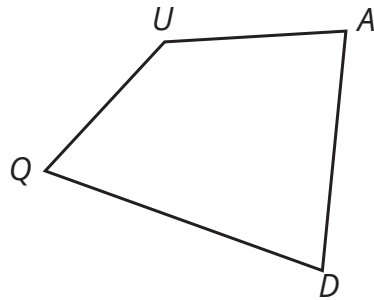
5. Describe how you could construct each named quadrilateral with the given diagonal.
- a. Parallelogram $WXYZ$ given diagonal $\overline{WY}$

b. Rhombus $RHOM$ given diagonal $\overline{RO}$

c. Kite $KITE$ given diagonal $\overline{KT}$

Midsegments of Quadrilaterals

Consider quadrilateral $QUAD$.



Remember...

Not all points on a segment bisector are midpoints. The intersection of the segment bisector and the segment is the midpoint.

1. Construct the segment bisector of each side of quadrilateral $QUAD$ and label each midpoint.
2. Draw the midsegments that connect the midpoints of adjacent sides. What shape does it appear you drew using the midsegments of quadrilateral $QUAD$?

Revisit the quadrilaterals you drew on patty paper in this lesson.

3. Draw the midsegments of each adjacent side for each of the quadrilaterals you drew on patty paper. Make a conjecture about the figure created by connecting adjacent midpoints of a quadrilateral.

4. Does your conjecture hold true for concave quadrilaterals? Use a drawing to justify your answer.

Let's explore one of the midsegments of a trapezoid.

5. Use the given parallel line segments and a straightedge to complete this question.



- Draw any trapezoid. Label the vertices.
- Construct the midsegment of your trapezoid that connects the non-parallel sides. Label the endpoints.
- Use your ruler to determine the length of each base and the length of the midsegment.
- Determine the average of the lengths of the bases. What do you notice?

.....

Remember...

A trapezoid is a quadrilateral with exactly **one** pair of parallel sides.

.....

- e. Compare your answer to part (d) with those of your classmates. Make a conjecture about the midsegment of a trapezoid that connects non-parallel sides.

6. How many possible midsegments are there for any quadrilateral?

ACTIVITY
2.5

Inscribed Quadrilaterals

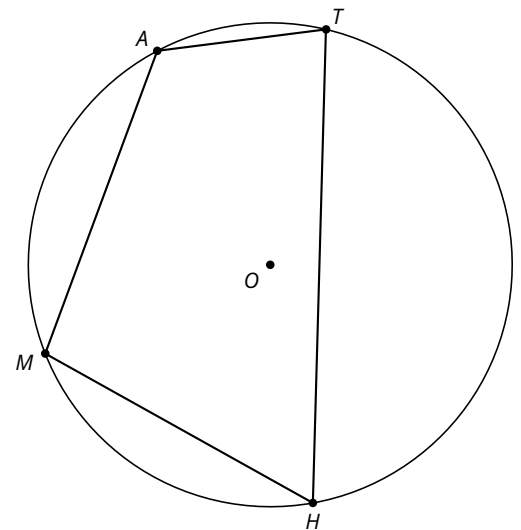
A **quadrilateral inscribed in a circle** is a quadrilateral whose vertices all lie on a single circle. Some of the quadrilaterals you drew in this lesson are inscribed quadrilaterals.

1. Identify any inscribed quadrilaterals you drew on your pieces of patty paper.
2. What is the sum of the measures of the opposite angles in the inscribed quadrilaterals you identified?

.....
Inscribed
quadrilaterals are
also called cyclic
quadrilaterals.
.....

Consider quadrilateral *MATH* inscribed in circle *O*.

3. Use a protractor to determine the measure of each interior angle of quadrilateral *MATH*. Write the measurements in the diagram.



4. Determine each sum.
 - a. $m\angle M + m\angle T$
 - b. $m\angle A + m\angle H$
5. Make a conjecture about the measures of opposite angles of an inscribed quadrilateral.
6. Could you inscribe the isosceles trapezoid you drew on your piece of patty paper in Activity 1 in a circle? Explain your reasoning.



Talk the Talk

Identifying the Properties of Quadrilaterals

Use the quadrilaterals you drew on your pieces of patty paper and the observations you made to complete the table.

1. Identify the properties that are always true for the given quadrilateral by placing an X in the appropriate box.

Side Relationships	Property	Parallelogram	Rectangle	Rhombus	Square	Isosceles Trapezoid	Kite
	a. Opposite sides are parallel.						
	b. Only one pair of opposite sides is parallel.						
	c. Opposite sides are congruent.						
	d. Only one pair of opposite sides is congruent.						
	e. All sides are congruent.						
	f. 2 pairs of consecutive sides are congruent.						

	Property	Parallelogram	Rectangle	Rhombus	Square	Isosceles Trapezoid	Kite
Angle Relationships	g. Opposite angles are congruent.						
	h. At least one pair of opposite angles is congruent.						
	i. All angles are right angles.						
Diagonal Relationships	j. Diagonals bisect each other.						
	k. Diagonals are perpendicular.						
	l. Diagonals are congruent.						

2. Use the given parallel line segments and a straightedge to complete this question.



- Draw any trapezoid. Label the vertices.
- Construct the midsegment of your trapezoid that connects the non-parallel sides. Label the endpoints.
- Use your ruler to determine the length of each base and the length of the midsegment.
- Determine the average of the lengths of the bases. Is the conjecture you made previously about the midsegment of a trapezoid that connects the midpoints of non-parallel sides still true when the orientation of the trapezoid is vertical instead of horizontal? Explain your reasoning.

Lesson 2 Assignment

Write

Define each term in your own words. Use the words *diagonal*, *interior angle*, and *midsegment* in your definitions.

1. kite
2. isosceles trapezoid
3. a quadrilateral inscribed in a circle

Remember

The diagonals of a parallelogram bisect each other and the diagonals of a rectangle are congruent. A square, rhombus, and kite have perpendicular diagonals.

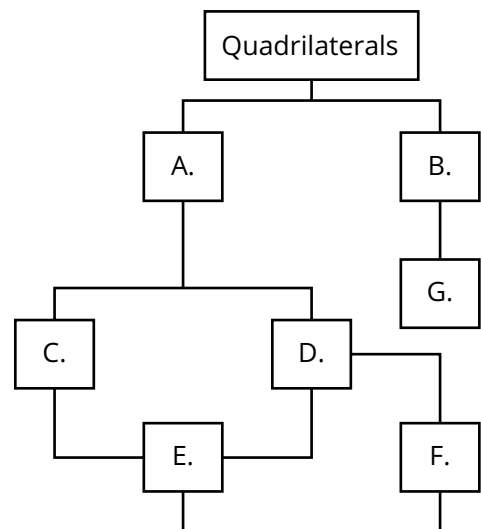
The opposite angles of parallelograms are congruent and the opposite angles of inscribed quadrilaterals are supplementary.

Practice

1. Determine which quadrilateral each letter in the diagram represents using the list shown.

Kites
Rectangles
Rhombi

Squares
Parallelograms
Trapezoid



2. State as many properties as you can about each quadrilateral.

a. Rectangle

.....

b. Isosceles trapezoid

Lesson 2 Assignment

c. Kite

d. Parallelogram

e. Rhombus

f. Square

3. Describe how to construct each quadrilateral using the given diagonal.

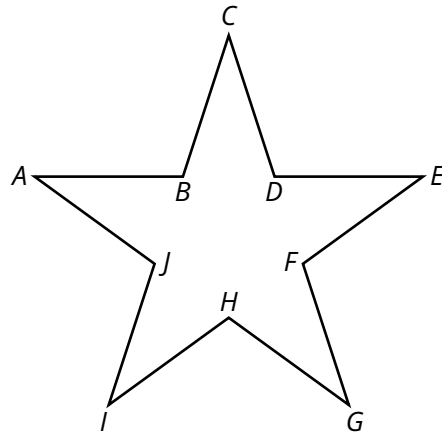
a. Square $WXYZ$ given diagonal $\overline{WY}$

b. Parallelogram $RSTU$ that is non-rectangular given diagonal $\overline{RT}$

Lesson 2 Assignment

Prepare

Use patty paper to identify a geometric figure that is congruent to each figure in the diagram. Describe the transformation(s) you used.



1. $\overline{AB}$

2. $\angle FGH$

3. $\angle AJI$

3

Constructing an Inscribed Regular Polygon

OBJECTIVES

- Write an explanation of how to use construction tools to create a congruent angle.
- Construct a regular hexagon inscribed in a circle using a variety of strategies.
- Construct a square inscribed in a circle using patty paper and using a compass and straightedge.
- Construct an equilateral triangle using a compass and straightedge.
- Construct an equilateral triangle inscribed in a circle.

KEY TERM

- inscribed polygon

.....

You have used construction tools to construct figures in a plane.

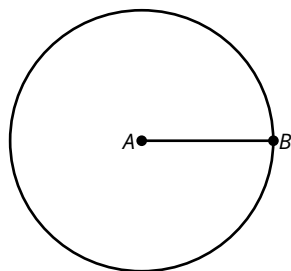
How can you construct a regular polygon inscribed in a circle?

Getting Started

Duped

.....
When you duplicate
a segment you
construct a segment
congruent to the
original segment.
.....

You can use your construction skills to duplicate a line segment inside a circle.

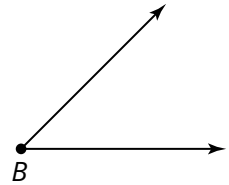


1. Consider circle A.
 - a. Construct $\overline{BC}$ congruent to $\overline{AB}$ in circle A so that point C lies on circle A.
 - b. Connect points A and C. What do you know about the length of $\overline{AC}$?
 - c. What type of triangle is $\triangle ABC$? Explain your reasoning.
 - d. What do you know about the angle measures of $\triangle ABC$?

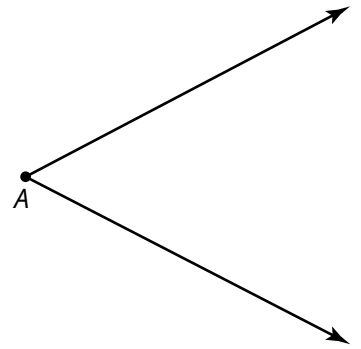
Duplicating an Angle

Not only can you duplicate a line segment using construction, remember that you can also duplicate an angle.

1. Construct $\angle C$ congruent to $\angle B$.

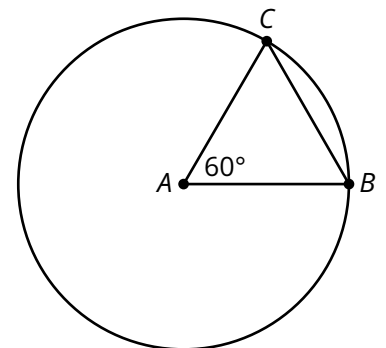


2. Construct an angle that is twice the measure of $\angle A$. Then explain how you performed the construction.



3. How is duplicating an angle similar to duplicating a line segment?
How is it different?

4. In the Getting Started, you constructed an equilateral triangle ABC . Duplicate the 60° angle shown to construct 5 additional equilateral triangles that share the center of the circle as a vertex. What do you notice about the polygon you have constructed?



Constructing an Inscribed Hexagon

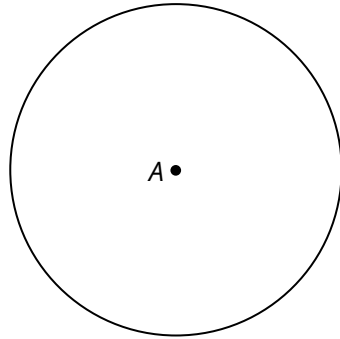
.....

An **inscribed polygon** is a polygon drawn inside another polygon or circle in which all the vertices of the interior polygon lie on the outer figure.

.....

In the Getting Started, you created an equilateral triangle by duplicating the length of the radius to create a chord. You can use what you know about duplicating a line segment to construct a regular hexagon inscribed in a circle using a compass and straightedge.

1. Use circle A to complete your construction.



- a. Draw and duplicate the radius of circle A.
- b. Create 6 chords around the circumference of circle A so that they are congruent to the radius connecting the endpoints of each consecutive chord. Explain your process.

.....

Remember ...

A regular polygon is a polygon with all sides congruent and all angles congruent.

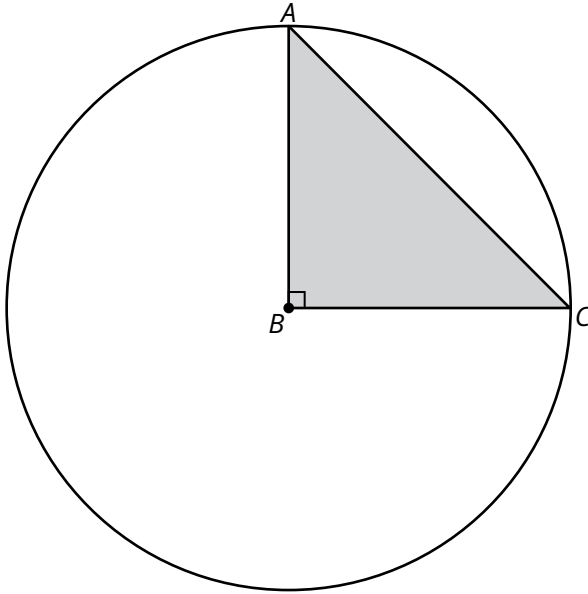
.....

The figure you constructed is a regular hexagon inscribed in a circle.

2. How can you duplicate an angle to verify that the inscribed hexagon is a regular hexagon?
3. What do you know about the measures of the interior angles of a regular hexagon?

Constructing an Inscribed Square

In a previous activity, you constructed a square given a side length. Let's now consider how to inscribe a square inside a circle. One way to create a square is to use transformations.



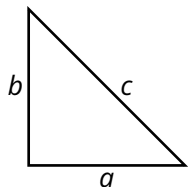
1. Circle B contains a right triangle, $\triangle ABC$, with points A and C on the circle and point B at the center. Describe each element using a term associated with a circle.
 - a. $\angle B$
 - b. $\overline{BC}$ and $\overline{BA}$
 - c. $\overline{AC}$

2. Use patty paper to rotate the right triangle to create square $ACDE$. How do you know all 4 vertices lie on the circle?

.....

Remember ...

Recall that the Pythagorean Theorem says that $a^2 + b^2 = c^2$.



The converse of the Pythagorean Theorem says that if $a^2 + b^2 = c^2$, then the triangle is a right triangle.

.....

3. Verify that you created a square inscribed in circle B . To do this, you need to show that all side lengths of the square are equal and each interior angle measures 90° .

a. Show that all side lengths are equal. Suppose $BC = x$. Determine each side length of the square. Label each length.

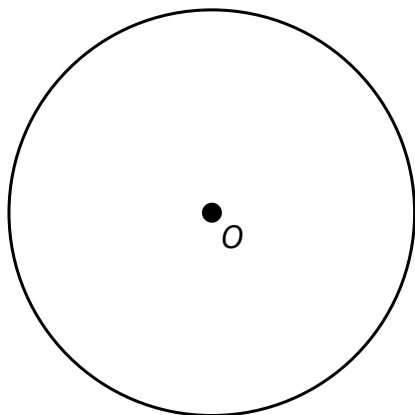
b. Verify that each interior angle measure of the square is 90° .

4. Consider $\overline{AD}$ and $\overline{CE}$.

a. Describe each line segment in terms of the circle and in terms of the square.

b. Describe the relationship between $\overline{AD}$ and $\overline{CE}$.

5. Use diagonals to construct a square inscribed in circle O. Describe your process.



Constructing and Inscribing an Equilateral Triangle

In this activity, you will construct an equilateral triangle. To perform the construction, you will use only a compass and straightedge and rely on the basic geometric constructions you have learned, such as duplicating a line segment.



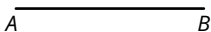
1. Maria claims that she can construct an equilateral triangle by constructing and then duplicating a line segment three times using a given segment and having the endpoints of all three line segments intersect. William thinks that Maria's method will not result in an equilateral triangle.

Is Maria correct? Explain your reasoning.

Remember ...

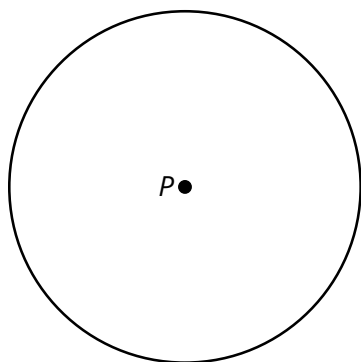
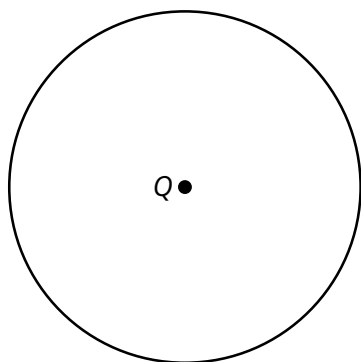
An equilateral triangle has three congruent sides and three congruent angles.

2. Construct an equilateral triangle using the given side length. Then, describe the steps you performed for the construction.

A  B

An inscribed equilateral triangle is an equilateral triangle inside a circle with all of its vertices touching the circle.

3. Show two different ways to construct an inscribed equilateral triangle using each circle given. Explain your process for each construction.



.....
Think About...

How can you use the process for constructing an inscribed regular hexagon to inscribe an equilateral triangle?

.....



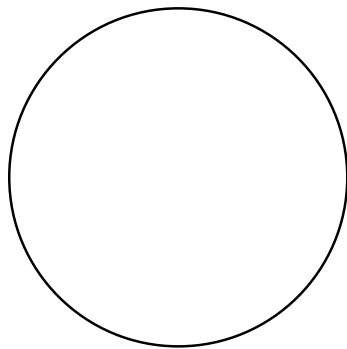
Talk the Talk

Playing the Angles

1. Use the constructions you have learned to construct an angle with a measure of 75° using only a compass and straightedge. Summarize your process.

You have constructed a regular quadrilateral (square), a regular hexagon, and a regular triangle (equilateral triangle) inscribed in a circle. What other regular polygons can you inscribe in a circle?

2. Inscribe a regular octagon in the circle provided, if possible.



Lesson 3 Assignment

Write

Explain how to construct a congruent angle using a compass and straightedge.

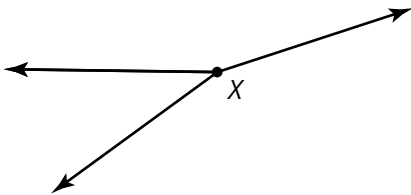
Remember

All regular polygons can be inscribed in a circle. Some regular polygons, such as an equilateral triangle, square, hexagon, and octagon, can be inscribed in a circle using a compass and straightedge.

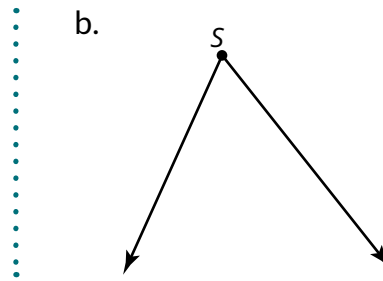
Practice

1. Duplicate each angle using construction tools.

a.



b.



Lesson 3 Assignment

2. Inscribe a regular hexagon inside a circle. Explain your process.

3. Construct a square inscribed in a circle. Then, explain how you know the figure is a square and how you know it is inscribed.

Lesson 3 Assignment

4. Construct a 15° angle using only a compass and straightedge. Summarize your steps.

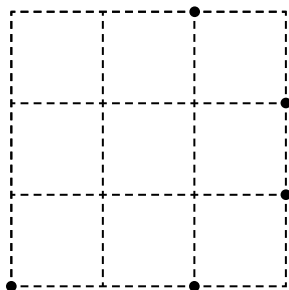
5. Inscribe an equilateral triangle inside a circle. Explain your process.

Lesson 3 Assignment

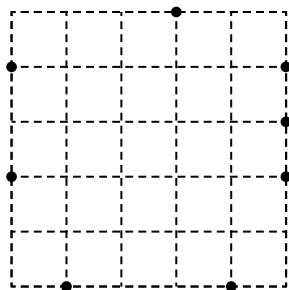
Prepare

The points represent possible vertices of a polygon. Identify and connect the vertices that form each given figure.

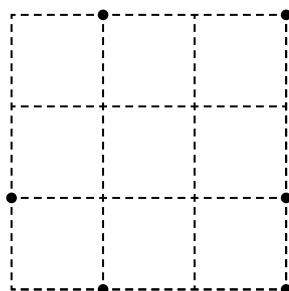
1. right triangle



2. isosceles triangle



3. isosceles right triangle



4

Conjectures About Triangles

OBJECTIVES

- Construct circles, parallel lines, and perpendicular bisectors.
- Identify and draw exterior angles of triangles and midsegments of triangles.
- Make and express conjectures - both verbally and in writing, about the sum of the interior angles of a triangle, exterior angles of a triangle, triangle side length inequalities, and midsegments of a triangle.
- Apply the Triangle Inequality theorem to determine whether three side lengths form a triangle.

KEY TERMS

- base angles
- exterior angle
- remote interior angles
- Triangle Inequality theorem

.....

You have made conjectures about line and angle relationships and about quadrilaterals using circles.

What conjectures can you make about triangles using circles?

Getting Started

Not an Illusion

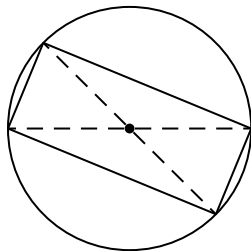
Previously, you created quadrilaterals and made conjectures about them. When you divide a quadrilateral along any one of its diagonals, you create two triangles.

Ask Yourself . . .

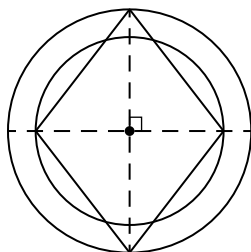
How can you use your conjectures from the previous lesson about the properties of quadrilaterals to describe the triangles formed?

1. Identify each quadrilateral. Then, classify the triangles formed by dividing the quadrilateral using each diagonal. Explain your reasoning.

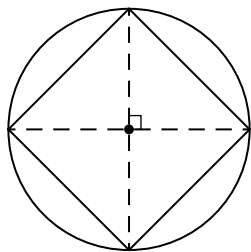
a.



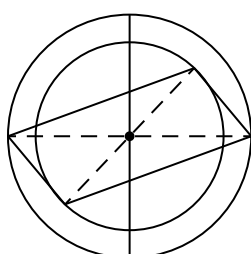
b.



c.



d.



ACTIVITY
4.1

Base Angles, Angle Pairs, and Converses

Previously, you made conjectures like this:

“If two parallel lines are cut by a transversal, then the corresponding angles are congruent.”

Remember, you can also make conjectures about the converse of a if-then statement. The converse is the statement that results from interchanging the hypothesis (the part after the “if”) and the conclusion (the part after the “then”) of the original statement.

1. Write the converse of the conjecture stated above about parallel lines and corresponding angles.
2. What is the validity of both the conditional statement and the converse? Explain your reasoning.

.....
Remember ...

You can also refer to an if-then statement as a conditional statement.

.....

Ask Yourself . . .

Are all conditional statements and their converses are biconditional statements?

Remember, when a conditional statement and its converse are both true, this is called a biconditional statement. A biconditional statement is a statement written in the form " p if and only if q ". It is a combination of both a conditional statement and the converse of that conditional statement.

3. Consider the conditional statement: If Ana goes out in the rain without an umbrella, she will get wet.

Write the converse and determine whether this is a biconditional statement.

.....

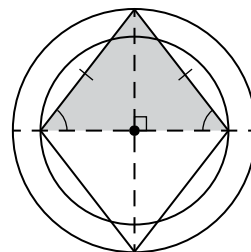
The congruent sides of an isosceles triangle are called the legs. The other side is called the base, and the angles between the base and the congruent sides are called **base angles**.

.....



4. Jacob observed that the angles opposite the congruent sides of an isosceles triangle are congruent. He conjectured that if two sides of a triangle are congruent, then the angles opposite the congruent sides are always congruent.

Do you think this conjecture is correct? Create isosceles triangles and use a protractor to measure angles to test Jacob's conjecture.



5. Write the converse of Jacob's conjecture. Do you think the converse is true? How could you test the converse conjecture?
6. Do you think you can write Jacob's conjecture as a biconditional statement? If so, write it as a biconditional statement. If not, explain your reasoning.

Testing a conjecture and proving it to be a theorem are not the same thing. Remember, a theorem has been proved true in all cases. A proof requires reasoning using definitions and theorems, whereas testing involves making observations, noting counterexamples, and using measuring tools.

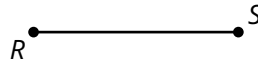
Triangle Sum and Exterior Angles

Remember...

A conjecture is a statement that is believed to be true but is not yet proven.

Let's construct a new diagram, investigate the relationships, and make a few more conjectures about triangles.

1. Construct circle R and circle S , each with radius $\overline{RS}$. Label the two points of intersection as point T and point V .



2. Draw line segments to form $\triangle RST$. Explain why the triangle you created is an equilateral triangle. What does this tell you about the relationship between point T and the endpoints of $\overline{RS}$?
3. Consider the conditional statement: If a point is on the perpendicular bisector of a line segment, then it is equidistant from the endpoints of the line segment.
 - a. Write the converse of the conditional statement.
 - b. Construct the perpendicular bisector of $\overline{RS}$. Label the point where the perpendicular bisector intersects the line segment as point Q .

- c. Determine the validity of the converse. If the converse is false, provide a counterexample. If it is true, use the converse to draw a conclusion about points T and V ?

4. Consider the angles formed with the construction of a parallel line.

- a. Construct $\overline{AR}$ parallel to $\overline{ST}$. Then, make a conjecture about the angles formed.

- b. Consider the central angles $\angle TRS$, $\angle TRA$, and $\angle ARB$ and their intercepted arcs: $\widehat{TS}$, $\widehat{TA}$, and $\widehat{AB}$. What is the sum of the measures of these central angles? What is the sum of the measures of the corresponding intercepted arcs?

- c. Use your construction to make a conjecture about the sum of the measures of the interior angles of a triangle? Explain your reasoning.

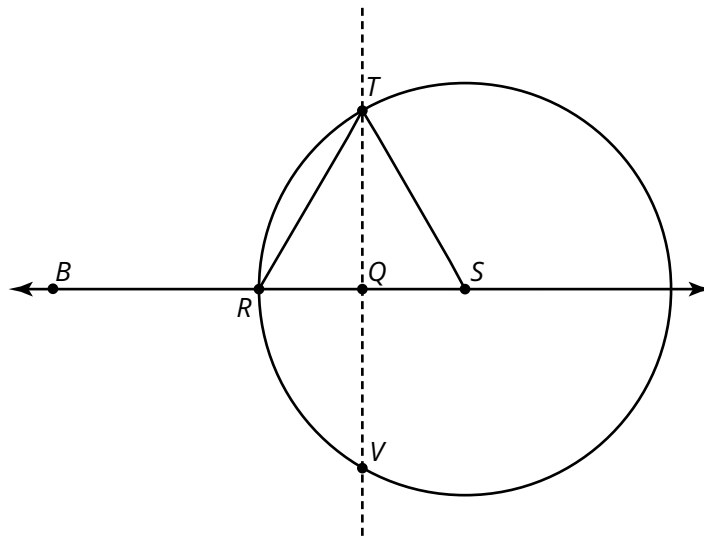
Ask Yourself . . .

What conjectures did you use to describe the angle and arc additions?

.....
Review the definition of **explain your reasoning** in the Academic Glossary.
.....

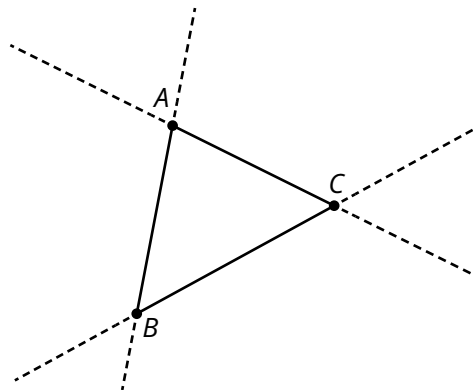
.....
 An **exterior angle** of a polygon is an angle that forms a linear pair with an interior angle of the polygon. The **remote interior angles** of a triangle are the two angles that are not adjacent to the specified exterior angle.

A portion of the diagram you used for your construction is shown. Consider $\triangle RST$. $\angle TRB$ is an *exterior angle* and $\angle RST$ and $\angle STR$ are *remote interior angles* of $\triangle RST$.



5. What conjecture can you make about the measures of the exterior angle, $\angle TRB$, and the two remote interior angles, $\angle RST$ and $\angle STR$?

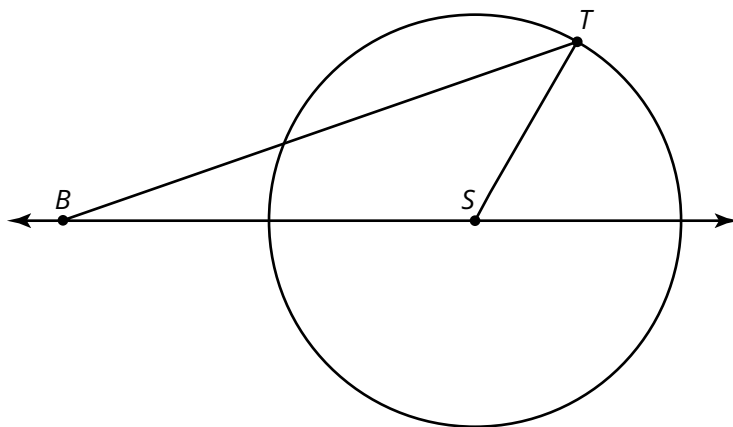
6. Each interior angle of a triangle has two corresponding exterior angles. Identify each exterior angle and use a protractor to test your conjecture from Question 5. What do you notice?



ACTIVITY
4.3

Triangle Inequality and Midsegments

Consider the diagram shown.



1. Measure the side lengths of $\triangle TBS$.

2. Consider the diagram.
 - a. Construct two congruent line segments to the radius $\overline{ST}$ to create two more triangles.

 - b. Name the triangles as $\triangle UBS$ and $\triangle XBS$.

Previously, when you studied triangles, you may have learned a conjecture that the sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Ask Yourself . . .

Can others follow and understand your process and reasoning?

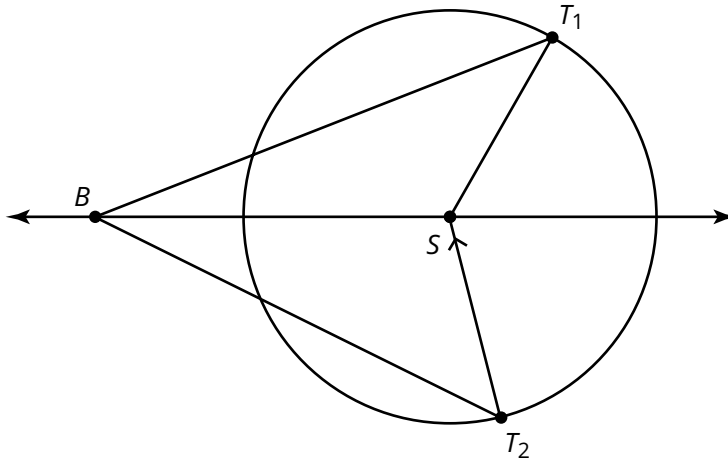
3. How can you use the diagram and the triangles you drew to demonstrate that this conjecture is true?
Explain your reasoning.

Once this conjecture is proven, it becomes a theorem. The **Triangle Inequality theorem** states that the sum of the lengths of any two sides of a triangle is greater than the length of the third side.

4. Use the Triangle Inequality theorem to determine whether it is possible to form a triangle using the set of segments with the given measures. Explain your reasoning.
- a. 5, 6, 8
 - b. 8, 9, 18
 - c. 6, 8, 14
 - d. 7, 7, 13

5. Consider the diagram that shows point T at two different locations on the circle that create triangles with side lengths ST and SB .

- a. For each triangle, determine and label the midpoint of $\overline{ST}$ as point X and label the midpoint of $\overline{BT}$ as point Y . Connect the midpoints to form a midsegment.



Use subscripts to denote the different locations of points X and Y depending on what triangle they are in.

- b. Measure and compare the midsegments and $\overline{SB}$. Measure and compare the angles formed by the midsegments and the angles formed at the base of each triangle. What do you notice? What conjecture can you make about the midsegment of each triangle?



Talk the Talk

Remember ...

A definition is an exact statement describing a geometric term using undefined terms or other defined terms. Three undefined terms in Euclidean geometry are point, line, and plane. A postulate is a statement accepted without proof.

You Will Only Come to This Conclusion

You have investigated different relationships to make and analyze several conjectures about triangles throughout this lesson. You used constructions to verify the Triangle Inequality theorem. You will prove other conjectures as theorems in upcoming lessons.

Remember a conjecture is a statement believed to be true based on observations. You must prove a conjecture is true for it to be fully accepted. You may use definitions, postulates, known theorems, and properties when proving a conjecture. A conjecture that has been proven true becomes a theorem.

1. Based on your observations, draw an example of each conjecture made in this lesson.

a. The base angles of an isosceles triangle are congruent.	
b. A point that lies on a perpendicular bisector of a line segment is equidistant from the endpoints of the line segment.	

c. The sum of the measures of the interior angles of a triangle is 180° .	
d. The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.	
e. The sum of the lengths of any two sides of a triangle is greater than the length of the third side.	
f. The midsegment of a triangle is one-half the measure and parallel to the third side.	

Lesson 4 Assignment

Write

If a conditional statement is true, can you assume the converse is also true?

Use examples to justify your reasoning.

Remember

The base angles of an isosceles triangle are congruent and a point that lies on a perpendicular bisector of a line segment is equidistant from the endpoints of the line segment.

The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles and the sum of the measures of the interior angles of a triangle is 180° .

The sum of the lengths of any two sides of a triangle is greater than the length of the third side and the midsegment of a triangle is one-half the measure and parallel to the third side.

Practice

1. Write a conjecture and its converse about the geometric objects in each part such that the validity of both statements is true. Then, write the statements as a biconditional statement.

a. base angles of an isosceles triangle and their opposite sides



b. the interior angles of a triangle

Lesson 4 Assignment

c. corresponding angles and alternate interior angles

d. the side lengths of a triangle

2. Draw examples of triangle midsegments. What conjecture can you make about the midsegment of a triangle?

Lesson 4 Assignment

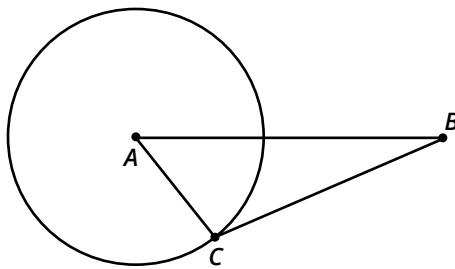
3. Use the Triangle Inequality theorem to determine whether it is possible to form a triangle using the set of segments with the given measures. Explain your reasoning.

a. 9, 5, 2

b. 20, 20, 6

c. 19, 35, 16

4.



- Construct two congruent line segments to the radius $\overline{AC}$ to create two more triangles.
- Name the triangles $\triangle EAB$ and $\triangle DAB$.
- How can you use the diagram and the triangles you drew by constructing congruent segments to verify the Triangle Inequality theorem?

Lesson 4 Assignment

Prepare

1. Describe how you can construct a perpendicular bisector.

2. Describe how you can construct an angle bisector.

5

Points of Concurrency

OBJECTIVES

- Construct the circumcenter, incenter, centroid, and orthocenter.
- Verbally describe how to use points of concurrency to construct both the inscribed circle and the circumscribed circle of a triangle.
- Understand that the three medians of a triangle meet at a point.

.....

You have constructed the perpendicular bisector of a line segment and the angle bisector of an angle.

What happens when you make these constructions on all sides or all angles of a triangle?

KEY TERMS

- concurrent
- point of concurrency
- circumcenter
- circumscribed circle
- incenter
- median
- centroid
- altitude
- orthocenter

Getting Started

It Comes Back Around

You have used circles to explore relationships involving triangles. Let's investigate what happens when you connect any three points on a circle to create different types of triangles.

1. Construct a circle and label the center. Then, draw any three points on the circle. Draw chords between the points to create a triangle.
2. What is the relationship between the center of the circle and each vertex of your triangle?
3. Compare your triangle with your classmates' triangles. What do you notice?
4. Draw three lines, each of which passes through the center of the circle and through a vertex of your triangle.

.....
The **point of concurrency** is the point at which concurrent lines, rays, or line segments intersect.
.....

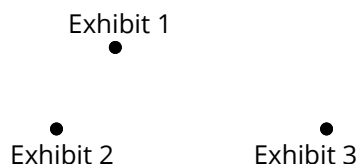
The lines you drew are *concurrent* lines. **Concurrent** lines, rays, and line segments are three or more lines, rays, or line segments intersecting at a single point.

5. Are there other sets of three lines that pass through the center of the circle? Explain your reasoning.

ACTIVITY
5.1

Investigating the Circumcenter

Jacob and Amir have been given the task of building an information kiosk for the zoo. They want the kiosk to be easily accessible from each of the various exhibits. Three of the exhibits at the zoo are shown.



1. Jacob suggests building the kiosk so that it is equidistant from Exhibit 3 and Exhibit 2.



- a. Amir says that for the kiosk to be equidistant from Exhibit 2 and Exhibit 3, it must go at the midpoint of the line segment between the two exhibits. Is Amir correct? Draw a diagram and explain your reasoning.

- b. Use a construction to show all possible locations of the kiosk so that it is equidistant from Exhibit 3 and Exhibit 2. Explain your reasoning.

2. Amir suggests building the kiosk so that it is equidistant from Exhibit 1 and Exhibit 2. Use a construction to show all possible locations of the kiosk so that it is equidistant from these two exhibits. Explain your reasoning.

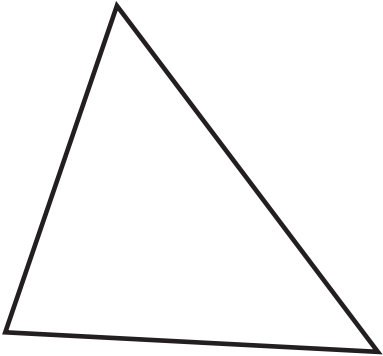
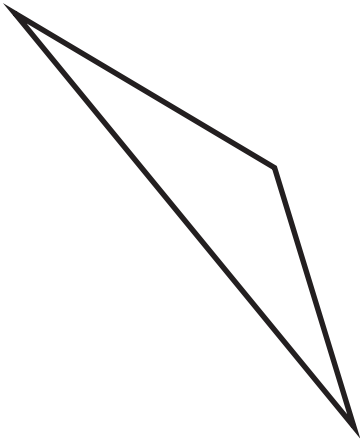
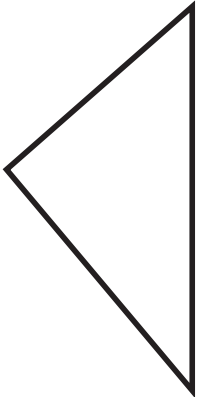
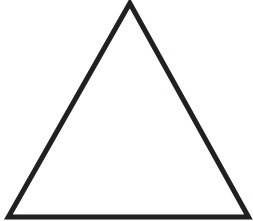
3. Jacob then wonders where the kiosk could be built so that it is equidistant from Exhibit 1 and Exhibit 3. Use a construction to show all possible locations of the kiosk so that it is equidistant from these two exhibits. Explain your reasoning.

4. Describe how to determine a location that is the same distance from all three exhibits. Is there more than one possible location that is equidistant from all three exhibits? Explain your reasoning.

5. Verify that the location you described in Question 4 is equidistant from each exhibit.

You constructed a point of concurrency using the perpendicular bisectors of a triangle.

6. Construct the three perpendicular bisectors of each triangle.

Perpendicular Bisectors	
Acute Triangle (Not Equilateral) 	Obtuse Triangle 
Right Triangle 	Equilateral Triangle 

7. Make a conjecture about the location of the intersection of the three perpendicular bisectors of any triangle. Is the intersection located inside, outside, or on the triangle?

The **circumcenter** is the point of concurrency of the three perpendicular bisectors of a triangle.

8. Consider the four triangles and the perpendicular bisectors you constructed.

- a. Measure the distance from the circumcenter to each vertex of the triangle. Is the circumcenter always, sometimes, or never equidistant from each vertex of the triangle? Explain your reasoning.

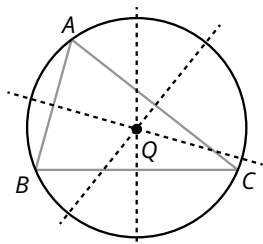
- b. Measure the distance along each perpendicular from the circumcenter to each side of the triangle. Is the circumcenter always, sometimes, or never equidistant from each side of the triangle? Explain your reasoning.

.....

A circumscribed

circle is a circle that passes through all the vertices of a polygon.

Circle Q is circumscribed around $\triangle ABC$.



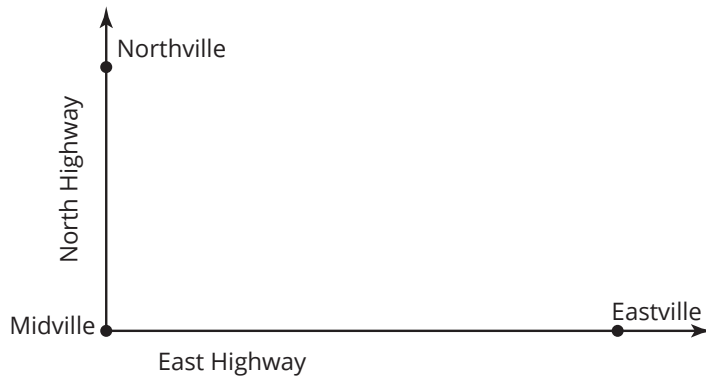
9. Revisit the triangles with perpendicular bisectors. Use a compass to construct the circumscribed circle of each triangle.

10. What does the circumcenter tell you about the relationship between circles and triangles? Explain your reasoning.
-

ACTIVITY
5.2

Investigating the Incenter

Two major highways leading out of Northville and Eastville meet in Midville.



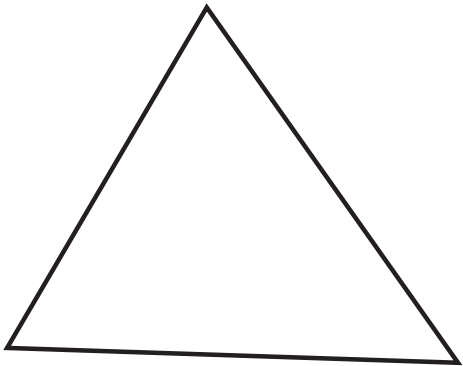
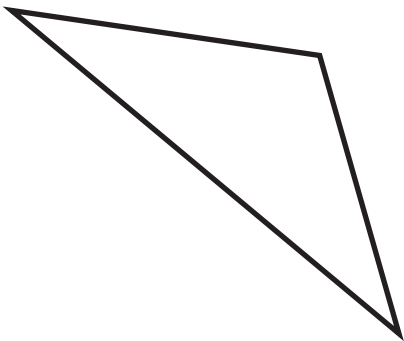
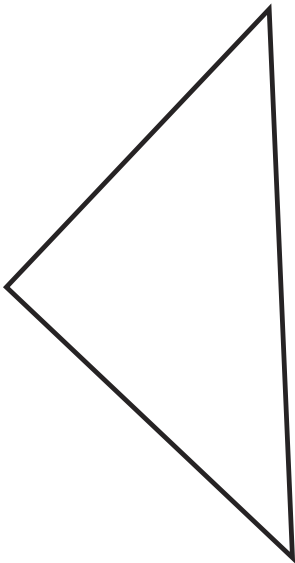
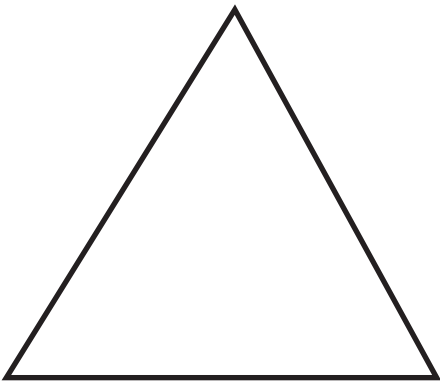
A developer wants to build a shopping center that is equidistant from each highway.

1. Use a construction to show all possible locations of the shopping center so that it is equidistant from each highway. Explain your reasoning.

The Northeast Highway will be built to connect Northville and Eastville. The developer decides to build the shopping center equidistant from the three highways.

2. Use constructions to determine whether there is a single point that is equidistant from the three highways. Explain your reasoning.

3. Construct the three angle bisectors of each triangle.

Angle Bisectors	
Acute Triangle (Not Equilateral) 	Obtuse Triangle 
Right Triangle 	Equilateral Triangle 

4. Make a conjecture about the location of the intersection of the three angle bisectors of any triangle. Is the intersection located inside, outside, or on the triangle?

The **incenter** is the point of concurrency of the three angle bisectors of a triangle.

5. Consider the four triangles and the angle bisectors you constructed.
- Measure the distance from the incenter to each vertex of the triangle. Is the incenter always, sometimes, or never equidistant from each vertex of the triangle? Explain your reasoning.
 - Measure the perpendicular distance from the incenter to each side of the triangle. Is the incenter always, sometimes, or never equidistant from each side of the triangle? Explain your reasoning.

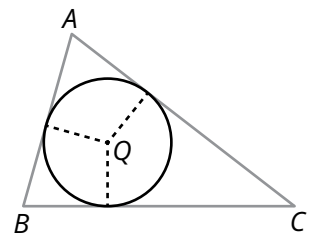
6. Revisit the triangles with angle bisectors. Use a compass to construct the inscribed circle of each triangle.

7. What does the incenter tell you about the relationship between circles and triangles? Explain your reasoning.

.....

Remember ...

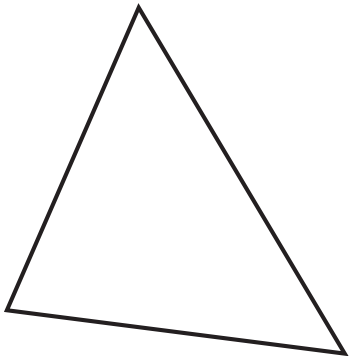
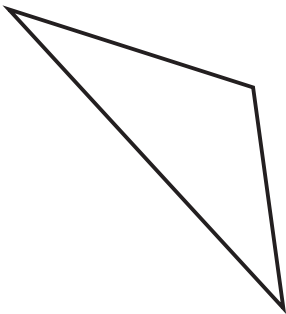
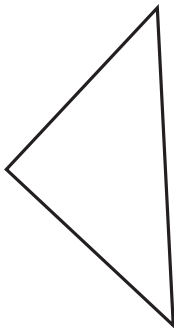
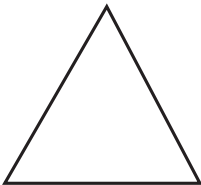
An inscribed circle is a circle drawn inside a polygon such that it is tangent to each side. Circle Q is inscribed in $\triangle ABC$.



Investigating the Centroid

A **median** of a triangle is a line segment that connects a vertex to the midpoint of the opposite side.

1. Construct the three medians of each triangle.

Medians	
Acute Triangle (Not Equilateral) 	Obtuse Triangle 
Right Triangle 	Equilateral Triangle 

2. Make a conjecture about the location of the intersection of the three medians of any triangle. Is the intersection located inside, outside, or on the triangle?

The **centroid** is the point of concurrency of the three medians of a triangle. It is also known as the center of gravity.

3. Consider the four triangles and the medians you constructed.
 - a. Measure the distance from the centroid to each vertex of the triangle. Is the centroid always, sometimes, or never equidistant from each vertex of the triangle? Explain your reasoning.

- b. Measure the perpendicular distance from the centroid to each side of the triangle. Is the centroid always, sometimes, or never equidistant from each side of the triangle? Explain your reasoning.

The centroid divides each median into two line segments.

4. Compare the distance from the centroid to the vertex and the distance from the centroid to the midpoint of the opposite side in each of the triangles in the graphic organizer. What is the ratio?

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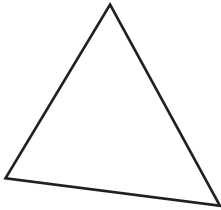
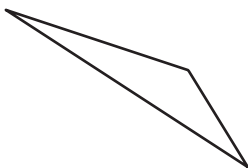
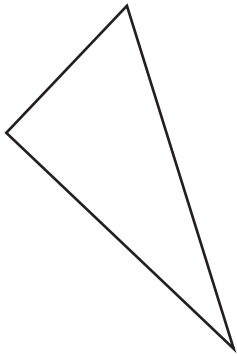
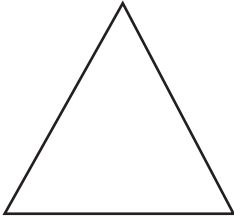
The center of gravity is the point in an object where its weight is evenly distributed, and it can balance in all directions.

.....

Investigating the Orthocenter

An **altitude** of a triangle is a line segment that is perpendicular to a side of the triangle and has one endpoint at the opposite vertex.

1. Construct the three altitudes of each triangle.

Altitudes	
Acute Triangle (Not Equilateral) 	Obtuse Triangle 
Right Triangle 	Equilateral Triangle 

2. Make a conjecture about the location of the intersection of the three altitudes of any triangle. Is the intersection located inside, outside, or on the triangle?

The **orthocenter** is the point of concurrency of the three altitudes of a triangle.

3. Consider the four triangles and the altitudes you constructed.
 - a. Measure the distance from the orthocenter to each vertex of the triangle. Is the orthocenter always, sometimes, or never equidistant from each vertex of the triangle? Explain your reasoning.
 - b. Measure the perpendicular distance from the orthocenter to each side of the triangle. Is the orthocenter always, sometimes, or never equidistant from each side of the triangle? Explain your reasoning.



Talk the Talk

I Concur

Look back at the points of concurrency you determined for the different triangles in this lesson.

1. Complete the table to describe the location of each point of concurrency for acute, obtuse, and right triangles.

Point of Concurrency	Acute Triangle	Obtuse Triangle	Right Triangle
Circumcenter			
Incenter			
Centroid			
Orthocenter			

2. Examine the circumcenter, incenter, centroid, and orthocenter for each equilateral triangle. What do you notice?
3. What is the minimum number of lines you need to construct to identify the circumcenter, incenter, centroid, and orthocenter of a triangle? Explain your reasoning.

4. Complete the table to describe the construction required to determine each point of concurrency.

Point of Concurrency	Located where the three _____ intersect.
Circumcenter	
Incenter	
Centroid	
Orthocenter	

5. Determine which point of concurrency is most appropriate in each situation.
- A flea market is situated on a triangular piece of land. Each entrance is located at one of the three vertices of triangle. Daniela wants to set up her merchandise at a location that is equidistant from all three entrances.
 - An artist is building a mobile with several metal triangles of various sizes. The triangles are connected to each other using steel rods and the rods are welded onto each triangle at a point which would allow the triangle to balance horizontally.
 - Koda's backyard is a triangular plot of land. He is using fencing to build a circular dog pen. He wants the dog pen to be as large as possible and needs to determine the location of the center of the circular dog pen.

Lesson 5 Assignment

Write

Describe the similarities and differences between each pair of terms.

1. *concurrent* and *point of concurrency*
2. *incenter* and *orthocenter*
3. *centroid* and *circumcenter*
4. *altitude* and *median*

Remember

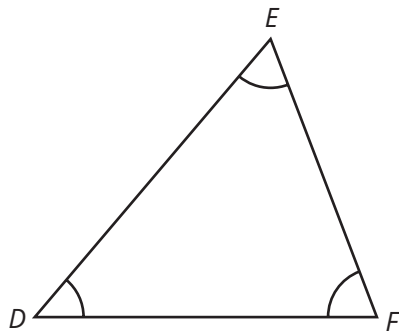
For every triangle:

- The circumcenter is the point of concurrency of the perpendicular bisectors of each side.
- The incenter is the point of concurrency of the angle bisectors.
- The centroid is the point of concurrency of the medians.
- The orthocenter is the point of concurrency of the altitudes.

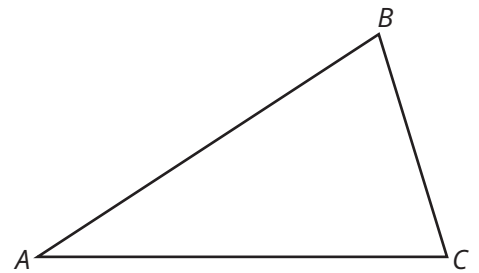
Practice

1. Use a compass and straightedge to perform each construction.

- a. Construct the incenter of $\triangle DEF$.

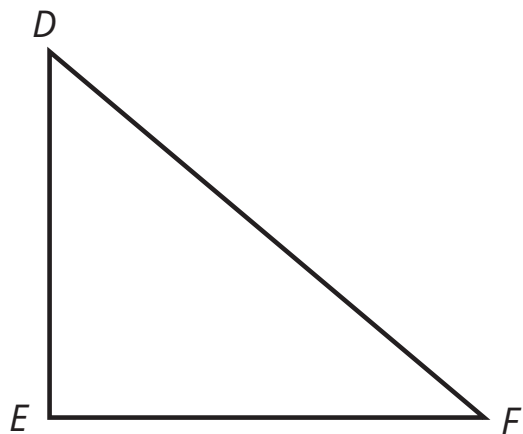


- b. Construct the circumcenter of $\triangle ABC$.

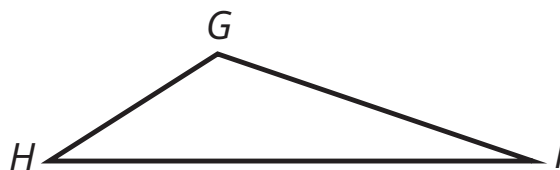


Lesson 5 Assignment

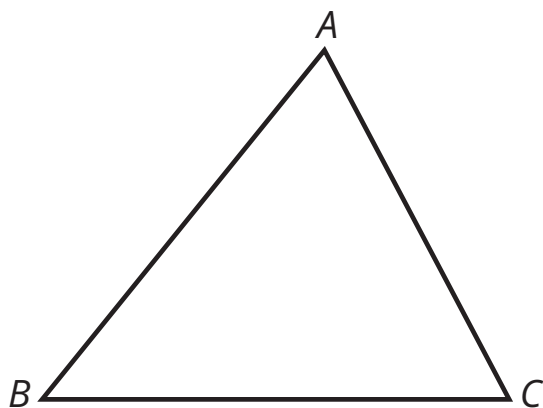
c. Construct the circumcenter of $\triangle DEF$.



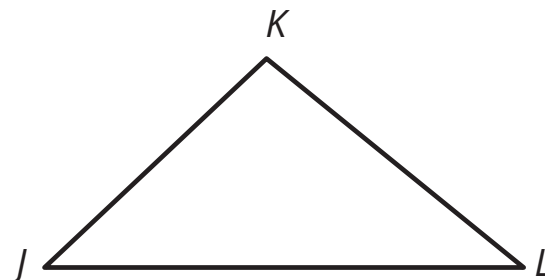
d. Construct the circumcenter of $\triangle GHI$.



e. Construct the centroid of $\triangle ABC$.



f. Construct the orthocenter of $\triangle JKL$.



Lesson 5 Assignment

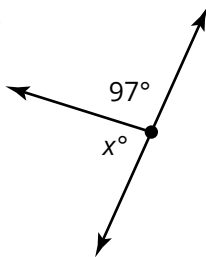
2. Write the term that best completes each statement.

- a. The incenter of a triangle is the point of concurrency of the _____ of a triangle.
- b. The circumcenter of a triangle is the point of concurrency of the _____ of a triangle.
- c. The centroid of a triangle is the point of concurrency of the _____ of a triangle.
- d. The orthocenter of a triangle is the point of concurrency of the _____ of a triangle.

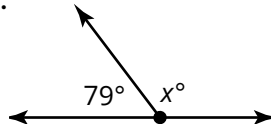
Prepare

Determine each unknown measure.

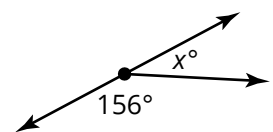
1.



2.



3.



Composing and Decomposing Shapes

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

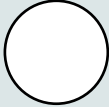
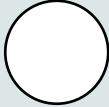
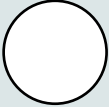
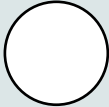
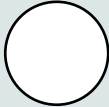
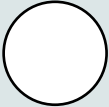
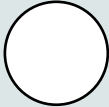
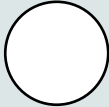
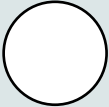
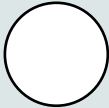
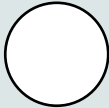
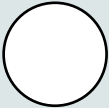
Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Composing and Decomposing Shapes* topic by:

TOPIC 1 <i>Composing and Decomposing Shapes</i>	Beginning of Topic	Middle of Topic	End of Topic
making conjectures about vertical angles, the angle pairs formed when parallel lines are intersected by a transversal, and the distance between the points on a perpendicular bisector of a line segment and the endpoints of the line segment.			
making conjectures about the sum of the interior angle measures of a triangle, the base angles of isosceles triangles, and the midsegments of triangles.			
making conjectures about the properties of quadrilaterals.			
showing that opposite angles in a inscribed quadrilateral are supplementary.			
defining the terms <i>inscribed</i> , <i>circumscribed</i> , <i>angle bisector</i> , and <i>perpendicular bisector</i> .			
defining inscribed polygons.			

continued on the next page

TOPIC 1 SELF-REFLECTION
continued

TOPIC 1 <i>Composing and Decomposing Shapes</i>	Beginning of Topic	Middle of Topic	End of Topic
constructing an equilateral triangle, a hexagon, and a triangle inscribed in a circle.			
constructing the circumcenter, incenter, centroid, and orthocenter of a triangle.			
circumscribing a circle about a triangle.			
recognizing which point of concurrency is most helpful when solving a given problem.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

- Describe a new strategy you learned in the *Composing and Decomposing Shapes* topic.
- What mathematical understandings from the topic do you feel you are making the most progress with?
- Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 1 SUMMARY

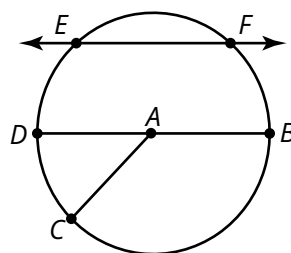
Composing and Decomposing Shapes

LESSON

1

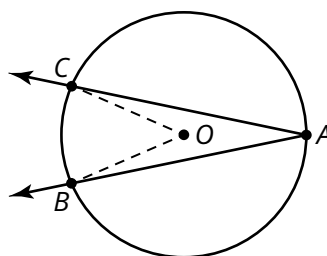
Using Circles to Make Conjectures

A circle is the set of all points in a plane equidistant from a given point, the center of the circle. A **diameter** ($\overline{DB}$) of a circle is a line segment passing through the center (point A) of the circle with endpoints on the circle. The **radius** ($\overline{AC}$, $\overline{AD}$, $\overline{AB}$) is the distance from the center of a circle to a point on the circle. A **central angle** is an angle with its vertex at the center of a circle, like $\angle DAC$. A **chord** ($\overline{EF}$) is a segment whose endpoints are points on a circle. A chord is formed by the intersection of the circle and a **secant** line, like $\overleftrightarrow{EF}$.



Any two points on a circle determine a major arc and a minor arc. The arc with the greater measure is the **major arc**, like $\widehat{DBC}$. The other arc is the **minor arc**, like $\widehat{DC}$.

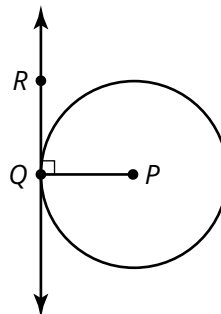
An **inscribed angle** is an angle whose vertex is on a circle and whose sides contain chords of the circle. In this circle, $\angle CAB$ is the inscribed angle. We can also say that $\widehat{CB}$ is an **intercepted arc** because the rays of the angle cross the circle at points C and B. The measure of the intercepted arc is twice that of the inscribed angle. The measure of the intercepted arc is also twice that of the central angle that shares the same intercepted arc.



For example, if $m\angle CAB = 35^\circ$, then $m\widehat{CB} = \angle COB = 70^\circ$.

A **tangent** to a circle is a line that intersects the circle in exactly one point, called the *point of tangency*.

For example, the tangent line $\overleftrightarrow{QR}$ forms a right angle with the radius $\overline{PQ}$ where the two intersect at the point of tangency, point Q.

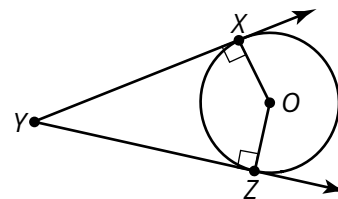


KEY TERMS

- central angle [ángulo central]
- major arc [arco mayor]
- minor arc [arco menor]
- secant [secante]
- chord [cuerda]
- inscribed angle [ángulo inscrito]
- intercepted arc [arco interceptado]
- tangent [tangente]
- circumscribed angle [ángulo circunscrito]
- radius [radio]
- diameter [diámetro]
- coincident [coincidente]
- interior angle of a polygon [ángulo interior de un polígono]
- kite
- isosceles trapezoid [trapezio isósceles]
- midsegment [segmento medio]
- quadrilateral inscribed in a circle [cuadrilátero inscrito en un círculo]
- convex polygon [polígono convexo]
- concave polygon [polígono cóncavo]
- inscribed polygon [polígono inscrito]
- base angles [ángulos de la base]
- exterior angle [ángulo exterior]
- remote interior angles [ángulos interiores remotos]
- Triangle Inequality theorem [Teorema de la desigualdad triangular]

A **circumscribed angle** has its two sides tangent to the circle with a vertex that is outside of the circle.

For example, $\angle XYZ$ is a circumscribed angle.



LESSON

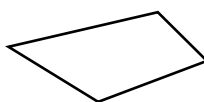
2

Conjectures About Quadrilaterals

A **convex polygon** contains all the line segments connecting any pair of its points. Polygons that are not convex are called **concave polygons**.

A quadrilateral may be convex or concave. The quadrilaterals you are most familiar with—trapezoids, parallelograms, rectangles, rhombi, and squares—are convex. A concave quadrilateral has one angle that is greater than 180° .

Convex



Concave



A quadrilateral has exactly two diagonals. The diagonals of any convex quadrilateral create two pairs of vertical angles and four linear pairs of angles. You can use the diameters of circles to draw the diagonals of different quadrilaterals as long as the two segments are not coincident. Two line segments are **coincident** if they lie exactly on top of each other.

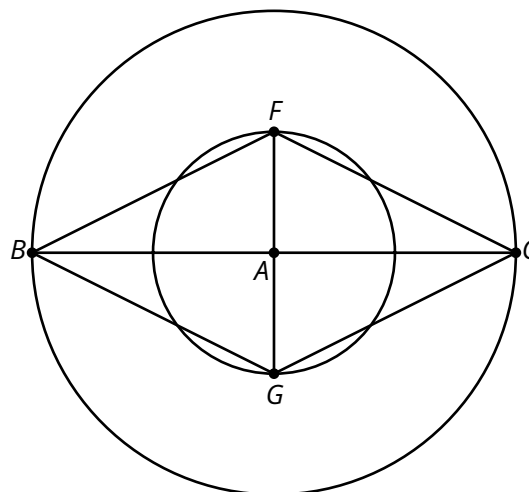
An **interior angle of a polygon** is an angle inside the polygon between two adjacent sides. The interior angles of special quadrilaterals have different relationships.

For example, the four interior angles of a rectangle are congruent while only opposite interior angles of a parallelogram are congruent.

A **kite** is a quadrilateral with two pairs of equal adjacent sides. If the diagonals of a quadrilateral are perpendicular, non-congruent, and only one bisects the other, it can only be classified as a kite.

For example, consider the quadrilateral formed using two concentric circles. The inner circle and outer circle have two diameters that bisect each other. Connecting points B, F, C, and G will form a kite that is also a rhombus.

A trapezoid is a quadrilateral with only one pair of parallel sides, known as the bases. The non-parallel sides are known as

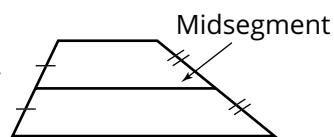


the legs. A special type of trapezoid is an isosceles trapezoid. An **isosceles trapezoid** is a trapezoid with congruent legs.

Property	Parallelogram	Rectangle	Rhombus	Square	Isosceles Trapezoid	Kite
Opposite sides are parallel.	X	X	X	X		
Only one pair of opposite sides is parallel.					X	
Opposite sides are congruent.	X	X	X	X		
Only one pair of opposite sides is congruent.					X	
Opposite angles are congruent.	X	X	X	X		
At least one pair of opposite angles is congruent.	X	X	X	X		X
Diagonals bisect each other.	X	X	X	X		
Diagonals are perpendicular.			X	X		X
Diagonals are congruent.		X		X	X	
All angles are right angles.		X		X		
All sides are congruent.			X	X		
2 pairs of consecutive sides are congruent.			X	X		X

A **midsegment** of a polygon is any line segment that connects two midpoints of the sides of the polygon.

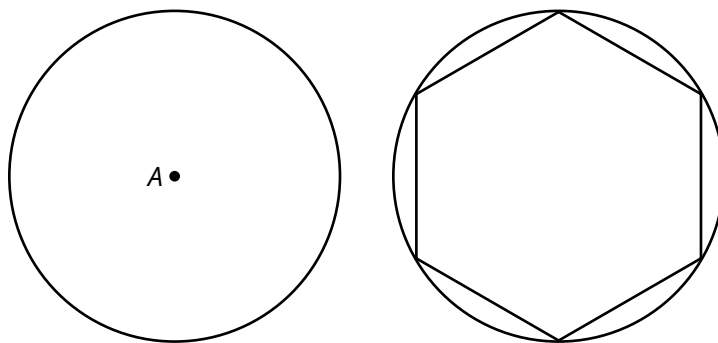
The *midsegment of a trapezoid* that connects the midpoints of the two non-parallel sides of the trapezoid is parallel to each base. The length of this midsegment is equal to half the sum of the lengths of its bases



A **quadrilateral inscribed in a circle** is a quadrilateral whose vertices all lie on a single circle. The opposite angles of a quadrilateral inscribed in a circle are supplementary. The sum of either pair of opposite angles is 180° .

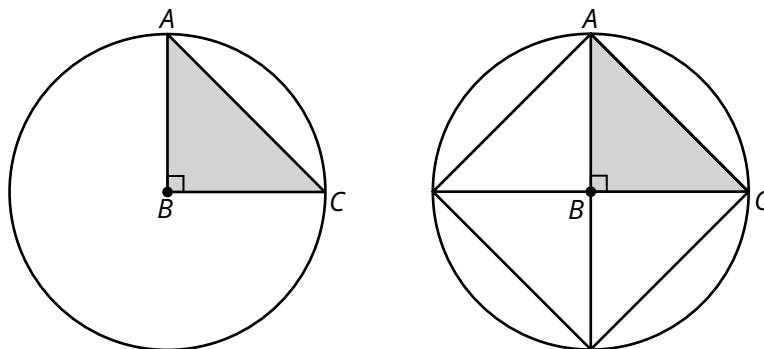
Constructing an Inscribed Regular Polygon

You can use what you know about duplicating a line segment to construct an inscribed polygon, such as a regular hexagon. An **inscribed polygon** is a polygon drawn inside another polygon or circle in which all of the vertices of the interior polygon lie on the outer figure. Each line segment that forms the hexagon is the same length as the radius of the circle. Use a compass to measure the radius. Then without changing the width of the compass, place the point on any point on the circle and construct an arc that intersects the circle. The point of intersection will be the endpoint of the line segment. Repeat the process on each new endpoint until the inscribed polygon has six sides.



To construct a square, you can use a transformation of an inscribed polygon.

For example, circle B contains a right triangle, $\triangle ABC$, with points A and C on the circle and point B at the center. To create a square, rotate $\triangle ABC$ to create an inscribed square with all 4 vertices touching the circle.



Since an equilateral triangle is composed of three equal line segments and three equal angles, you can also use patty paper or a compass and a straightedge to construct an equilateral triangle on its own or inscribed in a circle.

LESSON

4

Conjectures About Triangles

When you divide a quadrilateral along any one of its diagonals, you create two triangles.

An isosceles triangle has a vertex angle that is formed by the two congruent sides. The congruent sides of an isosceles triangle are called *legs*. The other side is called the *base*, and the angles between the base and the congruent sides are called **base angles**.

An equilateral triangle is a triangle in which all three sides are equal in length. All three angle measures of an equilateral triangle are also congruent.

The sum of the measures of the interior angles of a triangle is 180° .

An **exterior angle** of a polygon is an angle that forms a linear pair with an interior angle of the polygon.

For example, $\angle TRB$ is an exterior angle of $\triangle RST$.

The **remote interior angles** of a triangle are the two angles that are not adjacent to the specified exterior angle.

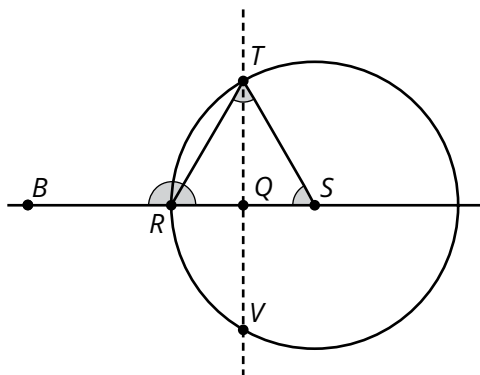
For example, $\angle RST$ and $\angle STR$ are remote interior angles of $\triangle RST$.

The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

For example, $m\angle RST + m\angle STR = m\angle TRB$

The **Triangle Inequality theorem** states that the sum of the lengths of any two sides of a triangle is greater than the length of the third side.

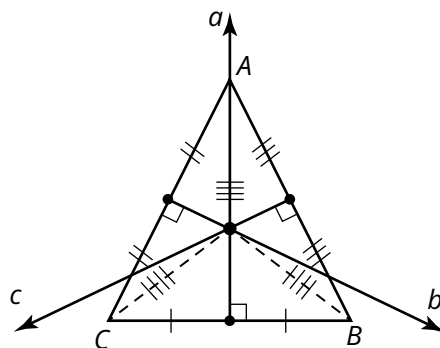
The *midsegment of a triangle* is one-half the measure of and parallel to the third side.



Points of Concurrence

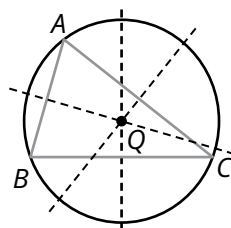
Concurrent lines, rays, and line segments are three or more lines, rays, or line segments that intersect at a single point. The **point of concurrency** is the point at which concurrent lines, rays, or segments intersect.

The **circumcenter** is the point of concurrency of the three perpendicular bisectors of a triangle. It is equidistant from the vertices.

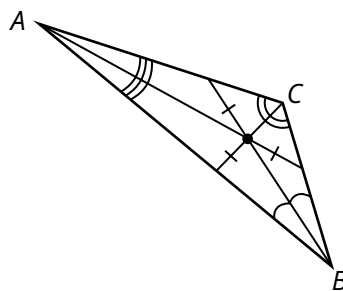


A **circumscribed circle** is a circle that passes through all the vertices of a polygon.

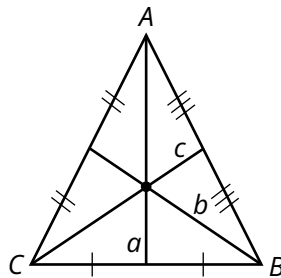
For example, circle Q is circumscribed around $\triangle ABC$.



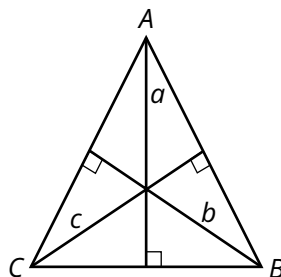
The **incenter** is the point of concurrency of the three angle bisectors of a triangle. It is equidistant from the sides.



A **median** of a triangle is a line segment that connects a vertex to the midpoint of the opposite side. The **centroid** is the point of concurrency of the three medians of a triangle. The centroid of a triangle divides each median into two parts so that the distance from the centroid to the vertex is twice the distance from the centroid to the midpoint of the opposite side.



An **altitude** of a triangle is a line segment that is perpendicular to a side of the triangle and has one endpoint at the opposite vertex. The **orthocenter** is the point of concurrency of the three altitudes of a triangle.

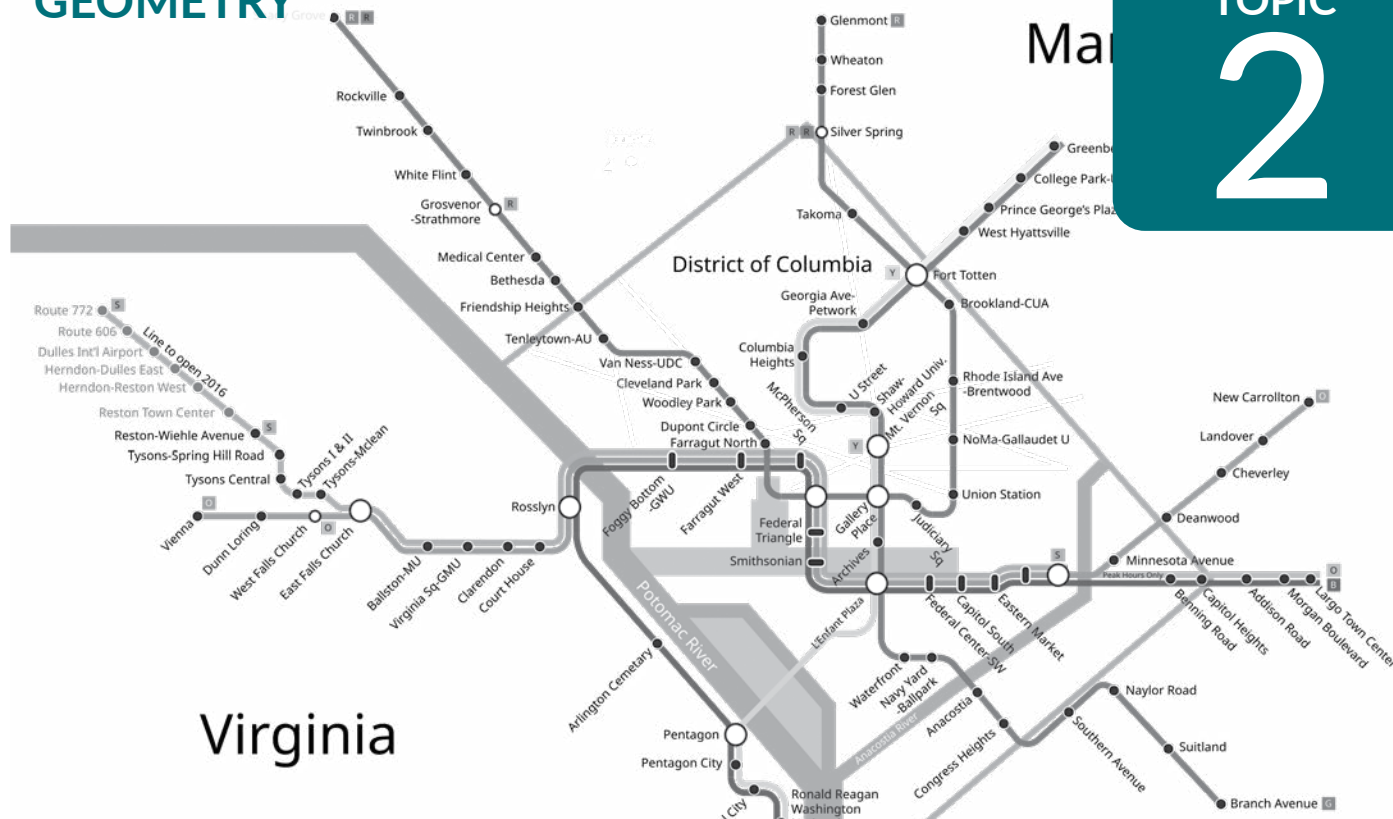


Points of concurrency are a valuable tool in constructing geometric figures. When circumscribing a triangle, you can use the circumcenter to identify the center of the circle. The circumcenter is equidistant from each of the vertices, and when the circle passes through each vertex, the perpendicular bisectors will become the radii. You can also use the incenter of a triangle to inscribe a circle.

Points of concurrency can lie on the interior of a triangle, the exterior of a triangle, or on the triangle.

	Acute Triangle	Obtuse Triangle	Right Triangle
Circumcenter	Interior	Exterior	On Hypotenuse
Incenter	Interior	Interior	Interior
Centroid	Interior	Interior	Interior
Orthocenter	Interior	Exterior	At Vertex of Right Angle

The incenter, circumcenter, centroid, and orthocenter for an equilateral triangle are the same point.



Public transportation routes aren't always grids. Sometimes they form more interesting shapes and angles.

Justifying Line and Angle Relationships

LESSON 1	Forms of Proof	495
LESSON 2	Proving Parallel Line Theorems	521
LESSON 3	Interior and Exterior Angles of Polygons	549
LESSON 4	Perpendicular Bisector and Isosceles Triangle Theorems	567
LESSON 5	Inverse, Contrapositive, Direct Proof, and Indirect Proof	593
LESSON 6	Angle Relationships Inside and Outside Circles	617



1

Forms of Proof

OBJECTIVES

- Recognize postulates as building blocks of proofs.
- Apply real number properties to angle measures and distances.
- Prove theorems involving angles both verbally and in writing.
- Complete a flow chart proof.
- Complete a two-column proof.
- Complete a paragraph proof.

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You have used observations to conjecture that vertical angles are congruent.

How can you use definitions, properties, and postulates to prove your conjecture?

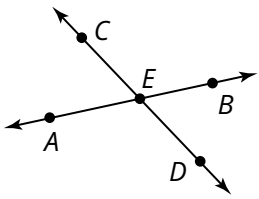
KEY TERMS

- addition property of equality
- subtraction property of equality
- substitution property of equality
- transitive property of equality
- flow chart proof
- two-column proof
- right angle congruence postulate
- congruent supplements theorem
- vertical angle theorem
- paragraph proof

Getting Started

Infinite Regression

Think back on the conjecture you made in a previous topic that vertical angles are congruent. You made this conjecture based on the angle measures formed by two perpendicular diameters of a circle because you knew each angle measured 90° . What happens when you do not know the angle measures formed by two intersecting lines? How do you prove that vertical angles are always congruent?



Consider $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ that intersect at point E .

1. What is the sum of $m\angle AEC$ and $m\angle CEB$?
2. How do you know that these pairs of angles are supplementary?
3. How do you know that $m\angle AEC + m\angle CEB = m\angle AEB$?
4. How do you know your reasoning to Question 3 is true?

These questions can seem to never end. That's why mathematicians have agreed upon postulates in Euclidean geometry. There needs to be a starting point from which you can prove your conjectures.

You previously made conjectures about the sum of the measures of two angles and of two arcs. These conjectures are actually postulates. You were introduced to the following postulates in a previous topic, and they will come in handy as you begin to prove your conjectures.

Remember ...

A postulate is a statement that is accepted without proof.

Postulate	Statement
angle addition postulate	If point D lies in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$.
linear pair postulate	If two angles form a linear pair, then the angles are supplementary.
segment addition postulate	If point B is on $\overline{AC}$ and between points A and C , then $AB + BC = AC$.

ACTIVITY 1.1

Properties of Real Numbers in Geometry

Many properties of real numbers can be applied in geometry. These properties are important when making conjectures and proving new theorems.

Property	Statement
addition property of equality	If a , b , and c are real numbers and $a = b$, then $a + c = b + c$.
subtraction property of equality	If a , b , and c are real numbers and $a = b$, then $a - c = b - c$.
reflexive property of equality	If a is a real number, then $a = a$.
substitution property of equality	If a and b are real numbers and $a = b$, then a can be substituted for b .
transitive property of equality	If a , b , and c are real numbers and $a = b$, and $b = c$, then $a = c$.

1. Give algebraic examples of each property using numbers and variables.

The addition property of equality can be applied to angle measures and distances.

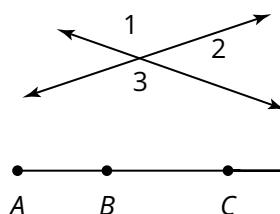
WORKED EXAMPLE

Angle measures:

If $m\angle 1 = m\angle 3$,
then $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 2$.

Distances:

If $AB = CD$, then $AB + BC = BC + CD$.

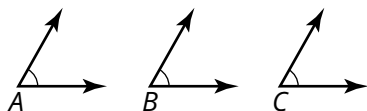
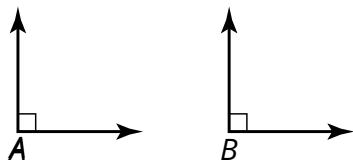
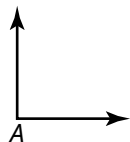
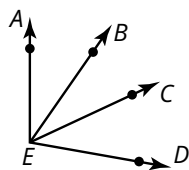


Notice the symbols used to represent measures of angles. Distance refers to the measurement of how far apart two points are.

.....

In Geometry, properties of equality apply to measurements or distances. You can apply the properties of congruence to geometric figures. Figures that are congruent have the same size and shape.

.....



2. Consider each diagram and write a statement that applies each property to angle measures.

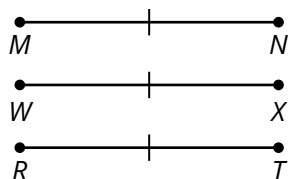
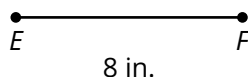
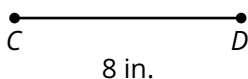
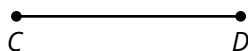
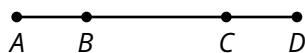
a. subtraction property of equality

b. reflexive property of equality

c. substitution property of equality

d. transitive property of equality

3. Consider each diagram and write a statement that applies each property to segment measures.



a. subtraction property of equality

b. reflexive property of equality

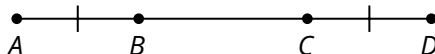
c. substitution property of equality

d. transitive property of equality

Forms of Proof

Remember a proof is a logical series of statements and corresponding reasons that starts with a hypothesis and arrives at a conclusion. There is more than one way to organize a proof.

The diagram shows four collinear points A , B , C , and D such that point B lies between points A and C , point C lies between points B and D , and $\overline{AB} \cong \overline{CD}$.



Consider the conditional statement: If $\overline{AB} \cong \overline{CD}$, then $\overline{AC} \cong \overline{BD}$. When you begin the proof process, you take the parts of a conditional statement and organize them into “Given” and “Prove” statements.

1. Amir organized the conditional statement as shown. Explain he decided which statement is given and which needs to be proved.

Amir

Given: $\overline{AB} \cong \overline{CD}$

Prove: $\overline{AC} \cong \overline{BD}$



You have used the Problem Solving Model to solve problems in previous courses. You can use this structure to guide your reasoning as you prove geometric theorems.

Notice and Wonder

What do you notice about the line segments in the diagram? Do you wonder whether there are more pairs of congruent segments?

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Organize and Mathematize

To organize your proof, analyze the conditional statement and what information is given and what information you need to prove.

.....

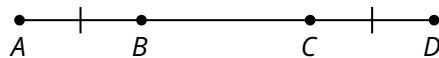
2. For each conditional statement, write the hypothesis as the “Given” and the conclusion as the “Prove.”

a. If $\angle DEG$ and $\angle GEF$ are a linear pair, then $m\angle DEG + m\angle GEF = 180^\circ$.

b. If $\angle ABD$ and $\angle DBC$ are complementary, then $\overrightarrow{BA} \perp \overrightarrow{BC}$.

c. If $\angle 2$ and $\angle 3$ are vertical angles, then $\angle 2 \cong \angle 3$.

You can reason about the proof of conditional statements using constructions. Consider the conditional statement from Question 1: If $\overline{AB} \cong \overline{CD}$, then $\overline{AC} \cong \overline{BD}$.



3. What constructions do you need to prove the conclusion of the conditional statement? Explain your reasoning.

4. Use the diagram and constructions to reason about the validity of the conditional statement.

You can also use the postulates and properties you have learned so far to create different forms of proof. A **flow chart proof** is a proof in which the statement and reason for each step are written in boxes. Arrows connect the boxes and indicate how each step and reason is generated from one or more other steps and reasons.

Predict and Analyze

Think about how you might reason from the given statement to the conclusion. Create or analyze diagrams and consider how they can be used to plan your proof.

A proof by construction uses a compass and a straight edge to create an example demonstrating how to reach the conclusion.

A flow chart does not always fall in a straight line of reasoning.

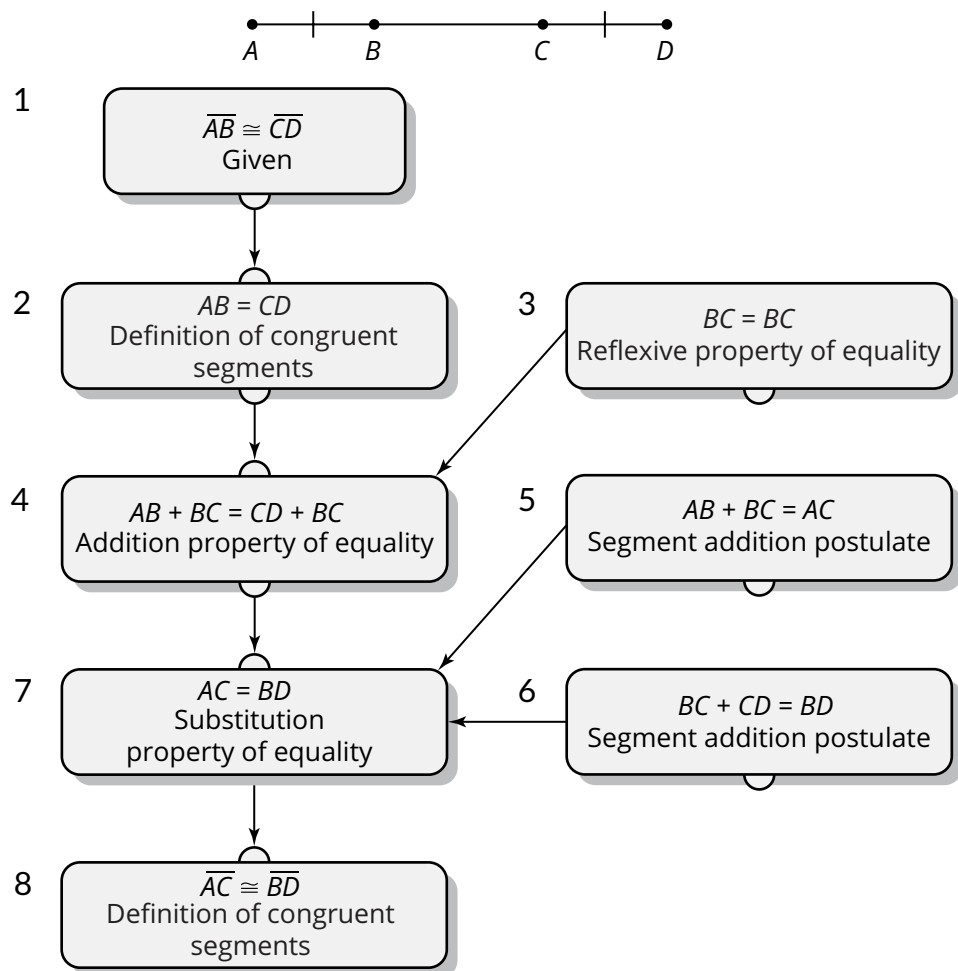
To complete a proof you take the given hypothesis of a statment as true and then use deductive reasoning to demonstrate that the conclution (prove) is true. If the conclusion does not follow from the true hypothesis then the conditional statement is false and you have not proven the conjecture.

Test and Interpret

Create a logical sequence of statements and reasons to prove the hypothesis.

WORKED EXAMPLE

You can complete a flow chart proof of the conditional statement: If $\overline{AB} \cong \overline{CD}$, then $\overline{AC} \cong \overline{BD}$.



5. Why is Step 2 of the flow chart proof critical in getting from the hypothesis to the conclusion?

6. Why does Step 8 follow Step 7?

7. Is there more than one correct way to order these statements and reasons? Explain your reasoning.

Ask Yourself . . .

What patterns do you notice?

A **two-column proof** is a proof in which the steps are written in the left column and the corresponding reasons are written in the right column. Each step and corresponding reason are numbered.

WORKED EXAMPLE

You can rewrite your flow chart proof in a two-column format.

Statements	Reasons
1. $\overline{AB} \cong \overline{CD}$	1. Given
2. $AB = CD$	2. Definition of congruent segments
3. $BC = BC$	3. Reflexive property of equality
4. $AB + BC = CD + BC$	4. Addition property of equality
5. $AB + BC = AC$	5. Segment addition postulate
6. $BC + CD = BD$	6. Segment addition postulate
7. $AC = BD$	7. Substitution property of equality
8. $\overline{AC} \cong \overline{BD}$	8. Definition of congruent segments

8. How does the two-column proof relate to the flow chart proof?

.....

"Q.E.D." is an abbreviation for the Latin phrase *quod erat demonstrandum*, which means *that which was to be demonstrated*. It is a notation often placed at the end of a proof to indicate its completion. President James Garfield created a unique proof of the Pythagorean Theorem. He placed Q.E.D. at the end of his proof.

.....



9. Alexander insists the angle in Figure 1 is larger than the angle in Figure 2. Fernando disagrees and insists both angles are the same size.

Figure 1

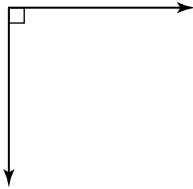
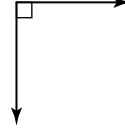
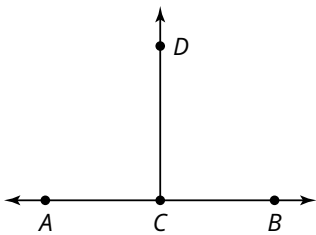


Figure 2



Who is correct? Justify your reasoning.

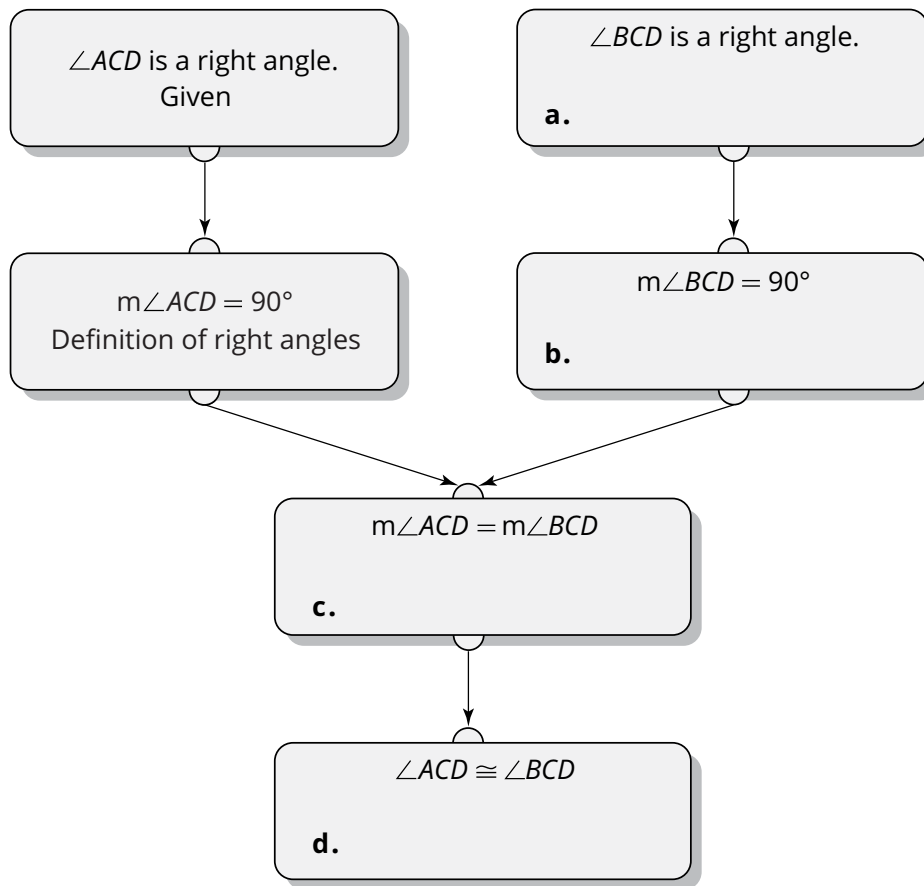


One of Euclid's first five postulates relates to right angles. The right angle congruence postulate states: "All right angles are congruent." While this statement is accepted without proof, definitions can be used to prove it.

Given: $\angle ACD$ and $\angle BCD$ are right angles.

Prove: $\angle ACD \cong \angle BCD$

10. Complete the flow chart of the right angle congruence postulate by selecting a reason from the word bank for each statement shown and writing it in the correct step.



.....

Reasons

Definition of
congruent angles

Definition of
right angles

Given

Transitive property
of equality

.....

11. Why is the order of the steps important in a flow chart or two-column proof?

ACTIVITY
1.3

Proofs of the Congruent Supplements Theorem

Considering what you know about angles, you might conjecture that if two angles are supplements of the same angle or of congruent angles, then the angles are congruent. Let's prove that this conjecture is true.



Ask Yourself . . .

How else can you represent this information?

1. Use the diagram to write the “Given” statements for the congruent supplements theorem. The “Prove” statement is provided.

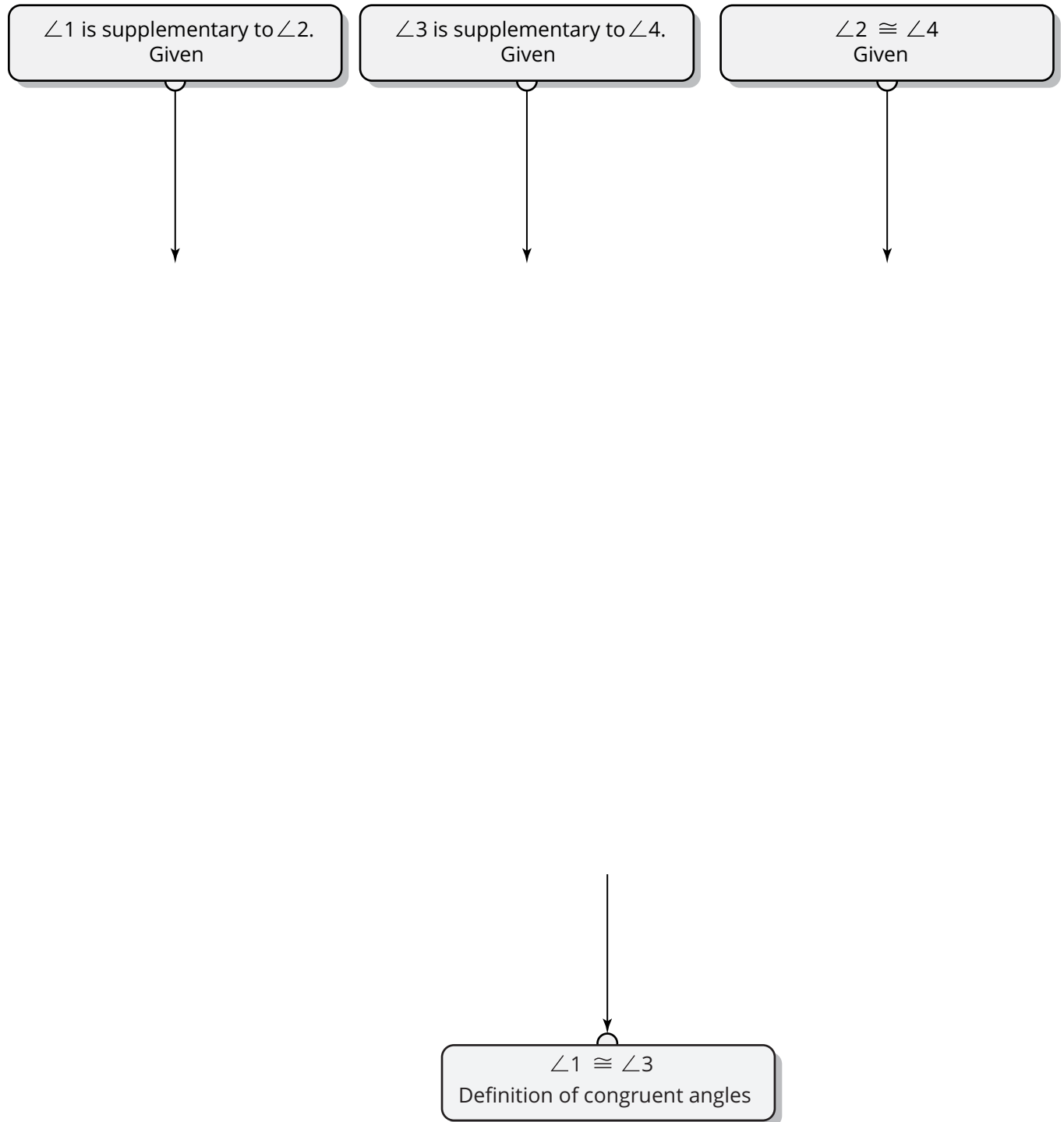
Given:

Given:

Given:

Prove: $\angle 1 \cong \angle 3$

2. Cut out and use the statements and reasons located at the end of the lesson to complete the flow chart to prove the conjecture.



3. Create a two-column proof of the conjecture. Each box of the flow chart you completed should appear as a row in the two-column proof.

Statements	Reasons
1. $\angle 1$ is supplementary to $\angle 2$.	1. Given
2. $\angle 3$ is supplementary to $\angle 4$.	2. Given
3. $\angle 2 \cong \angle 4$	3. Given
4.	4.
5.	5.
6.	6.
7.	7.
8.	8.
9.	9.
10. $\angle 1 \cong \angle 3$	10. Definition of congruent angles

Remember . . .

A conjecture that is accepted without proof is a postulate, but a conjecture that you prove using known definitions, properties, postulates, or theorems becomes a theorem.

Because you have proved that this conjecture is true, you may now refer to it as a theorem. The **congruent supplements theorem** states: “If two angles are supplements of the same angle or of congruent angles, then the angles are congruent.”

ACTIVITY 1.4

Proofs of the Vertical Angle Theorem

You can use the properties and postulates you know to prove the conjecture you made about vertical angles in a previous topic. You conjectured that vertical angles are congruent.

1. Use the diagram to write the “Prove” statements for your conjecture. The “Given” statements are provided.

Given: $\angle 1$ and $\angle 2$ are a linear pair.

Given: $\angle 2$ and $\angle 3$ are a linear pair.

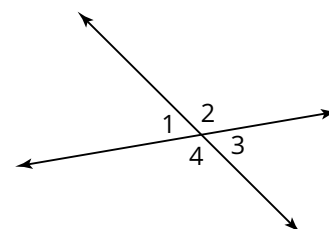
Prove:



Given: $\angle 2$ and $\angle 3$ are a linear pair.

Given: $\angle 3$ and $\angle 4$ are a linear pair.

Prove:

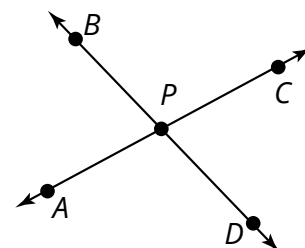


You have used reasoning through constructions, flow chart proofs, and two-column proofs to prove theorems. Another type of proof is a **paragraph proof**, in which the statements and corresponding reasons are written in complete sentences.

Analyze the paragraph proof for one of the prove statements.

.....

Two lines, two line segments, or a line and a line segment intersect to form vertical angles.



WORKED EXAMPLE

If $\angle 1$ and $\angle 2$ are a linear pair, then $\angle 1$ and $\angle 2$ are supplementary angles by the linear pair postulate. If $\angle 2$ and $\angle 3$ are a linear pair, then $\angle 2$ and $\angle 3$ are supplementary angles by the linear pair postulate. By the definition of supplementary angles, $m\angle 1 + m\angle 2 = 180^\circ$ and $m\angle 2 + m\angle 3 = 180^\circ$. By the substitution property of equality, $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$. By the reflexive property of equality, $m\angle 2 = m\angle 2$. By the subtraction property of equality, $m\angle 1 = m\angle 3$. Finally, $\angle 1 \cong \angle 3$ by the definition of congruent angles.

$\angle APB$ and $\angle CPD$ are vertical angles and $\angle BPC$ and $\angle DPA$ are vertical angles.

.....

- Can you state that $m\angle 1 + m\angle 2 = 180^\circ$ before stating that $\angle 1$ and $\angle 2$ are supplementary angles using the linear pair postulate? Explain your reasoning.
- What is the purpose of stating that $m\angle 2 = m\angle 2$ by the reflexive property of equality?
- How would the paragraph proof of the second prove statement of the vertical angle theorem be different?

Because your conjecture has been proved to be true, you can now refer to it as a theorem. The **vertical angle theorem** states: "Vertical angles are congruent."

Once a theorem has been proved, it can be used as a reason in another proof. Using theorems that have already been proved allows you to write shorter proofs.

5. Create a two-column proof for the other prove statement using the congruent supplements theorem.

Statements	Reasons

.....

Remember ...

The congruent supplements theorem states: "If two angles are supplements of the same angle, or of congruent angles, then the angles are congruent."

.....

Using Theorems to Determine Unknown Measures

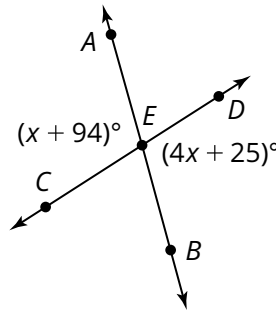
Keep a list of theorems that you prove throughout this course. It will be an excellent resource as you continue to make new conjectures and expand your system of geometry.

In this topic, you demonstrated that an accepted postulate is true using definitions and proved two theorems.

- The right angle congruence postulate: "All right angles are congruent."
- The congruent supplements theorem: "Supplements of congruent angles, or of the same angle, are congruent."
- The vertical angle theorem: "Vertical angles are congruent."

Once you have proved a theorem, you know that it is true in all instances. You can apply these theorems to determine unknown geometric measures.

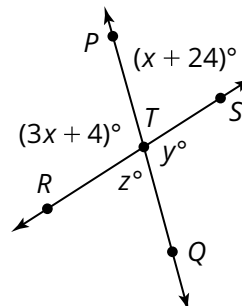
1. In the figure shown, $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ at point E . Determine $m\angle AED$. Explain how you determined the angle measure.



Ask Yourself ...

What is the name of the angle whose measure you want to know?

2. In the figure shown, $\overleftrightarrow{PQ}$ intersects $\overleftrightarrow{SR}$ at point T . Determine each angle measure. Explain how you determined the angle measure.



Ask Yourself ...

Did you justify your mathematical reasoning?



Talk the Talk

Goof-Proof

1. Choose the correct property or theorem for each statement.

a. $\angle H \cong \angle K$

$$\angle K \cong \angle M$$

$$\text{Therefore } \angle H \cong \angle M$$

b. $m\overline{MN} = m\overline{OP}$

$$m\overline{MN} + m\overline{RS} = m\overline{OP} + m\overline{RS}$$

c. $m\angle T = 34^\circ$

$$m\angle W = 34^\circ$$

$$\text{Therefore } m\angle T = m\angle W$$

d. $m\angle A + m\angle B = 180^\circ$

$$m\angle A + m\angle C = 180^\circ$$

$$\text{Therefore } \angle B \cong \angle C$$

e. $m\angle V = m\angle V$

f. $\angle 1 \cong \angle 2$

Property or Theorem

addition property

of equality

congruent supplements
theorem

reflexive property

of equality

substitution property

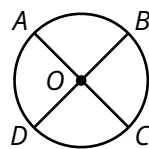
of equality

transitive property

of congruence

vertical angle theorem

Consider circle O with diameters $\overline{AC}$ and $\overline{BD}$.



2. What do you know about $\widehat{AB}$ and $\widehat{CD}$?

3. Use a paragraph proof to prove $\widehat{AB} \cong \widehat{CD}$.

Statements and Reasons for the congruent supplements theorem proof

$m\angle 1 + m\angle 2 = 180^\circ$
Definition of supplementary angles

$m\angle 3 + m\angle 4 = 180^\circ$
Definition of supplementary angles

$m\angle 2 = m\angle 4$
Definition of congruent angles

$m\angle 1 = m\angle 3$
Subtraction property of equality

$m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$
Substitution property of equality

$m\angle 1 + m\angle 2 = m\angle 3 + m\angle 2$
Substitution property of equality

Lesson 1 Assignment

Write

- Describe the differences between a flow chart proof, a two-column proof, and a paragraph proof in your own words.

Remember

A *proof* is a logical series of statements and corresponding reasons that starts with a hypothesis and arrives at a conclusion. There is more than one way to organize a proof.

Practice

- Identify the property that justifies each statement.

a. If $\overline{AB} \cong \overline{PR}$ and $\overline{PR} \cong \overline{ST}$, then $\overline{AB} \cong \overline{ST}$

b. If $\overline{JK} = 6$ centimeters and $\overline{CD} = 6$ centimeters, then $\overline{JK} = \overline{CD}$.

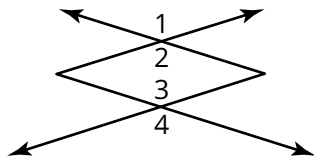
c. $\angle ABC \cong \angle ABC$

d. If $m\angle 3 = m\angle 1$, then $m\angle 3 + m\angle 2 = m\angle 1 + m\angle 2$.

- Enter the reasons to complete the two-column proof.

Given: $\angle 1 \cong \angle 4$

Prove: $\angle 2 \cong \angle 3$



Statements	Reasons
1. $\angle 1 \cong \angle 4$	1. Given
2. $\angle 4 \cong \angle 3$	2.
3. $\angle 1 \cong \angle 2$	3.
4. $\angle 1 \cong \angle 3$	4.
5. $\angle 2 \cong \angle 3$	5. Transitive property of congruence

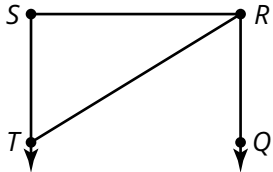
Lesson 1 Assignment

3. Write a paragraph proof to prove the statement.

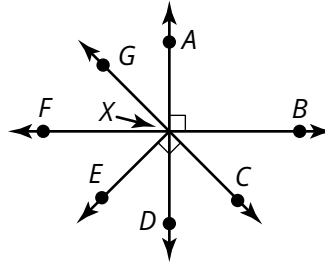
Given: $m\angle QRS = 90^\circ$

Given: $\angle RTS \cong \angle QRT$

Prove: $\angle RTS$ and $\angle TRS$ are complementary.

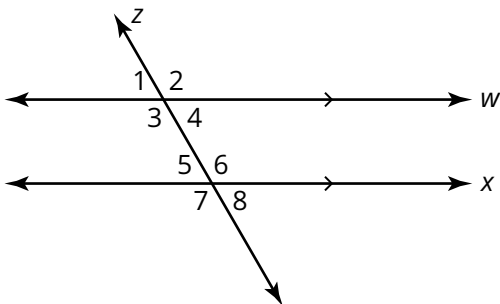


4. In the figure, $\angle GXF \cong \angle CXD$.



- What theorem tells you that $\angle AXG \cong \angle CXD$?
- What theorem tells you that $\angle EXF \cong \angle EXD$?
- What theorem tells you that $\angle GXD \cong \angle CXF$?

Prepare



- Identify the corresponding angles.

Lesson 1 Assignment

2. Identify the alternate interior angles.

3. Identify the vertical angles.

4. Identify the linear pairs.

2

Proving Parallel Line Theorems

OBJECTIVES

- Prove the corresponding angles theorem, the same-side interior angles theorem, the alternate interior angles theorem, and their converses.
- Demonstrate understanding of the alternate exterior angles theorem, the same-side exterior angles theorem, and their converses.
- Prove the perpendicular/parallel line theorem.
- Write a proof plan that connects and justifies mathematical statements through deductive reasoning.

.....

You already know about special angle pairs produced by parallel lines cut by a transversal.

How do you know that these relationships are true in all cases?

KEY TERMS

- corresponding angles theorem
- corresponding angles converse theorem
- same-side interior angles theorem
- alternate interior angles theorem
- alternate exterior angles theorem
- same-side exterior angles theorem
- alternate interior angles converse theorem
- same-side interior angles converse theorem
- alternate exterior angles converse theorem
- perpendicular/parallel line theorem

Getting Started

Remember...

A transversal is a line that intersects two or more other lines.

Corresponding angles are two non-adjacent angles that lie on the same side of the transversal, one angle on the outside of the lines and one angle on the inside of the lines.

Alternate interior angles are two non-adjacent angles that lie between the two lines and on opposite sides of a transversal.

Same-side interior angles are two non-adjacent angles that lie between the two lines on the same side of the transversal.

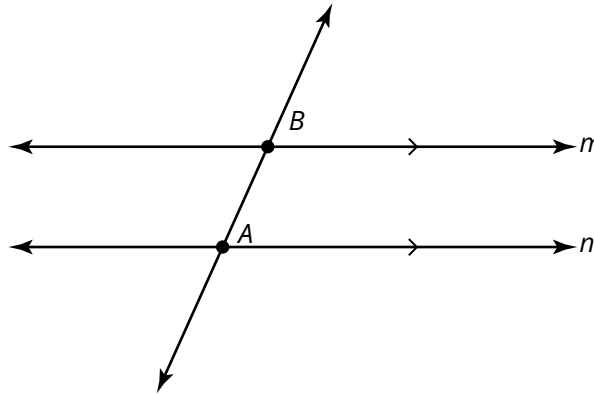
Alternate exterior angles are two non-adjacent angles that lie on the opposite side of the transversal, both angles lie outside of the lines.

Same-side exterior angles are two non-adjacent angles that lie on the same side of the transversal, both angles lie outside the lines.

Criss Cross Applesauce

You may recall that you can use translations to show that corresponding angles are congruent.

1. Consider parallel lines m and n with a pair of corresponding angles, $\angle A$ and $\angle B$.



- a. Use patty paper to translate one of the corresponding angles so that it lies on top of the other corresponding angle. Identify the line of translation and describe the distance and direction of the translation.
 - b. What does this translation demonstrate about corresponding angles?
2. Describe how you can use a patty paper translation and what you know about vertical angles to demonstrate that alternate interior angles are congruent.

3. Describe how you can use a patty paper translation and what you know about linear pairs to demonstrate that same-side interior angles are supplementary.
4. Describe how you can use patty paper translation and what you know about vertical angles to demonstrate that alternate exterior angles are congruent.
5. Describe how you can use a patty paper translation and what you know about linear pairs to demonstrate the same-side exterior angles are supplementary.

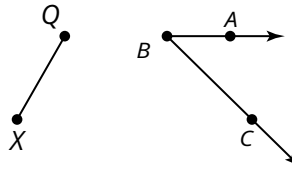


Corresponding Angles Theorem

Let's use translations to prove the conjecture that when parallel lines are cut by a transversal, corresponding angles are congruent. Consider these three properties of translations when formulating a proof:

- Line segments are mapped to line segments of the same length.
- Angles are mapped to angles of the same measure.
- A translated line is either identical to the original line or parallel to it.

1. The given diagram shows $\angle ABC$.



- Draw the translation of $\angle ABC$ along a line parallel to $\overline{QX}$ in the direction and distance from point Q to point X . Label the points A' , B' , and C' .
- Identify a pair of parallel lines. How do you know the lines are parallel?

.....
Remember ...

An auxiliary line is a line or line segment added to a diagram to help in solving or proving a concept.

- Draw transversal $\overline{BB'}$. Identify a pair of corresponding angles. How do you know the corresponding angles are congruent?

2. Use your work in Question 1 and the properties of translations to write a paragraph proof to demonstrate that two parallel lines intersected by a transversal create corresponding angles that are congruent.

Because your conjecture has been proved to be true, you can now refer to it as a theorem. The **corresponding angles theorem** states: “If two parallel lines are intersected by a transversal, then corresponding angles are congruent.”

Recall that the converse of a theorem is created by interchanging the hypothesis and conclusion of the original theorem.

3. Write the converse of the corresponding angles theorem.

4. How can constructions be used to demonstrate the validity of the converse conjecture?

.....

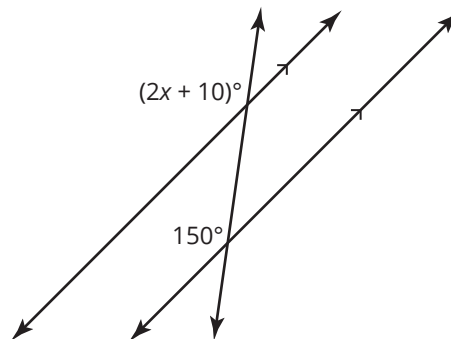
Remember ...

When a conditional statement and its converse are both true, this is called a biconditional statement. A biconditional statement is a statement written in the form “ p if and only if q .”

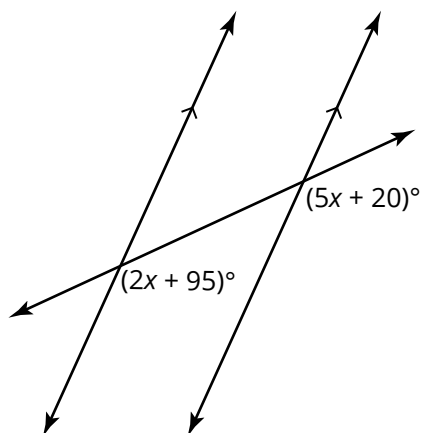
.....

Because your conjecture has been proved to be true, you can now refer to it as a theorem. The **corresponding angles converse theorem** states: “If two lines intersected by a transversal form congruent corresponding angles, then the lines are parallel.”

5. Determine the value of x .



6. Determine the measure of the angles.



Same-Side Interior Angles Theorem

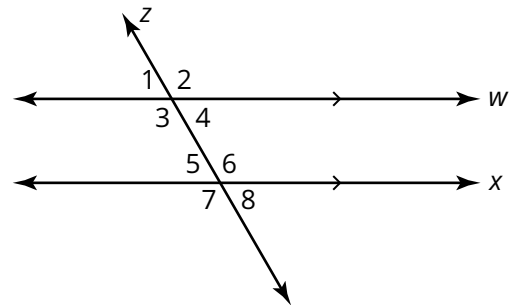
Now that you have proved the corresponding angles theorem, you can use this theorem in future reasoning. The theorem can be used, for example, to prove other relationships regarding angles formed by parallel lines and a transversal. Let's prove your conjecture that if two parallel lines are intersected by a transversal, then the interior angles on the same side of the transversal are supplementary.

Consider the Given and Prove statements and the diagram.

Given: $w \parallel x$ and z is a transversal.

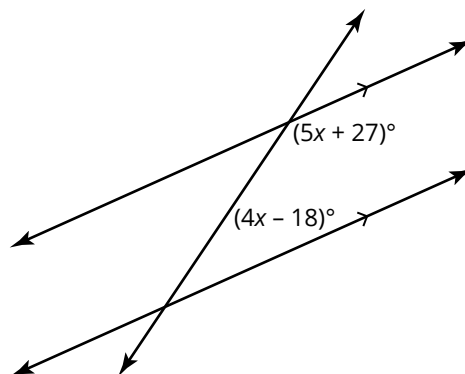
Prove: Same-side interior angles are supplementary angles.

1. Cut out and use the statements and reasons located at the end of the lesson to prove your conjecture. Organize the statements and reasons to form a flow chart proof of this theorem.



Because your conjecture has been proved to be true, you can now refer to it as a theorem. The **same-side interior angles theorem** states, "If two parallel lines are intersected by a transversal, then the interior angles on the same side of the transversal are supplementary."

2. Determine the value of x .



ACTIVITY
2.3

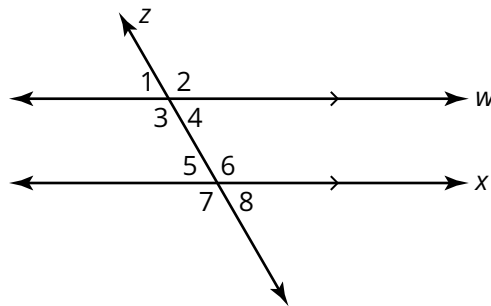
Alternate Interior Angles Theorem

A helpful technique you can use to write proofs is to start by writing a proof plan. When you write a proof plan, you use a combination of your own words and mathematical language to describe how you connect the given statement to the prove statement using postulates and theorems you have already proved. When you write a proof plan, you can test it and fill in missing steps before you write the actual proof.

Consider, for example, the conjecture you made earlier regarding alternate interior angles. You conjectured that if two parallel lines are intersected by a transversal, then the alternate interior angles are congruent.

Given: $w \parallel x$ and z is a transversal.

Prove: Alternate interior angles are congruent.



.....
Draw and Deduce

Use a proof plan to help you connect statements using deductive reasoning: "If I know this, then I know this."
.....

1. Analyze Gabriela's and Destiny's proof plans.

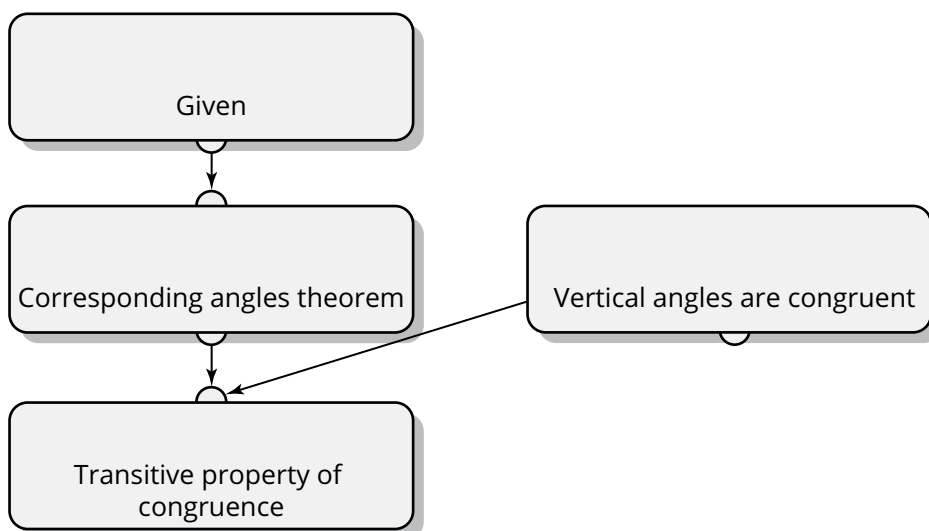


Gabriela	Destiny
I'll show that alternate interior angles 4 and 5 are congruent. It's given that the two lines are parallel and are cut by a transversal, so I know that angles 4 and 6 are supplementary, because they are same-side interior angles. And angles 5 and 6 are supplementary, because they are a linear pair. So, $m\angle 4 + m\angle 6 = 180^\circ$. But $m\angle 5 + m\angle 6$ is also equal to 180°. That means that angles 4 and 5 have to have the same measure.	I need to prove that angles 3 and 6 are congruent, which means they have the same measure. The Given states that the lines are parallel and cut by a transversal, so angles 2 and 6 are congruent, because they are corresponding. But angles 2 and 3 are congruent too, because they are vertical angles. If $\angle 3$ and $\angle 6$ are both congruent to the same angle $\angle 2$, then they must be congruent to each other.

a. Which proof plan is correct? Justify your answer.

b. List the definitions, postulates, and theorems Gabriela and Destiny use in their proof plans. Explain your reasoning.

- c. Use Destiny's proof plan to complete the flow chart proof of the alternate interior angles theorem by writing the statement for each reason.



- d. Use Gabriela's proof plan to create a two-column proof.

Statements	Reasons

Because your conjecture has been proved to be true, you can now refer to it as a theorem. The **alternate interior angles theorem** states: "If two parallel lines are intersected by a transversal, then the alternate interior angles are congruent."

You also conjectured about other relationships regarding angles formed by parallel lines and a transversal.

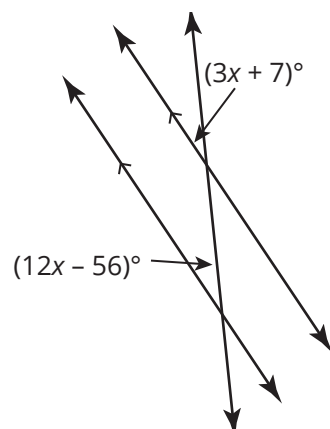
You conjectured that if two parallel lines are intersected by a transversal, then the alternate exterior angles are congruent.

You also conjectured that if two parallel lines are intersected by a transversal, then the same-side exterior angles are supplementary.

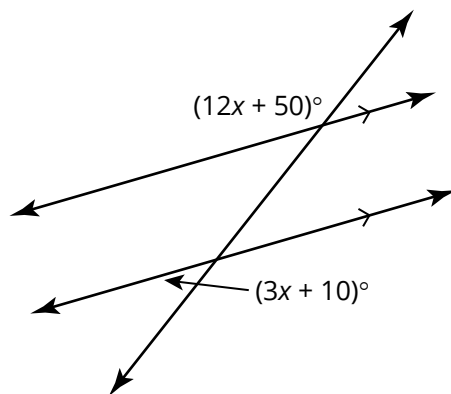
2. Draw diagrams for these theorems and then write a proof plan for each.

Because your proof plans demonstrate that the statements can be proved true, you can now refer to these as theorems. The **alternate exterior angles theorem** states: “If two parallel lines are intersected by a transversal, then the alternate exterior angles are congruent.” The **same-side exterior angles theorem** states: “If two parallel lines are intersected by a transversal, then the same-side exterior angles are supplementary.”

3. Determine the value of x .



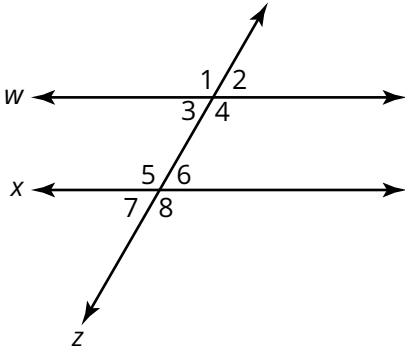
4. Determine the measure of the angles.



ACTIVITY
2.4

Parallel Line Converse Theorems

Earlier, you proved the corresponding angles converse theorem. You can use that theorem to prove the converse of other parallel line theorems.



1. Write the converse of the alternate interior angles theorem.

2. Use the diagram to write the given and prove statements for your converse statement. Then prove that this relationship exists in all cases.

Because your converse statement has been proved to be true, you can now refer to it as a theorem. The **alternate interior angles converse theorem** states: "If two lines intersected by a transversal form congruent alternate interior angles, then the lines are parallel."

3. Write the converse of the same-side interior angles theorem.

4. Use the diagram to write the given and prove statements for your converse statement. Then prove that this relationship exists in all cases.

Because your converse statement has been proved to be true, you can now refer to it as a theorem. The **same-side interior angles converse theorem** states: “If two lines intersected by a transversal form supplementary same-side interior angles, then the lines are parallel.”

5. Consider the exterior angles theorems.

a. Write the converse of the alternate exterior angles theorem. Draw a diagram and write a proof plan.

b. Write the converse of the same-side exterior angles theorem. Draw a diagram and write a proof plan.

Because your proof plan demonstrates that the converse statement can be proved true, you can now refer to it as a theorem. There are other converse theorems that you can show to be true using what you know. The **alternate exterior angles converse theorem** states: "If two lines intersected by a transversal form congruent alternate exterior angles, then the lines are parallel." The **same-side exterior angles converse theorem** states: "If two lines intersected by a transversal form supplementary same-side exterior angles, then the lines are parallel."

Remember that when the validity of a conditional statement and its converse are true, you can write the statement as a biconditional statement.

6. Write the alternate exterior angles theorem and its converse as a biconditional statement.

7. Write the same-side exterior angles theorem and its converse as a biconditional statement.

2.5

Perpendicular/Parallel Line Theorem

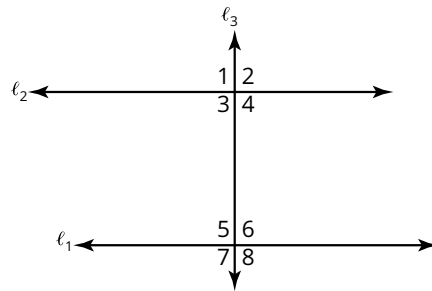
You can extend what you have proven about parallel line converse theorems to prove a theorem about parallel lines cut by a perpendicular transversal.



1. Daniela says that if two lines are perpendicular to the same line, then the two lines are parallel to each other. Is Daniela correct? If she is correct, complete the proof to justify the reasoning, or state why she is not correct.

Given: $\ell_1 \perp \ell_3$ and $\ell_2 \perp \ell_3$

Prove: $\ell_1 \parallel \ell_2$



Statements	Reasons

Because Daniela's conjecture has been proved to be true, you can now refer to it as a theorem. The **perpendicular/parallel line theorem** states: "If two lines are perpendicular to the same line, then the two lines are parallel to each other."



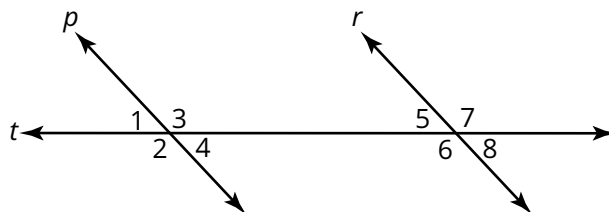
Talk the Talk

If 'M' Then Again

In this lesson you proved theorems about lines and angles.

- corresponding angles theorem
- corresponding angles converse theorem
- same-side interior angles theorem
- same-side interior angles converse theorem
- alternate interior angles theorem
- alternate interior angles converse theorem
- same-side exterior angles theorem
- same-side exterior angles converse theorem
- alternate exterior angles theorem
- alternate exterior angles converse theorem
- perpendicular/parallel line theorem

1. Determine the theorem that justifies each if-then statement.



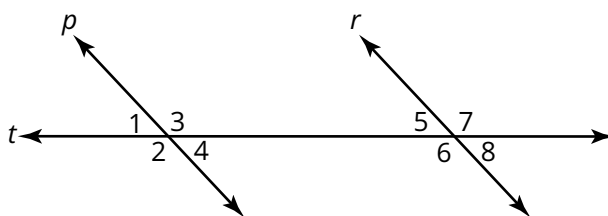
- If line p is parallel to line r , then $\angle 2 \cong \angle 7$.
- If line p is parallel to line r , then $\angle 4 \cong \angle 5$.
- If $\angle 3 \cong \angle 6$, the line p is parallel to line r .

- d. If line p is parallel to line r , then $\angle 1 \cong \angle 5$.
- e. If line p is parallel to line r , then $m\angle 4 + m\angle 6 = 180^\circ$.
- f. If line p is parallel to line r , then $m\angle 1 + m\angle 7 = 180^\circ$.
- g. If $\angle 3 + \angle 5 = 180^\circ$, then line p is parallel to line r .



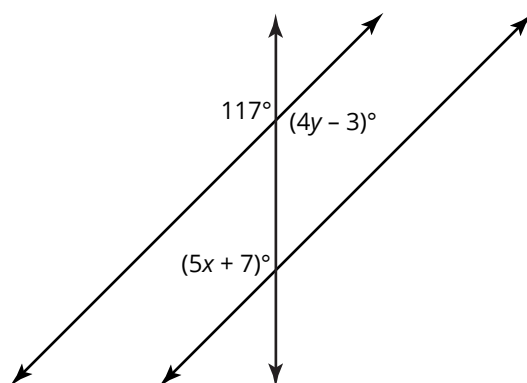
2. Olivia determined the measures of all eight angles labeled using the given information. Javier said she could only calculate the measure of four angles with certainty.

Given: $m\angle 4 = 37^\circ$

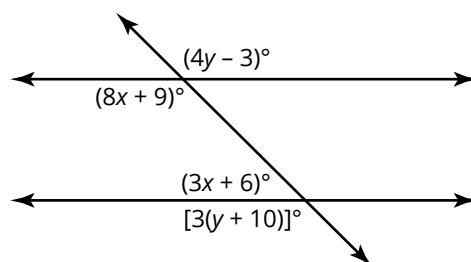


Who is correct? Explain your reasoning.

3. Determine the value of x and y .



4. Determine the value of x and y .



Statements and Reasons for the Same-Side Interior Angles Theorem Proof

	$\angle 1 \cong \angle 5$ Corresponding angles theorem
	$\angle 1$ and $\angle 3$ are supplementary. Linear pair postulate
	$m\angle 1 + m\angle 3 = 180$ Definition of supplementary angles
	$\angle 5$ and $\angle 3$ are supplementary. Definition of supplementary angles
	$\angle 1$ and $\angle 3$ are a linear pair. Definition of linear pair
	$m\angle 1 = m\angle 5$ Definition of congruent angles
	$w \parallel x$, z is a transversal Given
	$m\angle 5 + m\angle 3 = 180^\circ$ Substitution property of equality

Why is this page blank?

So you can cut out the statements and reasons on the other side.

Lesson 2 Assignment

Write

Write the converse of each postulate or theorem.

1. alternate interior angles theorem:
“If two parallel lines are intersected by a transversal, then alternate interior angles are congruent.”
2. alternate exterior angles theorem:
“If two parallel lines are intersected by a transversal, then alternate exterior angles are congruent.”
3. same-side interior angles theorem:
“If two parallel lines are intersected by a transversal, then same-side interior angles are supplementary.”
4. same-side exterior angles theorem:
“If two parallel lines are intersected by a transversal, then same-side exterior angles are supplementary.”

Remember

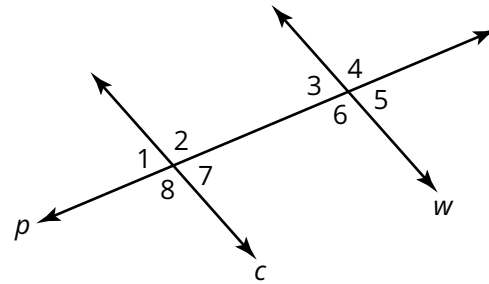
If two parallel lines are intersected by a transversal, then:

- Corresponding angles are congruent.
- Alternate interior angles are congruent.
- Alternate exterior angles are congruent.
- Same-side interior angles are supplementary.
- Same-side exterior angles are supplementary.

Lesson 2 Assignment

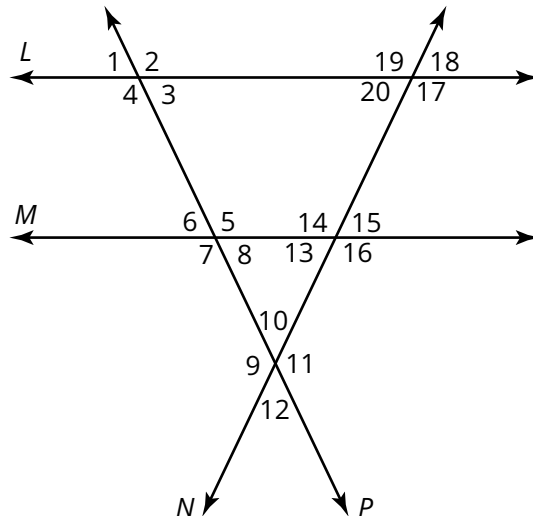
Practice

- Consider the diagram shown.
Determine which theorem leads to the conclusion that $c \parallel w$ for each statement.



- $\angle 3 \cong \angle 7$
 - $\angle 5$ and $\angle 8$ are supplementary
 - $\angle 4 \cong \angle 8$
 - $\angle 2$ and $\angle 3$ are supplementary
- Using the diagram in conjunction with postulates and theorems, determine the measure of all unknown angles.

Given: $\angle 2 > \angle 7 > \angle 19$, $m\angle 2 = 125^\circ$



Lesson 2 Assignment

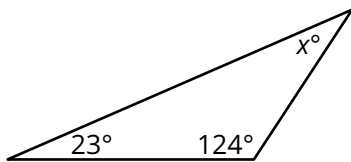
3. Write the alternate interior angles theorem and its converse as a biconditional statement.

4. Write the same-side interior angles theorem and its converse as a biconditional statement.

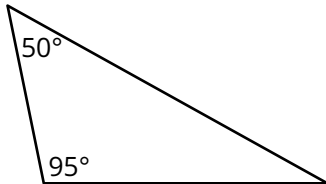
Prepare

Determine each unknown angle measure.

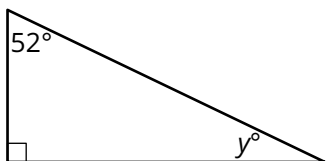
1.



2.



3.



3

Interior and Exterior Angles of Polygons

OBJECTIVES

- Prove, both verbally and in writing, that the measures of the interior angles of a triangle sum to 180 degrees.
- Derive a formula for the sums of the measures of interior angles of a polygon.
- Derive a formula for the sums of the measures of exterior angles of a polygon.

KEY TERMS

- triangle sum theorem
- exterior angle theorem

.....

You have made conjectures about the measures of the interior angles of triangles.

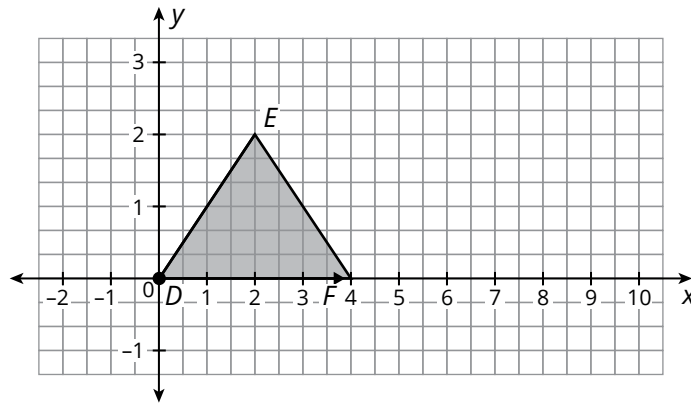
How can you prove theorems about interior and exterior angle measures in polygons with any number of sides?

Getting Started

Rotating and Translating

Let's make a conjecture about the sum of the interior angles of a triangle.

Consider triangle DEF .



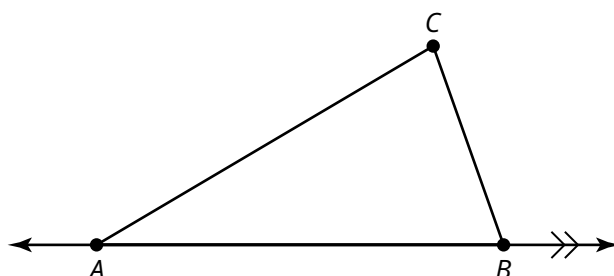
1. Draw the translation of $\triangle DEF$ along line DF in the direction and distance from point D to F . Then, rotate the triangle 180 degrees about the midpoint of translated segment $D'E'$. Label the vertices of the transformed triangle as D'' , E'' , and F'' .
2. Make a conjecture about the sum of the three angles of a triangle. Explain how the rotation and translation support your conjecture.

ACTIVITY 3.1

Triangle Sum and Exterior Angle Theorems

Let's use a line to help prove that the sum of the interior angles of a triangle is equal to 180° .

1. Consider the diagram shown.



- a. Draw an auxiliary line parallel to $\overleftrightarrow{AB}$ through point C. Identify the three adjacent angles at point C that form a line.
- b. Write a plan to prove that the sum of the interior angles of a triangle is 180° . Then write a paragraph proof regarding this relationship.

Remember ...

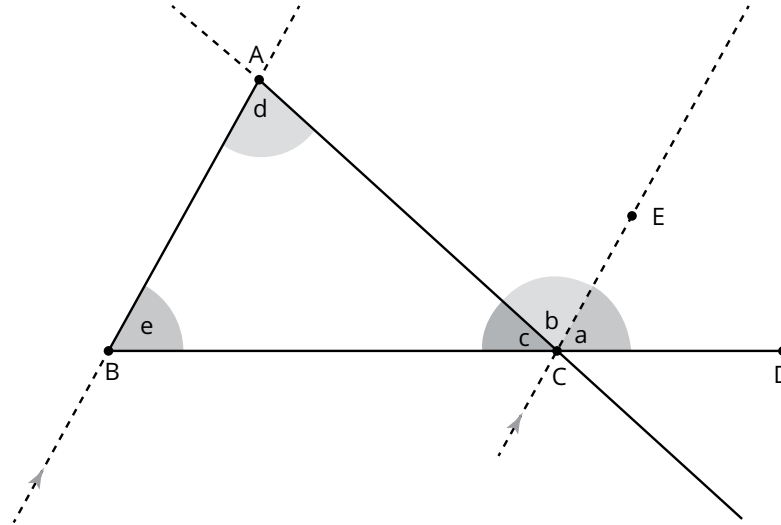
an auxiliary line is a line that is drawn to help complete a geometric proof.

Think About ...

You can now use the corresponding angles theorem and alternate interior angles theorem as reasons in your proofs because you have proved them!

Because you have proved this relationship is true, you can now refer to it as a theorem. The **triangle sum theorem** states: "The sum of the measures of the interior angles of a triangle is equal to 180° ."

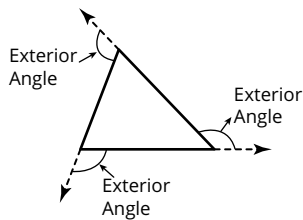
Consider another diagram which can be used to prove the triangle sum theorem.



2. Explain how the diagram demonstrates the triangle sum theorem.

..... It appears that a relationship exists among the angles of a triangle such that the measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

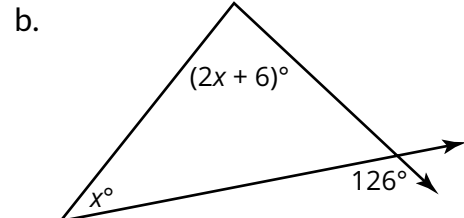
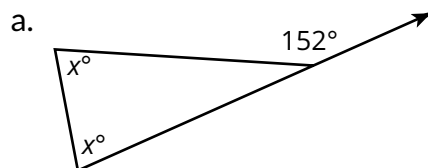
The remote interior angles of a triangle are the two angles that are not adjacent to the specified exterior angle.



3. Explain how the diagram demonstrates that this relationship exists.

Because you have demonstrated this relationship is true, you can now refer to it as a theorem. The **exterior angle theorem** states: "The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles."

..... 4. Solve for x in each diagram.



ACTIVITY
3.2

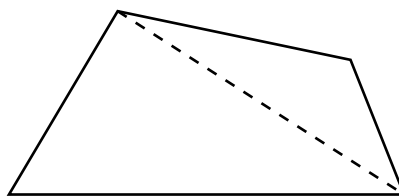
The Sum of the Measures of the Interior Angles of a Polygon

You have proved a theorem about the interior angles of a triangle. An interior angle of a polygon faces the inside of a polygon and is formed by consecutive sides of the polygon.

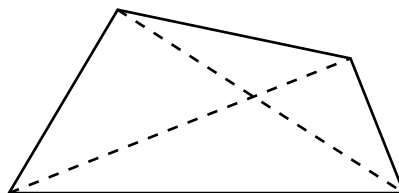


1. Ms. Jackson asked her class to determine the sum of the measures of the interior angles of a quadrilateral.

Madison drew a quadrilateral and added one diagonal as shown. He conjectured that the sum of the measures of the interior angles of a quadrilateral must be equal to 360° .



Ana drew a quadrilateral and added two diagonals as shown. She conjectured that the sum of the measures of the interior angles of a quadrilateral must be equal to 720° .



Who made the correct conjecture? Explain your reasoning.

You can use the triangle sum theorem to calculate the sum of the measures of the interior angles of other polygons.

2. Draw each polygon. Then calculate the sum of the measures of the interior angles of each polygon by drawing all possible diagonals from one vertex of the polygon. Complete the table for 3-, 4-, 5-, and 6-sided polygons.

a. quadrilateral

b. pentagon

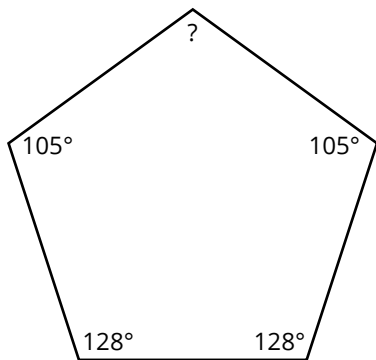
c. hexagon

Number of Sides of the Polygon	3	4	5	6	7	8	9	16
Number of Diagonals Drawn	0							
Number of Triangles Formed	1							
Sum of the Measures of the Interior Angles								

3. What pattern do you notice about the sum of the measures of the interior angles of a polygon as the number of sides of each polygon increases by 1?

4. Predict the number of possible diagonals drawn from one vertex and the number of triangles formed for 7-, 8-, 9-, and 16-sided polygons. Then, complete the table in Question 2. Explain your predictions.

5. For a polygon that has n sides, how many triangles are formed by drawing all diagonals from one vertex? Explain your reasoning.
6. What is the sum of the measures of the interior angles of an n -sided polygon? Explain your reasoning.
7. Use a formula to calculate the sum of the measures of the interior angles of a polygon with 32 sides.
8. When the sum of the measures of the interior angles of a polygon is 9540° , how many sides does the polygon have? Explain your reasoning.
9. Determine the measure of the unknown angle.



10. Calculate the measure of each interior angle of a decagon when each interior angle is congruent. How did you calculate your answer?

11. Complete the table.

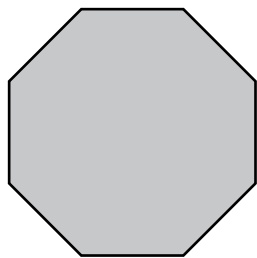
Number of Sides of the Regular Polygon	3	4	5	6	7
Sum of Measures of the Interior Angles					
Measure of Each Interior Angle					

12. For a regular polygon that has n sides, write a formula to calculate the measure of each interior angle.

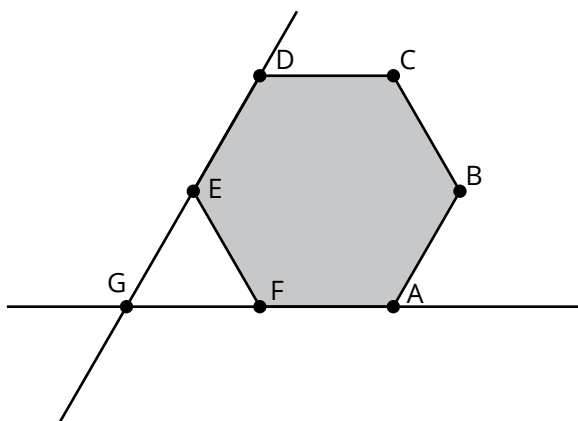
13. Use the formula to calculate the measure of each interior angle of a regular 100-sided polygon.

14. When the measure of each interior angle of a regular polygon is equal to 150° , determine the number of sides. How did you calculate your answer?

15. A stop sign is a regular octagon. What is the measure of each interior angle of the regular octagon?



16. Polygon $EDCBAF$ is a regular hexagon. Determine the measure of $\angle EGF$.



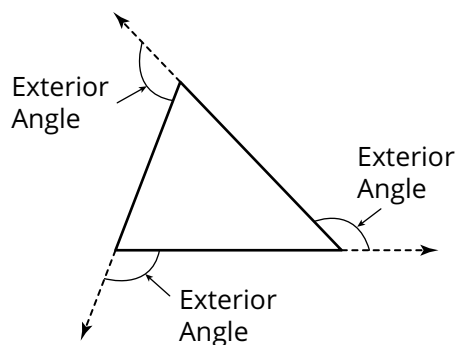
ACTIVITY
3.3

The Sum of the Measures of the Exterior Angles of a Polygon

.....
Think About...

These formulas work for
convex polygons, not
for concave polygons.
Can you figure out why?
.....

You wrote a formula for the sum of the measures of the interior angles of a polygon. Now let's write a formula for the sum of the measures of the exterior angles of a polygon.



1. Use the diagram, your formula for the sum of the measures of the interior angles of a polygon, and the linear pair postulate to calculate the sum of the measures of the exterior angles of a triangle.

Let’s explore the sum of the measures of the exterior angles of other polygons.

2. Draw each polygon. Then calculate the sum of the measures of the exterior angles of each polygon by extending each side of the polygon to locate an exterior angle at each vertex. Complete the table for 3-, 4-, 5-, and 6-sided polygons.

a. quadrilateral

b. pentagon

c. hexagon

Ask Yourself . . .

How can you organize and record your mathematical ideas?

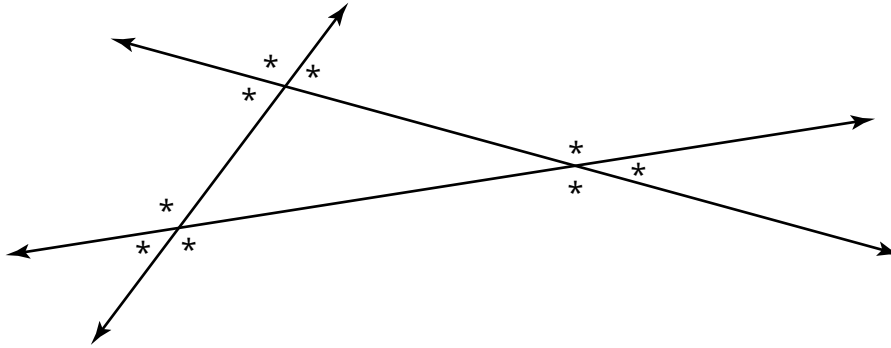
Number of Sides of the Polygon	3	4	5	6	7	15
Number of Linear Pairs Formed						
Sum of the Measures of Linear Pairs						
Sum of the Measures of Interior Angles						
Sum of the Measures of Exterior Angles						

3. What patterns do you notice?
4. Make predictions about the sum of the measures of the exterior angles of 7- and 15-sided polygons. Complete the table in Question 2.
5. What is the sum of the measures of the exterior angles of an n -sided polygon?
6. When the sum of the measures of the exterior angles of a polygon is 360° , how many sides does the polygon have? Explain your reasoning.
7. Calculate the measure of each angle and explain your reasoning.
 - a. each exterior angle of an equilateral triangle
 - b. each exterior angle of a regular pentagon
 - c. each exterior angle of a regular hexagon
8. When the measure of each exterior angle of a regular polygon is 18° , how many sides does the polygon have? Explain how you calculated your answer.

Talk the Talk

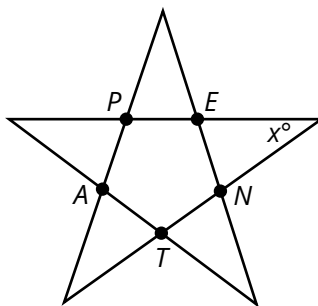
Peace Out

1. Maria announced to the class that she could calculate the sum of the measures of the starred angles in this diagram without knowing the measure of any specific angle. How is this possible? Using theorems or postulates, explain what Maria is thinking.



The Texas flag features a single five-pointed star. The flag, often called the "Lone Star Flag," was officially adopted in 1839, after Texas became an independent republic following a successful revolution against Mexico.

2. *PENTA* is a regular pentagon. Solve for x .

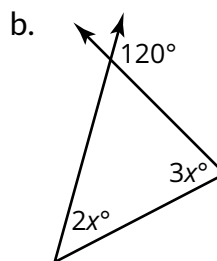
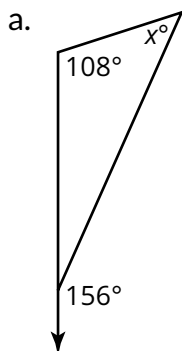


3. The Susan B. Anthony dollar coin minted in 1999 features a regular 11-gon, or hendecagon, inside a circle on both sides of the coin.



What is the measure of each interior angle of the regular hendecagon?

4. Solve for x in each diagram.



This coin was issued to recognize Anthony's contributions to the women's suffrage movement and the evolving role of women in American society. President Jimmy Carter signed the Susan B. Anthony Dollar Coin Act into law in 1978. The coin was first released in Rochester, New York, Anthony's home during her most active years.

Lesson 3 Assignment

Write

Complete each statement with the correct term.

1. The _____ states that the sum of the measures of the interior angles of a triangle is equal to 180° .
2. The sum of the measures of the _____ of a triangle is equal to the corresponding exterior angle.
3. A(n) _____ is a line drawn to help complete a proof.

Remember

The sum of the measures of the interior angles of a triangle is equal to 180° , and the sum of the measures of the interior angles of a quadrilateral is equal to 360° .

For a polygon with n sides, the sum of the measures of the interior angles is equal to $180(n - 2)$ degrees, and the sum of the measures of the exterior angles is equal to 360° .

Practice

1. Determine the measure of an interior angle for each given regular polygon.

a. regular nonagon

b. regular 15-gon

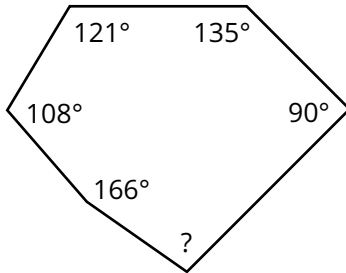
c. regular decagon

d. regular 47-gon

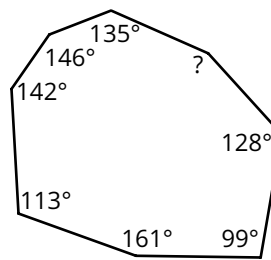
Lesson 3 Assignment

2. Determine the measure of the unknown angle in each figure.

a.



b.



3. For a regular polygon that has 30 sides, what is the measure of each exterior angle? Explain your reasoning.

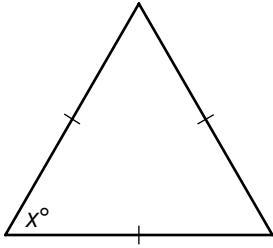
4. The degree measure of each exterior angle of a regular octagon is represented by the expression $7x - 4$. Solve for x .

Lesson 3 Assignment

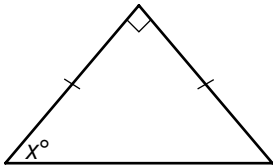
Prepare

Determine each unknown measure.

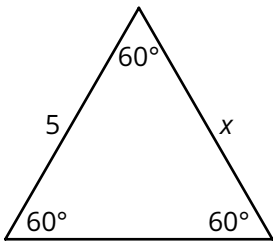
1.



2.



3.



4

Perpendicular Bisector and Isosceles Triangle Theorems

OBJECTIVES

- Prove and verbally explain how points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.
- Prove that the base angles of isosceles triangles are congruent.
- Demonstrate, both verbally and in writing, the 30° - 60° - 90° triangle theorem and the 45° - 45° - 90° triangle theorem by applying algebraic reasoning.

.....

You have conjectured about points on a perpendicular bisector and relationships between angles and sides of various triangles.

How can you use definitions and theorems to prove these conjectures?

KEY TERMS

- perpendicular bisector converse theorem
- isosceles triangle base angles theorem
- isosceles triangle base angles converse theorem
- 30° - 60° - 90° triangle theorem
- 45° - 45° - 90° triangle theorem
- rationalize the denominator

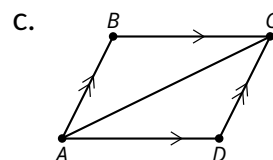
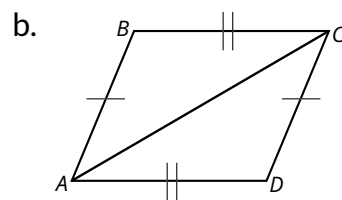
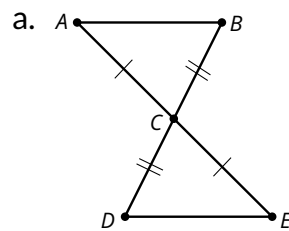
Getting Started

Mappings Matter

You have proven theorems related to triangle congruence. You know that two triangles are congruent when:

- Three corresponding sides are congruent (SSS).
- Two corresponding sides and the included angle are congruent (SAS).
- Two corresponding angles and the included side are congruent (ASA).

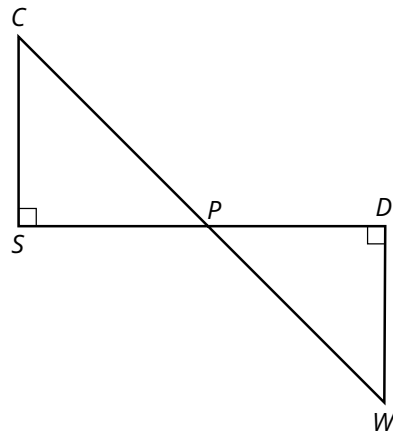
1. Use Side-Angle-Side (SAS), Side-Side-Side (SSS), or Angle-Side-Angle (ASA) to explain why the triangles in each pair are congruent. Explain your reasoning.



The Perpendicular Bisector Theorem

In a previous topic you learned the abbreviation CPCTC often used as a reason in proofs, which states corresponding parts of congruent triangles are congruent. This reason can be used only after you have proved that the triangles are congruent.

1. Consider $\triangle CPS$ and $\triangle WPD$ in the figure shown.
Suppose $\overline{CW}$ and $\overline{SD}$ bisect each other.



.....

Draw and Deduce

How can you mark the diagram to reflect the given information?

.....

- a. Explain how you can demonstrate that the triangles are congruent.
- b. Use CPCTC to identify the congruent corresponding parts of the two triangles.

You can use the triangle congruence theorems and the concept that corresponding angles and sides in two congruent triangles are congruent as key strategies to prove other geometric relationships.

.....
Remember ...

The perpendicular bisector of a line segment is the line that divides the line segment into 2 congruent parts and that intersects the line segment at a right angle.

.....

You have already proved that any point on a perpendicular line segment is equidistant from the segment's endpoints using transformations. You can also prove this relationship using congruent triangles.

2. Consider $\overline{AB}$.

- a. Construct the perpendicular bisector of $\overline{AB}$. Then, draw a line segment from any one point on the bisector to each endpoint of the line segment, A and B .



- b. Use your construction to write a proof plan to prove the perpendicular bisector theorem. Then complete the proof.

.....
Draw and Deduce

Can you first prove that the two triangles are congruent?

.....

The perpendicular bisector theorem states: "Points on a perpendicular bisector of a line segment are equidistant from the segment's endpoints."

3. Consider the converse of the perpendicular bisector theorem.

a. State the perpendicular bisector theorem as a conditional statement using if-then.

b. State the converse of perpendicular bisector theorem.

Analyze the proof of the converse of the perpendicular bisector theorem.

WORKED EXAMPLE

Given: Points Q and R are equidistant from point P .

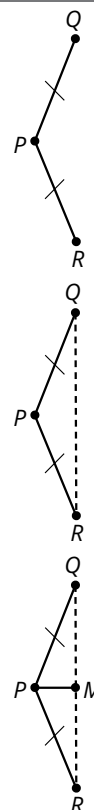
An auxiliary line segment, $\overline{QR}$, can be drawn to form an isosceles triangle, $\triangle RPQ$.

Construct the midpoint of $\overline{QR}$, point M . Another auxiliary line segment, $\overline{PM}$, can be drawn, connecting the midpoint with point P .

The two triangles, $\triangle PQM$ and $\triangle PRM$ are congruent triangles by SSS congruence.

This means that $\angle PMQ$ and $\angle PMR$ are congruent by CPCTC. Since these two angles are congruent and form a linear pair, they are both 90° angles.

Thus, point P lies on the perpendicular bisector of $\overline{QR}$.



4. Consider the Worked Example. How would you know that point P lies on the perpendicular bisector of $\overline{QR}$? Explain your reasoning.

Because this relationship has been proved true, you can now refer to it as a theorem. The **perpendicular bisector converse theorem** states: "If a point is equidistant from the endpoints of a line segment, then the point lies on the perpendicular bisector of the segment." This theorem can be useful when proving theorems about right triangles.

Isosceles Triangle Base Angles Theorem

CPCTC makes it possible to prove other theorems.

You have explored the relationship between the sides and angles of an isosceles triangle. Specifically, you conjectured that if the two sides of a triangle are congruent, then the angles opposite these sides are congruent. The isosceles triangle base angles theorem states: "If two sides of a triangle are congruent, then the angles opposite these sides are congruent."

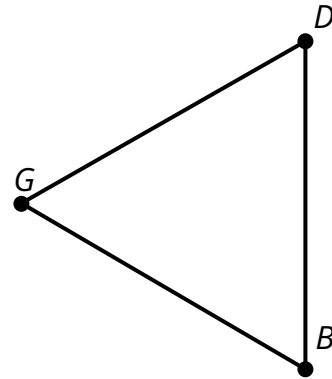
Think About...

You may want to draw a perpendicular bisector for these proofs. When you draw an auxiliary line, use the reason "construction."

1. Consider the diagram to prove your conjecture using a paragraph proof.

Given: $\overline{GB} \cong \overline{GD}$

Prove: $\angle B \cong \angle D$



Because you have proved this relationship is true, you can now refer to it as a theorem. The **isosceles triangle base angles theorem** states: "If two sides of a triangle are congruent, then the angles opposite these sides are congruent."

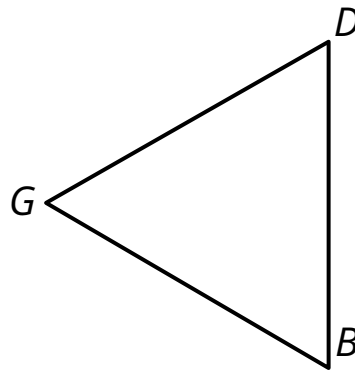
2. Consider the converse of the Isosceles Triangle Base Angles Theorem.

a. State the converse as a conjecture.

b. Consider the diagram to prove your conjecture using a paragraph proof.

Given: $\angle B \cong \angle D$

Prove: $\overline{GB} \cong \overline{GD}$

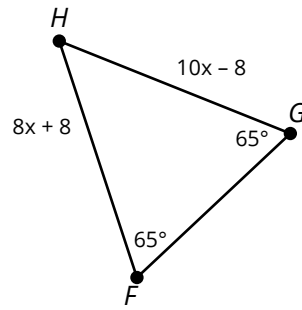


.....
Think About...

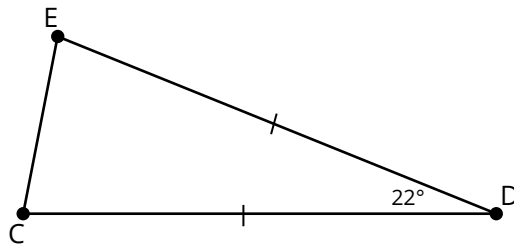
You can draw an auxiliary line which bisects angle G and passes through a point somewhere on line segment DB.
.....

Because you have proved this relationship is true, you can now refer to it as a theorem. The **isosceles triangle base angles converse theorem** states: "If two angles of a triangle are congruent, then the sides opposite these angles are congruent."

3. Determine the length of $\overline{HG}$



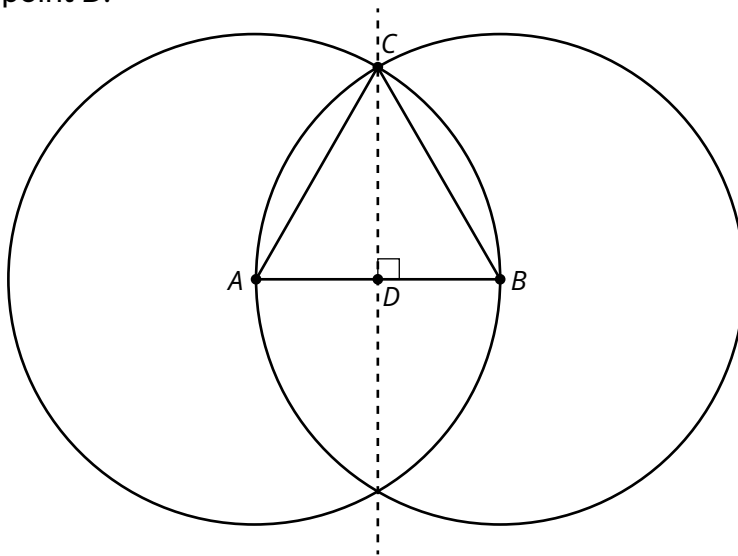
4. Determine the $m\angle DEC$.



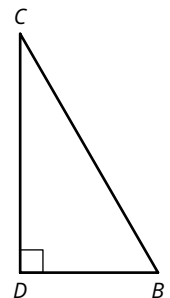
Special Right Triangles

When proving the perpendicular bisector theorem and theorems about isosceles triangles, you used auxiliary lines to create right triangles. There are geometric theorems specifically related to right triangles. Let's determine the relationship that exists among the measures of the angles and sides of a 30° - 60° - 90° triangle.

1. Consider the equilateral triangle constructed in the diagram. A perpendicular bisector of $\overline{AB}$ is also constructed, which intersects $\overline{AB}$ at point D .

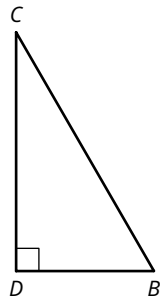


- a. Explain how you know that $\triangle ACB$ is an equilateral triangle.
- b. Explain how you know that $\triangle CDB$ is a 30° - 60° - 90° triangle. Label the angle measures in the diagram.



- c. Explain how you know that point C lies on the perpendicular bisector of $\overline{AB}$.
- d. What do you know about the lengths of $\overline{DB}$ and $\overline{CB}$?
2. Use what you know and the Pythagorean Theorem to demonstrate the 30° - 60° - 90° triangle theorem algebraically. Let a represent the length of the shorter leg, $\overline{DB}$.

3. Label the side measures in the diagram.



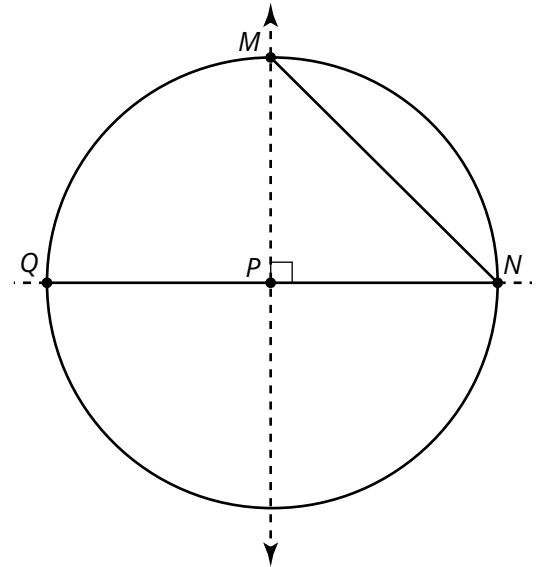
Because you have demonstrated this relationship to be true, you can now refer to it as a theorem. The **30° - 60° - 90° triangle theorem** states: “The length of the hypotenuse in a 30° - 60° - 90° triangle is 2 times the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.”

Let’s determine the relationship that exists among the measures of the angles and sides of a 45° - 45° - 90° triangle.

4. Consider the isosceles right triangle constructed in the diagram. Line MP is a perpendicular bisector of diameter $\overline{QN}$.

a. Explain how you know that $\triangle MPN$ is an isosceles right triangle.

b. Explain how you know that $\angle PMN$ and $\angle PNM$ are congruent angles. Label the angle measures in the diagram.



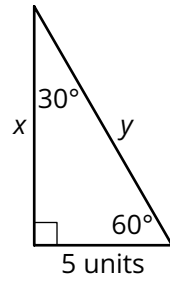
5. Use what you know and the Pythagorean Theorem to demonstrate the 45° - 45° - 90° triangle theorem algebraically. Let a represent the length of each congruent leg.

6. Label the side measures in the diagram.

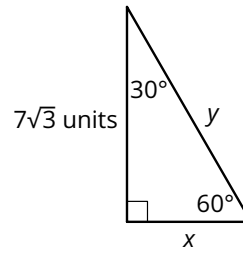
Because you have demonstrated this relationship to be true, you can now refer to it as a theorem. The **45° - 45° - 90° triangle theorem** states: "The length of the hypotenuse in a 45° - 45° - 90° triangle is $\sqrt{2}$ times the length of a leg."

7. Determine the length of sides x and y in each triangle.

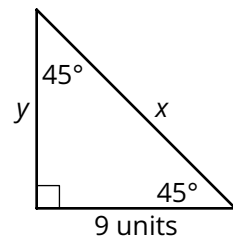
a.



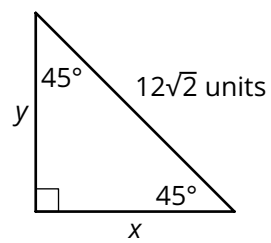
b.



c.



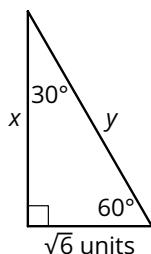
d.



Now that you have practiced identifying the unknown lengths based on the special relationships among the sides of the triangle, consider when the given side lengths are expressed in a different form.

WORKED EXAMPLE

Determine the unknown side lengths in each triangle.



$$x = \sqrt{3} \cdot \sqrt{6} \quad y = 2\sqrt{6} \text{ units}$$

$$x = \sqrt{3 \cdot 6}$$

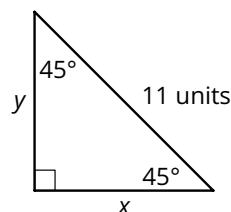
$$x = \sqrt{3 \cdot 3 \cdot 2}$$

$$x = \sqrt{3^2 \cdot 2}$$

$$x = 3\sqrt{2} \text{ units}$$

Remember a standard procedure involving radicals is to simplify the radical, which is the process of removing all perfect square numbers from under the radical symbol.

The product property of radicals states that $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$ when a and b are greater than 0.



$$x\sqrt{2} = 11$$

$$x = \frac{11}{\sqrt{2}} \quad y = \frac{11\sqrt{2}}{2} \text{ units}$$

$$x = \frac{11}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{11\sqrt{2}}{2} \text{ units}$$

Sometimes you might have to set up an equation and solve for a variable to determine a side length.

A standard math convention is to **rationalize the denominator**, which is the process of rewriting a fraction with no irrational numbers in the denominator.

To rationalize the denominator of a fraction involving radicals, multiply a fraction by a form of 1 so that the product in the denominator includes a perfect square radicand. Then rewrite, if possible.



8. Calculate the length of the legs of the isosceles triangle shown.

Angel says that the length of each side is $\sqrt{10}$ units.

$$a\sqrt{2} = 20$$

$$\frac{20}{\sqrt{2}} = \sqrt{\frac{20}{2}}$$

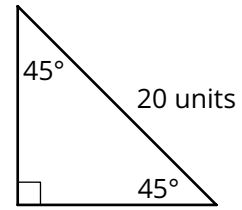
$$= \sqrt{10}$$

Luis says that the length of each side is $10\sqrt{2}$ units.

$$a\sqrt{2} = 20$$

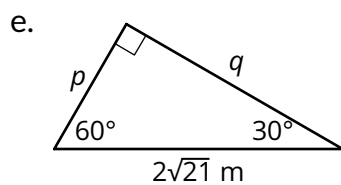
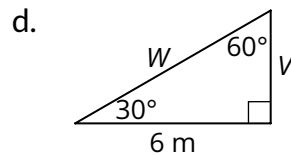
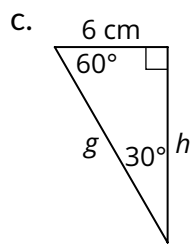
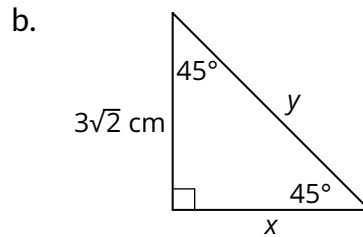
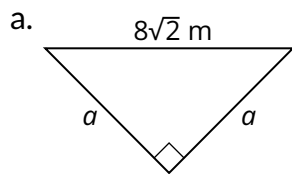
$$\frac{20}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{20\sqrt{2}}{2}$$

$$= 10\sqrt{2}$$

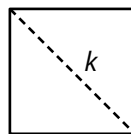


Determine who's correct and explain the error in the other student's work.

9. Solve for the unknown side lengths of each figure.



f. The perimeter of a square measures 32 cm.



Ask Yourself...

How can you use finding the unknown angles and sides of special triangles in everyday life?

PROBLEM SOLVING



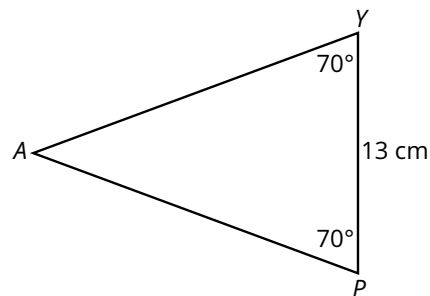
ACTIVITY

4.4

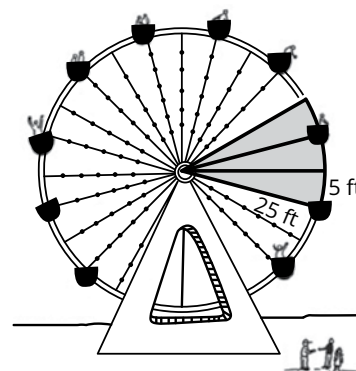
Solving Problems with Theorems

Use any theorems you have proven to solve each problem.
Explain each answer.

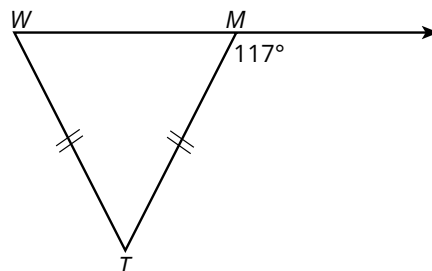
1. Calculate AP if the perimeter of $\triangle AYP$ is 43 cm.



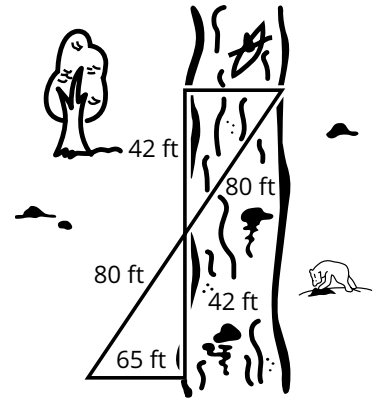
2. Lighting booms on a Ferris wheel consist of four steel beams that have cabling with light bulbs attached. These beams, along with three shorter beams, form the edges of three congruent isosceles triangles, as shown. Maintenance crews are installing new lighting along the four beams. Calculate the total length of lighting needed for the four beams.



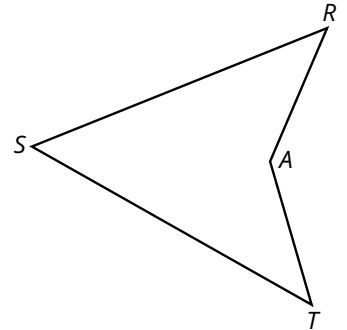
3. Calculate $m\angle T$.



4. What is the width of the river?



5. Given: $\overline{ST} \cong \overline{SR}$, $\overline{TA} \cong \overline{RA}$
Explain why $\angle T \cong \angle R$.



6. The Pacific 97 stamp is an isosceles right triangle shaped stamp made to celebrate the 1997 World Stamp Exhibition in San Francisco. Suppose the longest side of this stamp is 50 millimeters.

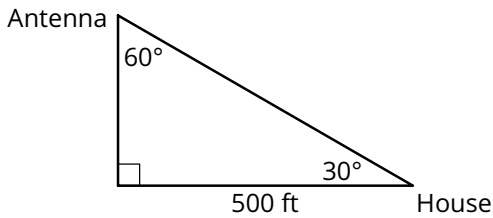


- a. Use the Pythagorean Theorem to determine the approximate length of the other sides of this stamp. Round your answer to the nearest tenth of a millimeter.
- b. Use the 45° - 45° - 90° triangle theorem to determine the approximate length of the other sides of this stamp. Round your answer to the nearest tenth of a millimeter.

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The World Stamp Exhibition event was held for the 150th anniversary of the start of the U.S. Postal Service in 1847.

.....



7. A broadcast antenna is situated on top of a tower. The signal travels from the antenna to your house so you can watch TV. Consider the diagram to calculate the height of the tower and the distance the signal travels.



8. This stamp was issued in the Netherlands. Suppose the length of each side of the Netherlands stamp is 40 millimeters. Use the 30° - 60° - 90° triangle theorem to determine the height of the stamp.



9. In 1929, Uruguay issued a triangular parcel post stamp with sides equal in measure. Suppose the height of the Uruguay stamp is 30 millimeters. Use the 30° - 60° - 90° triangle theorem to determine the length of the three sides of the stamp.

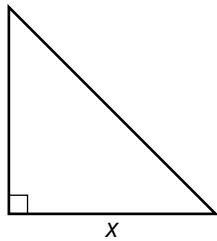


Talk the Talk

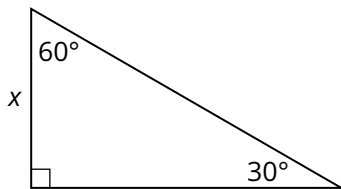
City of Bridges

1. Label the remaining sides of each triangle in terms of x .

a. 45° - 45° - 90° triangle



b. 30° - 60° - 90° triangle



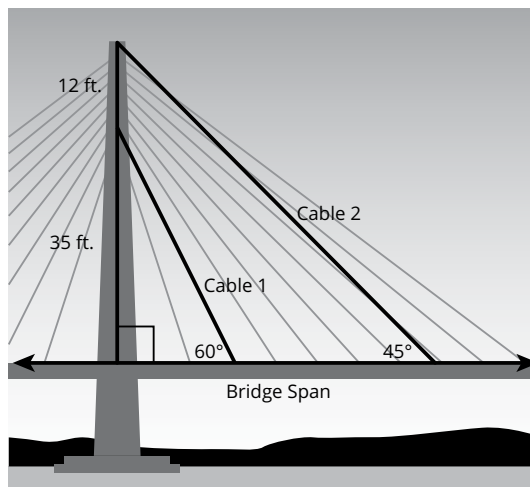
2. Given a 45° - 45° - 90° triangle, explain how to determine each unknown length.

a. The length of a leg given the length of the hypotenuse

b. The length of the hypotenuse given the length of a leg

3. Huyen is building a square bookshelf. She wants to add a diagonal support beam to the back to strengthen it. The diagonal divides the bookshelf into two 45° - 45° - 90° triangles. if each side of the bookshelf is 4 feet long, what must the length of the support beam be?

A cable stayed bridge has one or more towers erected above piers in the middle of the span. From these towers, cables stretch down diagonally (usually to both sides).



One cable forms a 60° angle with the bridge deck and is connected to the tower 35 feet above the bridge deck. A second cable forms a 45° angle with the bridge deck and is connected to the tower 12 feet above the point at which the first cable connects to the tower.

4. Calculate the length of both cables and the length of the bridge span from each cable to the tower.

Lesson 4 Assignment

Write

Provide an example to illustrate each term.

1. isosceles triangle base angle theorem
2. isosceles triangle base angle converse theorem
3. perpendicular bisector
4. 30° - 60° - 90° triangle theorem
5. 45° - 45° - 90° triangle theorem

Remember

The perpendicular bisector theorem states: "The points on a perpendicular bisector of a line segment are equidistant from the segment's endpoints."

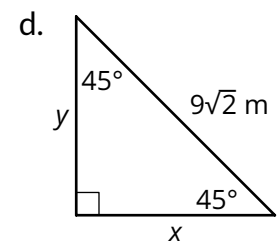
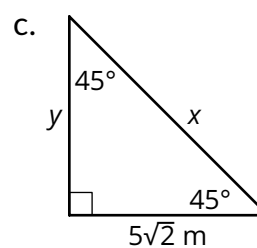
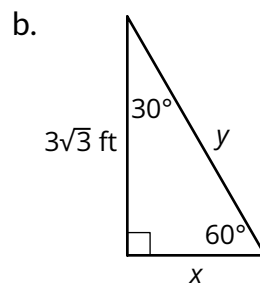
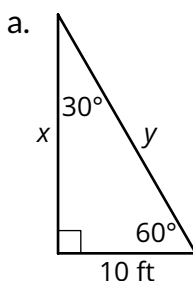
The isosceles triangle base angles theorem states: "If two sides of a triangle are congruent, then the angles opposite these sides are congruent."

The 30° - 60° - 90° triangle theorem states: "The length of the hypotenuse in a 30° - 60° - 90° triangle is 2 times the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg."

The 45° - 45° - 90° triangle theorem states: "The length of the hypotenuse in a 45° - 45° - 90° triangle is $\sqrt{2}$ times the length of a leg."

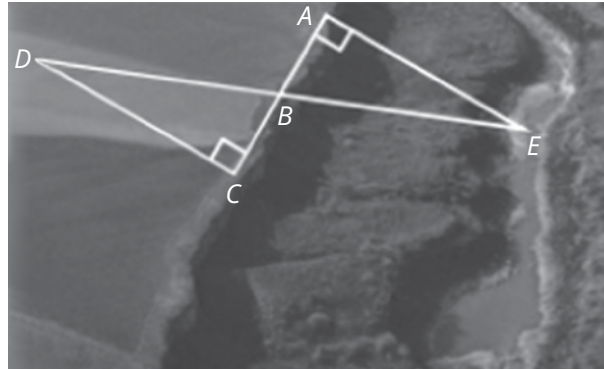
Practice

1. Solve for the unknown side lengths.

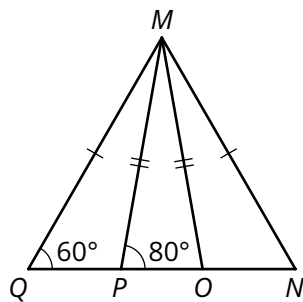


Lesson 4 Assignment

2. Olivia is hiking through the forest and she comes upon a canyon. She wants to know how wide the canyon is. She measures the distance between points A and B as 35 feet. Then, she measures the distance between points B and C as 35 feet. Finally, she measures the distance between points C and D as 80 feet. How wide is the canyon? Explain your reasoning.

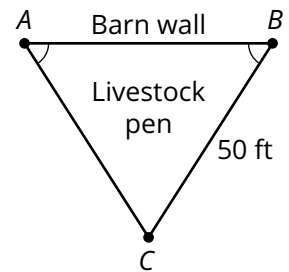


3. Explain why $m\angle NMO = 20^\circ$.



Lesson 4 Assignment

4. Daniela is building a livestock pen in the shape of a triangle. She is using one side of a barn for one of the sides of her pen and has already placed posts in the ground at points A, B, and C, as shown in the diagram. If she places fence posts every 10 feet, how many more posts does she need? Note: There will be no other posts placed along the barn wall.



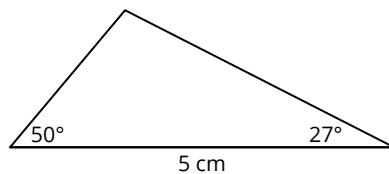
5. A mathematical society in India designed this stamp. The pyramidal design is an equilateral triangle. Suppose the height of the pyramidal design on the stamp is 42 millimeters. Determine the area of the pyramidal design on the stamp.



Lesson 4 Assignment

Prepare

1. Use a ruler and protractor to redraw this triangle increasing the 5 cm side to 7 cm.



2. How did increasing the length of the side affect the measure of the third angle?
3. How did increasing the length of the side affect the length of the other two sides?
4. Is the triangle you drew similar to the original triangle? Explain.

Lesson 4 Assignment

5. Use a ruler and protractor to draw a triangle that has two sides with the measure of 3 cm and 10 cm and an included angle with the measure of 70° .

6. Use a ruler and protractor to draw a triangle that has two sides with the measure of 3 cm and 10 cm and an included angle with the measure of 50° .

7. Without measuring, can you predict which triangle in Questions 5 and 6 have the longest third side?
Explain your reasoning.

8. Use a ruler to measure the third sides. Is your prediction correct?

5

Inverse, Contrapositive, Direct Proof, and Indirect Proof

OBJECTIVES

- Write the inverse and contrapositive of a conditional statement.
- Differentiate between direct and indirect proof.
- Use indirect proof.

KEY TERMS

- inverse
- contrapositive
- direct proof
- indirect proof or proof by contradiction
- hinge theorem
- hinge converse theorem

.....

You have determined the validity of conditional statements and their converses.

How can you determine the validity of the inverse and contrapositive of conditional statements?

Getting Started

The Inverse and the Contrapositive

.....
Recall that to state the converse, reverse the hypothesis and the conclusion.
.....

.....
Negating a statement is presenting the opposite of the statement.
.....

Every conditional statement written in the form “If p , then q ” has three additional conditional statements associated with it: the converse, the *contrapositive*, and the *inverse*. To state the **inverse**, negate the hypothesis and the conclusion. To state the **contrapositive**, negate the hypothesis and conclusion, and reverse them.

Conditional Statement	If p , then q .
Converse	If q , then p .
Inverse	If not p , then not q .
Contrapositive	If not q , then not p .

Recall that there are two reasons why a conclusion may be false. Either the assumed information is false, or the argument is not valid. In this problem, you will determine whether the inverse and contrapositive of some conditional statements are valid.

WORKED EXAMPLE

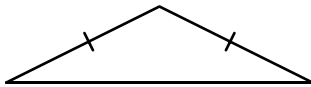
Consider the conditional statement:

If a triangle is an equilateral triangle, then the triangle is isosceles. The hypothesis is “a triangle is an equilateral triangle.” The conclusion is “the triangle is isosceles.”

The inverse of the conditional statement is:

If a triangle is not an equilateral triangle, then the triangle is not isosceles.

You can provide a counterexample as evidence that the inverse is not valid.



The triangle shown is not an equilateral triangle, because all three sides are not congruent, so the hypothesis of the inverse is true. However, the triangle shown is an isosceles triangle because two sides are congruent, so the conclusion of the inverse is false. Therefore, the inverse statement is not valid.

The contrapositive of the conditional statement is:

If a triangle is not isosceles, then the triangle is not an equilateral triangle. The contrapositive is valid. In a triangle that is not isosceles, no two sides are congruent. Therefore, it is not possible for all three sides to be congruent.

1. If a quadrilateral is a square, then the quadrilateral is a rectangle.

a. Hypothesis p :

b. Conclusion q :

c. Is the conditional statement true? Explain your reasoning.

d. Not p :

e. Not q :

f. Inverse:

g. Is the inverse true? Explain your reasoning.

h. Contrapositive:

i. Is the contrapositive true? Explain your reasoning.

2. If an integer is even, then the integer is divisible by two.

a. Hypothesis p :

b. Conclusion q :

c. Is the conditional statement true? Explain your reasoning.

d. Not p :

e. Not q :

f. Inverse:

g. Is the inverse true? Explain your reasoning.

h. Contrapositive:

i. Is the contrapositive true? Explain your reasoning.

3. If a polygon has six sides, then the polygon is a pentagon.

a. Hypothesis p :

b. Conclusion q :

c. Is the conditional statement true? Explain your reasoning.

d. Not p :

e. Not q :

f. Inverse:

g. Is the inverse true? Explain your reasoning.

h. Contrapositive:

i. Is the contrapositive true? Explain your reasoning.

4. If two lines intersect, then the lines are perpendicular.

a. Hypothesis p :

b. Conclusion q :

c. Is the conditional statement true? Explain your reasoning.

d. Not p :

e. Not q :

f. Inverse:

g. Is the inverse true? Explain your reasoning.

h. Contrapositive:

i. Is the contrapositive true? Explain your reasoning.

5. What do you notice about the validity of a conditional statement and the validity of its inverse?

6. What do you notice about the validity of a conditional statement and the validity of its contrapositive?

Proof by Contradiction

All of the proofs up to this point were *direct proofs*. A **direct proof** begins with the given information and works to the desired conclusion directly through the use of givens, definitions, properties, postulates, and theorems.

An **indirect proof**, or **proof by contradiction**, uses the contrapositive. If you prove the contrapositive true, then the original conditional statement is true. Begin by assuming the conclusion is false, and use this assumption to show one of the given statements is false, thereby creating a contradiction.

.....

Notice, you are trying to prove $\triangle CHA \not\cong \triangle CTA$. You assume the negation of this statement, $\triangle CHA \cong \triangle CTA$. This becomes the first statement in your proof, and the reason for making this statement is “assumption.”

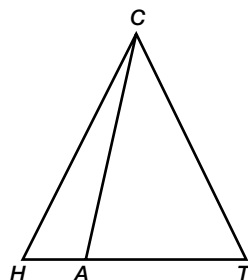
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WORKED EXAMPLE

Let's look at an example of an indirect proof.

Given: In $\triangle CHT$, $\overline{CH} \cong \overline{CT}$,
 $\overline{CA}$ does not bisect $\overline{HT}$

Prove: $\triangle CHA \not\cong \triangle CTA$



Statements	Reasons
1. $\triangle CHA \cong \triangle CTA$	1. Assumption
2. $\overline{CA}$ does not bisect $\overline{HT}$	2. Given
3. $\overline{HA} \cong \overline{TA}$	3. CPCTC
4. $\overline{CA}$ bisects $\overline{HT}$	4. Definition of bisect
5. $\triangle CHA \cong \triangle CTA$ is false	5. This is a contradiction. Step 4 contradicts step 2; the assumption is false
6. $\triangle CHA \not\cong \triangle CTA$ is true	6. Proof by contradiction

In step 5, the “assumption” is stated as “false.” The reason for making this statement is “contradiction.”

In an indirect proof:

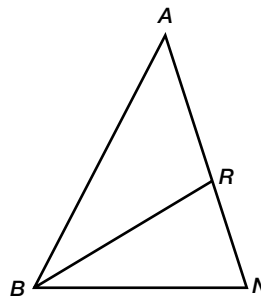
- State the assumption; use the negation of the conclusion, or “Prove” statement.
- Write the givens.
- Write the negation of the conclusion.
- Use the assumption, in conjunction with definitions, properties, postulates, and theorems, to prove a given statement is false, thus creating a contradiction.

Hence, your assumption leads to a contradiction; therefore, the assumption must be false. This proves the contrapositive.

1. Create an indirect proof of the following.

Given: $\overline{BR}$ bisects $\angle ABN$,
 $\angle BRA \not\cong \angle BRN$

Prove: $\overline{AB} \not\cong \overline{NB}$



Ask Yourself . . .

How can you organize and record your mathematical ideas?

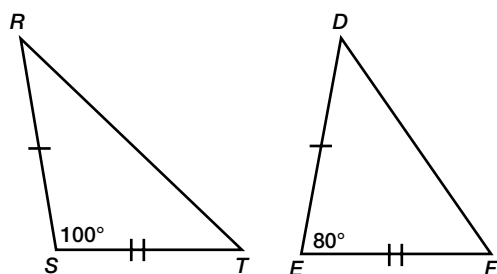
2. Create an indirect proof to show that a triangle cannot have more than one right angle.

ACTIVITY
5.2

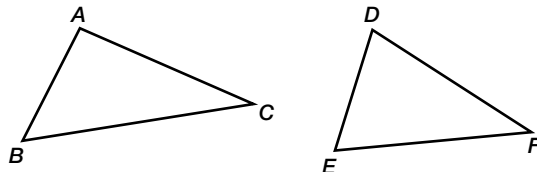
Hinge Theorem and Its Converse

The **hinge theorem** states: "If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first pair is larger than the included angle of the second pair, then the third side of the first triangle is longer than the third side of the second triangle."

In the two triangles shown, notice that $RS = DE$, $ST = EF$, and $\angle S > \angle E$. The hinge theorem says that $RT > DF$.



1. Use an indirect proof to prove the hinge theorem.



Given: $AB = DE$
 $AC = DF$
 $m\angle A > m\angle D$

Prove: $BC > EF$

Negating the conclusion, $BC > EF$, means that either BC is equal to EF , or BC is less than EF . Therefore, this theorem must be proven for both cases.

Case 1: $BC = EF$

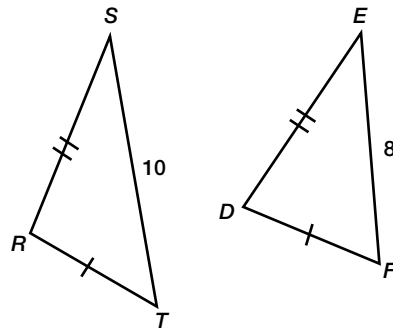
Case 2: $BC < EF$

a. Write the indirect proof for Case 1, $BC = EF$.

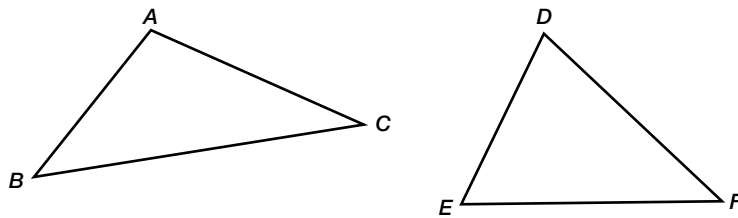
b. Write the indirect proof for Case 2, $BC < EF$.

The **hinge converse theorem** states: “If two sides of one triangle are congruent to two sides of another triangle and the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first pair of sides is larger than the included angle of the second pair of sides.”

In the two triangles shown, notice that $RT = DF$, $RS = DE$, and $ST > EF$. The hinge converse theorem guarantees that $m\angle R > m\angle D$.



2. Create an indirect proof to prove the hinge converse theorem.



Given: $AB = DE$

$AC = DF$

$BC > EF$

Prove: $m\angle A > m\angle D$

This theorem must be proven for two cases.

Case 1: $m\angle A = m\angle D$

Case 2: $m\angle A < m\angle D$

a. Create an indirect proof for Case 1, $m\angle A = m\angle D$.

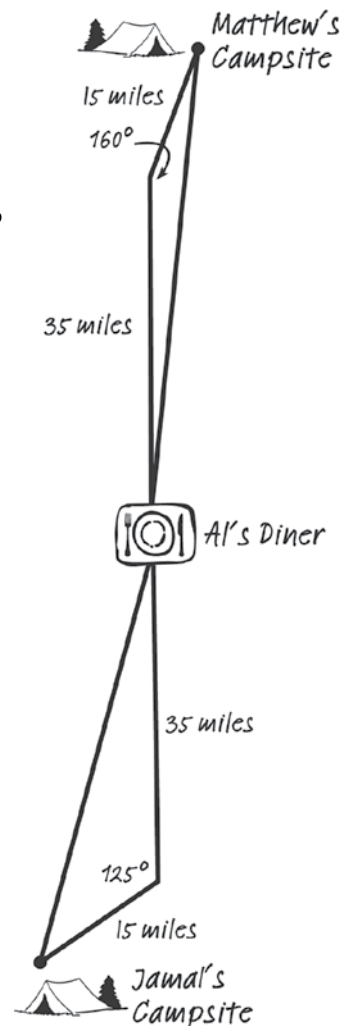
b. Create an indirect proof for Case 2, $m\angle A < m\angle D$.

Talk the Talk

It All Hinges on Reason

1. Matthew and Jamal's families are going camping for the weekend. Before heading out of town, they decide to meet at Al's Diner for breakfast. During breakfast, the boys try to decide which family will be further away from the diner "as the crow flies." "As the crow flies" is an expression based on the fact that crows, generally fly straight to the nearest food supply.

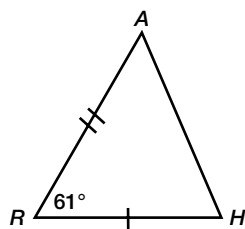
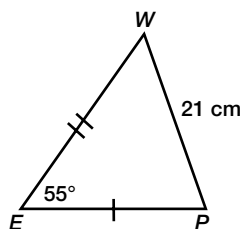
Matthew's family is driving 35 miles due north and taking an exit to travel an additional 15 miles northeast. Jamal's family is driving 35 miles due south and taking an exit to travel an additional 15 miles southwest. Use the diagram shown to determine which family is further from the diner. Explain your reasoning using the hinge theorem.



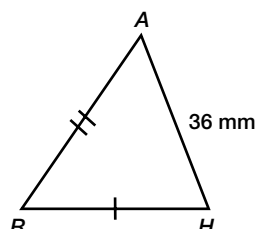
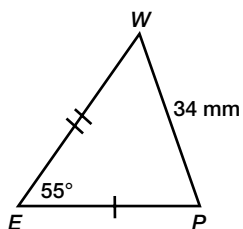
2. Which of the following is a possible length for AH : 20 cm, 21 cm, or 24 cm? Explain your choice.

Ask Yourself ...

What observations can you make?

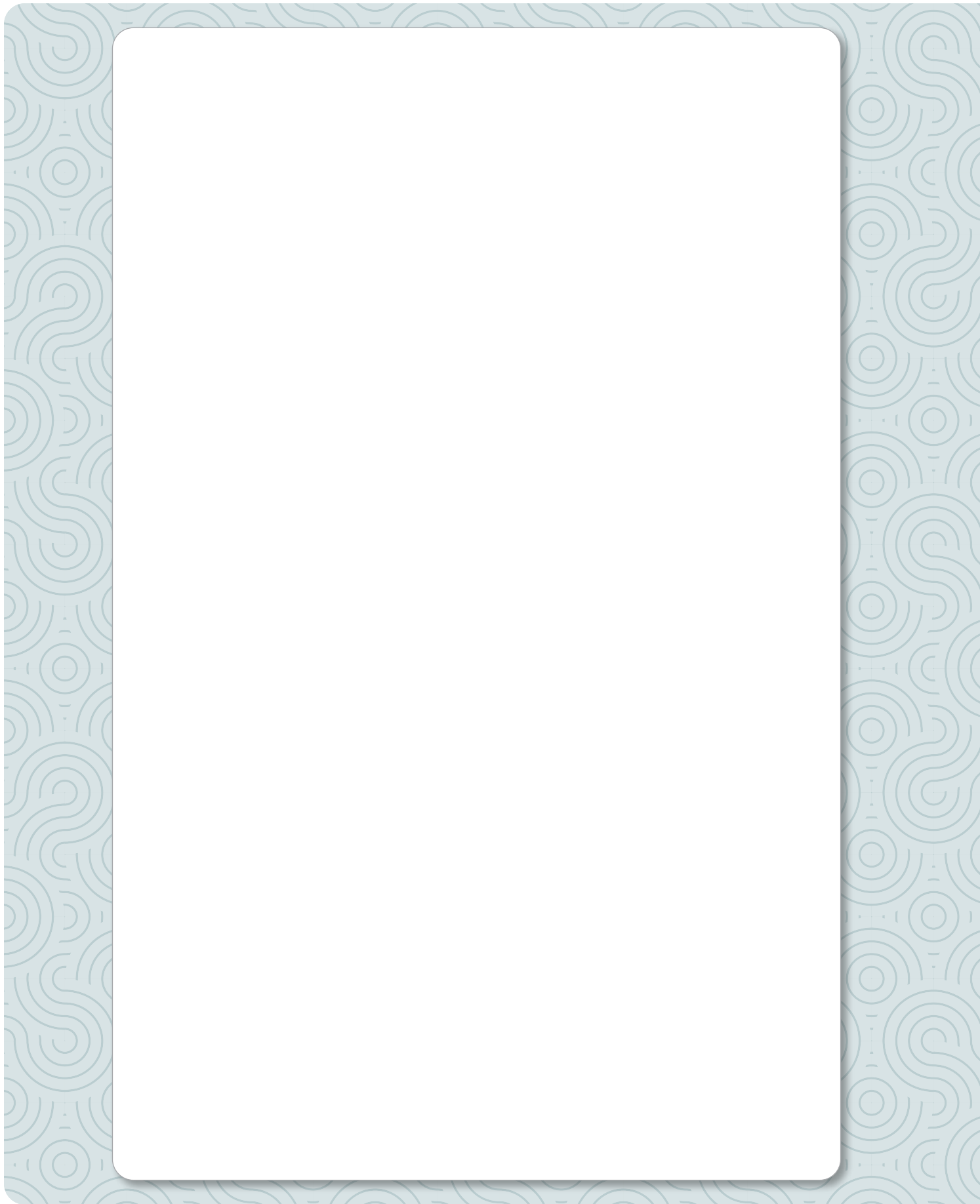


3. Which of the following is a possible angle measure for $\angle ARH$: 54° , 55° or 56° ? Explain your choice.



Consider the conditional statement: If a number is divisible by 4, then it is even.

4. Determine the validity of the conditional statement.
5. Write the inverse of the conditional statement and determine the validity.
6. Write the contrapositive of the conditional statement and determine the validity.



Lesson 5 Assignment

Write

Describe the hinge theorem and its converse in your own words.

Remember

Every conditional statement written in the form “If p , then q ” has three additional conditional statements associated with it.

Converse statement: if q then p .

Inverse statement: if not p , then not q .

Contrapositive statement: if not q , then not p .

When the conditional statement is true, the inverse may be true or false and the contrapositive is always true. When the conditional statement is false, the inverse may be true or false and the contrapositive is always false.

Lesson 5 Assignment

Practice

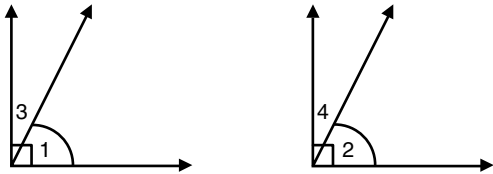
1. Consider the conditional statement “If a quadrilateral is a rectangle, then it is a parallelogram.”
 - a. Identify the hypothesis and the conclusion.
 - b. Is the conditional statement true? Explain.
 - c. Write the converse of the conditional statement. Is the converse true? Explain.
 - d. Write the inverse of the conditional statement. Is the inverse true? Explain.
 - e. Write the contrapositive of the conditional statement. Is the contrapositive true? Explain.

Lesson 5 Assignment

2. Consider the conditional statement “If a triangle is equilateral, then the triangle is equiangular.”
 - a. Identify the hypothesis and the conclusion.
 - b. Is the conditional statement true? Explain.
 - c. Write the converse of the conditional statement. Is the converse true? Explain.
 - d. Write the inverse of the conditional statement. Is the inverse true? Explain.
 - e. Write the contrapositive of the conditional statement. Is the contrapositive true? Explain.

Lesson 5 Assignment

3. Use an indirect two-column proof to show that the complements of congruent angles are congruent.



Given: $m\angle 1 = m\angle 2$, $m\angle 1 + m\angle 3 = 90^\circ$, $m\angle 2 + m\angle 4 = 90^\circ$

Prove: $m\angle 3 = m\angle 4$

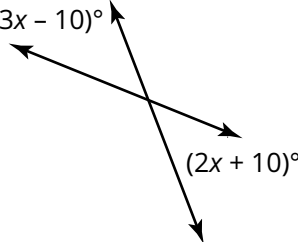
4. Write an indirect paragraph proof to show that an isosceles triangle cannot have a base angle that is a right angle.

Lesson 5 Assignment

Prepare

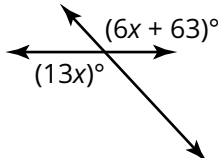
Determine the value of x .

1. $(3x - 10)^\circ$



$(2x + 10)^\circ$

2. $(6x + 63)^\circ$



$(13x)^\circ$

6

Angle Relationships Inside and Outside Circles

OBJECTIVES

- Determine the measures of arcs, central angles, and inscribed angles.
- Prove, both verbally and in writing, the inscribed angle theorem, the inscribed right triangle-diameter theorem, the inscribed quadrilateral-opposite angles theorem, the interior angles of a circle theorem and the exterior angles of a circle theorem.
- Determine the measures of angles formed by two chords, two secants, a tangent and a secant, or two tangents.
- Construct a tangent line to a circle from a point outside the circle.
- Apply the tangent to a circle theorem to solve problems.

.....

You have reasoned about and proved a number of angle relationships for lines and polygons.

How can these angle relationships help you reason about and prove theorems involving the angle and arc measures of circles?

KEY TERMS

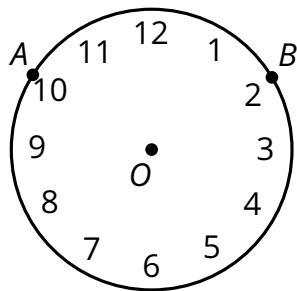
- degree measure of an arc
- adjacent arcs
- arc addition postulate
- inscribed angle theorem
- inscribed right triangle-diameter theorem
- inscribed quadrilateral-opposite angles theorem
- interior angles of a circle theorem
- exterior angles of a circle theorem
- tangent to a circle theorem

Getting Started

Look at the Time

Recall that the degree measure of a circle is 360° .

Each minor arc of a circle is associated with and determined by a specific central angle. The **degree measure of an arc** is the same as the degree measure of its central angle. For example, if $\angle PRQ$ is a central angle and $m\angle PRQ = 30^\circ$, then $m\widehat{PQ} = 30^\circ$.



1. Imagine the face of a clock. Consider that it is a circle with center O . Point A on the circle corresponds with the number 10 on the clock and point B corresponds with the number 2.
 - a. Draw a central angle using the given points. Identify the central angle you drew and its corresponding minor arc.
 - b. Without using a protractor, determine the central angle measure and the measure of the minor arc. Explain your reasoning.
 - c. Plot and label point C such that you would create major arc ACB and $\widehat{BC} \cong \widehat{AB}$. Explain your reasoning.

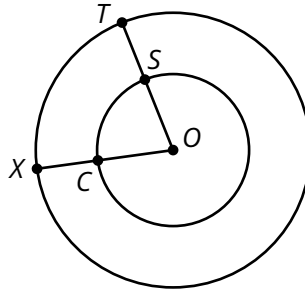
.....
Adjacent arcs are two arcs of the same circle sharing a common endpoint.
.....

The **arc addition postulate** states: "The measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs."

2. Consider circle O in Question 1.
- Plot and label point D on the circle such that it corresponds to the number 7 on the clock. Use the arc addition postulate to determine the measure of $\widehat{BD}$.
 - At which numbers on the clock could you plot point E such that $\widehat{BD} \cong \widehat{AE}$? Explain your reasoning.
3. If the measures of two central angles of the same circle (or congruent circles) are equal, are their corresponding minor arcs congruent? Explain your reasoning.
4. If the measures of two minor arcs of the same circle (or congruent circles) are equal, are their corresponding central angles congruent? Explain your reasoning.



5. Destiny explains to her classmate that $\widehat{SC}$ is congruent to $\widehat{TX}$. How did Destiny arrive at this conclusion? Is Destiny correct? Explain your reasoning.

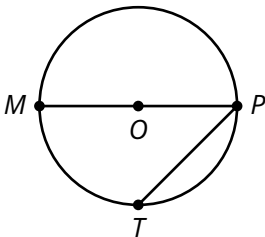
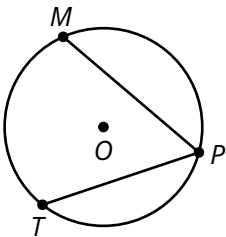
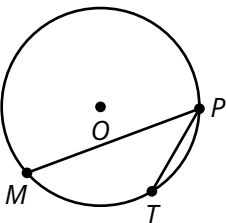


ACTIVITY
6.1

The Inscribed Angle Theorem

Recall the conjecture you made in a previous topic about the measure of an inscribed angle related to the measure of its intercepted arc. You reasoned through recorded observations that the measure of the inscribed angle is half the measure of the intercepted arc.

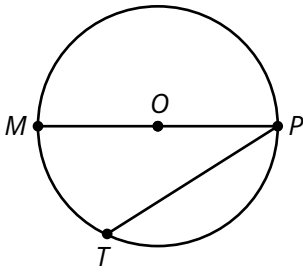
Inscribed angles formed by two chords can be drawn three different ways with respect to the center of the circle.

Case 1	Case 2	Case 3
$\angle MPT$ is inscribed in such a way that the center point lies on one side of the inscribed angle. 	$\angle MPT$ is inscribed in such a way that the center point lies on the interior of the inscribed angle. 	$\angle MPT$ is inscribed in such a way that the center point lies on the exterior of the inscribed angle. 

Remember ...

An inscribed angle is an angle whose vertex is on a circle and whose sides contain chords of the circle. An intercepted arc is a portion of the circumference of the circle located on the interior of the angle whose endpoints lie on the sides of an angle.

1. What do the intercepted arcs in each case have in common?
2. Is there another possible way inscribed angles can be drawn? Use an example to justify your answer.



To prove your conjecture that the measure of an inscribed angle is half the measure of the intercepted arc, you must prove each case. Let's consider Case 1.

Given: $\angle MPT$ is inscribed in circle O .

Point O lies on diameter $\overline{PM}$.

Prove: $m\angle MPT = \frac{1}{2} m\widehat{MT}$

You can use construction and algebraic reasoning to create a proof plan.

WORKED EXAMPLE

Let's consider how to prove that

$$m\angle MPT = \frac{1}{2} m\widehat{MT} \text{ for Case 1.}$$

Connect points O and T to form radius $\overline{OT}$.

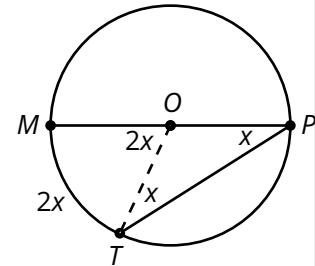
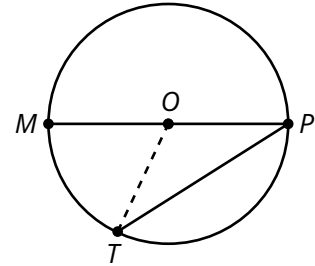
Label $m\angle MPT$ as x .

$m\angle OPT = x$ because $\triangle OTP$ is isosceles.

Then, $m\angle MOT = 2x$ and $m\widehat{MT} = 2x$.

Therefore, $m\widehat{MT} = 2(m\angle MPT)$ or

$$m\angle MPT = \frac{1}{2} m\widehat{MT}.$$



.....

You have used the addition and subtraction properties of equality as reasons in proofs. You can also use the multiplication and division properties of equality as reasons in proofs.

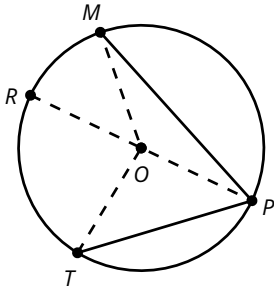
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3. Complete a formal proof for Case 1.

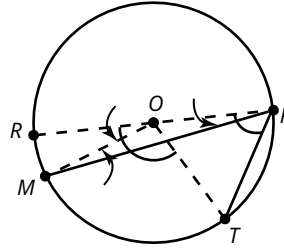
Statements	Reasons
1. $\angle MPT$ is inscribed in circle O . $m\angle MPT = x$ Point O lies on diameter $\overline{PM}$.	1. Given
2. Connect points O and T to form radius $\overline{OT}$.	2. Construction

4. Use algebraic reasoning to demonstrate your conjecture for the other two cases.

a. Case 2



b. Case 3



Notice and Wonder

Notice the constructions that have already been added to each diagram. How can they be helpful as you develop your reasoning?

Because you have proved your conjecture is true, you can now refer to it as a theorem. The **inscribed angle theorem** states: “The measure of an inscribed angle is half the measure of its intercepted arc.”

5. State the converse of the inscribed angle theorem. Do you think the converse is true?

ACTIVITY
6.2

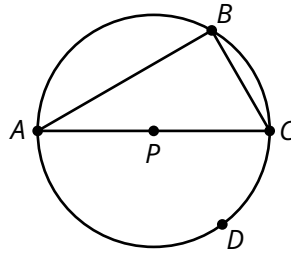
Theorems of Inscribed Polygons

Remember ...

An inscribed polygon is a polygon drawn inside a circle such that each vertex of the polygon touches the circle.

There is a special case of the inscribed angle theorem that applies to a specific type of inscribed polygon.

Consider $\triangle ABC$ that is inscribed in circle P .



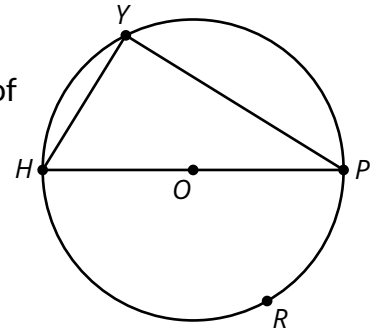
1. What kind of triangle is $\triangle ABC$? How do you know?
2. Write a conjecture about the type of triangle inscribed in a circle when one side of the triangle is a diameter.
3. Write the converse of the conjecture you wrote in Question 2. Do you think this statement is also true?

It would appear that $\triangle ABC$ is a right triangle and that $\angle B$ is a right angle. In general, it appears that if a triangle is inscribed in a circle such that one side of the triangle is a diameter of the circle, then the triangle is a right triangle.

4. Create a proof of this conjecture.

Given: $\triangle HYP$ is inscribed in circle O such that $\overline{HP}$ is the diameter of the circle.

Prove: $\triangle HYP$ is a right triangle.



Because you have proved that the relationship is true, you can now refer to it as a theorem. The **inscribed right triangle-diameter theorem** states: “If a triangle is inscribed in a circle such that one side of the triangle is a diameter of the circle, then the triangle is a right triangle.”

In a previous topic, you made a conjecture that the opposite angles of inscribed quadrilaterals are supplementary. This observation can be stated as a conjecture and then proved using the inscribed angle theorem.

5. Create a proof of this conjecture.

Given: Quadrilateral $QUAD$ is inscribed in circle O .

Prove: $\angle Q$ and $\angle A$ are supplementary angles.

$\angle U$ and $\angle D$ are supplementary angles.

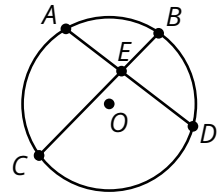
Because you have proved that this conjecture is true, you can now refer to it as a theorem. The **inscribed quadrilateral-opposite angles theorem** states: "If a quadrilateral is inscribed in a circle, then the opposite angles are supplementary."

ACTIVITY
6.3

Interior Angles of a Circle Theorem

The vertex of an angle can be located inside of a circle, outside of a circle, or on a circle. In this activity, you will explore angles with a vertex located inside of a circle.

1. Circle O with chords $\overline{AD}$ and $\overline{BC}$ is shown.



$$m\widehat{BD} = 70^\circ$$

$$m\widehat{AC} = 110^\circ$$

a. Consider $\angle BED$. How is this angle different from the angles that you have investigated so far? How is this angle the same?

b. Can you determine the measure of $\angle BED$ with the information you have so far? If so, how? Explain your reasoning.

c. Draw chord $\overline{CD}$. Use the information given in the figure to name the measures of any angles that you do know. Explain your reasoning.

d. How does $\angle BED$ relate to $\triangle CED$?

e. Write a statement to show the relationship between $m\angle BED$, $m\angle EDC$, and $m\angle ECD$.

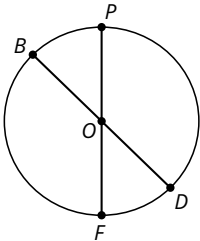
f. What is the measure of $\angle BED$?

g. Describe the measure of $\angle BED$ in terms of the measure of the arc intercepted by $\angle BED$ and the arc intercepted by the vertical angle of $\angle BED$.

2. Draw a circle with two chords that intersect inside it. Use measuring tools to determine the measure of an angle formed by the intersecting chords and the measures of the arcs that are intercepted by the angle and its vertical angle.

3. Consider the different measures you determined in the circles from Questions 1 and 2. Look for patterns and make a conjecture about the measure of an interior angle of a circle and the measures of the arcs that are intercepted by the angle and its vertical angle.

Let's first explore your conjecture for a circle where the vertex of the interior angle is at the center of the circle.



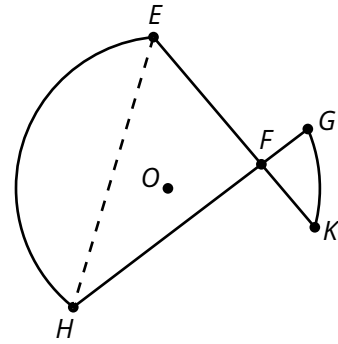
4. Consider circle O with diameters BD and PF . If $\angle BOP$ and $\angle FOD$ form vertical angles, then $m\angle BOP = \frac{1}{2}(m\widehat{BP} + m\widehat{FD})$. Use reasoning to demonstrate why this is true.

You've shown that the conjecture is true for two chords intersecting at the center of the circle. Now let's explore whether the conjecture holds true for any two chords that intersect in a circle.

5. Use algebraic reasoning to prove the conjecture.

Given: Chords $\overline{EK}$ and $\overline{GH}$ intersect at point F
in circle O .

Prove: $m\angle KFH = \frac{1}{2} (m\widehat{HK} + m\widehat{EG})$



.....
Notice and Wonder

An auxiliary chord is drawn from point E to point H . How does this chord help you reason about the measure of $\angle KFH$?
.....

Because you have proved that this conjecture is true, you can now refer to it as a theorem. The **interior angles of a circle theorem** states: "If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the interior of the circle, then the measure of the angle is half of the sum of the measures of the arcs intercepted by the angle and its vertical angle."

Exterior Angles of a Circle Theorem

In this activity, you will explore angles with a vertex located outside of a circle. Before you can prove the conjectures you will make about these types of angles, you should be able to construct a tangent line to a circle through a point outside the circle.

1. Construct tangent lines to a circle through a point outside of the circle.

Step 1: Draw a circle with center point C and locate point P outside of the circle.

Step 2: Draw $\overline{PC}$.

Step 3: Construct the perpendicular bisector of $\overline{PC}$.

Step 4: Label the midpoint of the perpendicular bisector of $\overline{PC}$ point M .

Step 5: Adjust the radius of your compass to the distance from point M to point C .

Step 6: Place the compass point on point M , and cut two arcs that intersect circle C .

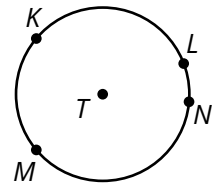
Step 7: Label the two points at which the arcs cut through circle C as point A and point B .

Step 8: Connect point P and A to form tangent line PA and connect point P and B to form tangent line PB .

Line PA and line PB are tangent to circle C .

Let's explore how the measure of an angle outside a circle, such as $\angle APB$ that you constructed, relates to the measures of the arcs its intercepts.

2. Circle T with points K , L , M , and N is shown.



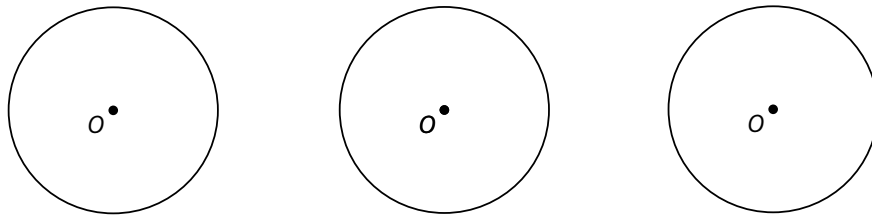
$$m\widehat{KM} = 80^\circ$$

$$m\widehat{LN} = 30^\circ$$

- a. Draw secants $\overleftrightarrow{KL}$ and $\overleftrightarrow{MN}$. Where do the secants intersect? Label this point as point P on the figure.
- b. Draw chord $\overline{KN}$. Can you determine the measure of $\angle KPM$ with the information you have so far? If so, how? Explain your reasoning.
- c. Use the information given in the figure to name the measures of any angles that you do know. Explain how you determined your answers.
- d. How does $\angle KPN$ relate to $\triangle KPN$?
- e. Write a statement to show the relationship between $m\angle KPN$, $m\angle NKP$, and $m\angle KNM$.
- f. What is the measure of $\angle KPN$?
- g. Describe the measure of $\angle KPM$ in terms of the measures of both arcs intercepted by $\angle KPM$.

An angle with a vertex located in the exterior of a circle can be formed by a secant and a tangent, two secants, or two tangents.

3. For each case, draw and label the exterior angle described. Then, determine the measure of the exterior angle and the measures of the arcs that are intercepted by the angle.
 - a. Case 1: exterior angle formed by a secant and a tangent
 - b. Case 2: exterior angle formed by two secants
 - c. Case 3: exterior angle formed by two tangents



4. Consider the different measures you determined in the circles from Questions 2 and 3. Look for patterns and make a conjecture about the measure of an exterior angle of a circle and the measure of the arcs that are intercepted by the angle.

It appears that if an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the exterior of the circle, then the measure of the angle is half of the difference of the measures of the arcs intercepted by the angle.

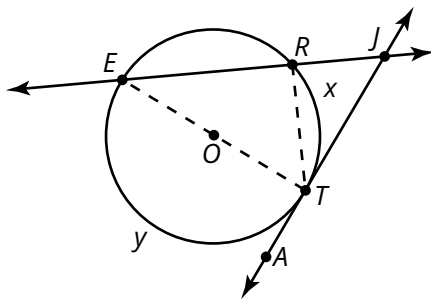
To prove this conjecture, you must prove each of the three cases.
Analyze the proof of Case 1 in the Worked Example.

WORKED EXAMPLE

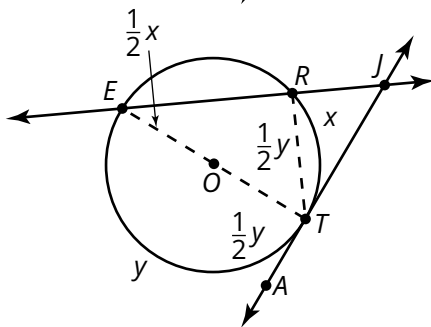
Case 1

Given: Secant $\overline{EX}$ and tangent $\overline{TJ}$ intersect at point J .

Prove: $m\angle EJT = \frac{1}{2}(m\widehat{ET} - m\widehat{RT})$



Label the measure of $\widehat{RT}$ as x and the measure of $\widehat{ET}$ as y .



This means $m\angle TER = \frac{1}{2}x$ and $m\angle ETA = \frac{1}{2}y$ since each is an inscribed angle of the intercepted arc.

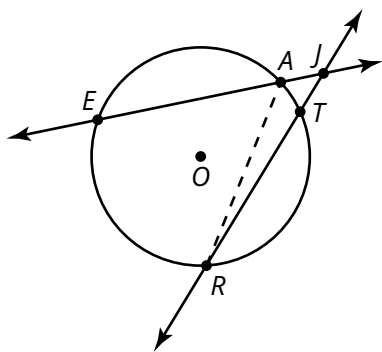
The exterior angle of $\triangle TEJ$ is $\angle ETA$, so $m\angle ETA = m\angle EJT + m\angle TER$.

Using substitution, $\frac{1}{2}y = m\angle EJT + \frac{1}{2}x$.

Therefore, $m\angle EJT = \frac{1}{2}y - \frac{1}{2}x$,

or $m\angle EJT = \frac{1}{2}(y - x)$.

$m\angle EJT = \frac{1}{2}(m\widehat{ET} - m\widehat{RT})$.

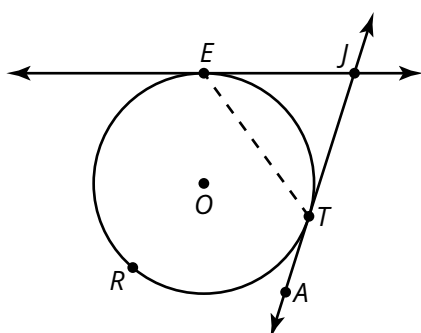


5. Reason algebraically to prove the remaining cases of the **exterior angles of a circle theorem**.

a. Case 2

Given: Secants $\overleftrightarrow{EJ}$ and $\overleftrightarrow{RJ}$ intersect at point J .

Prove: $m\angle EJR = \frac{1}{2}(m\widehat{ER} - m\widehat{AT})$



b. Case 3

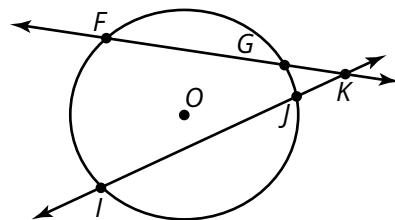
Given: Tangents $\overleftrightarrow{EJ}$ and $\overleftrightarrow{AJ}$ intersect at point J .

Prove: $m\angle EJT = \frac{1}{2}(m\widehat{ERT} - m\widehat{ET})$

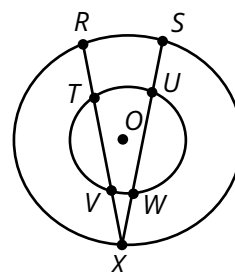
Because you have proved this relationship is true, you can now refer to it as a theorem. The **exterior angles of a circle theorem** states: "If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the exterior of the circle, then the measure of the angle is half of the difference of the measures of the arcs intercepted by the angle."

6. Use the diagram shown to determine the measure of each angle or arc.

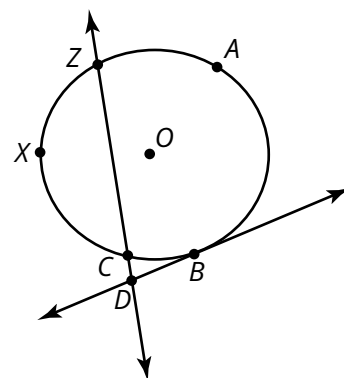
- a. Determine $m\widehat{FI}$.
 $m\angle K = 20^\circ$
 $m\widehat{GJ} = 80^\circ$



- b. Determine $m\angle X$.
 $m\widehat{VW} = 40^\circ$
 $m\widehat{TU} = 85^\circ$



- c. Determine $m\angle D$.
 $m\widehat{ZXC} = 120^\circ$
 $m\widehat{CB} = 30^\circ$



Tangent to a Circle Theorem

Remember...

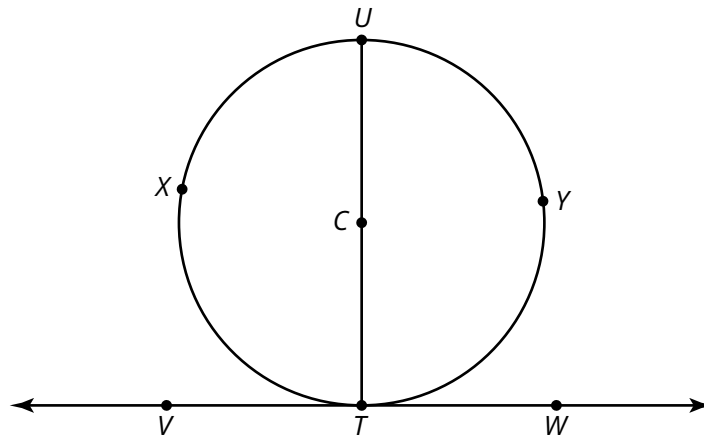
An inscribed angle is an angle whose measure is half the measure of its intercepted arc.

Remember...

A chord is a segment of a secant.

Think back on the first case of exterior angles you drew in the previous activity involving a secant and a tangent. There is a special case when the secant contains the diameter of the circle.

1. Consider $\angle UTV$ with vertex located on circle C . Line VW is drawn tangent to circle C at point T .



- a. Determine $m\widehat{UXT}$ and $m\widehat{UYT}$. Explain your reasoning.
- b. Determine $m\angle UTV$ and $m\angle UTW$. Explain your reasoning.
- c. Make a conjecture about the angles formed at the point of tangency.

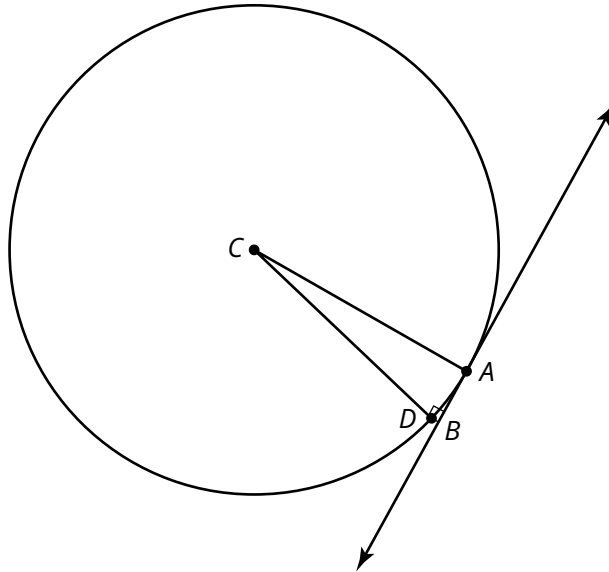
It appears that when a line is drawn tangent to a circle, the angles formed at the point of tangency are right angles, and therefore the radius drawn to the point of tangency is perpendicular to the tangent line.

This relationship can be proved and then stated as a theorem.

The proof of this relationship is completed by contradiction. Remember a proof by contradiction begins with an assumption that is the opposite of what you would like to prove. Using the assumption and its implications, you arrive at a contradiction. When this happens, the proof is complete.

WORKED EXAMPLE

Line segment CA is a radius of circle C . Point A is the point at which the radius intersects the tangent line.



Step 1: Assumption: The tangent line is not perpendicular to the radius $\overline{CA}$ of the circle.

Step 2: Point B , another point on the tangent line, is the point at which $\overline{CB}$ is perpendicular to the tangent line.

Step 3: Consider right triangle CBA with hypotenuse CA and leg CB , so $CA > CB$.

Step 4: Impossible!! $CB > CA$ because $CB = \text{length of radius } (CD) + DB$.

The assumption is incorrect; therefore, the tangent line is perpendicular to the radius $\overline{CA}$ of the circle.

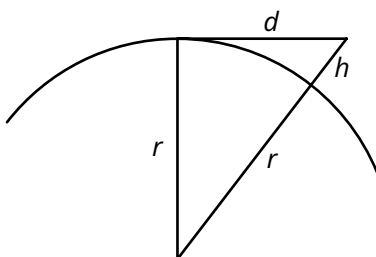
This completes the proof of the tangent to a circle theorem.

Because you have proved that this relationship is true, you can now refer to it as a theorem. The **tangent to a circle theorem** states: “A line drawn tangent to a circle is perpendicular to a radius of the circle drawn to the point of tangency.”

You can use the tangent to a circle theorem to solve problems.

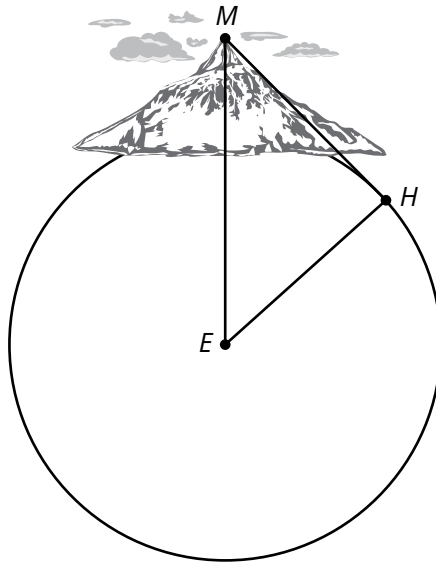
2. When you are able to see past buildings and hills or mountains—when you can look all the way to the horizon, how far is that? You can use the Pythagorean Theorem to help you tell.

Imagine you are standing on the surface of the Earth and you have a height of h . The distance to the horizon is given by d in the diagram shown, and r is the radius of Earth.



Using your height, create a formula you can use to determine how far away the horizon is.

3. Gabriela is standing at the top of Mount Everest, which has an elevation of 29,029 feet. Her eyes are 5 feet above ground level. The radius of Earth is approximately 3960 miles. How far can Gabriela see on the horizon?

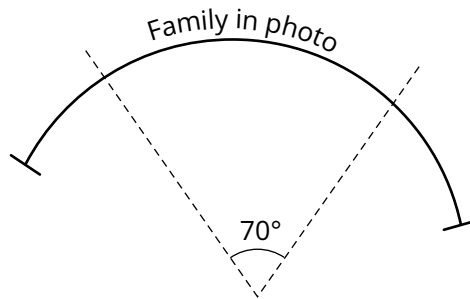


ACTIVITY
6.6

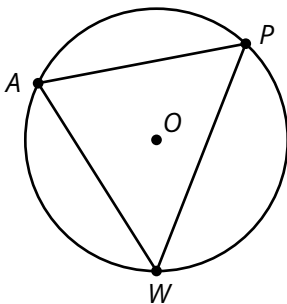
Determining Measures Inside and Outside Circles

You can apply the theorems you know involving circles to solve problems.

1. Hannah wants to take a family picture. Her camera has a 70° field of view, but to include the entire family in the picture, she needs to cover a 140° arc. Explain what Hannah needs to do to fit the entire family in the picture. Use the diagram to draw the solution.

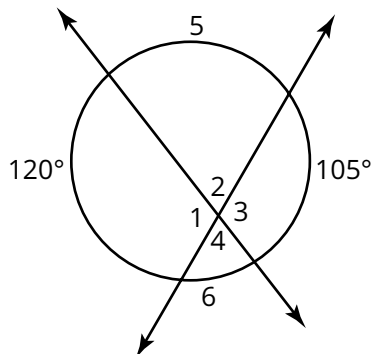


2. Yesterday, Ms. Sanchez taught her students how to determine the measure of an inscribed angle. William told his classmate that he finally understood why a circle is always 360° . Use $\triangle PAW$ drawn in the diagram to explain what William was thinking.



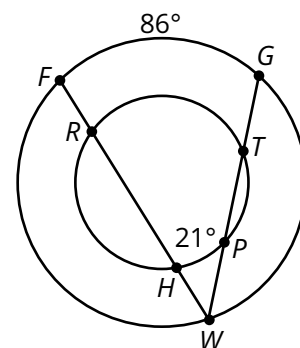


3. Madison says that using only the given information in the diagram, she can determine all the unknown measures of the circle. Is she correct? Justify your reasoning.

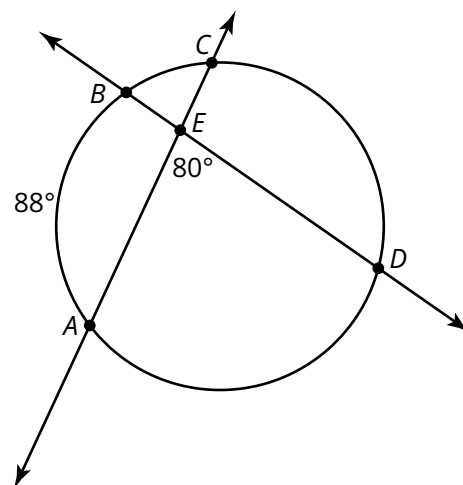


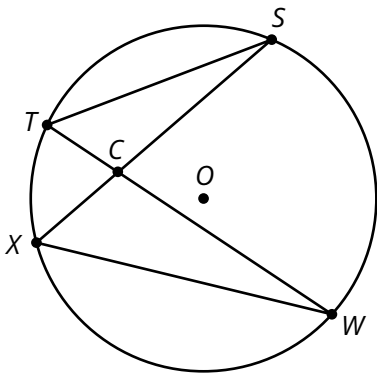
4. Apply the theorems you know to determine each measure.

- a. Determine $m\widehat{RT}$.
 $m\widehat{FG} = 86^\circ$
 $m\widehat{HP} = 21^\circ$

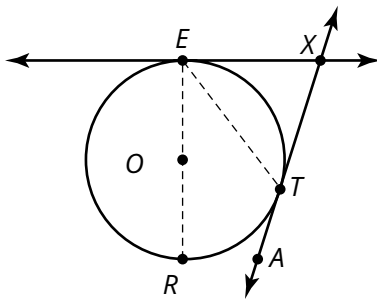


- b. Determine $m\widehat{CD}$.
 $m\widehat{AB} = 88^\circ$
 $m\angle AED = 80^\circ$

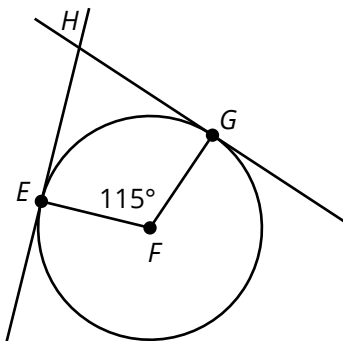




- c. Determine $m\angle TCX$.
 $m\angle TSX = 17^\circ$
 $m\angle STW = 50^\circ$



- d. Determine $m\angle EXT$.
 $m\angle RET = 40^\circ$



- e. Determine $m\angle EHG$.
 $m\angle EFG = 115^\circ$

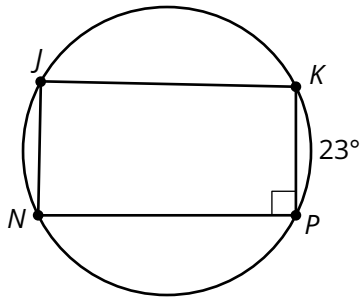


Talk the Talk

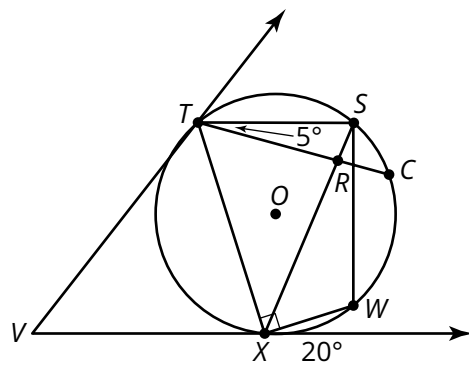
Circular Reasoning

1. What additional information can you conclude about the measures of the arcs and angles in each diagram? Explain your reasoning.

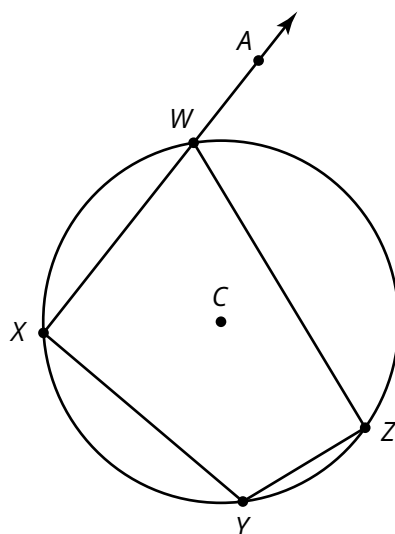
a.



b.



2. Angel raises his hand and informs Ms. Rodriguez that he has discovered another property related to the angles of an inscribed quadrilateral. Angel shows his teacher the diagram shown.



He claims that the measure of any exterior angle of the quadrilateral is equal to the measure of the opposite interior angle in the quadrilateral. In other words, $m\angle AWZ = m\angle Y$.

Explain Angel's reasoning.

Lesson 6 Assignment

Write

Define each theorem in your own words.

1. inscribed angle theorem
2. inscribed right triangle-diameter theorem
3. inscribed quadrilateral-opposite angles theorem
4. interior angles of a circle theorem
5. exterior angles of a circle theorem
6. tangent to a circle theorem

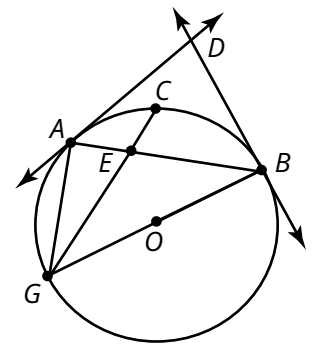
Remember

- The measure of an inscribed angle is half the measure of the arc it intercepts.
- The measure of an interior angle of a circle is half the sum of the measures of its intercepted arc and its vertical angle's intercepted arc.
- The measure of an exterior angle of a circle is half the difference of the measures of its intercepted arcs.
- A radius drawn to the point of tangency is perpendicular to the tangent line.

Practice

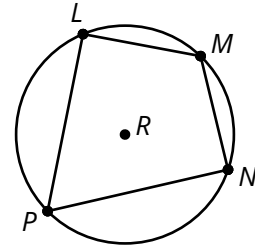
1. Consider circle O with diameter $\overline{GB}$. Line AD is tangent to circle O at point A and $\overleftrightarrow{DB}$ is tangent to circle O at point B . The measure of $\angle GBA$ is 38° . Chord $\overline{GC}$ bisects $\angle G$. Determine each measure.

- | | |
|--------------------|--------------------|
| a. $m\angle A$ | b. $m\angle AGB$ |
| c. $m\widehat{AG}$ | d. $m\widehat{AB}$ |
| e. $m\angle ADB$ | f. $m\widehat{AC}$ |

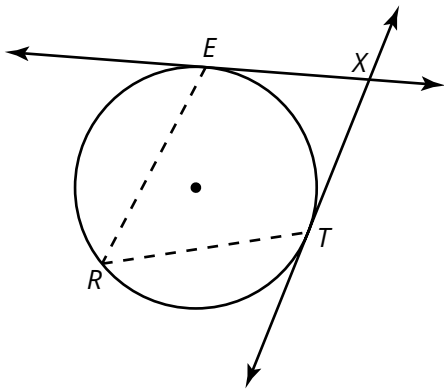


Lesson 6 Assignment

2. In the figure shown, quadrilateral $LMNP$ is inscribed in circle R , $m\angle P = 57^\circ$, and $m\angle L = m\angle N$. What are $m\angle M$, $m\angle L$, and $m\angle N$? Explain your reasoning.



3. Explain how knowing $m\angle ERT$ can help you determine $m\angle EXT$.



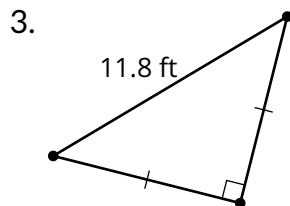
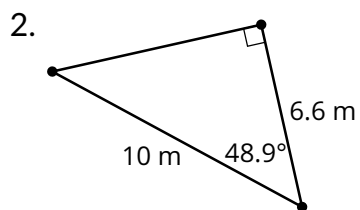
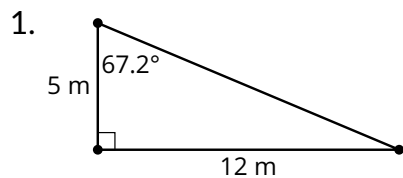
Lesson 6 Assignment

4. Construct a circle and label the center Z.
 - a. Construct a tangent line to circle Z through a point X. Label the point of tangency as point Y.
 - b. What is the measure of $\angle ZYX$? Explain your reasoning.

Lesson 6 Assignment

Prepare

Determine all the angle measures and side lengths of each right triangle.



Justifying Line and Angle Relationships

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Justifying Line and Angle Relationship* topic by:

TOPIC 2 <i>Justifying Line and Angle Relationships</i>	Beginning of Topic	Middle of Topic	End of Topic
using the properties of congruence and equality in geometric proofs.			
correctly interpreting diagrams, marking given information and recognizing what can and cannot be assumed.			
proving that vertical angles, alternate interior angles, and corresponding angles are congruent and that same-side interior angles are supplementary.			
proving that lines are parallel when alternate interior angles and corresponding angles are congruent or when same-side interior angles are supplementary.			
proving that points on a perpendicular bisector of a line segment are equidistant from the segment's endpoints.			
proving that the measures of the interior angles of a triangle sum to 180° and that the measure of an exterior angle of a triangle is equal to the sum of its remote interior angles.			
using the perpendicular bisector theorem to prove that base angles of an isosceles triangle are congruent.			

continued on the next page

TOPIC 2 SELF-REFLECTION *continued*

TOPIC 2 <i>Justifying Line and Angle Relationships</i>	Beginning of Topic	Middle of Topic	End of Topic
proving that the midsegment of a triangle is parallel to and half the length of the base.	☐	☐	☐
proving that the opposite sides of a parallelogram are parallel and congruent.	☐	☐	☐
proving that the diagonals of a parallelogram bisect each other.	☐	☐	☐
proving that the diagonals of a rectangle are congruent.	☐	☐	☐
recognizing and proving that an inscribed angle is half the measure of its intercepted arc and the central angle that intercepts the same arc.	☐	☐	☐
proving theorems related to the measure of angles formed at the intersection of chords, secants, and/or tangents.	☐	☐	☐
writing the inverse and contrapositive of a conditional statement.	☐	☐	☐
differentiating between direct and indirect proof.	☐	☐	☐
using indirect proof.	☐	☐	☐
constructing a tangent to a circle from a point outside of the circle.	☐	☐	☐
proving that the opposite angles of an inscribed quadrilateral are supplementary.	☐	☐	☐

continued on the next page

TOPIC 2 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Justifying Line and Angle Relationships* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 2 SUMMARY

Justifying Line and Angle Relationships Summary

LESSON

1

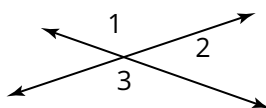
Forms of Proof

Many properties of real numbers can be applied in geometry. These properties are important when making conjectures and proving new theorems.

Property	Statements
addition property of equality	If a , b , and c are real numbers and $a = b$, then $a + c = b + c$.
subtraction property of equality	If a , b , and c are real numbers and $a = b$, then $a - c = b - c$.
reflexive property of equality	If a is a real number, then $a = a$.
substitution property of equality	If a and b are real numbers and $a = b$, then a can be substituted for b .
transitive property of equality	If a , b , and c are real numbers, $a = b$, and $b = c$, then $a = c$.

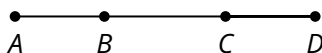
Angle measures:

If $m\angle 1 = m\angle 3$, then $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 2$.



Distances:

If $AB = CD$, then $AB + BC = CD + BC$.

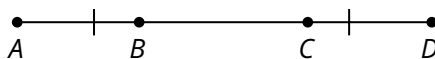


KEY TERMS

- addition property of equality
- subtraction property of equality
- substitution property of equality [propiedad de sustitución de la igualdad]
- transitive property of equality [propiedad transitiva de la igualdad]
- flow chart proof
- two-column proof [prueba de dos columnas]
- right angle congruence postulate
- congruent supplement theorem [teorema del suplemento congruente]
- vertical angle theorem
- paragraph proof [prueba en párrafo]
- corresponding angles theorem [teorema de los ángulos correspondientes]
- corresponding angles converse theorem

- same-side interior angles theorem
- alternate interior angles theorem [teorema de los ángulos interiores alternos]
- alternate exterior angles theorem [teorema de los ángulos exteriores alternos]
- same-side exterior angles theorem
- alternate interior angles converse theorem
- same-side interior angles converse theorem
- alternate exterior angles converse theorem
- same-side exterior angles converse theorem
- perpendicular/parallel line theorem [teorema de líneas perpendiculares y paralelas]
- triangle sum theorem [teorema de la suma del triángulo]
- exterior angle theorem [teorema del ángulo exterior]
- perpendicular bisector converse theorem
- isosceles triangle base angles theorem [teorema de los ángulos de la base del triángulo isósceles]
- isosceles triangle base angles converse theorem

Consider the diagram which shows four collinear points A, B, C, and D such that point B lies between points A and C, point C lies between points B and D, and $\overline{AB} \cong \overline{CD}$.



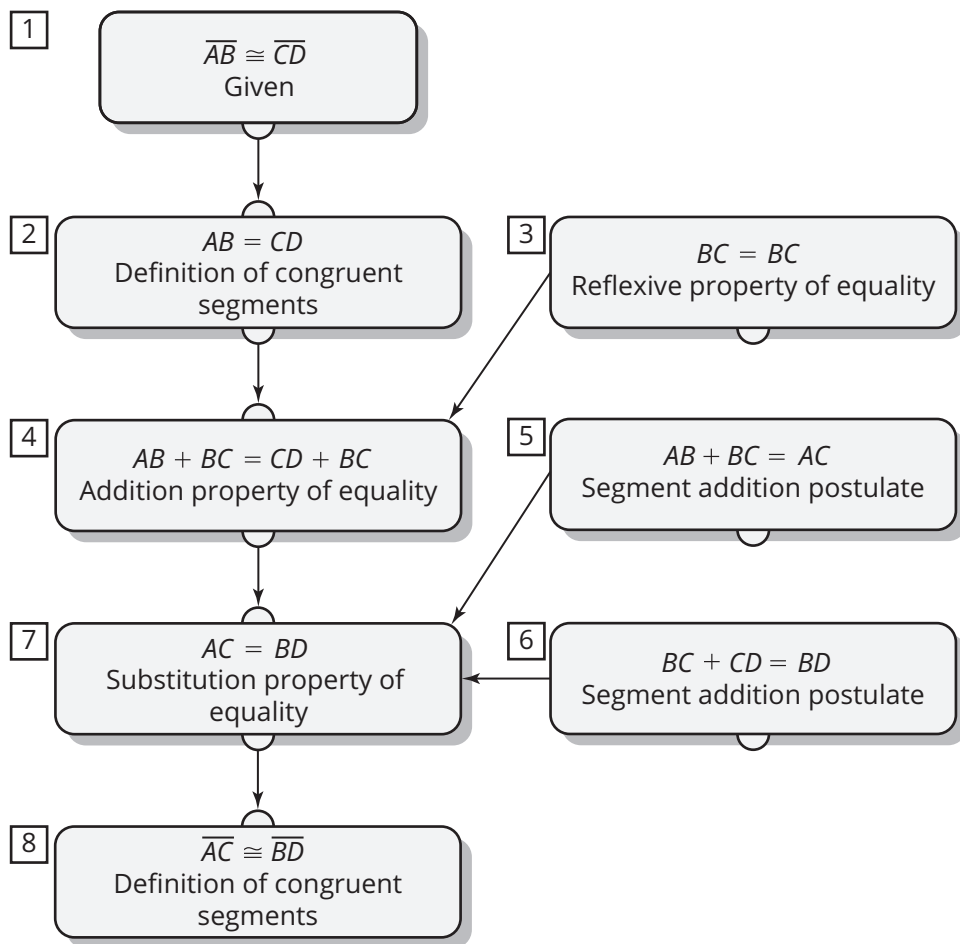
Consider the conditional statement: If $\overline{AB} \cong \overline{CD}$, then $\overline{AC} \cong \overline{BD}$. When you begin the proof process, you take the parts of a conditional statement and organize them into “Given” and “Prove” statements.

Given: $\overline{AB} \cong \overline{CD}$

Prove: $\overline{AC} \cong \overline{BD}$

You can also use the postulates and properties you have learned so far to create different forms of proofs.

A **flow chart proof** is a proof in which the steps and reasons for each step are written in boxes. Arrows connect the boxes and indicate how each step and reason is generated from one or more other steps and reasons.

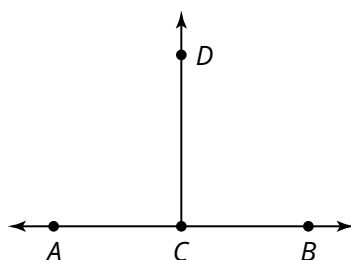


A **two-column proof** is a proof in which the steps are written in the left column and the corresponding reasons are written in the right column. Each step and corresponding reason are numbered.

Statements	Reasons
1. $\overline{AB} \cong \overline{CD}$	1. Given
2. $AB = CD$	2. Definition of congruent segments
3. $BC = BC$	3. Reflexive property of equality
4. $AB + BC = CD + BC$	4. Addition property of equality
5. $AB + BC = AC$	5. Segment addition postulate
6. $BC + CD = BD$	6. Segment addition postulate
7. $AC = BD$	7. Substitution property of equality
8. $\overline{AC} \cong \overline{BD}$	8. Definition of congruent segments

The **right angle congruence postulate** states: “All right angles are congruent.”

For example, consider $\overrightarrow{CD}$ that is perpendicular to $\overline{AB}$. Both $\angle ACD$ and $\angle BCD$ are right angles. Therefore, $\angle ACD \cong \angle BCD$.



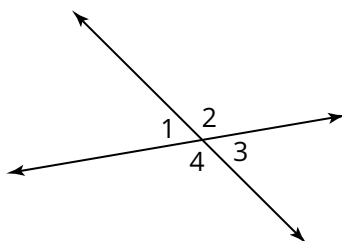
The **congruent supplement theorem** states: “If two angles are supplements of the same angle or of congruent angles, then the angles are congruent.”

For example, if $\angle 1$ is supplementary to $\angle 2$, $\angle 3$ is supplementary to $\angle 4$, and $\angle 2 \cong \angle 4$, then $\angle 1 \cong \angle 3$.



The **vertical angle theorem** states: “Vertical angles are congruent.”

For example, if $\angle 1$ and $\angle 2$ are a linear pair, $\angle 2$ and $\angle 3$ are a linear pair, $\angle 3$ and $\angle 4$ are a linear pair, and $\angle 4$ and $\angle 1$ are a linear pair, then $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$.



Once you have proved a theorem, you know that it is true in all instances. You can apply these theorems when determining unknown geometric measures.

- 30° – 60° – 90° triangle theorem [teorema del triángulo 30° – 60° – 90°]
- 45° – 45° – 90° triangle theorem [teorema del triángulo 45° – 45° – 90°]
- rationalize the denominator [racionalizar el denominador]
- inverse [inverso]
- contrapositive [contrapositivo]
- direct proof [prueba directa]
- indirect proof or proof by contradiction [prueba indirecta o prueba indirecta contradicción]
- hinge theorem
- hinge converse theorem
- degree measure of an arc
- arc addition postulate
- adjacent arcs [arcos adyacentes]
- inscribed angle theorem [teorema del ángulo inscrito]
- inscribed right triangle-diameter theorem
- inscribed quadrilateral-opposite angles theorem [teorema del cuadrilátero inscrito-ángulos opuestos]

- interior angles of a circle theorem [teorema de los ángulos interiores de un círculo]
- exterior angles of a circle theorem [teorema de los ángulos exteriores de un círculo]
- tangent to a circle theorem [teorema de la tangente a un círculo]

For example, given $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ at point E , you can determine $m\angle AED$.

$$x + 94 = 4x + 25$$

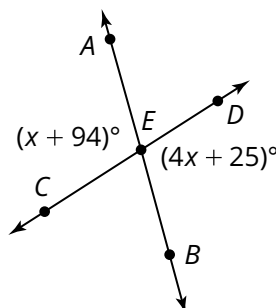
$$94 = 3x + 25$$

$$69 = 3x$$

$$x = 23$$

$$m\angle AEC = 117^\circ$$

$$m\angle AED = 180^\circ - 117^\circ = 63^\circ$$



LESSON

2

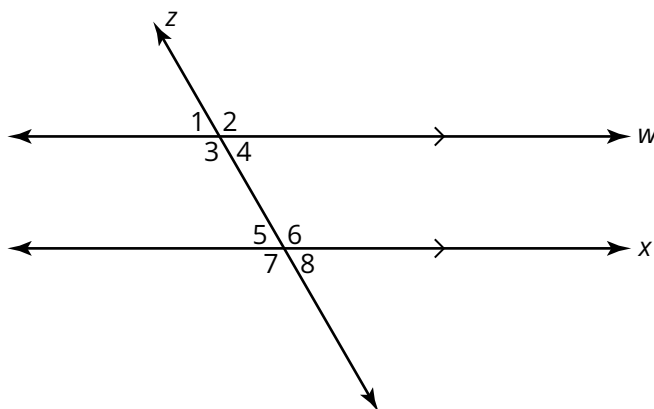
Proving Parallel Line Theorems

The **corresponding angles theorem** states: "If two parallel lines are intersected by a transversal, then corresponding angles are congruent."

For example, $\angle 1 \cong \angle 5$, $\angle 2 \cong \angle 6$, $\angle 3 \cong \angle 7$, and $\angle 4 \cong \angle 8$.

The **corresponding angles converse theorem** states: "If two lines intersected by a transversal form congruent corresponding angles, then the lines are parallel."

The **same-side interior angles theorem** states: "If two parallel lines are intersected by a transversal, then the interior angles on the same side of the transversal are supplementary." For example, $m\angle 3 + m\angle 5 = 180^\circ$ and $m\angle 4 + m\angle 6 = 180^\circ$.



The **alternate interior angles theorem** states: "If two parallel lines are intersected by a transversal, then the alternate interior angles are congruent." For example, $\angle 3 \cong \angle 6$ and $\angle 4 \cong \angle 5$.

The **alternate exterior angles theorem** states: “If two parallel lines are intersected by a transversal, then the alternate exterior angles are congruent.” For example, $\angle 1 \cong \angle 8$ and $\angle 2 \cong \angle 7$.

The **same-side exterior angles theorem** states: “If two parallel lines are cut by a transversal, then the same-side exterior angles are supplementary.” For example, $m\angle 1 + m\angle 7 = 180^\circ$ and $m\angle 2 + m\angle 8 = 180^\circ$.

The **alternate interior angles converse theorem** states: “If two lines intersected by a transversal form congruent alternate interior angles, then the lines are parallel.”

The **same-side interior angles converse theorem** states: “If two lines intersected by a transversal form supplementary same-side interior angles, then the lines are parallel.”

The **alternate exterior angles converse theorem** states: “If two lines intersected by a transversal form congruent alternate exterior angles, then the lines are parallel.”

The **same-side exterior angles converse theorem** states: “If two lines intersected by a transversal form supplementary same-side exterior angles, then the lines are parallel.”

The **perpendicular/parallel line theorem** states: “If two lines are perpendicular to the same line, then the two lines are parallel to each other.”

LESSON

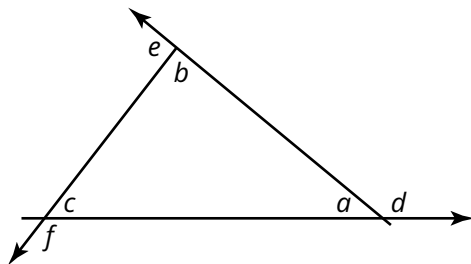
3

Interior and Exterior Angles of Polygons

The **triangle sum theorem** states that the sum of the interior angles of a triangle is equal to 180° .

The **exterior angle theorem** states: “The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.”

For example, $m\angle d = m\angle b + m\angle c$, $m\angle e = m\angle a + m\angle c$, and $m\angle f = m\angle a + m\angle b$.



The formula for the sum of the measures of the interior angles of a polygon $= 180(n - 2)^\circ$, where n is the number of sides of the polygon.

In a regular polygon we can also use this formula to determine the measure of each angle given the sum of the interior angles. The interior angle measure of a regular polygon = $\frac{180(n-2)}{n}$.

For any polygon we can say that the sum of the measures of the exterior angles is 360° , or one full revolution of a shape.

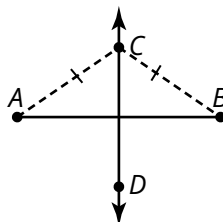
In a regular polygon we can also calculate the measure of each individual exterior angle. The exterior angle measure of a regular polygon = $\frac{360^\circ}{n}$, where n is the number of sides.

LESSON 4

Perpendicular Bisector and Isosceles Triangle Theorems

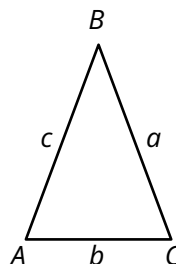
The **perpendicular bisector converse theorem** states: “If a point is equidistant from the endpoints of a line segment, then the point lies on the perpendicular bisector of the segment.”

For example, if $AC = BC$, then point C lies on the perpendicular bisector, $\overleftrightarrow{CD}$.



The **isosceles triangle base angles theorem** states: “If two sides of a triangle are congruent, then the angles opposite these sides are congruent.”

For example, in $\triangle ABC$, if side a is congruent to side c , then $\angle A \cong \angle C$.



The **isosceles triangle base angles converse theorem** states: “If two angles of a triangle are congruent, then the sides opposite these angles are congruent.”

For example, if $\angle A \cong \angle C$, then side a is congruent to side c .

There are geometric theorems specifically related to right triangles.

For example, the **30°-60°-90° triangle theorem** states: “The length of the hypotenuse in a 30°-60°-90° triangle is 2 times the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.”

The **45°-45°-90° triangle theorem** states: “The length of the hypotenuse in a 45°-45°-90° triangle is $\sqrt{2}$ times the length of a leg.”

A standard procedure involving radicals is to simplify the radical, which is the process of removing all perfect square numbers from under the radical symbol.

$$\begin{aligned}\sqrt{18} &= \sqrt{9} \cdot \sqrt{2} \\ &= \sqrt{3^2} \cdot \sqrt{2} \\ &= 3\sqrt{2}\end{aligned}$$

A standard math convention is to **rationalize the denominator**, which is a process of rewriting a fraction with no irrational numbers in the denominator.

$$\frac{11}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = 11\frac{\sqrt{2}}{2}$$

LESSON 5

Inverse, Contrapositive, Direct Proof, and Indirect Proof

To state the *inverse* of a conditional statement, negate both the hypothesis and the conclusion. To state the *contrapositive* of a conditional statement, negate both the hypothesis and the conclusion and then reverse them.

Conditional Statement: If p , then q .

Inverse: If not p , then not q .

Contrapositive: If not q , then not p .

For example,

Conditional Statement: If a triangle is equilateral, then it is isosceles.

Inverse: If a triangle is not equilateral, then it is not isosceles.

Contrapositive: If a triangle is not isosceles, then it is not equilateral.

In an **indirect proof**, or proof by contradiction, first write the givens. Then, write the negation of the conclusion. Then, use that assumption to prove a given statement is false, thus creating a contradiction. Hence, the assumption leads to a contradiction, therefore showing that the assumption is false. This proves the contrapositive.

For example, use indirect proof to prove a triangle cannot have more than one obtuse angle.

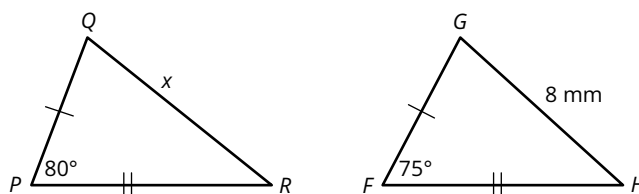
Given: Triangle DEF

Prove: A triangle cannot have more than one obtuse angle.

Given $\triangle DEF$, assume that $\triangle DEF$ has two obtuse angles. So, assume $m\angle D = 91^\circ$ and $m\angle E = 91^\circ$. By the Triangle Sum Theorem, $m\angle D + m\angle E + m\angle F = 180^\circ$. By the substitution property of equality, $91^\circ + 91^\circ + m\angle F = 180^\circ$, and by subtraction, $m\angle F = -2^\circ$. But it is not possible for a triangle to have a negative angle, so this is a contradiction. This proves that a triangle cannot have more than one obtuse angle.

The **hinge theorem** states: “If two sides of one triangle are congruent to two sides of another triangle and the included angle of the first pair is larger than the included angle of the second pair, then the third side of the first triangle is longer than the third side of the second triangle.”

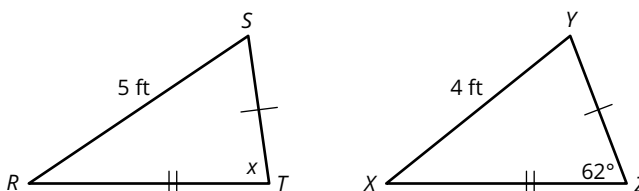
For example, consider $\triangle PQR$ and $\triangle FGH$.



$QR > GH$, so $x > 8$ millimeters.

The **hinge converse theorem** states: “If two sides of one triangle are congruent to two sides of another triangle and the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first pair of sides is larger than the included angle of the second pair of sides.”

Consider $\triangle RST$ and $\triangle XZY$.



$m\angle T > m\angle Z$, so $x > 62^\circ$.

LESSON 6

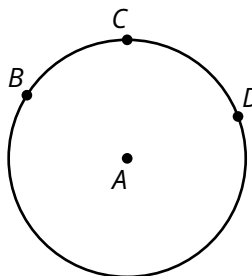
Angle Relationships Inside and Outside Circles

Each minor arc of a circle is associated with, and determined by, a specific central angle. The **degree measure of an arc** is the same as the degree measure of its central angle.

For example, if $\angle PRQ$ is a central angle and $m\angle PRQ = 30^\circ$, then $m\widehat{PQ} = 30^\circ$.

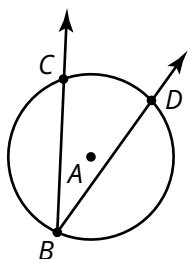
The **arc addition postulate** states: “The measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs.” **Adjacent arcs** are two arcs of the same circle sharing a common endpoint.

For example, $\widehat{BC}$ is adjacent to $\widehat{CD}$ and $m\widehat{BC} + m\widehat{CD} = m\widehat{BD}$.



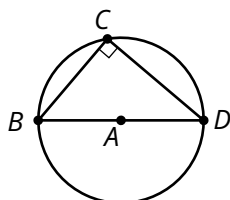
The **inscribed angle theorem** states: “The measure of an inscribed angle is half the measure of its intercepted arc.”

For example, $m\angle CBD = \frac{1}{2}m\widehat{CD}$.



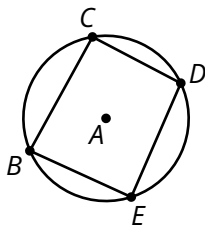
The **inscribed right triangle-diameter theorem** states: “If a triangle is inscribed in a circle such that one side of the triangle is a diameter of the circle, then the triangle is a right triangle.”

For example, $\triangle BCD$ is inscribed in circle A and side $\overline{BD}$ is also a diameter of circle A. Therefore, $\triangle BCD$ is a right triangle.



The **inscribed quadrilateral-opposite angles theorem** states: “If a quadrilateral is inscribed in a circle, then the opposite angles are supplementary.”

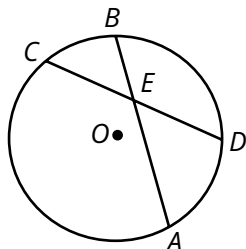
For example, quadrilateral $BCDE$ is inscribed in circle A. Therefore, $m\angle B + m\angle D = 180^\circ$ and $m\angle C + m\angle E = 180^\circ$.



The vertex of an angle can be located inside of a circle, outside of a circle, or on a circle.

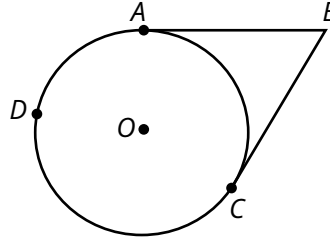
The **interior angles of a circle theorem** states: “If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the interior of the circle, then the measure of the angle is half of the sum of the measures of the arcs intercepted by the angle and its vertical angle.”

For example, $m\angle AED = \frac{1}{2}(m\widehat{AD} + m\widehat{BC})$.



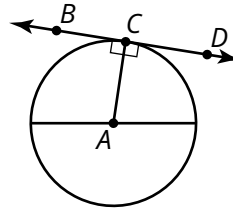
The **exterior angles of a circle theorem** states: “If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the exterior of the circle, then the measure of the angle is half of the difference of the measures of the arcs intercepted by the angle.”

For example, $m\angle ABC = \frac{1}{2}(m\widehat{ADC} - m\widehat{AC})$.



The **tangle to a circle theorem** states: “A line drawn tangent to a circle is perpendicular to a radius of the circle drawn to the point of tangency.”

For example, $\overleftrightarrow{BD}$ is tangent to circle A at point C. Therefore, $\overleftrightarrow{BD}$ is perpendicular to $\overline{AC}$.





Congruence theorems are all about analyzing triangles in terms of component sides and angles – especially right triangles.

Using Congruence Theorems

LESSON 1	Using Triangle Congruence to Determine Relationships Between Segments	667
LESSON 2	Using Congruent Triangles to Determine Relationships Between Chords and Tangents	679
LESSON 3	Properties of Quadrilaterals	701



1

Using Triangle Congruence to Determine Relationships Between Segments

OBJECTIVES

- Prove, in writing, the Hypotenuse-Leg congruence theorem.
- Use SSS, SAS, ASA, and/or AAS triangle congruence as evidence to justify the Leg-Leg congruence theorem and the Leg-Angle congruence theorem.
- Apply right triangle congruence theorems to solve problems.

KEY TERMS

- Hypotenuse-Leg (HL) congruence theorem
- Leg-Leg (LL) congruence theorem
- Leg-Angle (LA) congruence theorem

.....

You know about triangle congruence theorems, such as Side-Side-Side and Side-Angle-Side.

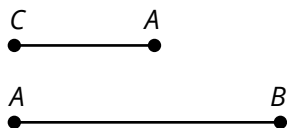
What congruence theorems apply to right triangles?

Getting Started

Pulling Your Leg

Let's investigate a conjecture about right triangles.

1. Consider right triangle ABC with right angle C .
 - a. Construct a right triangle using $\overline{CA}$ as a leg and $\overline{AB}$ as the hypotenuse. Then, write the steps you performed to construct the triangle.



- b. How does the length of side $\overline{CB}$ compare to the lengths of your classmates' sides $\overline{CB}$?
 - c. Use a protractor to measure $\angle A$ and $\angle B$ in $\triangle ABC$. How do the measures of these angles compare to the measures of your classmates' $\angle A$ and $\angle B$?
 - d. Is your triangle congruent to your classmates' triangles? Why or why not?

Hypotenuse-Leg Congruence Theorem

Many congruence theorems apply to all triangles. There are also theorems that apply only to right triangles. You have already proved the Hypotenuse-Angle congruence theorem. Methods for proving that two right triangles are congruent are somewhat shorter. You can prove that two right triangles are congruent using only two measurements.

1. Explain why only two pairs of corresponding parts are needed to prove that two right triangles are congruent.
2. Are all right angles congruent? Explain your reasoning.

Through an example, you have used construction and measurements to demonstrate that all right triangles with a given leg length and hypotenuse length are congruent. Now, let's prove this conjecture is true in all cases.

3. Complete the two-column proof of this conjecture.

Given:

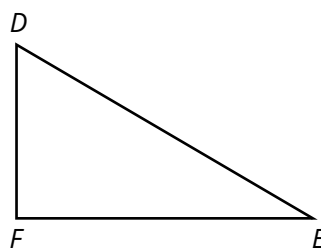
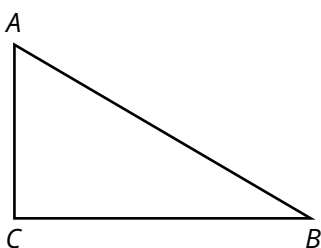
Right triangle ABC with right angle C .

Right triangle DEF with right angle F .

$$\overline{AC} \cong \overline{DF}$$

$$\overline{AB} \cong \overline{DE}$$

Prove: $\triangle ABC \cong \triangle DEF$



Draw and Deduce

Mark the diagram with the given information to visualize what you know.

Statements	Reasons
a. Right triangle ABC with right angle C . Right triangle DEF with right angle F .	
b. $\angle C \cong \angle F$	
c. $\overline{AC} \cong \overline{DF}$ and $\overline{AB} \cong \overline{DE}$	
d. $AC = DF$ and $AB = DE$	
e. $(AC)^2 = (DF)^2$ and $(AB)^2 = (DE)^2$	
f. $(AC)^2 + (CB)^2 = (AB)^2$ and $(DF)^2 + (FE)^2 = (DE)^2$	
g. $(AC)^2 + (CB)^2 = (DF)^2 + (FE)^2$ $(AC)^2 + (CB)^2 = (AC)^2 + (FE)^2$	
h. $(CB)^2 = (FE)^2$	
i. $CB = FE$	
j. $\overline{CB} \cong \overline{FE}$	
k. $\triangle ABC \cong \triangle DEF$	

4. Michael used algebra to reason about the conjecture. Explain why Michael's reasoning is correct. Show your work.

Michael



So, $a^2 + x^2 = y^2$, and $b^2 + x^2 = y^2$. So $a = b$, and the triangles are congruent by SSS or SAS.

The **Hypotenuse-Leg (HL) congruence theorem** states: "If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent."

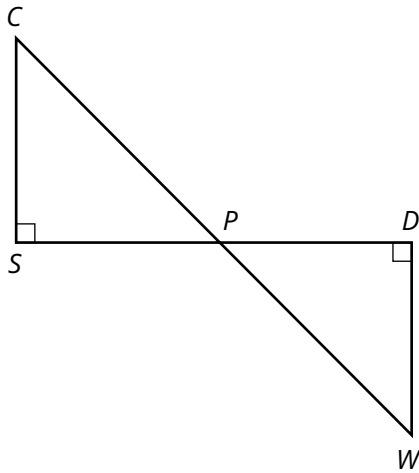
Applying Triangle Congruence Theorems

You have learned about the Hypotenuse-Angle congruence theorem and the Hypotenuse-Leg congruence theorem. There are two other right angle theorems that can be proved with similar reasoning. The **Leg-Leg (LL) congruence theorem** states: “If the two corresponding shorter legs of two right triangles are congruent, then the two triangles are congruent.” The **Leg-Angle (LA) congruence theorem** states: “If the leg and an acute angle of one right triangle are congruent to the corresponding leg and acute angle of another right triangle, then the triangles are congruent.”

1. Explain how the LL congruence theorem and the LA congruence theorem are each equivalent to another triangle congruence theorem that you know.

Determine whether there is enough information to prove that the two triangles are congruent. If so, name the congruence theorem used.

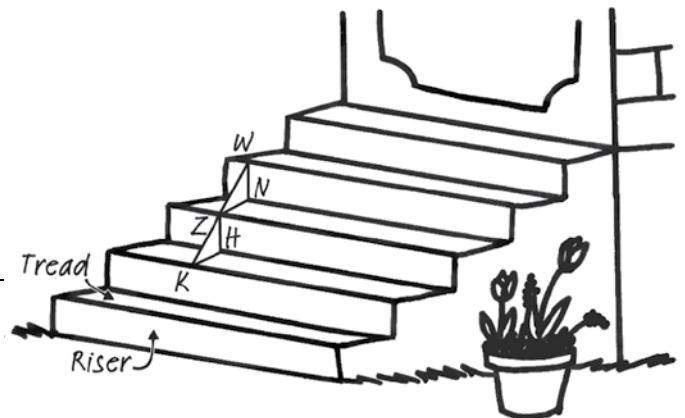
2. If $\overline{CS} \perp \overline{SD}$, $\overline{WD} \perp \overline{SD}$, and P is the midpoint of $\overline{CW}$, is $\triangle CSP \cong \triangle WDP$?



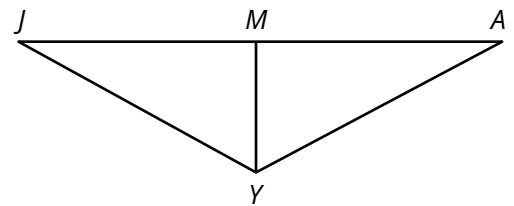
Ask Yourself ...

What tools or strategies can you use to solve this problem?

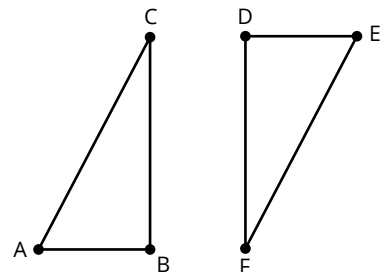
3. Olivia always trips on the third step and she thinks that step may be a different size. The contractor told her that all the treads and risers are perpendicular to each other. Is that enough information to state that the steps are the same size? In other words, if $\overline{WN} \perp \overline{NZ}$ and $\overline{ZH} \perp \overline{HK}$, is $\triangle WNZ \cong \triangle ZHK$?



4. If $\overline{JA} \perp \overline{MY}$ and $\overline{JY} \cong \overline{AY}$, is $\triangle JYM \cong \triangle AYM$?



5. If $\overline{BC} \perp \overline{BA}$, $\overline{FD} \perp \overline{DE}$, $\overline{AB} \cong \overline{ED}$, and $\overline{BC} \cong \overline{DF}$, is $\triangle ABC \cong \triangle EDF$?





Talk the Talk

Congruence Theorems to the Rescue

You have many different theorems you can use to prove that two triangles are congruent depending on what you know. Let's consider how the right triangle congruence theorems are related to the triangle congruence theorems associated with all triangles.

1. List the four triangle congruence theorems associated with right triangles.
2. List the four triangle congruence theorems associated with all triangles.
3. Associate each theorem in Question 1 with one or more related theorems in Question 2. Draw examples to justify your answers.

Ask Yourself ...

What patterns do you notice?

Lesson 1 Assignment

Write

Describe how each right triangle congruence theorem compares to one or more triangle congruence theorems in your own words.

1. Hypotenuse-Leg (HL) congruence theorem
2. Leg-Leg (LL) congruence theorem
3. Leg-Angle (LA) congruence theorem

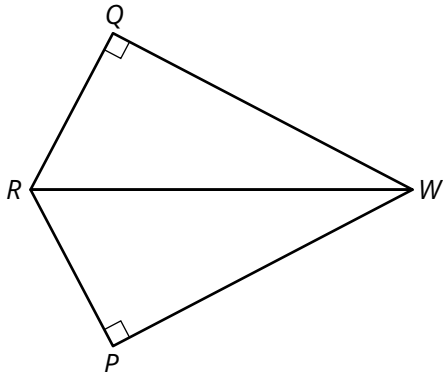
Remember

The Hypotenuse-Leg congruence theorem states that if the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent.

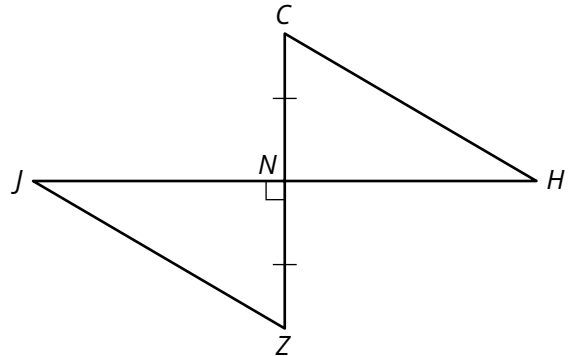
Practice

1. Determine the information that is needed to use the indicated theorem to show that the triangles are congruent.

- a. $\triangle RQW \cong \triangle RPW$ by HL



- b. $\triangle JNZ \cong \triangle HNC$ by LA

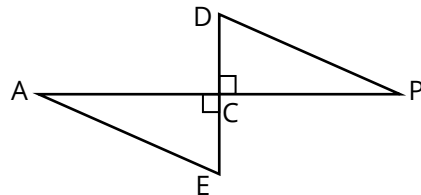


Lesson 1 Assignment

2. For each figure, determine if there is enough information to prove that the two triangles are congruent. If so, name the right triangle congruence theorem used.

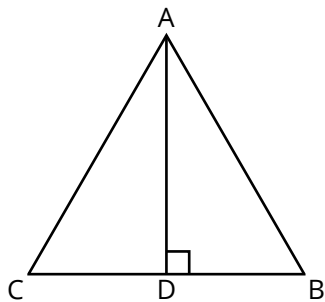
- a. Given: $\overline{DC} \cong \overline{EC}$, $\overline{PC} \cong \overline{AC}$, and $\angle DCP$ and $\angle ECA$ are right angles.

Is $\triangle DCP \cong \triangle ECA$?



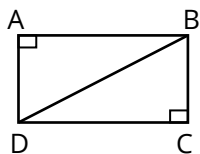
- b. Given $\overline{AD} \perp \overline{CB}$

Is $\triangle ADC \cong \triangle ADB$?



- c. Given $\overline{AD} \cong \overline{BC}$ and $\angle A$ and $\angle C$ are right angles.

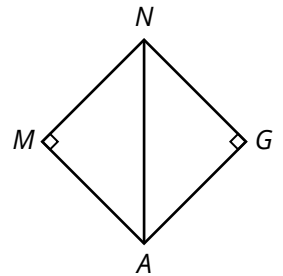
Is $\triangle BAD \cong \triangle DCB$?



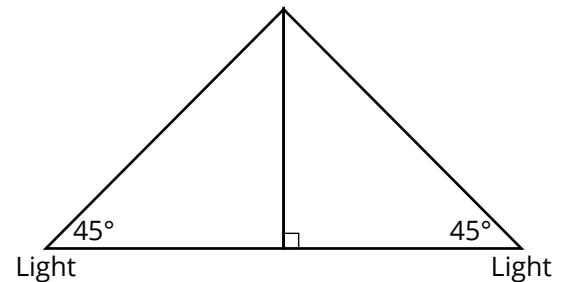
Lesson 1 Assignment

d. Given $\overline{NA}$ bisects $\angle MNG$, $\angle M$ and $\angle G$ are right angles.

Is $\triangle AMN \cong \triangle AGN$?

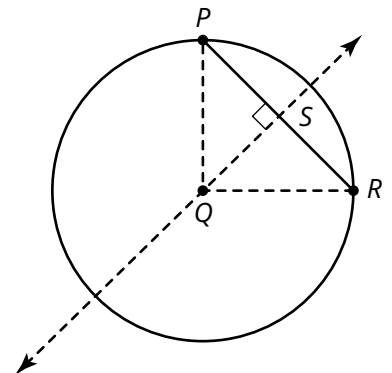


3. At the baseball stadium, a flag is illuminated by two lights on either side of the flagpole. Each light is aimed at an angle of 45 degrees from the ground. Is there enough information to determine that the two triangles formed by the flagpole, the ground, and the lights are congruent?



Prepare

Identify any relevant geometric segments and/or angles and write the relationships shown in the diagram.



2

Using Congruent Triangles to Determine Relationships Between Chords and Tangents

OBJECTIVES

- Analyze a proof of the tangent segment theorem.
- Prove the diameter-chord theorem in a written argument using precise mathematic language.
- Prove the equidistant chord theorem and its converse in a written argument using precise mathematic language.
- Prove the congruent chord-congruent arc theorem and its converse in a written argument using precise mathematic language.

.....

You have analyzed relationships among angles and chords.

What relationships exist between chord lengths and other parts of a circle?

KEY TERMS

- tangent segment theorem
- tangent circles
- diameter-chord theorem
- equidistant chord theorem
- equidistant chord converse theorem
- congruent chord-congruent arc theorem
- congruent chord-congruent arc converse theorem

Getting Started

.....
Borger Cordmarked-
style pottery
was made by the
peoples known to
archaeologists as
the Antelope Creek
culture who lived
along the Canadian
River in what is
today the north-
central part of the
Texas Panhandle
between A.D. 1100
and A.D. 1450.
.....

Putting the Pieces Together

An archaeologist during an excavation discovered a fragment of a plate. The archaeologist realized that in order to determine the date and use of the plate he would need to know the exact diameter of the plate.



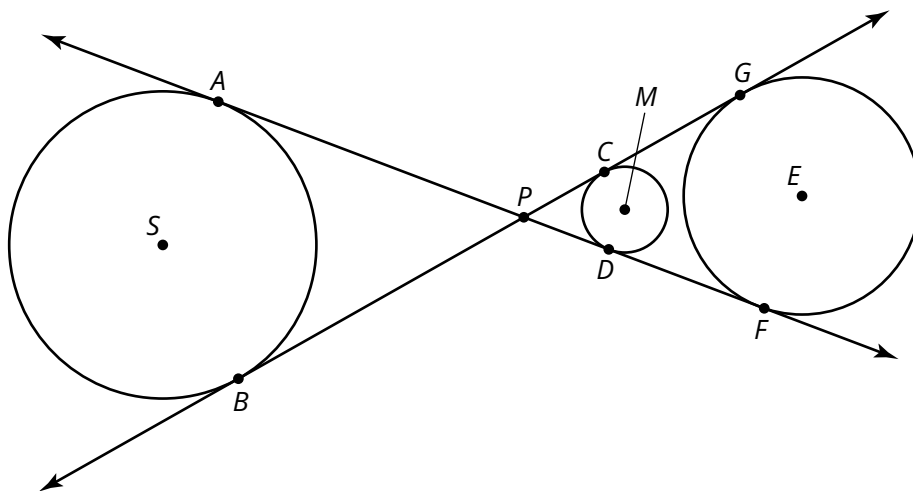
The archaeologist gets out a piece of paper, a compass, a straightedge, a ruler, and a marker.

1. What do you think the archaeologist did?

ACTIVITY
2.1

Tangent Segments

Consider point P located outside of circles M , E , and S . Lines AF and BG are drawn tangent to the circles as shown.



1. Identify the two tangent segments drawn from point P associated with each circle. Then, compare the lengths of the two segments.

a. circle S

b. circle M

c. circle E

.....
Remember ...

A tangent segment is a line segment formed by connecting a point outside of the circle to a point of tangency.

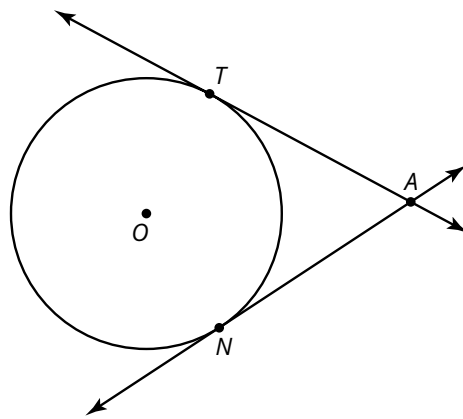
.....
Think About ...

Which tools can you use to compare lengths?

It appears that tangent segments drawn from the same point on the exterior of a circle are congruent.

WORKED EXAMPLE

Consider this proof plan of the tangent segment theorem.



$\overleftrightarrow{AT}$ is tangent to circle O at point T.
 $\overleftrightarrow{AN}$ is tangent to circle O at point N.

Connect points O and T to form radius $\overline{OT}$. Connect points O and N to form radius $\overline{ON}$.
 Connect points O and A to form $\overline{OA}$.

Given: $\overleftrightarrow{AT}$ is tangent to circle O at point T.
 $\overleftrightarrow{AN}$ is tangent to circle O at point N.

Prove: $\overline{AT} \cong \overline{AN}$

$$\begin{aligned}\overline{OT} &\cong \overline{ON} \\ \overline{ON} &\perp \overleftrightarrow{AN} \\ \overline{OT} &\perp \overleftrightarrow{AT}\end{aligned}$$

$\angle ONA$ and $\angle OTA$ are right angles.
 $\triangle ONA$ and $\triangle OTA$ are right triangles.

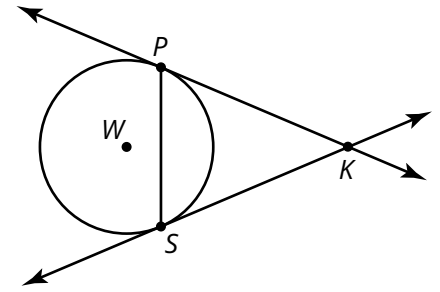
$$\begin{aligned}\overline{OA} &\cong \overline{OA} \\ \triangle ONA &\cong \triangle OTA \\ \overline{AT} &\cong \overline{AN}\end{aligned}$$

2. How do you know that $OT = ON$?

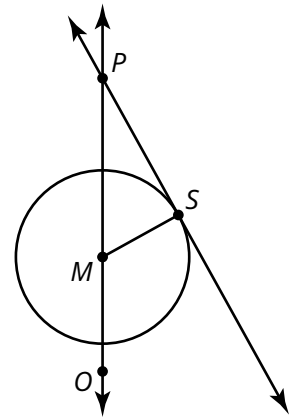
3. How do you know that $\triangle ONA \cong \triangle OTA$?

The **tangent segment theorem** states: "If two tangent segments are drawn from the same point on the exterior of a circle, then the tangent segments are congruent."

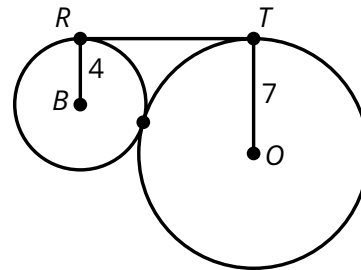
4. In the figure, $\overleftrightarrow{KP}$ and $\overleftrightarrow{KS}$ are tangent to circle W and $m\angle PKS = 46^\circ$. Calculate $m\angle KPS$. Explain your reasoning.



5. In the figure, $\overleftrightarrow{PS}$ is tangent to circle M and $m\angle SMO = 119^\circ$. Calculate $m\angle MPS$. Explain your reasoning.



6. In the figure,
- Circle B and circle O are tangent circles.
 - The length of radius $\overline{BR}$ is 4.
 - The length of radius $\overline{OT}$ is 7.
 - Segment RT is a common tangent.



Calculate the length of segment RT .

Tangent circles

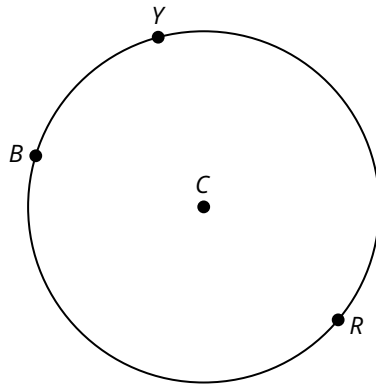
are circles that lie in the same plane and intersect at exactly one point.

Chords and their perpendicular bisectors lead to several interesting conclusions. In this lesson, you will make conjectures about the relationships among some special segments of circles and prove theorems about these geometric relationships.

.....
Think About . . .

Which point of concurrency is associated with the perpendicular bisectors of a triangle?
.....

1. Consider circle C with points B , Y , and R .
 - a. Draw chords $\overline{YR}$, $\overline{BR}$, and $\overline{BY}$ and construct the perpendicular bisector of each chord.



- b. Make a conjecture about the relationship between the perpendicular bisectors of a chord and the center point of the circle?

The perpendicular bisector of a chord appears to also bisect the chord's intercepted arc. This observation can be proved and stated as a theorem.

2. Complete the flow chart proof to prove this relationship between the perpendicular bisector of a chord and the chord's intercepted arc.

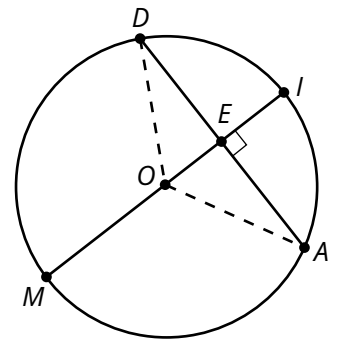
Given: $\overline{MI}$ is a diameter of circle O .
 $\overline{MI} \perp \overline{DA}$

Prove: $\overline{MI}$ bisects $\overline{DA}$.
 $\overline{MI}$ bisects $\widehat{DA}$.

$\overline{MI}$ is a diameter of circle O . $\overline{MI} \perp \overline{DA}$.
 Given



Connect points O and D to form radius $\overline{OD}$.
 Connect points O and A to form radius $\overline{OA}$.
 Construction



Ask Yourself ...

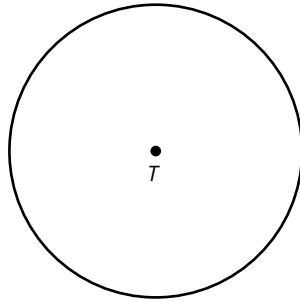
Did you make a plan to solve the problem?

.....

The **diameter-chord theorem** states: "If a circle's diameter is perpendicular to a chord, then the diameter bisects the chord and bisects the arc determined by the chord."

.....

3. Use circle T to draw two congruent chords that are not parallel to each other and do not pass through the center point of the circle.
- a. Construct the perpendicular bisector of each chord.



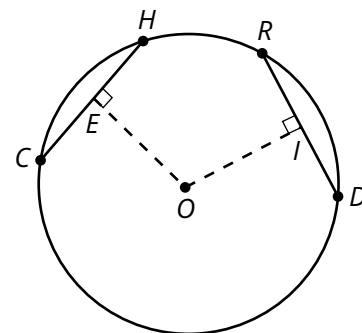
- b. Choose a construction tool and compare the distance each chord is from the center point of the circle.

Congruent chords appear to be equidistant from the center point of the circle. This observation can be proved and stated as a theorem.

4. Prove this relationship regarding chords.

Given: $\overline{CH} \cong \overline{DR}$
 $\overline{OE} \perp \overline{CH}$
 $\overline{OI} \perp \overline{DR}$

Prove: $\overline{CH}$ and $\overline{DR}$ are equidistant from center O .



Statements	Reasons
1. $\overline{CH} \cong \overline{DR}$ $\overline{OE} \perp \overline{CH}$ $\overline{OI} \perp \overline{DR}$	1. Given
2. Connect points O and H , O and C , O and D , and O and R to form radii $\overline{OH}$, $\overline{OC}$, $\overline{OD}$, and $\overline{OR}$, respectively.	2. Construction

.....
 Here's a hint. You need to show $OE = OI$.

.....
 The **equidistant chord theorem** states: "If two chords of the same circle or congruent circles are congruent, then they are equidistant from the center of the circle."

Let's conjecture that the converse of this theorem is also true. That is, if two chords of the same circle or congruent circles are equidistant from the center of the circle, then the chords are congruent.

.....

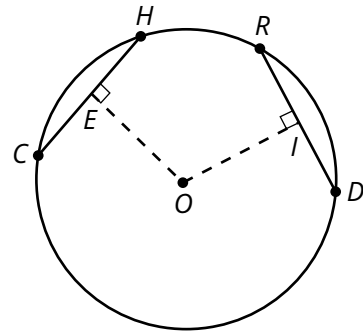
Connect points O and H , O and C , O and D , and O and R to form radii $\overline{OH}$, $\overline{OC}$, $\overline{OD}$, and $\overline{OR}$.

.....

5. Complete the paragraph proof to prove this converse statement is true in all cases.

Given: $OE = OI$
 $(\overline{CH}$ and $\overline{DR}$ are equidistant from the center point.)
 $\overline{OE} \perp \overline{CH}$
 $\overline{OI} \perp \overline{DR}$

Prove: $\overline{CH} \cong \overline{DR}$

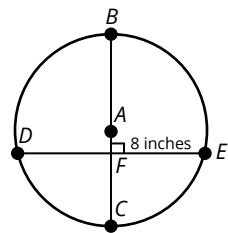


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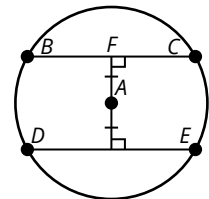
The **equidistant chord converse theorem** states: "If two chords of the same circle or congruent circles are equidistant from the center of the circle, then the chords are congruent."

.....

6. $\overline{BC}$ is a diameter of circle A. Determine the length of $\overline{DF}$ and identify the theorem used to determine the length.



7. If $DE = 18$ inches, determine the length of $\overline{FC}$ and identify the theorem used to determine the length.

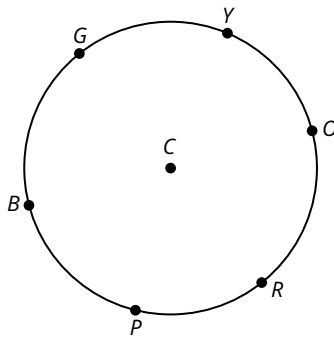


ACTIVITY 2.3

Chords and Arcs

The color wheel is made of three different kinds of colors: primary, secondary, and tertiary. Primary colors (red, blue, and yellow) are the colors you start with. Secondary colors (orange, green, and purple) are created by mixing two primary colors. Tertiary colors (red-orange, yellow-orange, yellow-green, blue-green, blue-purple, red-purple) are created by mixing a primary color with a secondary color.

1. Consider circle C shown with 6 points equally spaced along its circumference.



- a. Draw two congruent chords.
 - b. Draw four radii by connecting the endpoints of each chord with the center point of the circle.
2. Make a conjecture about the two central angles and the two intercepted minor arcs formed by each pair of radii.

.....

Color theory is a set of rules that is used to create color combinations. A color wheel is a visual representation of color theory.

.....

It appears that if two chords of the same circle or congruent circles are congruent, then their corresponding arcs are congruent. Let's prove this conjecture.

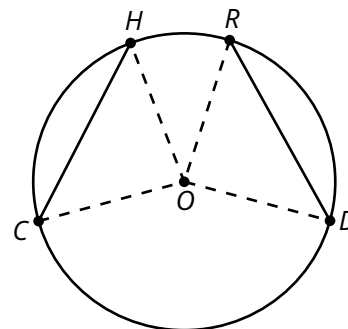
Remember...

Don't forget to write to include the construction that was made in the paragraph proof.

- Write a paragraph proof to prove this conjecture relating chords and their corresponding arcs.

Given: $\overline{CH} \cong \overline{DR}$

Prove: $\widehat{CH} \cong \widehat{DR}$



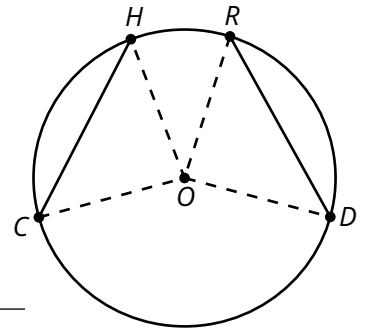
The **congruent chord-congruent arc theorem** states: "If two chords of the same circle or congruent circles are congruent, then their corresponding arcs are congruent."

The **congruent chord-congruent arc converse theorem** states: "If arcs of the same circle or congruent circles are congruent, then their corresponding chords are congruent."

4. Prove the congruent chord-congruent arc converse theorem.

Given: $\widehat{CH} \cong \widehat{DR}$

Prove: $\overline{CH} \cong \overline{DR}$



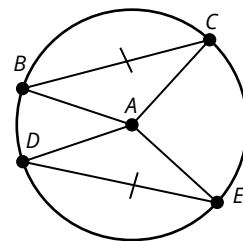
Statements	Reasons
1. $\widehat{CH} \cong \widehat{DR}$	1. Given
2. Connect points O and H, O and C, O and D, and O and R to form radii $\overline{OH}$, $\overline{OC}$, $\overline{OD}$, and $\overline{OR}$, respectively.	2. Construction

.....

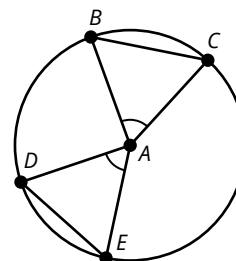
When you prove the converse of a conditional statement you are demonstrating the validity of the statement and proving the converse theorem.

.....

5. If $m\angle BAC = 133^\circ$, determine the $m\angle DAE$ and identify the theorem used to determine the measure.



6. If the length of $\overline{BC}$ is 15 millimeters, determine the length of $\overline{DE}$ and identify the theorem used to determine the length.



Talk the Talk

Broken Plate

1. Write the equidistant chord theorem and its converse as a biconditional statement.
2. Write the congruent chord-congruent arc theorem and its converse as a biconditional statement.
3. Why can you write the equidistant chord theorem and its converse and the congruent chord-congruent arc theorem and its converse as biconditional statements?



Consider the problem from the Getting Started.

4. Describe how the archaeologist can determine the diameter of the plate with the broken piece. Then, show your work on the broken plate shown.



Lesson 2 Assignment

Write

Match each definition with its corresponding term.

- | | |
|---|--|
| 1. tangent segment theorem | a. If two chords of the same circle or congruent circles are congruent, then their corresponding arcs are congruent. |
| 2. diameter-chord theorem | b. If two chords of the same circle or congruent circles are congruent, then they are equidistant from the center of the circle. |
| 3. equidistant chord theorem | c. If two arcs of the same circle or congruent circles are congruent, then their corresponding chords are congruent. |
| 4. equidistant chord converse theorem | d. If two chords of the same circle or congruent circles are equidistant from the center of the circle, then the chords are congruent. |
| 5. congruent chord-congruent arc theorem | e. If a diameter of a circle is perpendicular to a chord, then the diameter bisects the chord and bisects the arc determined by the chord. |
| 6. congruent chord-congruent arc converse theorem | f. If two tangent segments are drawn from the same point on the exterior of a circle, then the tangent segments are congruent. |

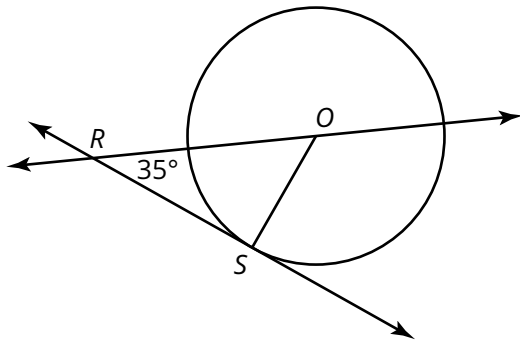
Lesson 2 Assignment

Remember

Relationships between chord lengths and other parts of a circle can be explained using several theorems including the tangent segment theorem, the diameter-chord theorem, the equidistant chord theorem and its converse, and the congruent chord-congruent arc theorem and its converse.

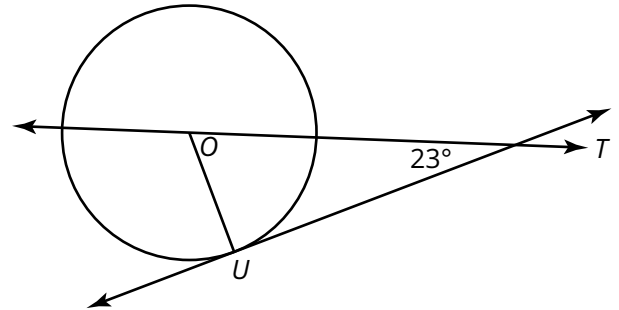
Practice

1. Calculate the measure of each angle. Explain your reasoning.
 - a. If $\overline{RS}$ is a tangent segment and $\overline{OS}$ is a radius, what is the measure of $\angle ROS$?

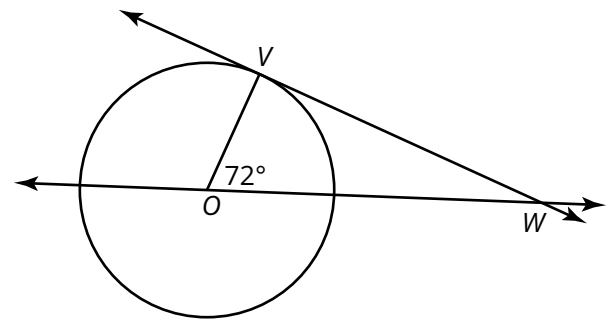


Lesson 2 Assignment

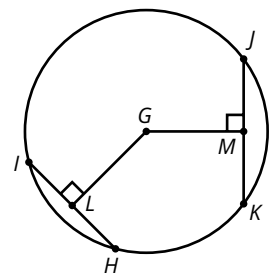
- b. If $\overline{UT}$ is a tangent segment and $\overline{OU}$ is a radius, what is the measure of $\angle TOU$?



- c. If $\overline{VW}$ is a tangent segment and $\overline{OV}$ is a radius, what is the measure of $\angle VWO$?

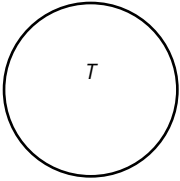


2. In circle G, $MG = 1.84$ centimeters, $GL = 1.98$ centimeters, $m\angle GLH = 90^\circ$, and $m\angle GMK = 90^\circ$. Determine which chord is longer, $\overline{IH}$ or $\overline{JK}$. Explain your reasoning.



Lesson 2 Assignment

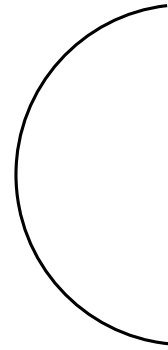
3. Use circle T to complete parts (a) through (e).



- a. Draw an inscribed right angle in circle T . Label each point where the angle intersects the circle. What is the name of the right angle?
- b. Draw the chord determined by the inscribed right angle. What is the name of the chord?
- c. Draw a second inscribed right angle in circle T . Label each point where the angle intersects the circle. What is the name of the second right angle?
- d. Draw the chord determined by the second inscribed right angle. What is the name of the chord?
- e. Describe the relationship between the arcs that correspond to the chords you named in parts (b) and (d). Explain your reasoning.

Lesson 2 Assignment

4. The figure shows a section of a circle. Draw two chords and construct their perpendicular bisectors to locate the center of the circle. Explain your work.



Prepare

Draw a quadrilateral with the given property. Do not draw the same quadrilateral twice.

1. congruent angles
2. congruent diagonals
3. congruent sides

3

Properties of Quadrilaterals

OBJECTIVES

- Use pre-reading strategies to make connections and organize ideas in order to prove the properties of a parallelogram, a rhombus, a rectangle, a square, a trapezoid, and a kite.
- Prove a quadrilateral is a parallelogram, rectangle, rhombus, or square.
- Solve problems by reading and applying the properties of a parallelogram, a rhombus, a rectangle, a square, a trapezoid, and a kite.

KEY TERMS

- parallelogram/congruent-parallel side theorem
- base angles of a trapezoid
- trapezoid midsegment theorem

You previously made conjectures about the properties of quadrilaterals using recorded observations.

How can you use the theorems you have learned thus far to prove that quadrilaterals have the properties you reasoned about?

Getting Started

Making Connections

Recall the different properties of quadrilaterals you investigated in an earlier topic. In this activity you will consider how different properties are associated with one or more quadrilaterals.

1. Locate the Organization of Quadrilaterals Flow Chart at the end of this lesson. Write the name of each quadrilateral inside its figure. Use markers to identify congruent sides and parallel sides so that the given label applies.
2. Locate and cut out the Properties of Quadrilaterals at the end of this lesson. Glue each property next to the figure to which it applies. If the property applies to more than one quadrilateral, glue it next to the most inclusive shape to which it applies.
3. How did you determine which property to glue within each shape category?

Ask Yourself . . .

On the Organization of Quadrilaterals Flow Chart, how do the arrows represent associations between the shapes?

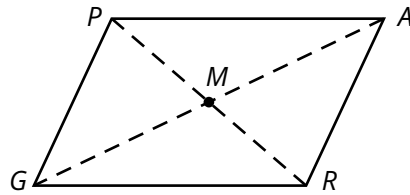
4. Is there a property associated with each category?
If not, explain why.

ACTIVITY
3.1

Properties of Parallelograms and Rhombi

A parallelogram is defined as a quadrilateral with both pairs of opposite sides parallel. Through investigation you learned that opposite sides of a parallelogram are congruent. How can you prove this is true for all parallelograms?

1. Consider parallelogram $PARG$ with diagonals $\overline{PR}$ and $\overline{AG}$ intersecting at point M .



- a. To prove both pairs of opposite sides of a parallelogram are congruent, which two triangles could you prove congruent? Explain your reasoning.
- b. How can you use the definition of a parallelogram to determine congruent angles in the triangles?
- c. What do the triangles share in common?

Now that you have proved that opposite sides of a parallelogram are congruent, you can use this property as a valid reason in future proofs.

2. Write a flow chart, two-column, or paragraph proof to prove that opposite sides of a parallelogram are congruent. Use parallelogram *PARG* for your proof.

Given: Parallelogram $PARG$ with diagonals $\overline{PR}$ and $\overline{AG}$ intersecting at point M

Prove: $\overline{PG} \cong \overline{RA}$ and $\overline{GR} \cong \overline{AP}$

Statements	Reasons

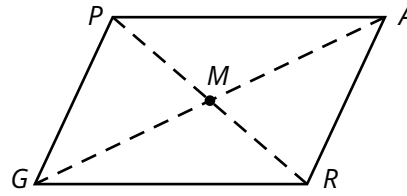
Now let's consider how to prove that pairs of opposite angles of a parallelogram are congruent. There is not enough information from your previous proof to conclude that both pairs of opposite angles are congruent.

3. What additional angles would you need to show congruent to prove that opposite angles of a parallelogram are congruent?

4. Use parallelogram *PARG* to prove that opposite angles of a parallelogram are congruent.

Given: Parallelogram $PARG$ with
diagonals $\overline{PR}$ and $\overline{AG}$
intersecting at point M

Prove: $\angle GPA \cong \angle ARG$

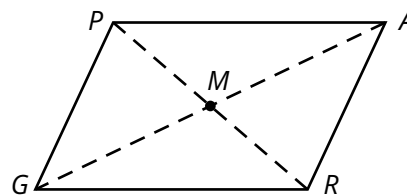


Draw and Deduce

Which relationships have you proved that can be used in this proof?

Statements	Reasons

5. Prove that the diagonals of a parallelogram bisect each other using what you have already proved about the angles and sides of the parallelogram in Question 1.



Draw and Deduce

Which segments in parallelogram $PARG$ need to be proved congruent? Which triangles can help you prove their congruency?



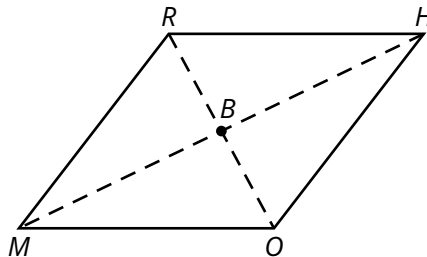
6. Michael told his math teacher that he thinks a quadrilateral is a parallelogram if only one pair of opposite sides is known to be both congruent and parallel.
Is Michael correct? Use the diagram from Question 1 to either prove or disprove his conjecture.

Ask Yourself ...

How can you prove that a pair of lines are parallel?

The **parallel/congruent-parallel side theorem** states: "If one pair of opposite sides of a quadrilateral is both congruent and parallel, then the quadrilateral is a parallelogram."

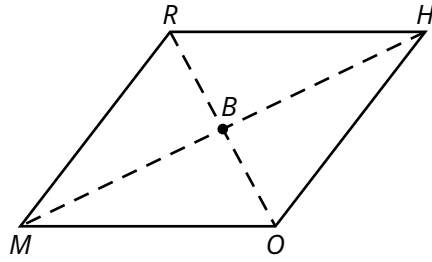
A rhombus is a parallelogram with all sides congruent. Can this classification be proved?



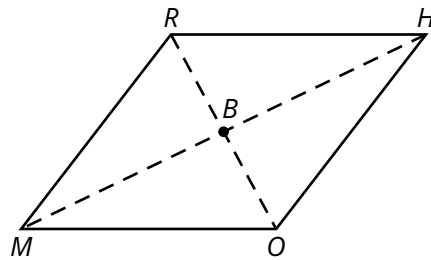
7. Prove that rhombus $RHOM$ is a parallelogram.

8. Since a rhombus is a parallelogram, what properties hold true for all rhombi?

9. Prove that the diagonals of a rhombus are perpendicular.
Use the rhombus $RHOM$.



10. Prove that the diagonals of a rhombus bisect the vertex angles.
Use the rhombus $RHOM$.

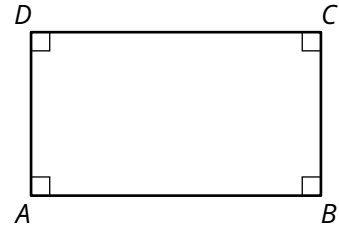


ACTIVITY
3.2

Properties of Rectangles and Squares

A rectangle is a parallelogram with all angles congruent. Can you prove that this is true for all rectangles?

1. Consider rectangle $ABCD$.
 - a. What is the measure of each angle?
Explain your reasoning.

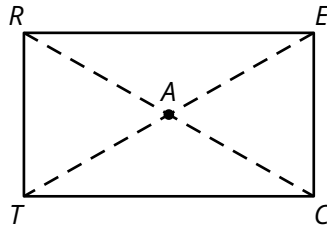


- b. What do the angle measures tell you about the relationships between adjacent sides?
 - c. What theorem can you use to prove $\overline{DA} \parallel \overline{CB}$ and $\overline{DC} \parallel \overline{AB}$?
Is this enough information to conclude rectangle $ABCD$ is a parallelogram? Explain your reasoning.
 - d. Based on your reasoning, what properties hold true for all rectangles?

2. Prove that the diagonals of a rectangle are congruent.

Given: Rectangle $RECT$ with diagonals $\overline{RC}$ and $\overline{ET}$ intersecting at point A

Prove: $\overline{RC} \cong \overline{TE}$



Statements	Reasons

A square is a parallelogram with all angles and all sides congruent.



3. Huyen says you have to prove that the diagonals of a square are perpendicular and congruent, and they bisect the vertex angles. Abby says you don't have to prove these properties of a square. Who is correct? Explain your reasoning.

Properties of Trapezoids

A trapezoid is a quadrilateral with exactly one pair of parallel sides. The parallel sides are called the bases and its non-parallel sides are called the legs. The **base angles of a trapezoid** are either pair of angles that share a base as a common side.

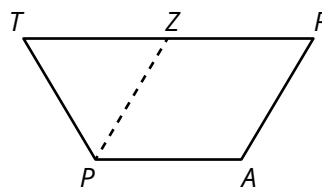
Recall that an isosceles trapezoid is a trapezoid with congruent non-parallel sides. You conjectured that the base angles of an isosceles trapezoid are congruent.

WORKED EXAMPLE

You can prove that the base angles of an isosceles trapezoid are congruent.

Given: Isosceles Trapezoid $TRAP$ with
 $\overline{TR} \parallel \overline{PA}$, $\overline{TP} \cong \overline{RA}$

Prove: $\angle T \cong \angle R$



Given isosceles trapezoid $TRAP$ with $\overline{TR} \parallel \overline{PA}$ and $\overline{TP} \cong \overline{RA}$ by construction, quadrilateral $ZRAP$ is a parallelogram by the definition of a parallelogram. Since, opposite sides of a parallelogram are congruent $\overline{ZP} \cong \overline{RA}$. Given $\overline{TP} \cong \overline{RA}$, $\overline{TP} \cong \overline{ZP}$ by the transitive property of congruence. By the definition of an isosceles triangle, $\triangle TPZ$ is an isosceles triangle. By the isosceles triangle base angles theorem, $\angle T \cong \angle TZP$. By the corresponding angles theorem, $\angle TZP \cong \angle R$. By the transitive property of congruence, $\angle T \cong \angle R$.

1. Analyze the Worked Example.
 - a. How does constructing $\overline{ZP}$ parallel to $\overline{RA}$ help lead to the conclusion?
 - b. Why is it important to state that $\angle TZP \cong \angle R$?
2. You must also prove $\angle A \cong \angle TPA$. Show $\angle A \cong \angle TPA$.

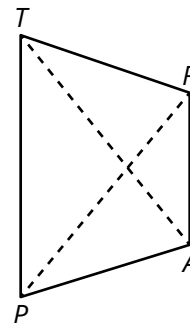


3. Maria insists that if a trapezoid with one pair of parallel sides has one pair of congruent base angles, then the trapezoid must be isosceles. She thinks proving two pairs of base angles are congruent is not necessary. Is Maria correct? Use reasoning and the trapezoid from the Worked Example to justify your answer.

4. Prove that the diagonals of an isosceles trapezoid are congruent.

Given: Isosceles trapezoid $TRAP$ with $\overline{TP} \parallel \overline{RA}$, $\overline{TR} \cong \overline{PA}$, and diagonals $\overline{TA}$ and $\overline{PR}$

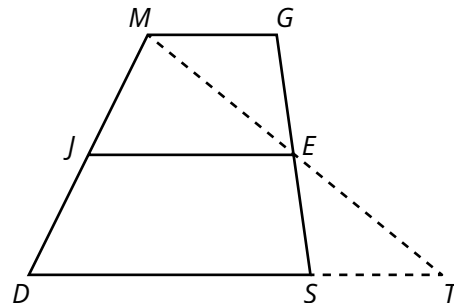
Prove: $\overline{TA} \cong \overline{PR}$



Statements	Reasons

You also conjectured that the length of the midsegment connecting the legs of a trapezoid is half the sum of its base lengths, and that the midsegment is parallel to each base.

To prove this conjecture regarding the midsegment of a trapezoid, it is necessary to connect points M and E of trapezoid $MDSG$ to form $\overline{ME}$, and then extend $\overline{ME}$ until it intersects the extension of $\overline{DS}$ at point T .



5. Complete the Prove statement and answer each question to plan to prove the trapezoid midsegment theorem.

- a. Given: $MDSG$ is a trapezoid with bases $\overline{MG}$ and $\overline{DS}$.
 J is the midpoint of $\overline{MD}$.
 E is the midpoint of $\overline{GS}$.

Prove:

Remember . . .

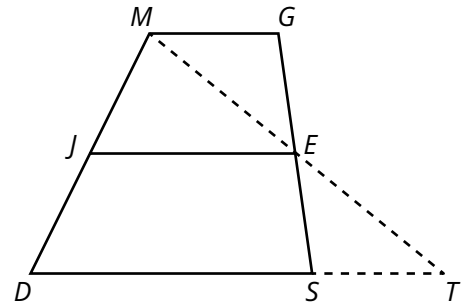
A midsegment of a polygon is a segment formed by connecting two midpoints of its sides.

b. Is $\angle MGE$ congruent to $\angle TSE$? Explain your reasoning.

c. Is $\triangle MGE$ congruent to $\triangle TSE$? Explain your reasoning.

d. How does $\overline{JE}$ relate to $\triangle MDT$? Explain your reasoning.

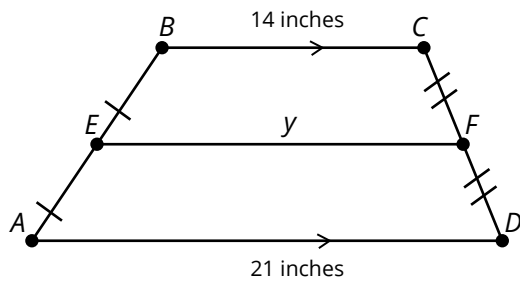
e. Use the midsegment of a triangle theorem and reasoning to prove that $\overline{JE} \parallel \overline{MG}$ and $\overline{JE} \parallel \overline{DS}$.



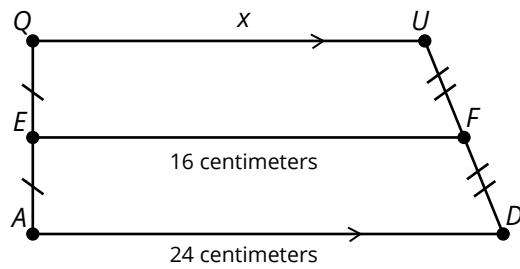
f. Use the midsegment of a triangle theorem and reasoning to prove that $JE = \frac{1}{2}(MG + DS)$.

The **trapezoid midsegment theorem** states: “The midsegment that connects the legs of a trapezoid is parallel to each of the bases and its length is one half the sum of the lengths of the bases.”

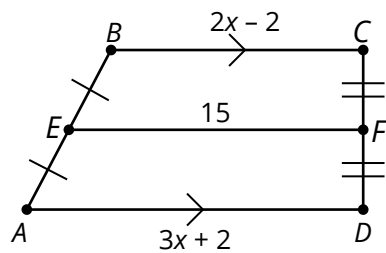
6. Determine the value of y .



7. Determine the value of x .



8. Determine the length of the two bases.



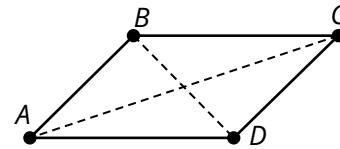
Using Opposite Sides, Opposite Angles, and Diagonals

You proved properties about the sides, angles, and diagonals of quadrilaterals. Use these properties to prove that a quadrilateral is a parallelogram, rhombus, rectangle, or square.

1. Write a flow chart proof to prove the relationship.

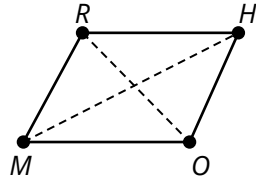
Given: Quadrilateral $ABCD$ with
 $\overline{BC} \cong \overline{AD}$ and $\overline{AB} \cong \overline{DC}$

Prove: Quadrilateral $ABCD$ is a
parallelogram.



2. Given: Quadrilateral $RHOM$
 $\overline{MH}$ bisects $\angle H$ and $\angle M$.
 $\overline{RO}$ bisects $\angle R$ and $\angle O$.

Prove: Quadrilateral $RHOM$ is a rhombus.



- Write a plan to prove that $\overline{RH} \cong \overline{RM}$.
- Write a plan to prove that $\overline{RH} \cong \overline{HO}$.
- How do you know that $\overline{RM} \cong \overline{HO}$?
- Is $\overline{RM} \cong \overline{OM}$? Explain your reasoning.
- How do you know that $\overline{RH} \cong \overline{OM}$?
- Is Quadrilateral $RHOM$ a parallelogram? Explain your reasoning.
- Is Quadrilateral $RHOM$ a rhombus? Explain your reasoning.

3. Complete the reasons in the two-column proof to prove the relationship.

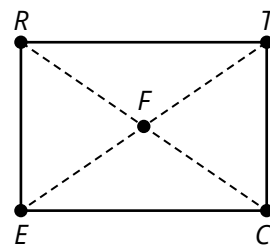
Given: Quadrilateral $RECT$

$$\overline{RC} \cong \overline{ET}$$

$\overline{RC}$ bisects $\overline{ET}$

$\overline{ET}$ bisects $\overline{RC}$

Prove: Quadrilateral $RECT$ is a rectangle.



Statements	Reasons
$\overline{RC}$ bisects $\overline{ET}$	
$\overline{ET}$ bisects $\overline{RC}$	
$\overline{RF} \cong \overline{FC}$	
$\overline{EF} \cong \overline{FT}$	
$\angle RFE \cong \angle CFT$	
$\triangle RFE \cong \triangle CFT$	
$\angle ERF \cong \angle FCT$	
$\overline{RE} \parallel \overline{CT}$	
$\angle RFT \cong \angle CFE$	
$\triangle CFE \cong \triangle RFT$	
$\angle TRF \cong \angle ECR$	
$\overline{RT} \parallel \overline{EC}$	
Quadrilateral $RECT$ is a parallelogram	
$\overline{RC} \cong \overline{ET}$	
$\overline{EC} \cong \overline{RT}$	
$\overline{RE} \cong \overline{ER}$	
$\triangle RET \cong \triangle ERC$	

$$\angle ERT \cong \angle REC$$

$\angle ERT$ and $\angle REC$ are
supplementary angles
 $m\angle ERT + m\angle REC = 180^\circ$

$$m\angle ERT = m\angle REC$$

$$m\angle ERT + m\angle ERT = 180^\circ$$

$$m\angle ERT = 90^\circ$$

$$m\angle REC = 90^\circ$$

$$\angle CTR \cong \angle REC$$

$$\angle ERT \cong \angle TCE$$

$$m\angle CTR = m\angle REC$$

$$m\angle ERT = m\angle TCE$$

$$m\angle CTR = 90^\circ$$

$$m\angle TCE = 90^\circ$$

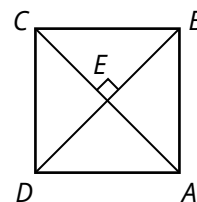
Quadrilateral $RECT$ is a
rectangle.

4. Given: $\overline{AC} \perp \overline{BD}$

$$\overline{AC} \cong \overline{BD}$$

$$AE = EC = BE = ED$$

Prove: Quadrilateral $ABCD$ is a square.



a. Write a plan to prove $\triangle BEC \cong \triangle BEA \cong \triangle AED \cong \triangle DEC$.

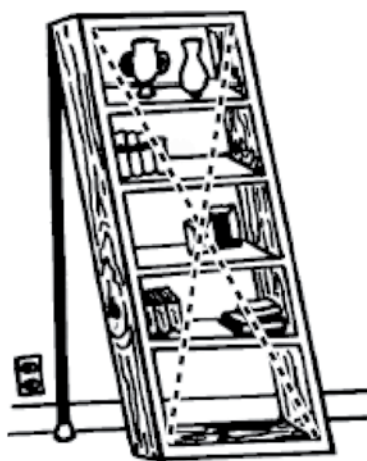
- b. Explain why $\overline{BC} \cong \overline{BA} \cong \overline{AD} \cong \overline{DC}$.
- c. Is Quadrilateral $ABCD$ a parallelogram? A rhombus? Explain your reasoning.
- d. What additional information do you need to show that Quadrilateral $ABCD$ is a square?
- e. Write a plan to prove that $\angle CBA$ and $\angle DAB$ are supplementary.
- f. Use the given information $\overline{AC} \cong \overline{BD}$ to write a plan to prove that $\triangle CBA \cong \triangle DAB$.
- g. Why is $\angle CBA \cong \angle DAB$?
- h. Write a plan to prove that the four angles of the quadrilateral are right angles.

ACTIVITY
3.5

Using Properties to Solve Problems

Use what you know about the properties of quadrilaterals to solve each problem.

1. Hannah is making a square mat for a picture frame. How can she make sure the mat is a square using only a ruler?
2. Destiny is putting together a bookcase. It came with diagonal support bars that are to be screwed into the top and bottom on the back of the bookcase. Unfortunately, the instructions were lost and Destiny does not have the directions or a measuring tool. She has a screwdriver, a marker, and a piece of string. How can Destiny attach the supports to make certain the bookcase will be a rectangle and the shelves are parallel to the ground?



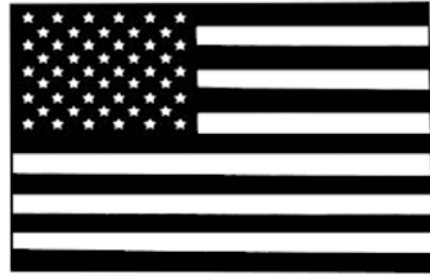
PROBLEM SOLVING



Ask Yourself . . .

How can you use properties of quadrilaterals in everyday life?

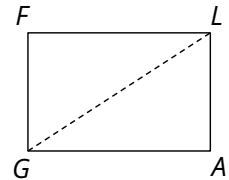
3. Ms. Brown told her geometry students that anyone bringing in a picture of a flag with a parallelogram would get extra credit.



- a. Malik brought in a picture of the United States flag. The teacher handed Malik a ruler and told him to prove the quadrilateral that contains the stars is a parallelogram. What are two ways Malik could prove the specified quadrilateral is a parallelogram?

.....
 You can write a
 flow chart, two-column,
 or paragraph proof to
 prove the statement.

- b. Ms. Brown told Malik he needed to prove that if a quadrilateral has opposite sides congruent, then the quadrilateral is a parallelogram. Use the diagram to help Malik prove the statement.



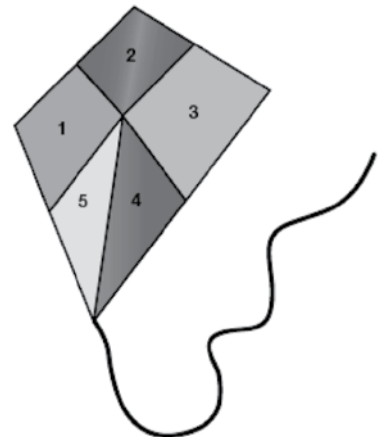
Given: Quadrilateral $FLAG$ with diagonal
 $\overline{GL}$, $\overline{FL} \cong \overline{GA}$, $\overline{LA} \cong \overline{GF}$

Prove: Quadrilateral $FLAG$ is a parallelogram.

4. Ms. Brown held up two different lengths of rope shown and a piece of chalk. She asked her students if they could use this rope and chalk to construct a rhombus on the blackboard. Gabriela raised her hand and said she could construct a rhombus with the materials. Ms. Brown handed Gabriela the chalk and rope. What did Gabriela do?



5. Could quadrilaterals 1, 2, and 3 on the kite shown be squares if the kite is not a parallelogram? Explain your reasoning.



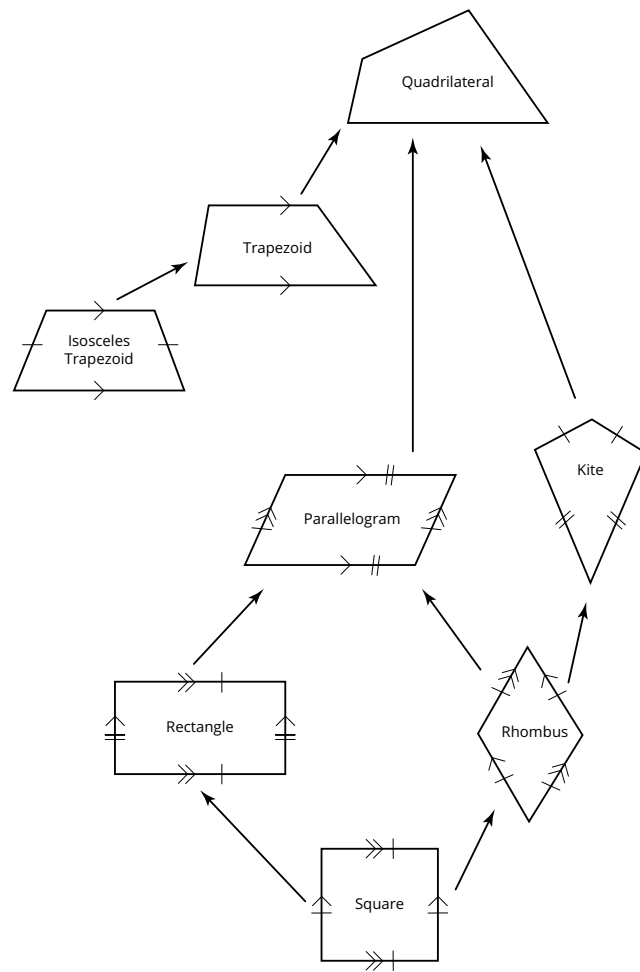


Talk the Talk

Name That Shape

1. Mr. Lopez said he was thinking of a quadrilateral and wanted his students to name the quadrilateral. He said he would answer a few yes-no questions about the properties of the quadrilateral to give them a hint. What questions would you ask Mr. Lopez?

Organization of Quadrilaterals Flow Chart



Properties of Quadrilaterals



two diagonals	two pairs of adjacent congruent sides	opposite angles are congruent
four sides	diagonals bisect each other	all sides are congruent
diagonals are congruent	all angles are congruent	opposite sides are congruent
diagonals are perpendicular	exactly one pair of parallel sides	diagonals are congruent

Lesson 3 Assignment

Write

Define each term in your own words.

1. parallelogram/congruent-parallel side theorem
2. base angles of a trapezoid
3. trapezoid midsegment theorem

Remember

- Properties of a parallelogram: opposite sides are parallel and congruent; opposite angles are congruent; diagonals bisect each other
- Properties of a rhombus: all sides are congruent; diagonals are perpendicular and bisect the vertex angles
- Properties of a rectangle: angles and diagonals are congruent
- Properties of isosceles trapezoids: base angles are congruent; diagonals are congruent
- Properties of a kite: two pairs of adjacent, congruent sides; at least one pair of opposite angles congruent; one diagonal bisects the other and the vertex angles; diagonals are perpendicular

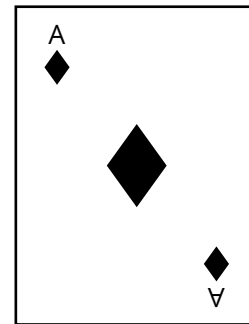
Practice

1. The neighbors in a rural community get together for a barn-raising. The first step is to build the rectangular base of the barn. One neighbor explains to everyone how diagonals can be used to verify the base is rectangular. What could the neighbor have said?

Lesson 3 Assignment

2. Luis tells you he is thinking of a quadrilateral that is either a square or a rhombus, but not both. He wants you to guess which quadrilateral he is thinking of and allows you to ask one question about the quadrilateral. Which question should you ask?

3. Consider the Ace of Diamonds playing card shown. The large diamond in the center of the playing card is a quadrilateral. Classify the quadrilateral based only on each piece of given information.



- a. The diagonals of the quadrilateral bisect each other.
- b. The four sides of the quadrilateral are congruent.
- c. The four angles and the four sides of the quadrilateral are congruent.
- d. The diagonals of the quadrilateral bisect the vertex angles.

Lesson 3 Assignment

- e. The four angles of the quadrilateral are congruent.
- f. The opposite sides of the quadrilateral are both congruent and parallel.
- g. The opposite angles of the quadrilateral are congruent.

Prepare

Compare the ratios in each pair.

1. $4 : 3$ $3 : 5$

2. $\frac{5}{1}$ $\frac{1}{5}$

3. $\frac{2}{3}$ $\frac{4}{6}$

4. 3 to 4 3 to 2

Using Congruence Theorems

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

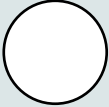
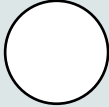
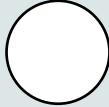
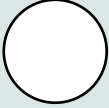
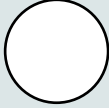
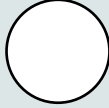
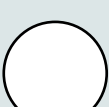
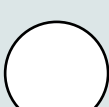
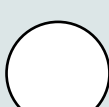
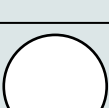
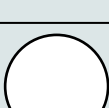
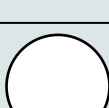
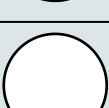
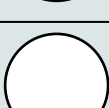
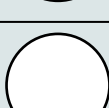
Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Using Congruence Theorems* topic by:

TOPIC 3: <i>Using Congruence Theorems</i>	Beginning of Topic	Middle of Topic	End of Topic
proving the Hypotenuse-Leg congruence theorem.			
using triangle congruence theorems to justify the Leg-Leg and Leg-Angle congruence theorems.			
applying right triangle congruence theorems to solve problems.			
using the tangent segment theorem to solve problems.			
proving the diameter-chord theorem, the equidistant chord theorem and its converse, and the congruent chord-congruent arc theorem and its converse.			
solving real-world problems using relationships among chords in a circle.			
classifying types of quadrilaterals.			

continued on the next page

TOPIC 3 SELF-REFLECTION
continued

TOPIC 3: <i>Using Congruence Theorems</i>	Beginning of Topic	Middle of Topic	End of Topic
proving properties about sides, diagonals, and angles of parallelograms, rhombi, rectangles, squares, and trapezoids.			
using the properties of special quadrilaterals in proofs.			
proving the segment joining the midpoints of two sides of a trapezoid with only one pair of parallel sides is parallel to the bases and one-half the sum of their lengths.			
proving a quadrilateral is a parallelogram, rectangle, rhombus, or square using opposite sides, opposite angles, and diagonals.			
applying the properties of parallelograms to solve real-world problems.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

- Describe a new strategy you learned in the *Using Congruence Theorems* topic.
- What mathematical understandings from the topic do you feel you are making the most progress with?

TOPIC 3 SELF-REFLECTION *continued*

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

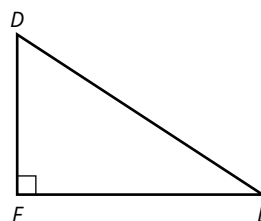
Using Congruence Theorems Summary

LESSON 1

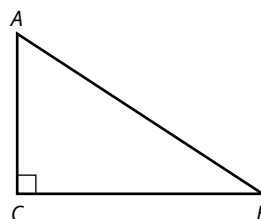
Using Triangle Congruence to Determine Relationships Between Segments

Many congruence theorems apply to all triangles. There are also theorems that only apply to right triangles. Methods for proving that two right triangles are congruent are somewhat shorter. You can prove that two right triangles are congruent using only two measurements.

The **Hypotenuse-Leg (HL) congruence theorem** states: “If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent.” For example, given $\overline{AC} \cong \overline{DF}$ and $\overline{AB} \cong \overline{DE}$, then $\triangle ABC \cong \triangle DEF$.



The **Leg-Leg (LL) congruence theorem** states: “If the two corresponding shorter legs of two right triangles are congruent, then the two triangles are congruent.” For example, given $\overline{AC} \cong \overline{DF}$ and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.



The **Leg-Angle (LA) congruence theorem** states: “If the leg and an acute angle of one right triangle are congruent to the corresponding leg and acute angle of another right triangle, then the triangles are congruent.” For example, given $\overline{AC} \cong \overline{DF}$ and $\angle A \cong \angle D$, then $\triangle ABC \cong \triangle DEF$.

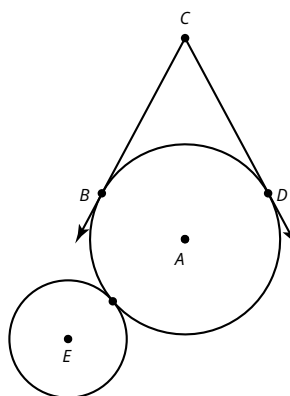
KEY TERMS

- Hypotenuse-Leg congruence theorem
- Leg-Leg congruence theorem
- Leg-Angle congruence theorem
- tangent segment theorem [teorema del segmento tangente]
- tangent circles [círculos tangentes]
- diameter chord theorem [teorema de diámetro y cuerda]
- equidistant chord theorem [teorema de las cuerdas equidistantes]
- equidistant chord converse theorem
- congruent chord-congruent arc theorem [teorema de cuerda congruente-arco congruente]
- congruent chord-congruent arc converse theorem
- parallelogram/congruent parallel sides theorem [teorema del paralelogramo/lados paralelos congruentes]
- base angles of a trapezoid [ángulos base de un trapecio]
- trapezoid midsegment theorem [teorema del segmento medio del trapecio]

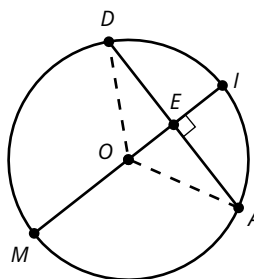
Using Congruent Triangles to Determine Relationships Between Chords and Tangents

Recall that a tangent segment is a line segment formed by connecting a point outside of the circle to a point of tangency. The **tangent segment theorem** states that if two tangent segments are drawn from the same point on the exterior of a circle, then the tangent segments are congruent. For example, $\overline{CB}$ is tangent to circle A at point B and $\overline{CD}$ is tangent to circle A at point D. Therefore, $\overline{CB} \cong \overline{CD}$.

Tangent circles are circles that lie in the same plane and intersect at exactly one point. For example, circle A and circle E are tangent circles.

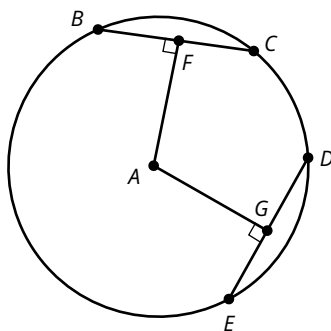


The **diameter-chord theorem** states: "If a circle's diameter is perpendicular to a chord, then the diameter bisects the chord and bisects the arc determined by the chord." For example, in circle O, diameter $\overline{IM} \perp$ chord $\overline{DA}$. Therefore, $\overline{DE} \cong \overline{AE}$ and $\widehat{DI} \cong \widehat{AI}$.



The **equidistant chord theorem** states: "If two chords of the same circle or congruent circles are congruent, then they are equidistant from the center of the circle." For example, in circle A, chord $\overline{BC} >$ chord $\overline{DE}$. Therefore, $AF = AG$.

The **equidistant chord converse theorem** states: “If two chords of the same circle or congruent circles are equidistant from the center of the circle, then the chords are congruent.”



The **congruent chord-congruent arc theorem** states: “If two chords of the same circle or congruent circles are congruent, then their corresponding arcs are congruent.” For example, in circle A, $\widehat{BC} \cong \widehat{DE}$.

The **congruent chord-congruent converse theorem** states: “If two arcs of the same circle or congruent circles are congruent, then their corresponding chords are congruent.”

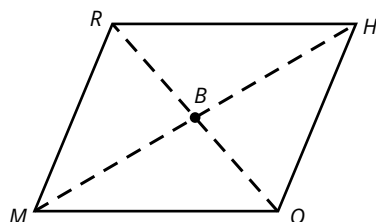
LESSON

3

Properties of Quadrilaterals

A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

The **parallelogram/congruent-parallel side theorem** states: “If one pair of opposite sides of a quadrilateral is both congruent and parallel, then the quadrilateral is a parallelogram.” For example, given quadrilateral $RHOM$, $\overline{RM} \cong \overline{OH}$, and $\overline{RM} \parallel \overline{OH}$, then quadrilateral $RHOM$ is a parallelogram.

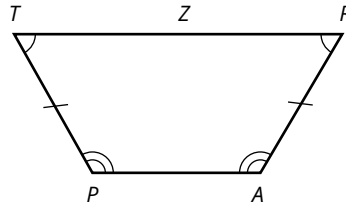


A rectangle is a quadrilateral with opposite sides congruent and all angles congruent. A rectangle is a parallelogram. A square is a quadrilateral with all angles and all sides congruent. A square is a rectangle.

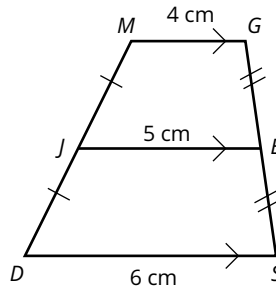
A trapezoid is a quadrilateral with exactly one pair of parallel sides.

The parallel sides are called the bases and its non-parallel sides are called the legs. The **base angles of a trapezoid** are either pair of angles that share a base as a common side. An isosceles trapezoid is a trapezoid with congruent non-parallel sides. The base angles of an isosceles trapezoid are congruent.

For example, quadrilateral $TRAP$ is an isosceles trapezoid with bases $\overline{TR}$ and $\overline{PA}$. Therefore, $\angle T \cong \angle R$ and $\angle A \cong \angle P$.



The length of the midsegment connecting the legs of a trapezoid is half the sum of its base lengths. The **trapezoid midsegment theorem** states: “The midsegment that connects the legs of a trapezoid is parallel to each of the bases and its length is one half the sum of the lengths of the bases.” For example, in trapezoid $MGSD$, $\overline{JE}$ is a midsegment connecting the legs of the trapezoid. Therefore, it is parallel to each of the bases and $JE = \frac{1}{2}(MG + DS)$.



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