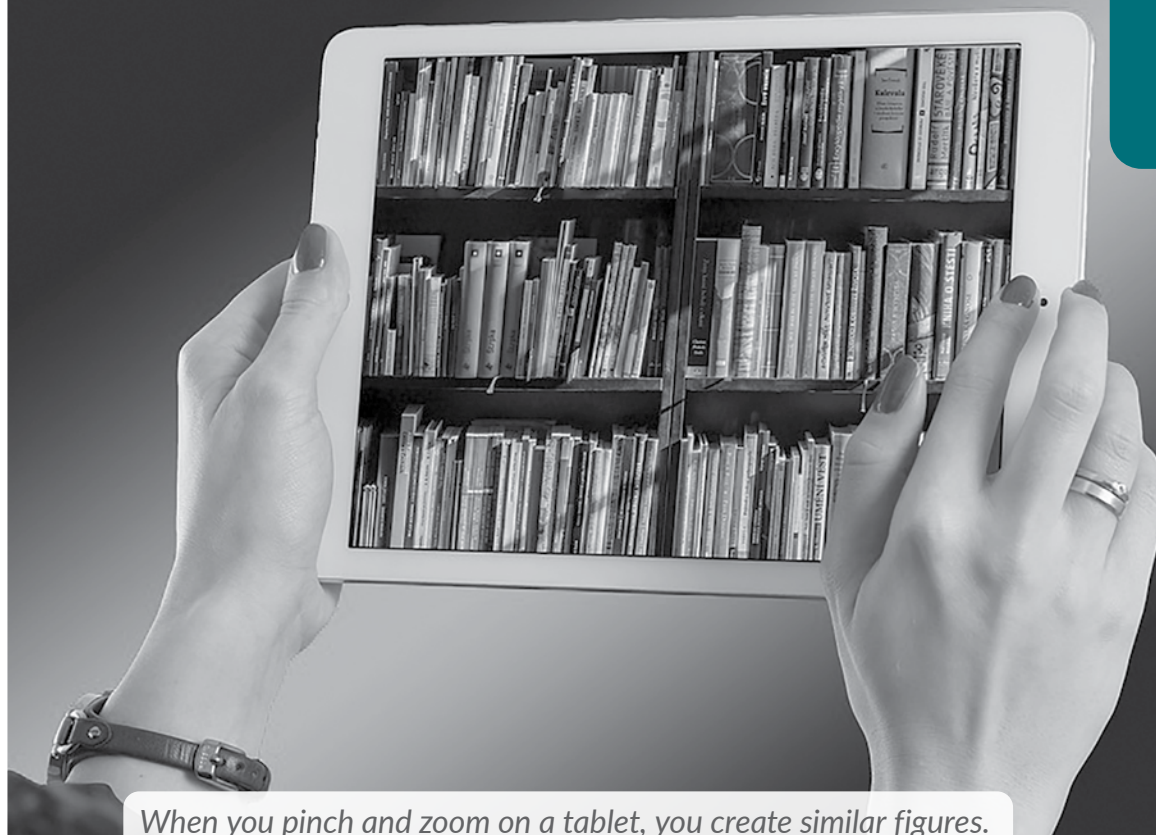


Investigating Proportionality

TOPIC 1	Similarity	743
TOPIC 2	Trigonometry	887





When you pinch and zoom on a tablet, you create similar figures.

Similarity

LESSON 1	Dilating Figures to Create Similar Figures	745
LESSON 2	Establishing Triangle Similarity Criteria	773
LESSON 3	Theorems About Proportionality	791
LESSON 4	More Similar Triangles	825
LESSON 5	Application of Similar Triangles	839
LESSON 6	Partitioning Segments in Given Ratios	855



1

Dilating Figures to Create Similar Figures

OBJECTIVES

- Determine the image of a given two-dimensional figure under a composition of dilations.
- Dilate geometric figures to show enlargements, reductions, and congruent figures.
- Identify the scale factors of dilations and produce dilations at a given scale factor.
- Demonstrate that triangles are similar using transformations, both orally and in writing.
- Use similarity transformations to determine whether two figures are similar.

KEY TERMS

- dilation
- similar figures
- similar triangles

.....

In previous courses, you learned about dilations, similarity, and transformations.

How can you use similarity transformations to demonstrate that two figures are similar?

Getting Started

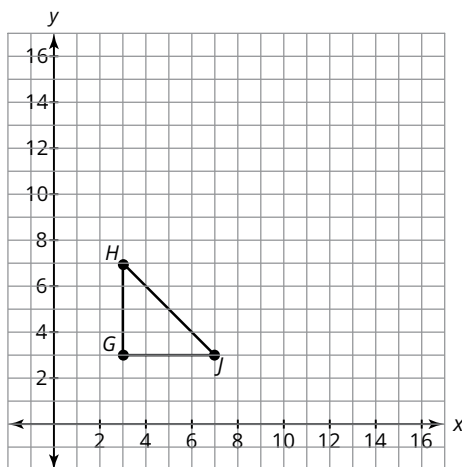
Dilating on the Coordinate Plane

.....
A **dilation** is a transformation of the figure in which the figure stretches or shrinks with respect to a fixed point, or center of dilation.
.....
.....

To show a dilation, I need to know the center of dilation and the coordinates of the pre-image.
.....

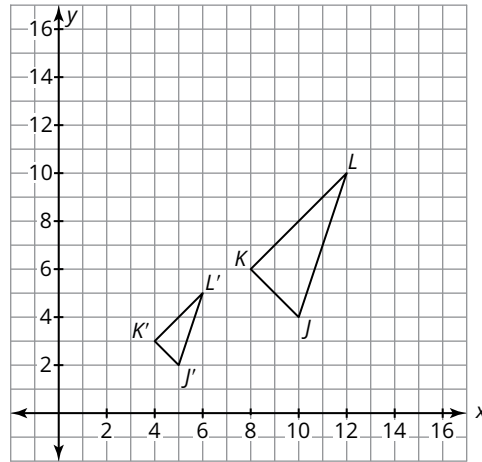
You can use your compass and a straightedge to perform a dilation. Consider $\triangle GHJ$ shown on the coordinate plane. You will dilate the triangle by using the origin as the center and by using a scale factor of 2.

1. How will the distance from the center of dilation to a point on the image of $\triangle G'H'J'$ compare to the distance from the center of dilation to a corresponding point on $\triangle GHJ$? Explain your reasoning.



2. For each vertex of $\triangle GHJ$, draw a ray that starts at the origin and passes through the vertex.
3. Use the duplicate segment construction to locate the vertices of $\triangle G'H'J'$.
4. List the coordinates of the vertices of $\triangle GHJ$ and $\triangle G'H'J'$. How do the coordinates of the image compare to the coordinates of the pre-image?

5. Triangle $J'K'L'$ is a dilation of $\triangle JKL$. The center of dilation is the origin.

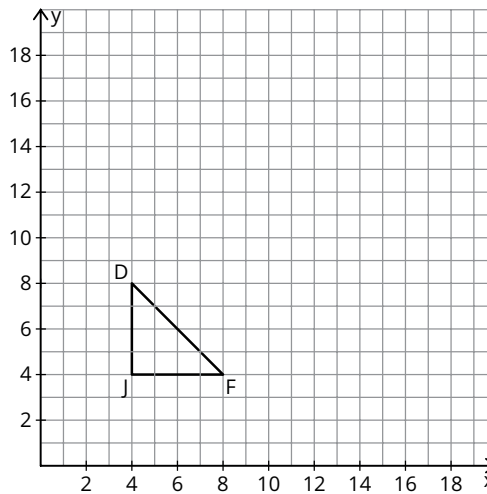


- List the coordinates of the vertices of $\triangle JKL$ and $\triangle J'K'L'$. How do the coordinates of the image compare to the coordinates of the pre-image?
 - What is the scale factor of the dilation? Explain your reasoning.
 - How do you think you can use the scale factor to determine the coordinates of the vertices of an image?
 - Use coordinate notation to describe the dilation.
6. Use coordinate notation to describe the dilation of point (x, y) when the center of dilation is at the origin using a scale factor of k .

Multiple Dilations

You can also perform multiple dilations, or a composition of dilations, on the coordinate plane using any point as the center of the dilation.

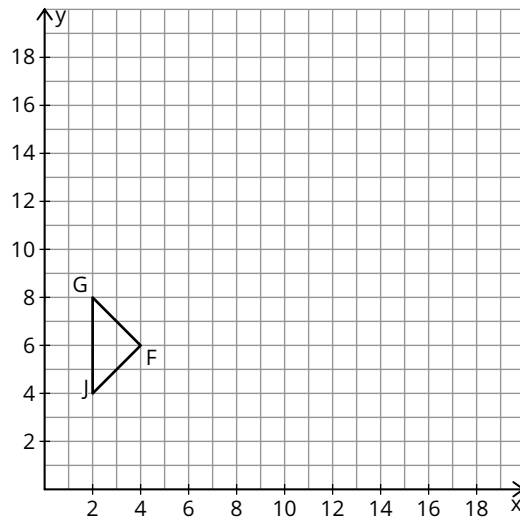
1. Consider $\triangle JDF$ on the coordinate plane shown.



- Show a dilation of $\triangle JDF$ by a factor of 3, using the point $(2, 2)$ as the center of dilation. Label the image as $\triangle J'D'F'$.
 - Show a dilation of $\triangle J'D'F'$ by a factor of $\frac{1}{2}$, using the same point $(2, 2)$ as the center of dilation. Label the new image as $\triangle J''D''F''$.
2. Ben says that dilating $\triangle JDF$ by a factor of 3 and then dilating the image by a factor of $\frac{1}{2}$ results in an image of the original triangle that is dilated by a factor of $\frac{3}{2}$, or 1.5.

Do you think Ben is correct? Explain your reasoning.

3. Consider $\triangle JGF$ on the coordinate plane shown.

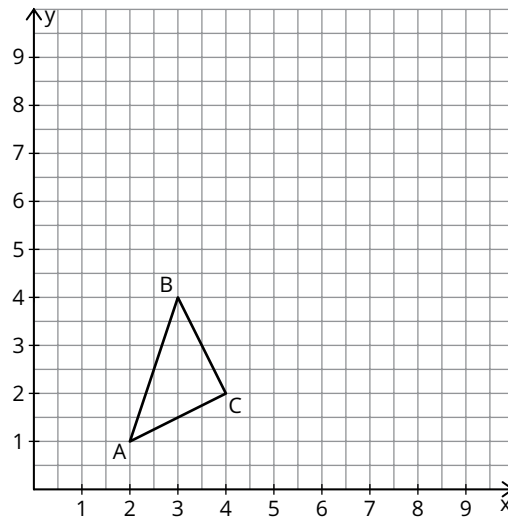


Ask Yourself . . .

What tools or strategies can you use to solve this problem?

- Using the origin as the center of dilation, show a dilation of $\triangle JGF$ by a factor of 2. Label the image as $\triangle J'G'F'$.
- Show a dilation of $\triangle J'G'F'$ by a factor of $\frac{1}{4}$. Label the image as $\triangle J''G''F''$.
- Write the composition of dilations using a single factor. Explain your reasoning.
- Write the composition of dilations using a single factor.

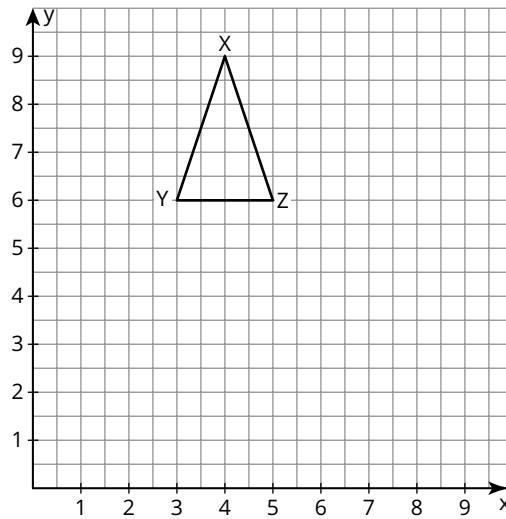
4. Use a composition of dilations to show a dilation of $\triangle ABC$ by a factor of 1.25 with the origin as the center of dilation. Label the image as $\triangle A''B''C''$. Show your work and explain your reasoning.



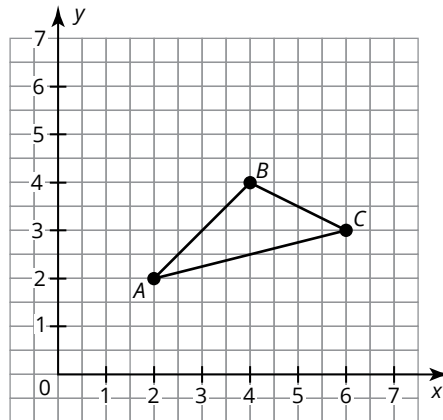
.....
Can you use
transformations
to determine
coordinates when the
center of dilation is
not the origin?
.....

5. Describe how you can determine the coordinates of an image after a composition of dilations when:
- the center of dilation is at the origin.
 - the center of dilation is not at the origin.

6. Use a composition of dilations to show a dilation of $\triangle XYZ$ by a factor of $\frac{2}{3}$ with the center of dilation at $(3, 2)$. Label the image as $\triangle X''Y''Z''$. Show your work and explain your reasoning.



7. An image is produced by applying a dilation by a factor of a , then another dilation by a factor of b .
- If the center of dilation is the origin, describe how to determine the coordinates of the pre-image. Explain your reasoning.
 - If the center of dilation is not the origin, what additional steps are needed to determine the coordinates of the pre-image.
8. Dilate the pre-image $\triangle ABC$ by a scale factor of 3 and a center at the origin, followed by a translation 2 units to the right and 1 unit down. Describe the transformation using coordinate notation, then, determine the coordinates of the image $\triangle A'B'C'$.



Dilating Figures by a Scale Factor

You have volunteered to help at the children's booth at an art festival. The children will use an image of the Alamo made out of poster board and craft sticks. Then, they will use a flashlight to create a shadow. Your job is to show the children how to use a flashlight and a wall to make their shadow.



1. How does the size of the shadow compare to the size of the object made out of poster board and craft sticks?
2. How does the shape of the shadow compare to the shape of the object made out of poster board and craft sticks?
3. Do you think that the shadow is a transformation of the object? Why or why not?

PROBLEM SOLVING

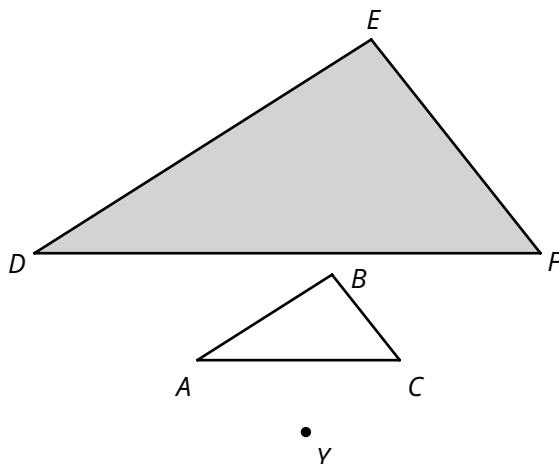


.....

The Alamo, located in San Antonio, Texas is one of the most famous historic sites in the United States. It is best known for a battle that occurred during the Texas Revolution in 1836, when a group of roughly 200 Texans made their stand against the Mexican army.

.....

Consider $\triangle ABC$, $\triangle DEF$, and point Y . Imagine that point Y is the flashlight and $\triangle DEF$ is the shadow of $\triangle ABC$. Thus, $\triangle DEF$ is a dilation of $\triangle ABC$.



4. Draw $\overline{YD}$, $\overline{YE}$, and $\overline{YF}$ on the figure shown. These line segments show the path of the light from the flashlight.

.....
Remember...

A *ratio* is a comparison of two numbers using division.

The ratio 4 : 3 can be written as $\frac{4}{3}$ or as 4 to 3.

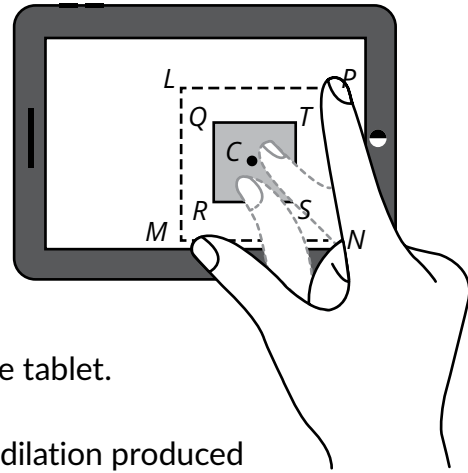
5. Use a metric ruler to determine the actual lengths of $\overline{YA}$, $\overline{YB}$, $\overline{YC}$, $\overline{YD}$, $\overline{YE}$, and $\overline{YF}$ to the nearest tenth of a centimeter.

Then express the ratios $\frac{YD}{YA}$, $\frac{YE}{YB}$, and $\frac{YF}{YC}$ as decimals. What do you notice about the ratios?

6. Use a protractor to measure the corresponding angles in the triangles. What can you conclude?
7. In any dilation:
 - a. How do the measures of the corresponding angles in the image and pre-image compare?
 - b. How do the ratios of the lengths of the corresponding sides in the image and pre-image compare?
8. What is the center of the dilation in the model of the shadow triangle?

The ratios you wrote to describe the corresponding side lengths of the original figure and the dilated figure are all equal to the same value. This value is the scale factor of the dilation.

A dilation can produce an enlargement, a reduction, or a congruent figure. For example, when you pinch and zoom on a tablet computer, you can create dilations.



9. Consider the dilations shown on the tablet.

- a. Estimate the scale factor of the dilation produced when you “pinch” the screen to create square $QRST$ given the pre-image of square $LMNP$. Show your work.

.....
Think About . . .

What is the center of dilation?

- b. Estimate the scale factor of the dilation produced when you “zoom” to create square $LMNP$ given the pre-image of square $QRST$. Show your work.

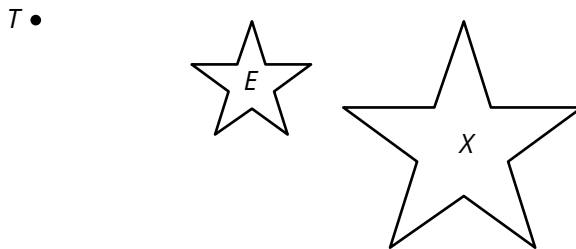
10. How does the image compare to the pre-image when:

- a. The scale factor is greater than 0 and less than 1?

- b. The scale factor is greater than 1?

A dilation produces *similar figures*. **Similar figures** are geometric figures where all pairs of corresponding angles are congruent and the lengths of all corresponding sides are proportional.

11. Use point T as the center of dilation.



a. Use a metric ruler and what you know about dilations to demonstrate that Figure E is similar to Figure X . Explain your reasoning.

b. What do you know about the angles in Figures E and X ? Explain your reasoning.

12. What scale factor will produce congruent figures? Provide an example to justify your answer.

13. How can you produce congruent figures using dilations that are mapped onto itself both on and off the coordinate plane? Explain your reasoning.

Using Similarity Transformations to Demonstrate Similarity

Ask Yourself ...

What is the difference between a *dilation* and a *rigid motion*?

.....

Think About ...

What tools can help you with this?

.....

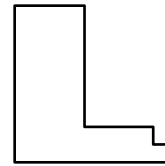
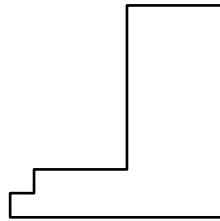
Two or more geometric figures are similar if a sequence of dilations and/or rigid motions maps one figure onto the other.

1. Demonstrate that the figures in each pair are similar using dilations and rigid motions. Explain your process and describe the transformations you used.

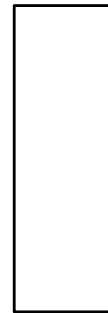
Figure A

Figure B

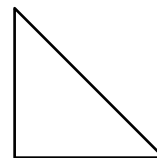
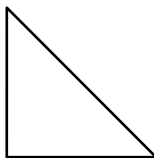
a.



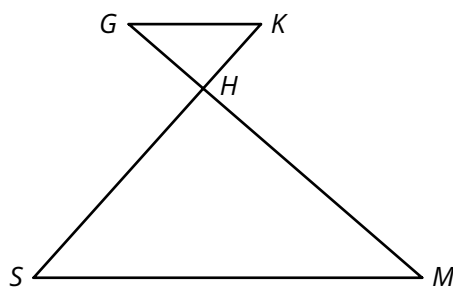
b.



c.



2. Consider the diagram shown and the three separate cases.



- a. Suppose $\overline{GK}$ is parallel to $\overline{SM}$.
Describe a sequence of transformations that maps one triangle onto the other triangle.
- b. Suppose $\angle G \cong \angle S$.
Describe a sequence of transformations that maps one triangle onto the other triangle.

.....

Remember ...

If $ab = cd$, then you

know that $\frac{a}{c} = \frac{d}{b}$ or

$$\frac{a}{d} = \frac{c}{b}.$$

I can show this is true

using the division

property of equality.

For example, when

$ab = cd$, I can use the

division property of

equality and divide

both sides by b , to

show that $a = \frac{cd}{b}$.

I can use the division

property of equality

again and divide both

sides by d to show

that $\frac{a}{d} = \frac{c}{b}$.

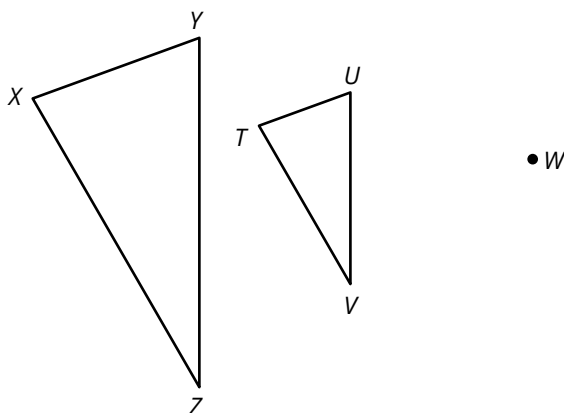
.....

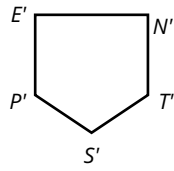
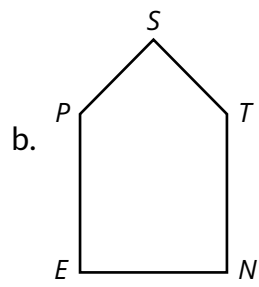
c. Suppose $(\overline{KH})(\overline{SH}) = (\overline{GH})(\overline{MH})$.

Describe a sequence of transformations that maps one triangle onto the other triangle.

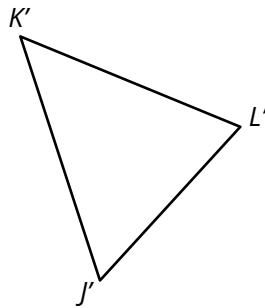
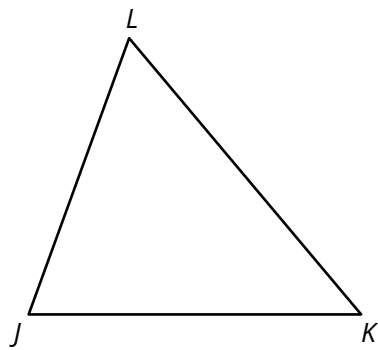
3. Use the definition of similarity in terms of transformations to determine whether each pair of figures are similar. Justify your reasoning.

a.

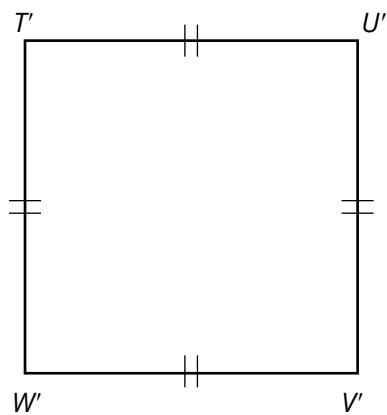
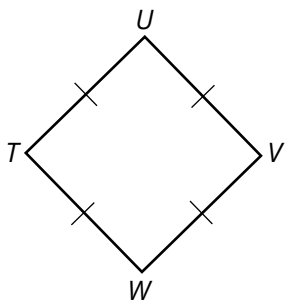




c.



d.



Using Geometric Theorems to Demonstrate Similarity

Similar triangles are triangles where all pairs of corresponding angles are congruent and the lengths of all corresponding sides are proportional. Similar triangles have the same shape but not always the same size.

.....
The symbol $\sim$
represents similarity.
.....

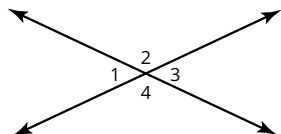
1. $\triangle HRY \sim \triangle JPT$.

Sketch a diagram that illustrates this similarity statement and list all of the pairs of congruent angles and all of the proportional sides.

.....

Remember ...

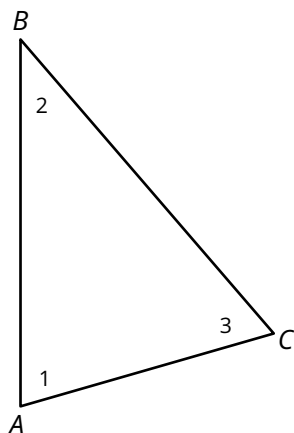
The vertical angle theorem states that vertical angles are congruent.



$$m\angle 1 = m\angle 3$$

$$m\angle 2 = m\angle 4$$

The triangle sum theorem states that the sum of the interior angle measures of any triangle is equal to 180° .

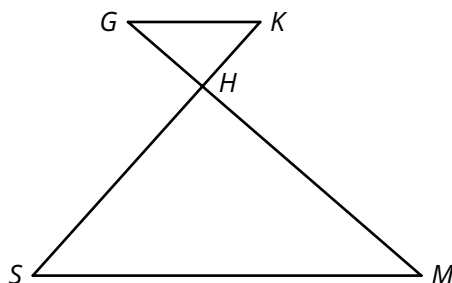


$$m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$$

Pairs of congruent angles are formed when parallel lines are crossed by a transversal: corresponding angles, alternate interior angles, and alternate exterior angles.

.....

2. Consider the figure shown. What conditions are necessary to show $\triangle GHK$ is similar to $\triangle MHS$?



3. Consider three separate cases for the diagram shown in Question 2. For each conjecture, determine whether this given information is enough to prove that the two triangles are similar. Explain why you think they are similar or provide a counterexample if you think the triangles are not similar.

a. When $4GH = MH$, then $\triangle GHK \sim \triangle MHS$.

b. When $\overline{GK}$ is parallel to $\overline{MS}$, then $\triangle GHK \sim \triangle MHS$.

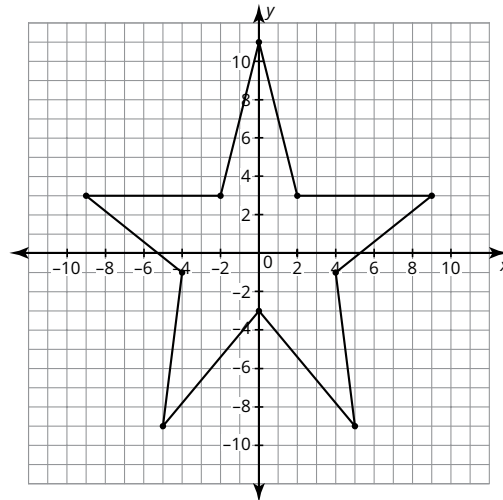
c. When $\angle G \cong \angle S$, then $\triangle GHK \sim \triangle SHM$.



Talk the Talk

Ad Astra per Aspera

You have created dilations of geometric figures to produce similar figures. How can you show dilations on the coordinate plane?



1. How would you determine the new coordinates of the star when it is dilated from the origin by a scale factor of B ?
2. Suppose the star is dilated from the origin by a scale factor of 3. Identify the new coordinates of each vertex.
3. Suppose the star is dilated from the origin by a scale factor of 0.5, followed by a translation 4 units to the left and 2 units up. Identify the new coordinates of each vertex.

Lesson 1 Assignment

Write

Define the term *similar triangles* in your own words.

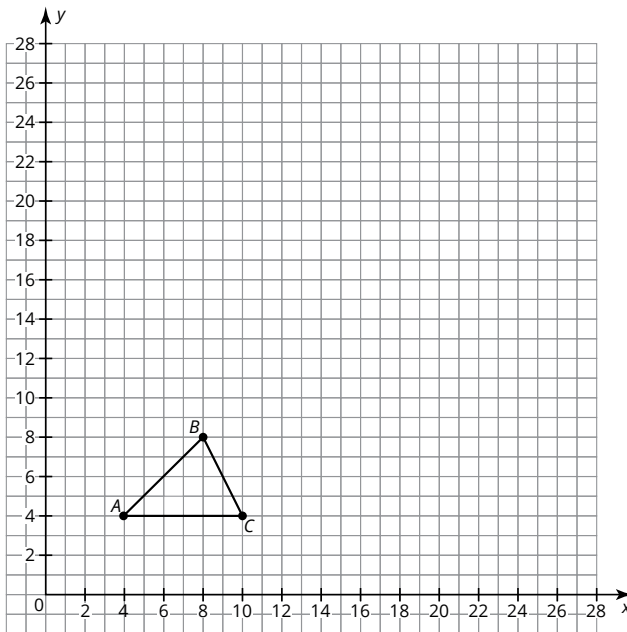
Remember

A *dilation* is a transformation that enlarges, reduces, or keeps congruent a pre-image to create a similar image.

When the scale factor of a dilation is greater than 1, the dilation is an enlargement. When the scale factor is between 0 and 1, the dilation is a reduction. When the scale factor is exactly 1, the dilation produces congruent figures.

Practice

1. Consider $\triangle ABC$ on the coordinate plane shown.



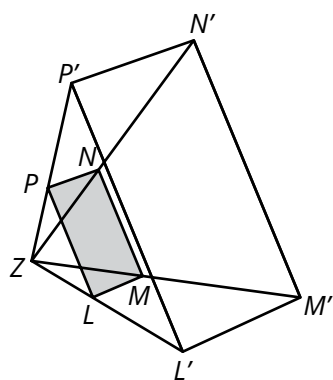
- Using the origin as the center of dilation, show a dilation of $\triangle ABC$ by a factor of $\frac{1}{4}$. Label the image $\triangle A'B'C'$.
- Using the origin as the center of dilation, show a dilation of $\triangle A'B'C'$ by a factor of 10. Label the image $\triangle A''B''C''$.

Lesson 1 Assignment

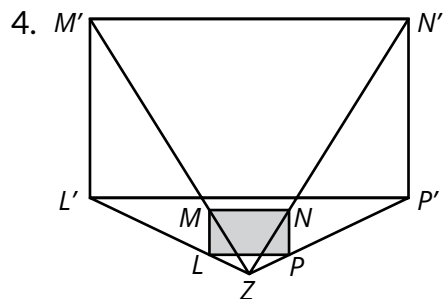
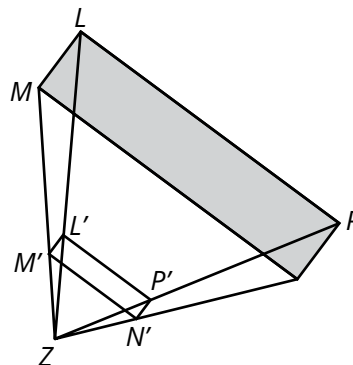
- c. Write the composition of dilations using a single factor. Explain your reasoning.

Given each pre-image and image, determine the scale factor.

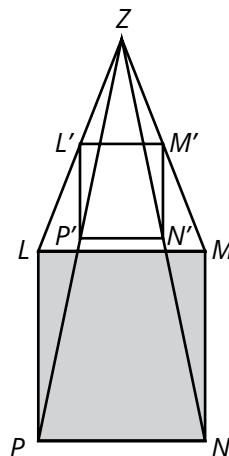
2.



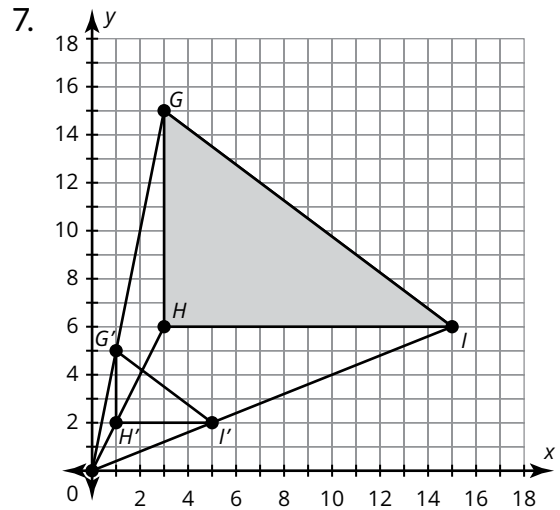
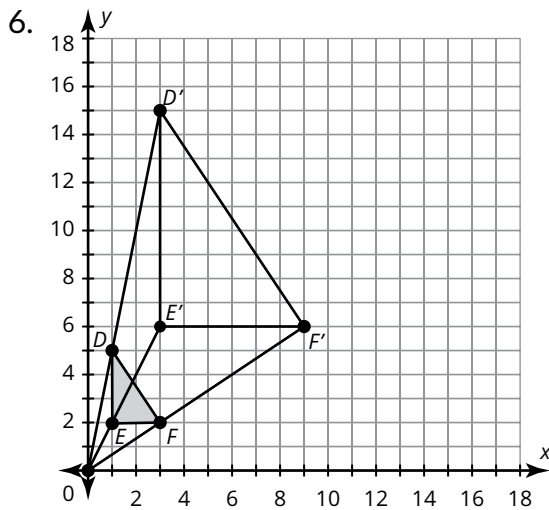
3.



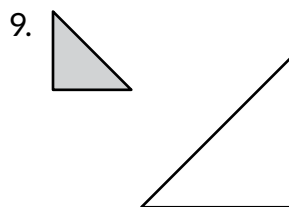
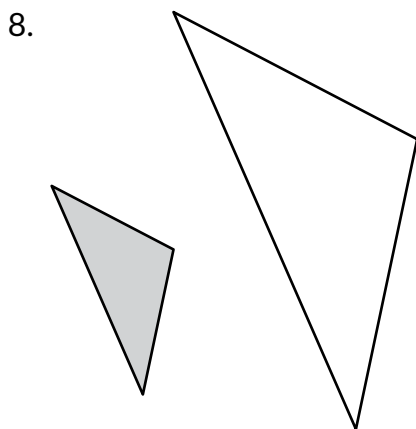
5.



Lesson 1 Assignment

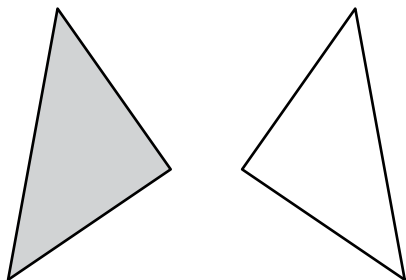


Use the definition of similarity in terms of transformations to determine whether each pair of figures are similar. Justify your reasoning.

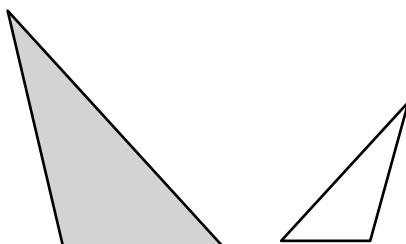


Lesson 1 Assignment

10.



11.



Lesson 1 Assignment

Prepare

The first set of three ordered pairs forms a triangle. Determine whether the second three corresponding ordered pairs form a similar triangle. Justify your answers.

1. $\{(2, 2), (4, 2), (4, 4)\}$
 $\{(4, 4), (8, 4), (8, 8)\}$

2. $\{(2, 3), (4, 5), (3, 9)\}$
 $\left\{\left(\frac{2}{3}, 1\right), \left(\frac{4}{3}, \frac{5}{3}\right), (1, 3)\right\}$

3. $\{(0, 1), (2, 7), (7, 4)\}$
 $\{(0, 4), (8, 28), (28, 16)\}$

2

Establishing Triangle Similarity Criteria

OBJECTIVES

- Write an explanation to establish the Angle-Angle (AA) similarity criterion for two triangles to be similar.
- Write an explanation to establish the Side-Side-Side (SSS) similarity criterion for two triangles to be similar.
- Write an explanation to establish the Side-Angle-Side (SAS) similarity criterion for two triangles to be similar.

.....

You have learned about SSS and SAS congruence.

What is the minimum amount of information about side lengths and angles needed to determine whether two triangles are similar?

KEY TERMS

- Angle-Angle (AA) similarity theorem
- Side-Side-Side (SSS) similarity theorem
- Side-Angle-Side (SAS) similarity theorem

Some Wonky Polygons

1. Draw an example of two polygons that have congruent corresponding angles, but do not have proportional corresponding sides.
2. Draw an example of two polygons that have proportional corresponding sides, but do not have congruent corresponding angles.
3. Explain why the polygons in each pair are not similar.

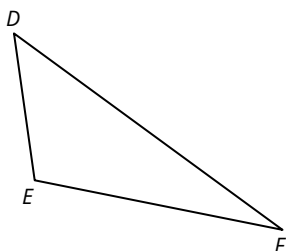
ACTIVITY
2.1

Establishing the AA Similarity Criterion

You can conclude that two triangles are similar if you are able to prove that all of their corresponding angles are congruent and all of their corresponding sides are proportional.

Let's use constructions to see if you can use fewer pairs of angles or fewer pairs of sides to show that triangles are similar.

1. Construct $\triangle D'E'F'$ using only $\angle D$ and $\angle E$ in $\triangle DEF$ as shown. Make all the corresponding side lengths of $\triangle D'E'F'$ different from the side lengths of $\triangle DEF$.



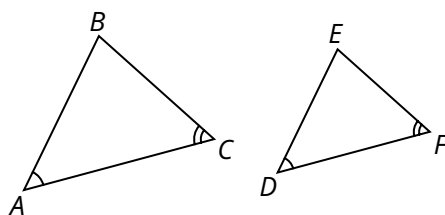
2. Measure the angles and sides of $\triangle D'E'F'$ and $\triangle DEF$. Are the two triangles similar? Explain your reasoning.

.....
Think About...

Did everyone
construct the same
triangle?

3. In $\triangle DEF$ and $\triangle D'E'F'$, two pairs of corresponding angles are congruent. Determine whether this is sufficient information to conclude that the triangles are similar.

The **Angle-Angle (AA) similarity theorem** states: “If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.”

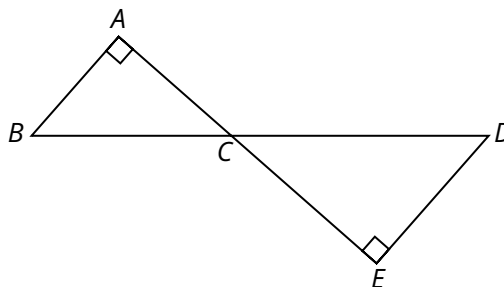


If $m\angle A = m\angle D$ and $m\angle C = m\angle F$, then $\triangle ABC \sim \triangle DEF$.

4. Explain why this similarity theorem is Angle-Angle instead of Angle-Angle-Angle.

WORKED EXAMPLE

You can apply the Angle-Angle similarity theorem to verify that two triangles are similar.

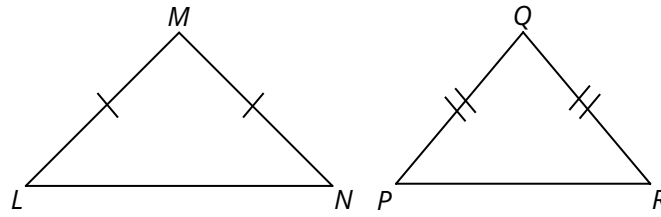


For $\triangle ABC$ and $\triangle EDC$, you know that:

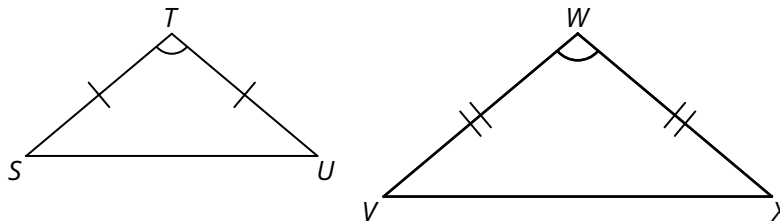
- $\angle A \cong \angle E$ because the two angles are right angles, and all right angles are congruent.
- $\angle ACB \cong \angle ECD$ because the two angles are vertical angles, and vertical angles are congruent.

The Angle-Angle similarity theorem tells you that because $\triangle ABC$ and $\triangle EDC$ have two pairs of congruent angles, they are similar.

5. The triangles shown are isosceles triangles. Do you have enough information to show that the triangles are similar? Explain your reasoning.



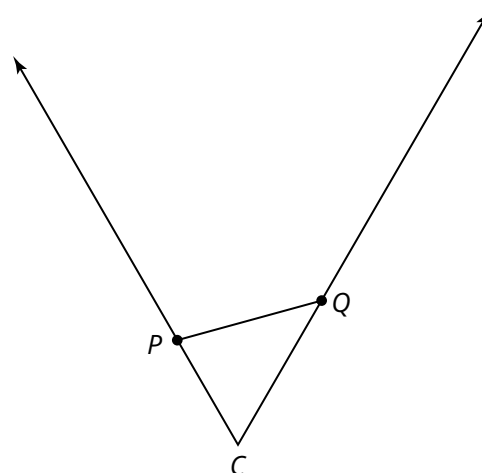
6. The triangles shown are isosceles triangles. Do you have enough information to show that the triangles are similar? Explain your reasoning.



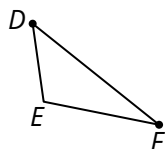
ACTIVITY
2.2

Establishing the SSS Similarity Criterion

The diagram shows $\overline{PQ}$ and a center of dilation C . The rays of the dilation and $\overline{PQ}$ form a triangle, $\triangle CPQ$.



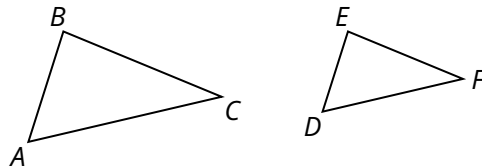
1. Lucia says that any line segment drawn from $\overrightarrow{CP}$ to $\overrightarrow{CQ}$ will produce a dilation of $\overline{PQ}$, which means that $\triangle C'P'Q'$ will be similar to $\triangle CPQ$. Juliana disagrees. She says that the line segment drawn from $\overrightarrow{CP}$ to $\overrightarrow{CQ}$ must be parallel to $\overline{PQ}$ for the triangles to be similar. Who is correct? Explain your thinking.



2. Construct $\triangle D'E'F'$ by doubling the lengths of sides $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$. Construct the new side lengths separately, and then construct the triangle. Do not duplicate angles.

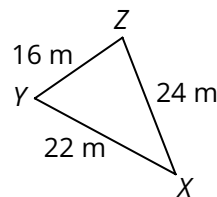
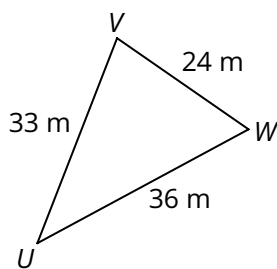
3. Measure the angles and sides of $\triangle D'E'F'$ and $\triangle DEF$. Are the two triangles similar? Explain your reasoning.
4. Three pairs of corresponding sides are proportional. Determine whether this is sufficient information to conclude that the triangles are similar.

The **Side-Side-Side (SSS) similarity theorem** states: "If all three corresponding sides of two triangles are proportional, then the triangles are similar."



$$\text{If } \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}, \text{ then } \triangle ABC \sim \triangle DEF.$$

5. Determine whether $\triangle UVW$ is similar to $\triangle XYZ$. If the triangles are similar, identify the proportional sides and congruent angles. Then, use symbols to write a similarity statement.



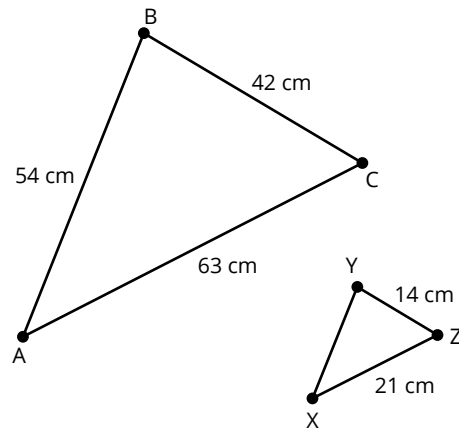
.....

Review the definition
of **describe** in the
Academic Glossary.

.....

6. Describe how transformations could be used to determine whether two triangles are similar when all pairs of corresponding sides are proportional.

7. $\triangle ABC \sim \triangle XYZ$
Determine the length of $\overline{XY}$.

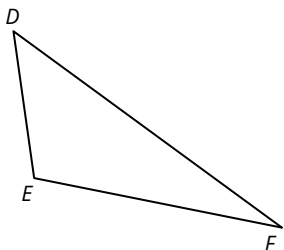


ACTIVITY
2.3

Establishing the SAS Similarity Criterion

Let's investigate another way to determine whether triangles are similar by analyzing included angles. Recall that an included angle is an angle formed by two consecutive sides of a figure and an included side is a line segment between two consecutive angles of a figure.

1. Construct $\triangle D'E'F'$ by duplicating an angle and doubling the length of the two sides that make up that angle. Construct the new side lengths separately, and then construct the triangle.



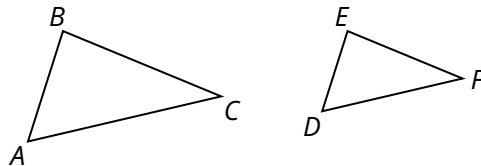
2. Measure the angles and sides of $\triangle D'E'F'$ and $\triangle DEF$. Are the two triangles similar? Explain your reasoning.
3. Two pairs of corresponding sides are proportional, and the corresponding included angles are congruent. Determine whether this is sufficient information to conclude that the triangles are similar.

PROBLEM SOLVING



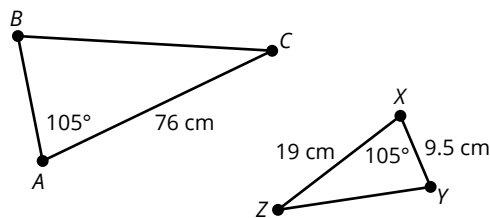
4. Describe how transformations could be used to determine whether two triangles are similar when two pairs of corresponding sides are proportional and the included angles are congruent.

The **Side-Angle-Side (SAS) similarity theorem** states: “If two of the corresponding sides of two triangles are proportional and the included angles are congruent, then the triangles are similar.”



If $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle A \cong \angle D$, then $\triangle ABC \sim \triangle DEF$.

5. $\triangle ABC \sim \triangle XYZ$
Determine the length of $\overline{AB}$.



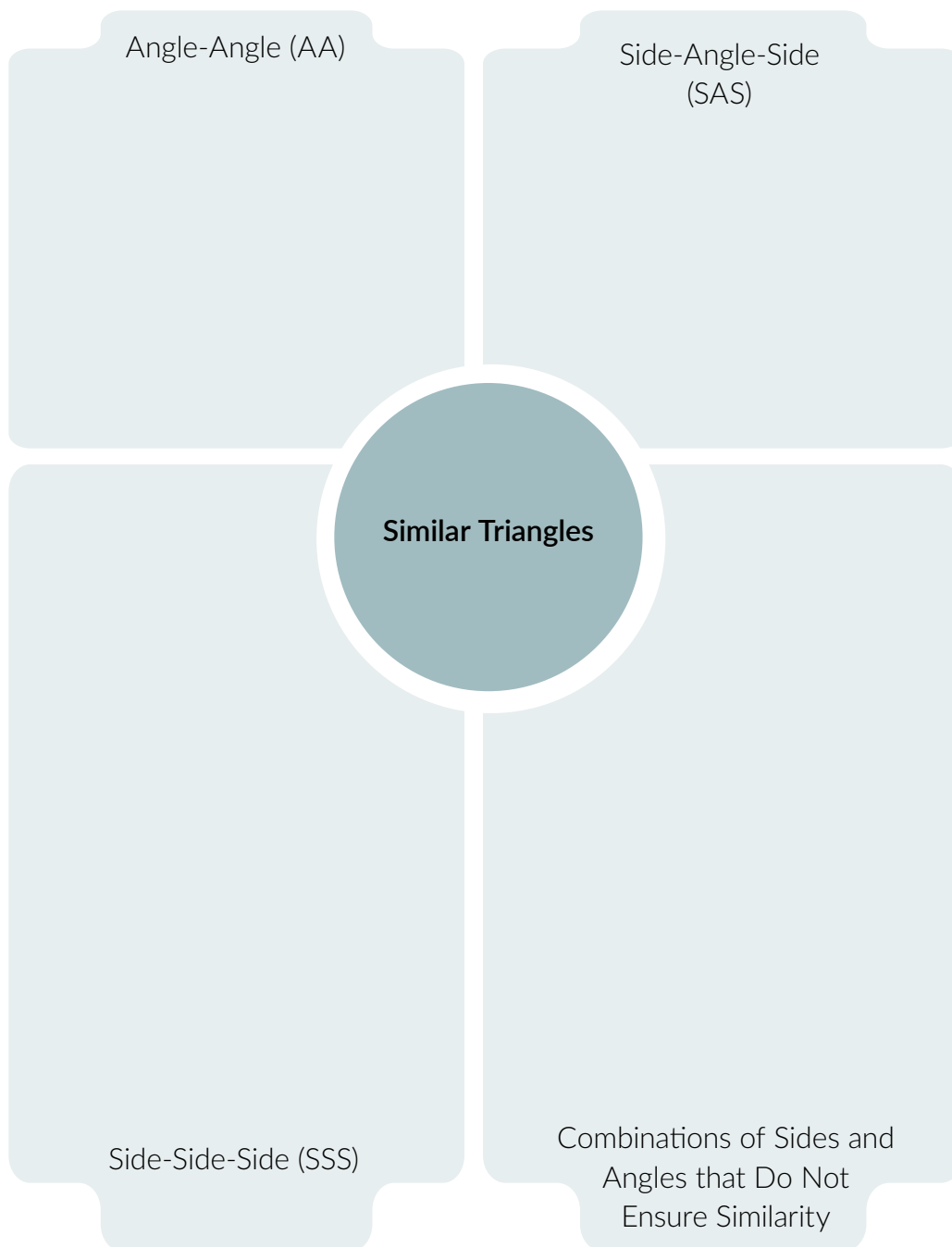


Talk the Talk

Organize and Compare

Omar is thinking of a triangle and he wants everyone in his class to draw a similar triangle.

1. Complete the graphic organizer to describe the minimum criteria he could provide in each case.



Lesson 2 Assignment

Write

Give an example of each term. Include a sketch with each example.

1. Angle-Angle similarity theorem
2. Side-Side-Side similarity theorem
3. Side-Angle-Side similarity theorem
4. included angle
5. included side

Remember

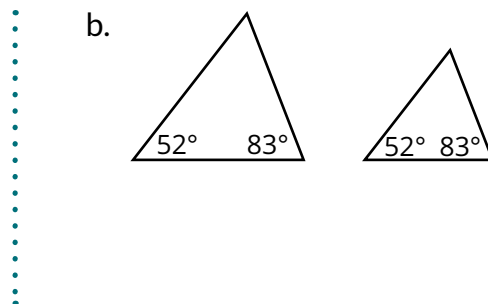
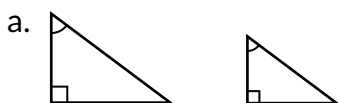
The Angle-Angle similarity theorem states: "If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar."

The Side-Side-Side similarity theorem states: "If all three corresponding sides of two triangles are proportional, then the triangles are similar."

The Side-Angle-Side similarity theorem states: "If two of the corresponding sides of two triangles are proportional and the included angles are congruent, then the triangles are similar."

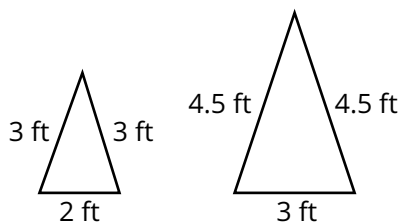
Practice

1. Determine whether the triangles in each pair are similar. Explain your reasoning.

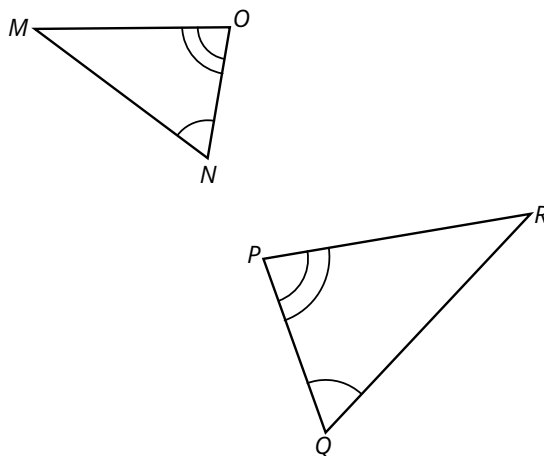


Lesson 2 Assignment

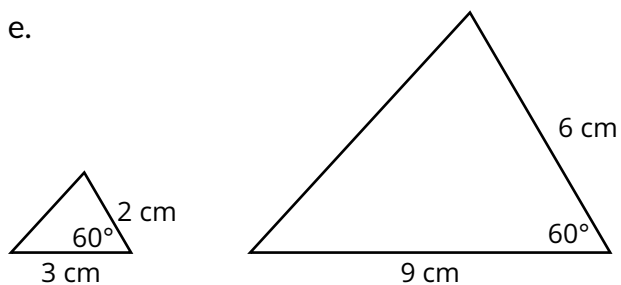
c.



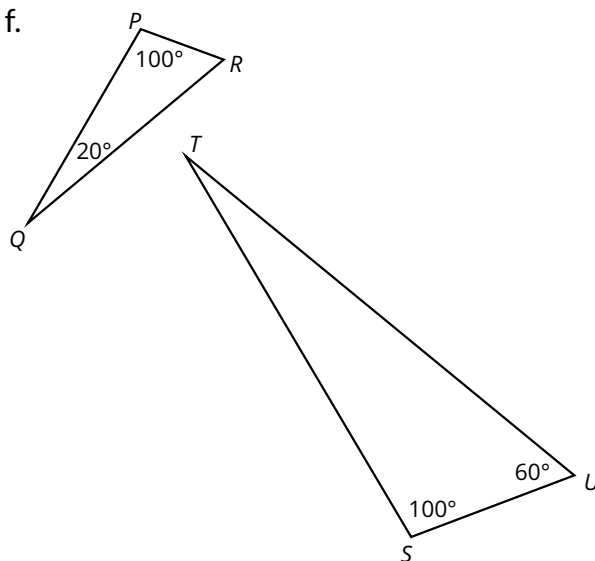
d.



e.

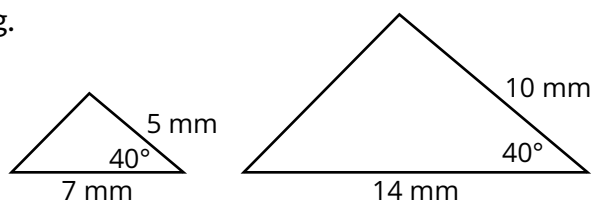


f.

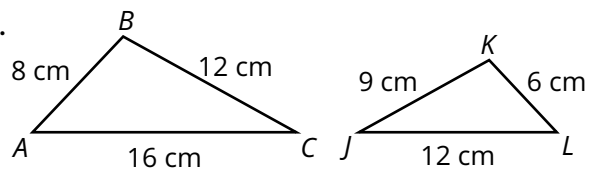


Lesson 2 Assignment

g.

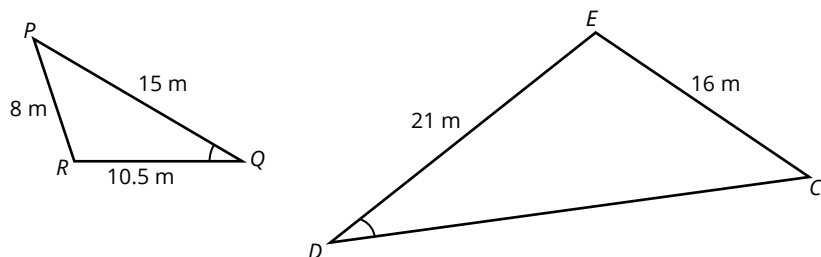


h.



2. $\triangle PQR \sim \triangle CED$

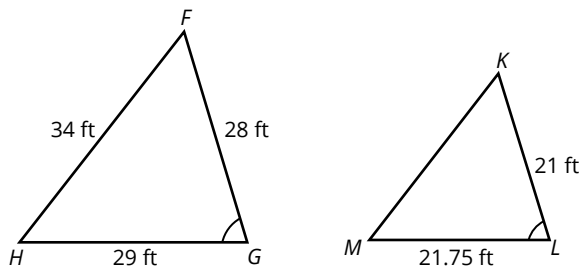
Determine the length of $\overline{DC}$.



Lesson 2 Assignment

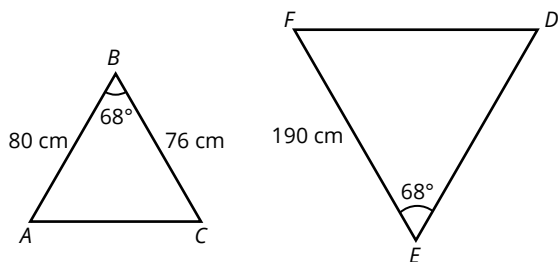
3. $\triangle FGH \sim \triangle KLM$

Determine the length of $\overline{KM}$.



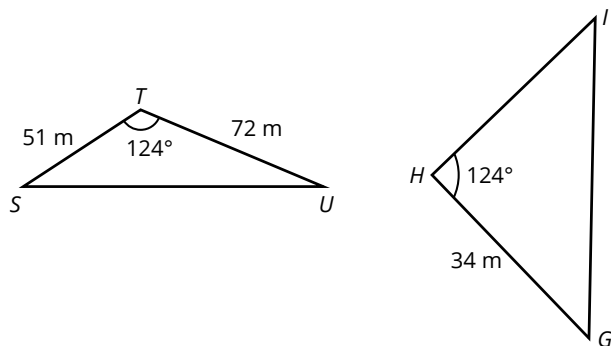
4. $\triangle ABC \sim \triangle DEF$

Determine the length of $\overline{DE}$.



5. $\triangle STU \sim \triangle GHI$

Determine the length of $\overline{HI}$.



Lesson 2 Assignment

Prepare

Solve each proportion.

1. $\frac{x}{22} = \frac{10}{4}$

2. $\frac{5}{23} = \frac{n}{5}$

3. $\frac{16}{16} = \frac{8}{y}$

4. $\frac{22}{4} = \frac{p}{10}$

5. Identify the relationship between the proportions in Questions 1 and 4.

3

Theorems About Proportionality

OBJECTIVES

- Prove the proportional side theorem with a written justification.
- Prove the Triangle Proportionality theorem with a written justification.
- Prove the converse of the Triangle Proportionality theorem with a written justification.
- Prove the proportional segments theorem associated with parallel lines with a written justification.
- Prove the triangle midsegment theorem with a written justification.

.....

You have used similarity transformations to decide when two figures are similar.

What theorems can you prove involving similarity?

KEY TERMS

- angle bisector/proportional side theorem
- Triangle Proportionality theorem
- converse of the Triangle Proportionality theorem
- proportional segments theorem
- triangle midsegment theorem

Getting Started

A Translation Dilation Connection

Recall this diagram from the previous lesson, *Establishing Triangle Similarity Criteria*. The diagram originally showed $\overline{PQ}$ and center of dilation C . The rays of the dilation and $\overline{PQ}$ form $\triangle CPQ$. The diagram now shows $\overleftrightarrow{PQ}$, which contains $\overline{PQ}$.

1. Complete each step.

.....
Remember ...

Any translation of a line produces an identical line or a parallel line.
.....

- Use point C as the center of dilation to draw a dilation of point P on $\overrightarrow{CP}$ so that $\overline{CP'}$ is twice the length of $\overline{CP}$.
- Draw the translation of $\overleftrightarrow{PQ}$ in the direction of $\overrightarrow{PP'}$ a distance equal to PP' .
- Use point C as the center of dilation to draw a dilation of point Q on $\overrightarrow{CQ}$ so that $\overline{CQ'}$ is twice the length of $\overline{CQ}$.
- Draw the translation of $\overleftrightarrow{PQ}$ in the direction of $\overrightarrow{QQ'}$ a distance equal to QQ' .

2. What do you notice about $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P'Q'}$? Explain your thinking.

Dilating Lines

The game of marbles is played in a circle. The goal is to knock your opponents' marbles outside of the circle by flicking a shooter marble at the other marbles in the circle. The shooter marble is often larger than the other marbles.

Antonio placed a shooter marble near three smaller marbles as shown.

1. Draw another row of three marbles above the first row of marbles using a dilation factor of 2 with the shooter marble as the center of the dilation. Explain how you located the positions of each additional marble. Label the distances between the marbles in the first row and in the second row.

2. Describe the relationship between the first and second rows of marbles.

.....

Think About...

Can you describe the 3 smaller marbles as being collinear?

.....

.....

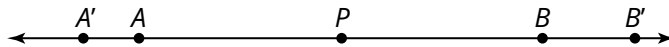
Clay marbles have been discovered in Egypt, Native American burial sites, Aztec pyramids, and they have even been used by NASA in weather balloons. In the United States, National Marble Day is celebrated on July 23rd.

.....

3. Use a ruler to compare the length of the line segments connecting the marbles on the first row to the length of the line segments connecting the marbles on the second row.

4. What can you conclude about dilating a line that does not pass through the center of a dilation?

5. Consider $\overleftrightarrow{AB}$. How could you show a dilation of this line using point P as the center of dilation?



- a. What happens to the line when the line is dilated through center point P such that point B is dilated to point B' ?
- b. What happens to the line when the line is dilated through center point P such that point A is dilated to point A' ?
- c. What can you conclude about dilating a line passing through the center of dilation?

ACTIVITY
3.2

Proving the Angle Bisector/ Proportional Side Theorem

When an interior angle of a triangle is bisected, you can observe proportional relationships among the sides of the triangles formed. You will be able to prove that these relationships apply to all triangles.

1. Consider $\triangle ABC$.

- a. Use construction tools to draw a segment that bisects $\angle A$ and intersects $\overline{BC}$. Label the point of intersection D .

- b. Measure the lengths of $\overline{AB}$ and $\overline{AC}$ to the nearest millimeter. What is the ratio of the length of $\overline{AB}$ to the length of $\overline{AC}$?

- c. Measure the lengths of $\overline{BD}$ and $\overline{CD}$ to the nearest millimeter. What is the ratio of the length of $\overline{BD}$ to the length of $\overline{CD}$?

- d. What do you notice about the ratios you determined in parts (b) and (c)?

2. Consider $\angle C$.

- a. Use construction tools to draw a segment that bisects $\angle C$ and intersects $\overline{AB}$. Label the point of intersection E .

- b. Determine the ratio of the length of $\overline{CB}$ to the length of $\overline{CA}$.

.....
Write all ratios
in this activity as
decimals to the
nearest hundredths.
.....

- c. Determine the ratio of the length of $\overline{BE}$ to the length of $\overline{AE}$.
- d. What do you notice about the ratios you determined in parts (b) and (c)?
3. Consider $\angle B$.
- a. Use construction tools to draw a segment that bisects $\angle B$ and intersects $\overline{AC}$. Label the point of intersection F .
- b. Determine the ratio of the length of $\overline{BA}$ to the length of $\overline{BC}$.
- c. Determine the ratio of the length of $\overline{AF}$ to the length of $\overline{CF}$.
- d. What do you notice about the ratios you determined in parts (b) and (c)?
4. Using your answers from Questions 1 through 3, make a conjecture about the angle bisectors of a triangle and the side lengths.

When an interior angle of a triangle is bisected, you can observe proportional relationships among the sides of the triangles formed. You will be able to prove that these relationships apply to all triangles.

5. Consider the given statement and diagram to prove that a bisector of an angle divides the opposite side into two segments whose lengths are proportional to the lengths of the sides adjacent to the angle.

Given: $\overline{AD}$ bisects $\angle BAC$

Prove: $\frac{AB}{AC} = \frac{BD}{CD}$

- a. Draw an auxiliary line parallel to $\overline{AB}$ through point C. Then, extend $\overline{AD}$ until it intersects the auxiliary line. Label the point of intersection, point E.

.....
Remember ...

An *auxiliary line* is a line or line segment added to a diagram to help in solving or proving a concept.
.....

b. Complete the proof to prove the conjecture.

Statements	Reasons
	1. Given
	2. Construction
	3. Definition of angle bisector
4. $\angle BAE \cong \angle CEA$	
	5. Transitive property of congruence
	6. Isosceles triangle base angles converse theorem
	7. Definition of congruent segments
	8. Alternate interior angles theorem
9. $\triangle DAB \sim \triangle DEC$	
10. $\frac{AB}{EC} = \frac{BD}{CD}$	
11. $\frac{AB}{AC} = \frac{BD}{CD}$	

Because your conjecture has been proved to be true, you can now refer to it as a theorem. The **angle bisector/proportional side theorem** states: "A bisector of an angle in a triangle divides the opposite side into two segments whose lengths are in the same ratio as the lengths of the sides adjacent to the angle."

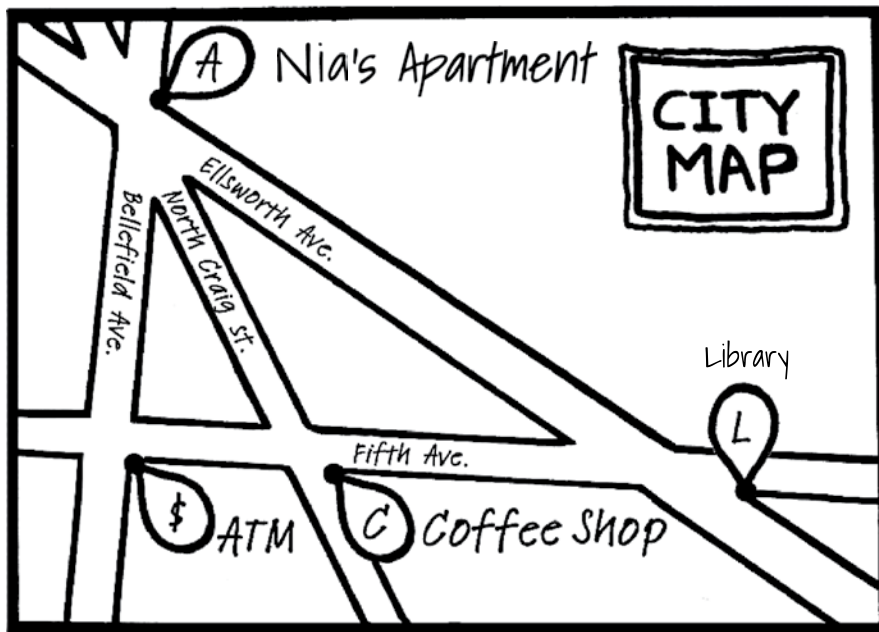
ACTIVITY 3.3

Applying the Angle Bisector/ Proportional Side Theorem

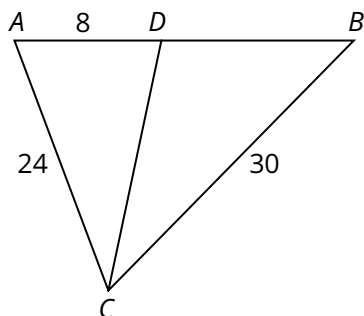
On the map shown, North Craig Street bisects the angle formed between Bellefield Avenue and Ellsworth Avenue.

- The distance from the ATM to the Coffee Shop is 300 feet.
- The distance from the Coffee Shop to the Library is 500 feet.
- The distance from Nia's apartment to the Library is 1200 feet.

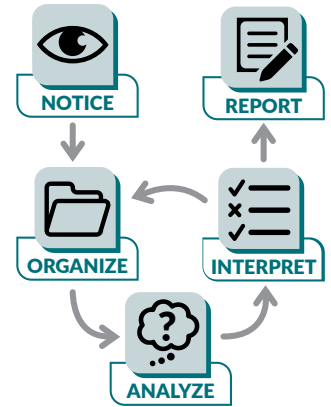
1. Determine the distance from Nia's apartment to the ATM.



2. $\overline{CD}$ bisects $\angle C$. Solve for $\overline{DB}$.

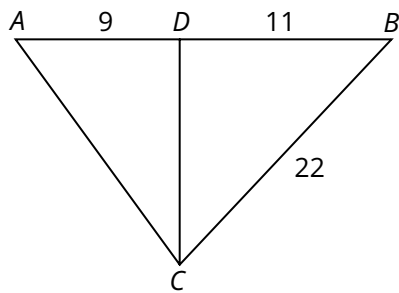


PROBLEM SOLVING

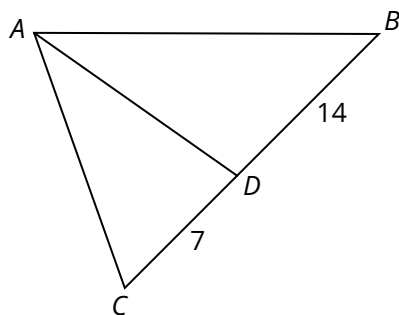


Diagrams in this activity are not drawn to scale. Pay attention to the diagram labels.

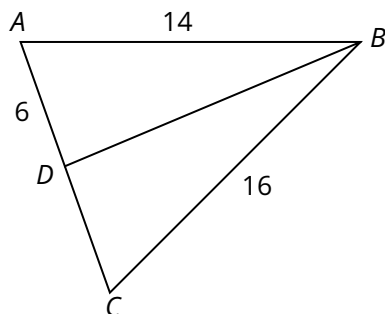
3. $\overline{CD}$ bisects $\angle C$. Solve for $\overline{CA}$.



4. $\overline{AD}$ bisects $\angle A$. $AC + AB = 36$. Solve for $\overline{AC}$ and $\overline{AB}$.



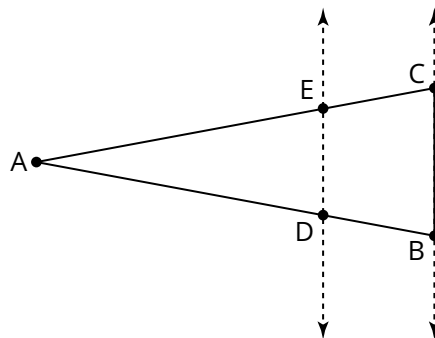
5. $\overline{BD}$ bisects $\angle B$. Solve for $\overline{AC}$.



ACTIVITY
3.4

Triangle Proportionality Theorem

Consider the diagram.
It shows a dilation of $\overline{DE}$ to $\overline{BC}$, using point A as the center of dilation.



Think About...

A diagram showing the dilation of a line segment looks a lot like a triangle.

1. How do you know that each relationship exists?

a. $\overline{ED} \parallel \overline{CB}$

b. $\frac{AC}{AE} = \frac{AB}{AD}$



2. Xavier says that no matter where he draws a line intersecting both sides $\overline{AC}$ and $\overline{AB}$, as long as the line is parallel to $\overleftrightarrow{CB}$, it will create similar triangles.

Is Xavier correct? Test Xavier's conjecture and explain your reasoning.

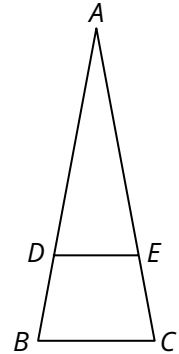
3. Suppose you draw $\overleftrightarrow{PQ}$, parallel to $\overline{CB}$, intersecting $\overline{AC}$ at point P and intersecting $\overline{AB}$ at point Q .

What can you say about $\overline{AP}$, $\overline{AC}$, $\overline{AQ}$, and $\overline{AB}$?

4. Cut out the statements and reasons at the end of the lesson. Then, rearrange them to create a proof to demonstrate that if a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally.

Given: $\overline{BC} \parallel \overline{DE}$

Prove: $\frac{BD}{DA} = \frac{CE}{EA}$



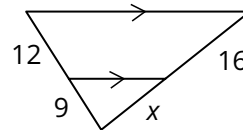
Statements	Reasons

Because the conjecture has been proved to be true, you can now refer to it as a theorem. The **Triangle Proportionality theorem** states: “If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally.”

You can apply the Triangle Proportionality theorem to solve problems.

WORKED EXAMPLE

In the diagram shown, 4 segments are formed by the parallel line that passes through the interior of the triangle.



To determine x , you can use what you know about the Triangle Proportionality theorem to set up and solve a proportion:

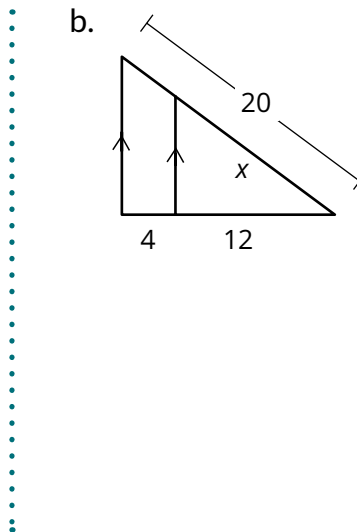
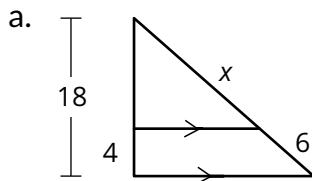
$$\frac{9}{12} = \frac{x}{16}$$

$$12x = 144$$

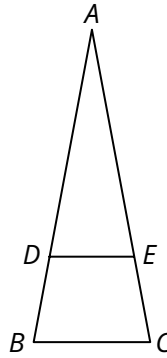
$$x = 12$$

The unknown length, x , is 12 units.

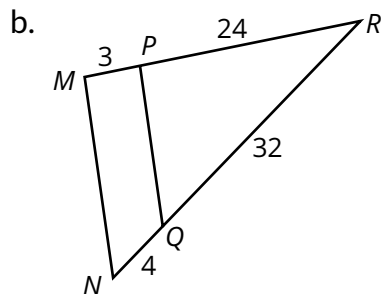
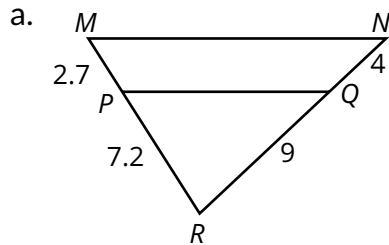
5. Determine each unknown value. Show your work.



The **converse of the Triangle Proportionality theorem** is also true.
It states: "If a line divides two sides of a triangle proportionally, then it is parallel to the third side."



6. Use $\triangle ABC$ and the converse of the Triangle Proportionality theorem to write the conditional statement that applies to the triangle.
7. Determine whether $\overline{MN}$ is parallel to $\overline{PQ}$ in each figure. Explain your reasoning.



ACTIVITY
3.5

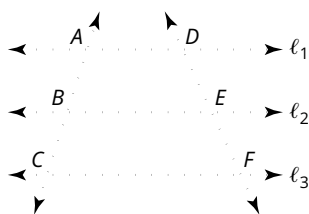
Proportional Segments Theorem

Use the diagram and follow the steps to prove the conjecture that if three parallel lines are intersected by two transversals, then they divide the transversals proportionally.

Given: $\ell_1 \parallel \ell_2 \parallel \ell_3$

Prove: $\frac{AB}{BC} = \frac{DE}{EF}$

- Through any two points there is exactly one line. Draw an auxiliary line segment $\overline{CD}$ to form $\triangle ACD$ and $\triangle FDC$. Label the intersection of $\overline{CD}$ and ℓ_2 as point H .



- Using the Triangle Proportionality theorem and $\triangle ACD$, what can you conclude?
- Using the Triangle Proportionality theorem and $\triangle FDC$, what can you conclude?
- What property of equality will justify the prove statement?

Because the conjecture has been demonstrated to be true, you can now refer to it as a theorem. The **proportional segments theorem** states: "If three parallel lines intersect two transversals, then they divide the transversals proportionally."

.....

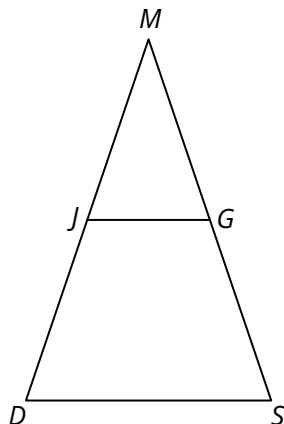
You have proved the Triangle Proportionality theorem. Let's consider if there are any proportional relationships when three parallel lines are intersected by two transversals.

.....

Triangle Midsegment Theorem

In a previous topic, you conjectured that the midsegment of a triangle is parallel to the third side of the triangle and is half its measure.

1. Use the diagram to prove this conjecture.



Given: J is the midpoint of $\overline{MD}$

G is the midpoint of $\overline{MS}$

Prove: $\overline{JG} \parallel \overline{DS}$

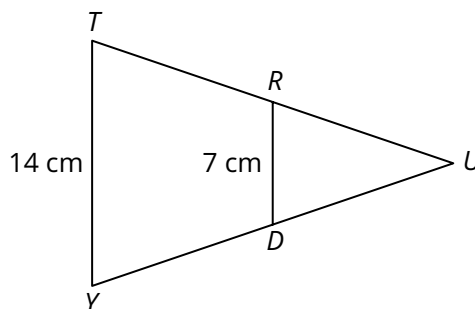
$$JG = \frac{1}{2} DS$$

Statements	Reasons

Because the conjecture has been proved to be true, you can now refer to it as a theorem. The **triangle midsegment theorem** states: “The midsegment of a triangle is parallel to the third side of the triangle and is half the measure of the third side of the triangle.”

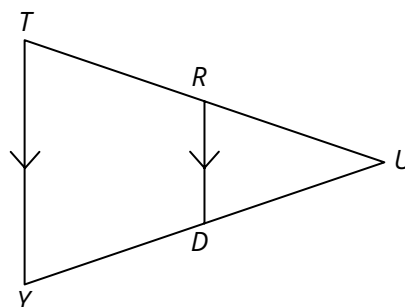


2. Ms. Tanaka asked her students to determine whether $\overline{RD}$ is a midsegment of $\triangle TUY$, given $\overline{TY} = 14$ cm and $\overline{RD} = 7$ cm.



Miguel told Gracie that by using the triangle midsegment theorem, he could conclude that $\overline{RD}$ is a midsegment. Is Miguel correct? How should Gracie respond if Miguel is incorrect?

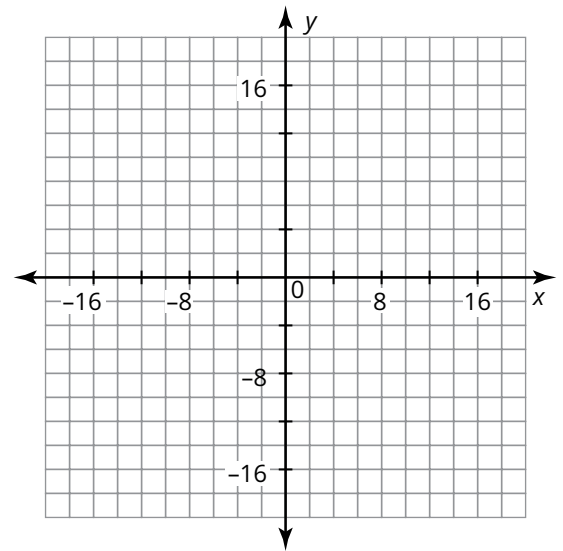
3. Ms. Tanaka drew a second diagram on the board and asked her students to determine whether $\overline{RD}$ is a midsegment of $\triangle TUY$ given $\overline{RD} \parallel \overline{TY}$.



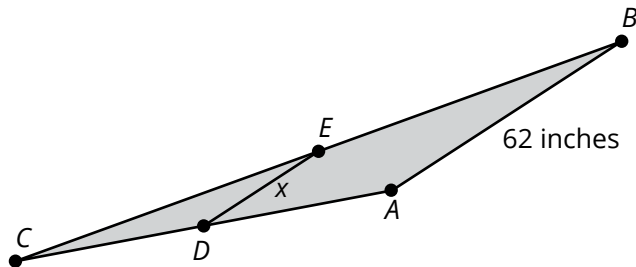
Gracie told Miguel that by using the triangle midsegment theorem, she could conclude that $\overline{RD}$ is a midsegment. Is Gracie correct? How should Miguel respond if Gracie is incorrect?

In $\triangle BJG$, the midpoint of $\overline{BJ}$ is $F(-3, 5)$. The midpoint of $\overline{BG}$ is $A(-6, -5)$. The midpoint of $\overline{GJ}$ is $R(6, -5)$.

4. Use the triangle midsegment theorem to determine the coordinates of the vertices of $\triangle BJG$. Show all of your work.



5. $\overline{DE}$ is a midsegment of $\triangle ABC$. Determine the value of x .

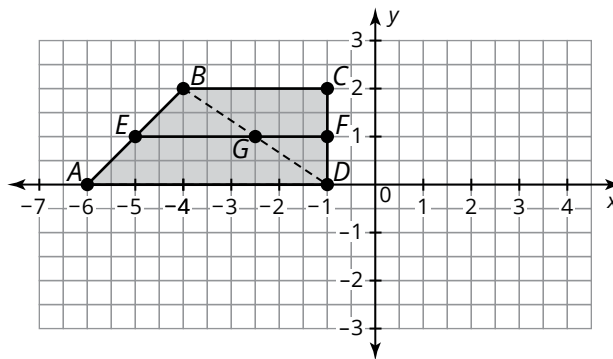


Remember that you have also studied the trapezoid midsegment theorem. The triangle and trapezoid midsegment theorem both show how midsegments have a special relationship to their parallel sides. The triangle midsegment theorem states that the midsegment is parallel to the third side of the triangle and half of the length, while the trapezoid midsegment theorem states the midsegment is parallel to the two bases and half the sum of the lengths of the two bases.

You can also see how the triangle midsegment theorem and trapezoid midsegment theorem are related on the coordinate plane.

WORKED EXAMPLE

Consider the trapezoid on the coordinate plane:



$\overline{EF}$ is the midsegment of trapezoid $BCDA$.

According to the trapezoid midsegment theorem $EF = \frac{1}{2}(BC + AD)$.

Drawing the diagonal $\overline{BD}$, creates two triangles.

The midsegment of $\triangle ABD$ is $\overline{EG}$ and the midsegment of $\triangle BDC$ is $\overline{FG}$.

Using the triangle midsegment theorem:

$$FG = \frac{1}{2}BC$$

$$EG = \frac{1}{2}AD$$

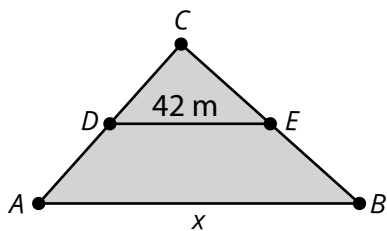
By the segment addition postulate, $EF = FG + EG$.

Using the substitution property of equality, $EF = \frac{1}{2}BC + \frac{1}{2}AD$.

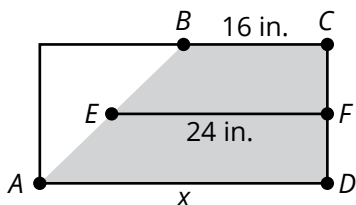
Finally, by the distributive property of equality, $EF = \frac{1}{2}(BC + AD)$.

6. Use the triangle midsegment theorem or the trapezoid midsegment theorem to determine the value of x .

a. $\overline{DE}$ is the midsegment of $\triangle ACB$



b. $\overline{EF}$ is the midsegment of trapezoid $ABCD$.



Medians of a Triangle

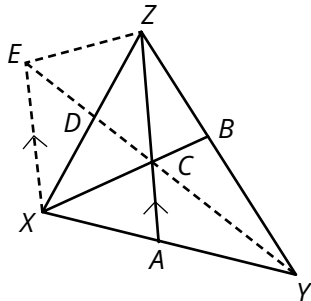
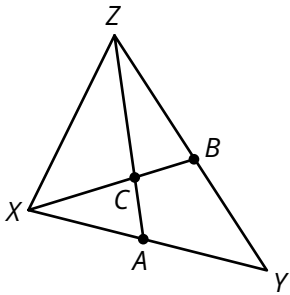
You have previously constructed the three medians of a triangle to determine the point of concurrency called the *centroid*. You will now prove that the medians of a triangle meet at a point. This proof has several parts and is divided into sections to make it easier to understand.

1. Use the diagram to prove this conjecture.

Given: $\triangle XYZ$ and medians $\overline{XB}$ and $\overline{ZA}$ that intersect at point C .

Prove: The three medians of a triangle are concurrent.

In this proof, you will demonstrate that the line segment from the third vertex, Y , that passes through point C is a median.



Statements	Reasons
	1. Given
2. Point A is the midpoint of $\overline{XZ}$.	
3. $\overline{XA} \cong \overline{AZ}$	
4. $XA = AZ$	

Construct auxiliary lines to create triangles necessary to continue with the proof.

5. $\overrightarrow{YC}$ intersects $\overline{XZ}$ at point D .
A line through point X parallel to $\overline{ZA}$ intersects $\overrightarrow{YC}$ at point E .
 $\overline{EZ}$ is drawn to create $\triangle YZE$.

Establish that point C is a midpoint of $\overline{YE}$.

$$6. \frac{EC}{CY} = \frac{XA}{AY}$$

$$7. \frac{EC}{CY} = \frac{XA}{XA}$$

$$8. \frac{EC}{CY} = 1$$

$$9. EC = CY$$

$$10. \overline{EC} \cong \overline{CY}$$

11. C is the midpoint of $\overline{EY}$.

Show that quadrilateral $EXCZ$ is a parallelogram.

12. Point B is the midpoint of $\overline{YZ}$.

13. $\overline{AC}$ is the midsegment of $\triangle XYE$.
 $\overline{BC}$ is the midsegment of $\triangle YZE$.

14. $\overline{CB} \parallel \overline{EZ}$.

15. Quad $EXCZ$ is a parallelogram.

.....
 You know from steps 2, 12, and 13 that $\overline{AC}$ connects the midpoints of two sides of $\triangle XYE$ and that $\overline{BC}$ connects the midpoints of two sides of $\triangle YZE$.

Prove that the three medians are concurrent.

$$16. \overline{XD} \cong \overline{ZD}$$

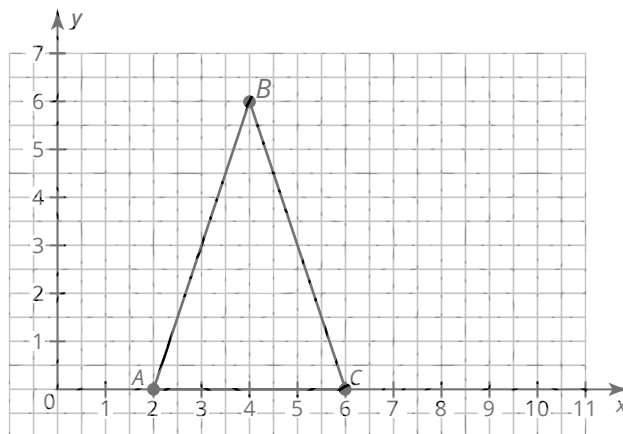
17. D is the midpoint of $\overline{XZ}$.

18. $\overline{YD}$ is the median of $\triangle XYZ$.

19. The three medians of a triangle are concurrent.

Now that the medians of a triangle are concurrent has been proven, you can apply this knowledge to solve problems.

2. Consider $\triangle ABC$ shown on the coordinate plane.



- Use the midpoint formula to determine the midpoint of $\overline{AC}$.
Label the midpoint as point D .
- Use the midpoint formula to determine the midpoint of $\overline{AB}$.
Label the midpoint as point E .
- Use the midpoint formula to determine the midpoint of $\overline{BC}$.
Label the midpoint as point F .
- Use a straightedge to draw the three medians of the triangle.
Label the point of concurrency of the medians as point G .
- Identify the coordinates of the point of concurrency.



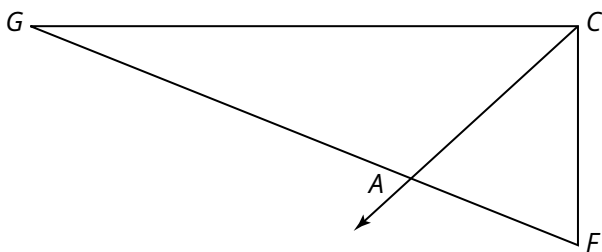
Talk the Talk

Name That Theorem

Name the theorem that applies to each question.

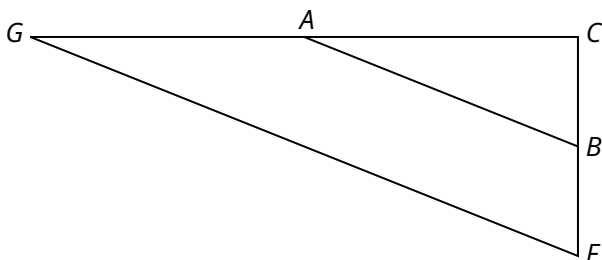
1. Given: $\overrightarrow{CA}$ bisects $\angle GCF$

Prove: $\frac{CG}{CF} = \frac{GA}{FA}$



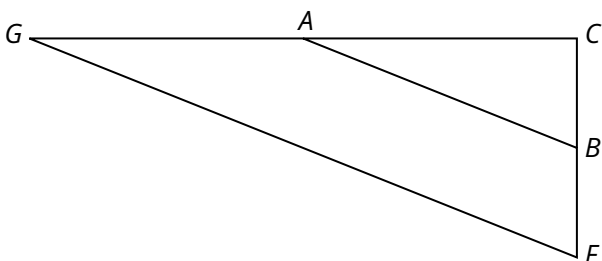
2. Given: $\overline{AB} \parallel \overline{GF}$

Prove: $\frac{GA}{AC} = \frac{FB}{BC}$



3. Given: $\frac{GA}{AC} = \frac{FB}{BC}$

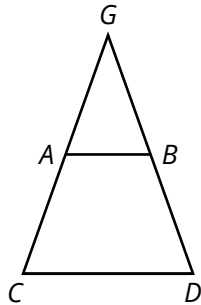
Prove: $\overline{AB} \parallel \overline{GF}$



4. Given: A is the midpoint of $\overline{GC}$.
 B is the midpoint of $\overline{GD}$.

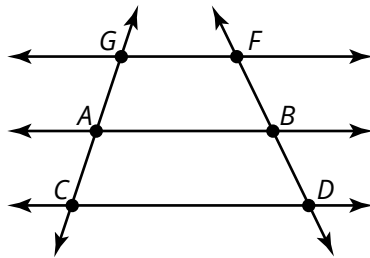
Prove: $\overline{AB} \parallel \overline{CD}$

$$AB = \frac{1}{2}CD$$



5. Given: $\overleftrightarrow{GF} \parallel \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

Prove: $\frac{GA}{AC} = \frac{FB}{BD}$



Statements and Reasons for the Triangle Proportionality theorem Proof



$\triangle ADE$ is similar to $\triangle ABC$.	Corresponding sides of similar triangles are proportional.
$\frac{BD}{DA} = \frac{CE}{EA}$	Corresponding angles theorem
$\angle AED \cong \angle C$	Given
$\overline{BC} \parallel \overline{DE}$	Corresponding angles theorem
$\frac{BA}{DA} = \frac{CA}{EA}$	AA similarity theorem
$BA = BD + DA$ and $CA = CE + EA$	substitution property of equality
$\frac{BD + DA}{DA} = \frac{CE + EA}{EA}$	Segment addition postulate
$\angle ADE \cong \angle B$	Simplify

Why is this page blank?

So you can cut out the statements and reasons on the other side.

Lesson 3 Assignment

Write

Write a biconditional statement for the Triangle Proportionality theorem. Include a sketch to demonstrate your understanding.

Remember

The angle bisector/proportional side theorem states: "A bisector of an angle in a triangle divides the opposite side into two segments whose lengths are in the same ratio as the lengths of the sides adjacent to the angle."

The Triangle Proportionality theorem states: "If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally."

The converse of the Triangle Proportionality theorem states: "If a line divides the two sides proportionally, then it is parallel to the third side."

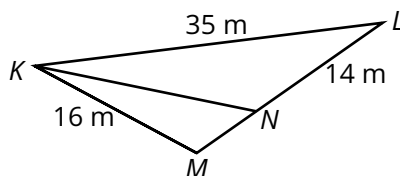
The proportional segments theorem states: "If three parallel lines intersect two transversals, then they divide the transversals proportionally."

The triangle midsegment theorem states: "The midsegment of a triangle is parallel to the third side of the triangle and half the measure of the third side of the triangle."

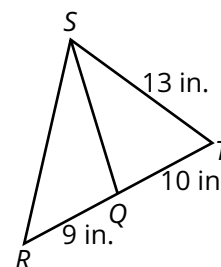
Practice

1. Calculate the indicated length in each figure.

a. $\overline{KN}$ bisects $\angle K$. Calculate $\overline{MN}$.

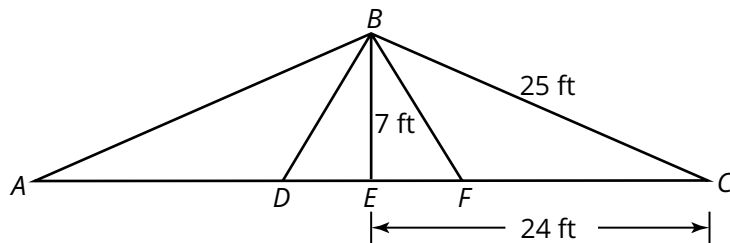


b. $\overline{SQ}$ bisects $\angle S$. Calculate $\overline{SR}$.

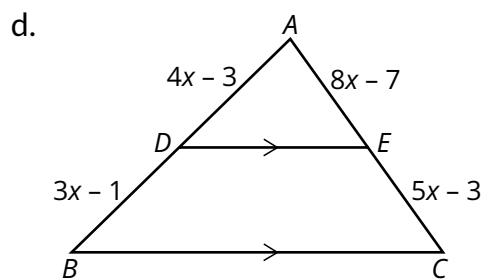
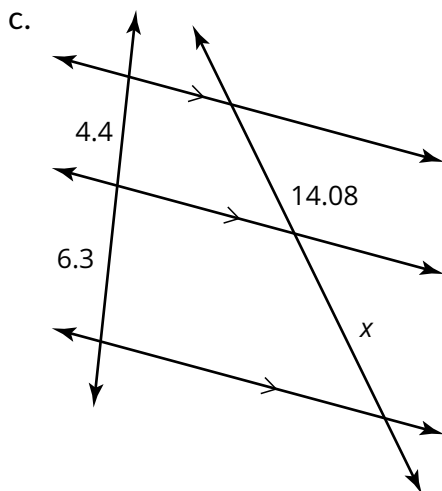
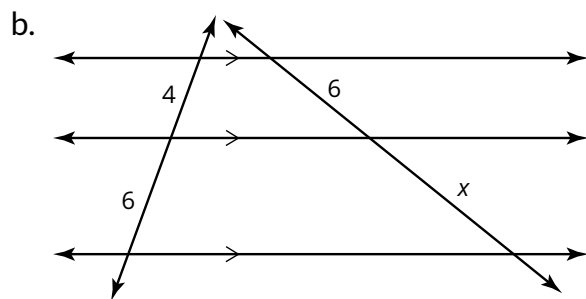
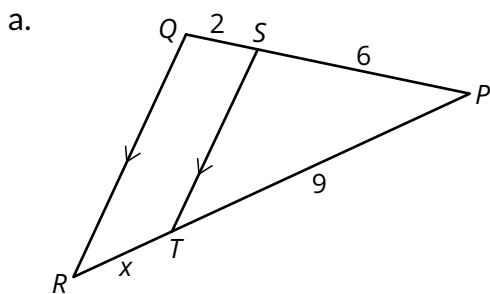


Lesson 3 Assignment

2. The figure shows a truss on a bridge. $\overline{BF}$ bisects $\angle CBE$. Use this information to calculate $\overline{EF}$ and $\overline{CF}$.



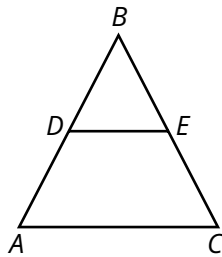
3. Determine the value of x in each figure.



Lesson 3 Assignment

4. Use the diagram and given information to write two statements that can be justified using the Triangle Midsegment Theorem.

a.

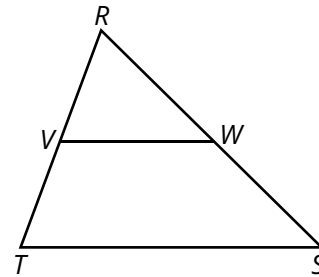


Given: $\triangle ABC$

D is the midpoint of $\overline{AB}$.

E is the midpoint of $\overline{BC}$.

b.

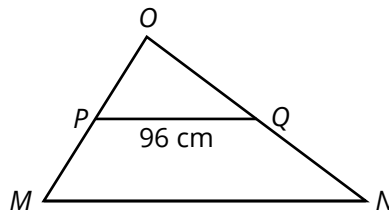


Given: $\triangle RST$

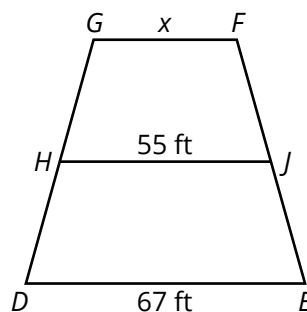
V is the midpoint of $\overline{RT}$.

W is the midpoint of $\overline{RS}$.

5. $\overline{PQ}$ is the midsegment of $\triangle MON$. What is MN ?

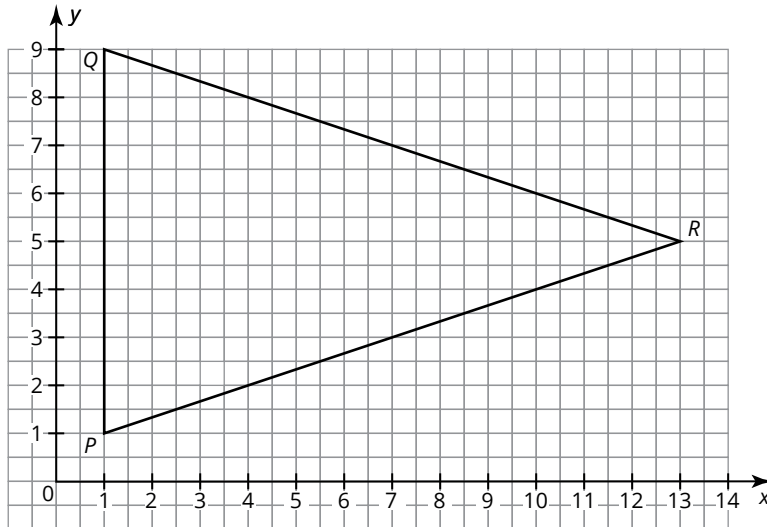


6. $\overline{HJ}$ is the midsegment of trapezoid $DEFG$. What is the value of x ?



Lesson 3 Assignment

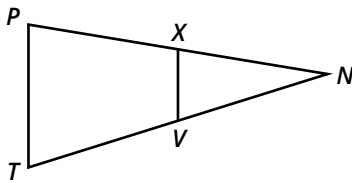
7. Consider $\triangle PQR$ on the coordinate plane.



- Use the midpoint formula to determine the midpoint of $\overline{PQ}$. Label the midpoint as point A.
- Use the midpoint formula to determine the midpoint of $\overline{QR}$. Label the midpoint as point B.
- Use the midpoint formula to determine the midpoint of $\overline{RP}$. Label the midpoint as point C.
- Use a straightedge to draw the three medians of the triangle. Label the point of concurrency of the medians as point T .
- Identify the coordinates of centroid T .

Lesson 3 Assignment

8a. Use the diagram and given statement to prove the Triangle Proportionality theorem.

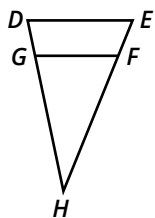


Given: $\overline{PT} \parallel \overline{XV}$

Prove: $\frac{PX}{XN} = \frac{TV}{VN}$

Statements	Reasons
1. $\overline{PT} \parallel \overline{XV}$	
2. $\angle NXV \cong \angle P$	
3. $\angle NVT \cong \angle T$	
4. $\triangle NXV \cong \triangle NPT$	
5. $\frac{PN}{XN} = \frac{TN}{VN}$	
6. $PN = PX + XN$ and $TN = TV + VN$	
7. $\frac{PX + XN}{XN} = \frac{TV + VN}{VN}$	
8. $\frac{PX}{XN} = \frac{TV}{VN}$	

b. Use the diagram and given statement to prove the converse of the Triangle Proportionality theorem.



Given: $\frac{DG}{GH} = \frac{EF}{FH}$

Prove: $\overline{DE} \parallel \overline{GF}$

Lesson 3 Assignment

Statements	Reasons
1. $\frac{DG}{GH} = \frac{EF}{FH}$	
2. $\angle H \cong \angle H$	
3. $\frac{DG}{GH} + \frac{GH}{GH} = \frac{EF}{FH} + \frac{FH}{FH}$	
4. $\frac{DG + GH}{GH} = \frac{EF + FH}{FH}$	
5. $DG + GH = DH$ and $EF + FH = EH$	
6. $\frac{DH}{GH} = \frac{EH}{FH}$	
7. $\triangle DHE \cong \triangle GHF$	
8. $\angle EDH \cong \angle FGH$	
9. $\overline{DE} \parallel \overline{GF}$	

Prepare

Solve each equation for x.

1. $\frac{10}{x} = \frac{7}{x-3}$

⋮

2. $\frac{x}{2} = \frac{8}{x}$

⋮

3. $\frac{x}{7} = \frac{6}{x}$

4

More Similar Triangles

OBJECTIVES

- Explore the relationships created when an altitude is drawn to the hypotenuse of a right triangle.
- Prove the right triangle/altitude similarity theorem with written justification.
- Use the geometric mean to solve for unknown lengths.
- Prove the Pythagorean Theorem using similar triangles with a written justification.

KEY TERMS

- right triangle/altitude similarity theorem
- geometric mean
- right triangle altitude/hypotenuse theorem
- right triangle altitude/leg theorem

You have proven the similarity of triangles using the AA similarity theorem.

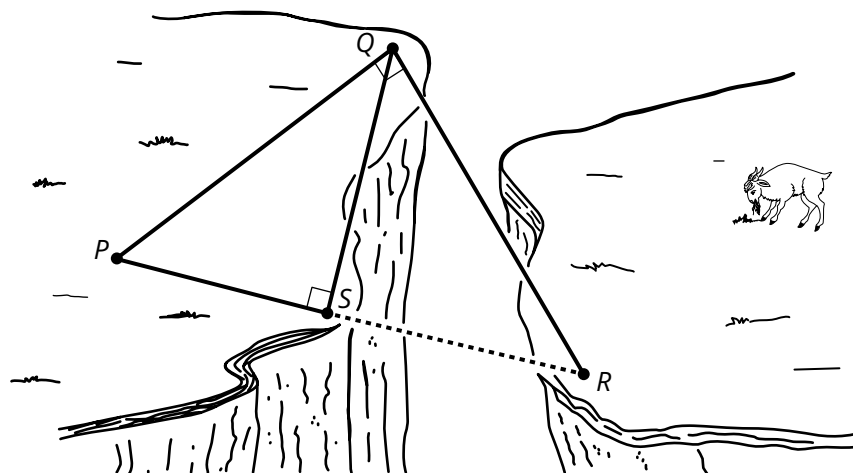
How can you use this theorem to prove that the triangles created from an altitude within a right triangle are similar?



Getting Started

Bridging the Gap

A bridge is needed to cross over a canyon. The dotted line segment connecting points S and R represents the bridge. The distance from point P to point S is 45 yards. The distance from point Q to point S is 130 feet. How long is the bridge?

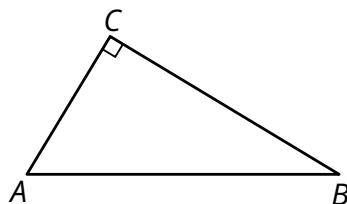


Did you know that the 2nd largest canyon system in the United States is the Palo Duro Canyon? The canyon extends from Canyon, TX to Silverton, TX.

To determine the length of the bridge, you must first explore what happens when an altitude is drawn to the hypotenuse of a right triangle. Let's investigate this relationship and then you can answer the question posed about the length of the bridge.

When an altitude is drawn to the hypotenuse of a right triangle, it forms two smaller triangles. All three triangles have a special relationship.

1. Construct an altitude to the hypotenuse in the right triangle ABC . Label the altitude CD .



Remember ...

An altitude is a perpendicular segment that indicates the height of a figure. It is drawn from a vertex to the opposite side or to the line that contains the opposite side.

-

5. Write the corresponding sides of each pair of triangles as proportions.

6. Write the corresponding angles of each pair of triangles as congruent statements.

The **right triangle/altitude similarity theorem** states: “If an altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other.”

Geometric Mean

Let's explore the relationship between the altitude of the right triangle you constructed in the Getting Started and its hypotenuse and legs.

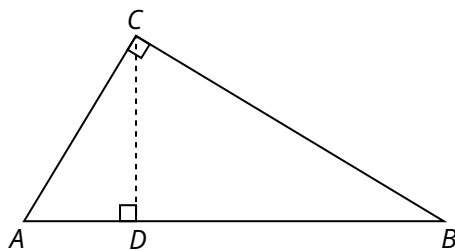
When an altitude of a right triangle is constructed from the right angle to the hypotenuse, three similar right triangles are created. This altitude is a *geometric mean*. The **geometric mean** of two positive numbers a and b is the positive number x such that $\frac{a}{x} = \frac{x}{b}$.

Two theorems are associated with the altitude to the hypotenuse of a right triangle.

The **right triangle altitude/hypotenuse theorem** states: "The measure of the altitude drawn from the vertex of the right angle of a right triangle to its hypotenuse is the geometric mean between the measures of the two segments of the hypotenuse."

The **right triangle altitude/leg theorem** states: "If the altitude is drawn to the hypotenuse of a right triangle, each leg of the right triangle is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg."

1. Use the diagram from the Getting Started to write a proportion to demonstrate each theorem.



- a. right triangle altitude/hypotenuse theorem

b. right triangle altitude/leg theorem

You can identify the geometric mean to help you solve problems.

WORKED EXAMPLE

For $\triangle ABC$, the geometric mean is the altitude length of 12.

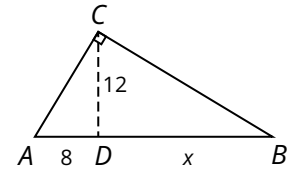
First, set up the proportion with the altitude length as the geometric mean, and then solve for x .

$$\frac{8}{12} = \frac{12}{x}$$

$$\frac{2}{3} = \frac{12}{x}$$

$$x = 18$$

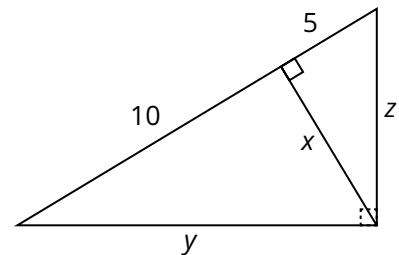
The length of $\overline{DB}$ is 18 units.



Remember ...

You can never assume that geometric figures are drawn to scale. Pay attention to diagram labels.

2. In the triangle shown, solve for x , y , and z .

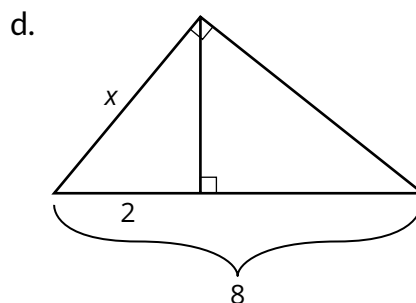
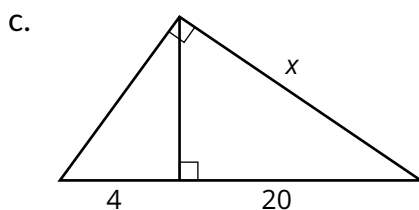
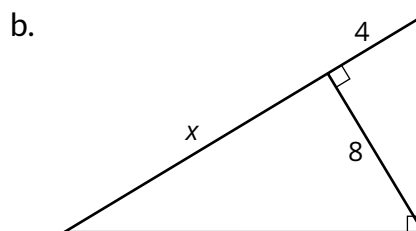
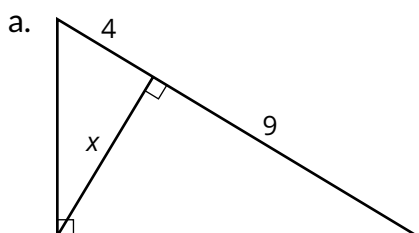


3. Solve for the length of the bridge in the Getting Started using the geometric mean.

Using Geometric Mean

The geometric mean and the theorems defined in the previous activity can be used to set up proportions to determine unknown lengths for a right triangle if you are given certain information.

1. Consider each diagram and solve for the unknown length.



Proving the Pythagorean Theorem with Similar Triangles

Remember ...

AC is a numeric value that represents the distance from point A to point C.

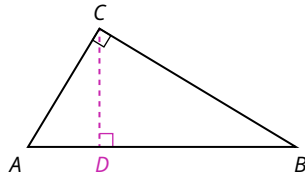
The right triangle/altitude similarity theorem can be used to prove the Pythagorean Theorem.

1. Use the diagram and follow the steps to prove the Pythagorean Theorem.

Given: $\triangle ABC$ with right angle C

Prove: $(AC)^2 + (CB)^2 = (AB)^2$

- a. Construct altitude $\overline{CD}$ to hypotenuse $\overline{AB}$.



- b. Write a proportional statement to describe the relationship between the longest leg and hypotenuse of $\triangle ABC$ and $\triangle CBD$. Then, rewrite the proportional statement as a product.
- c. Write a proportional statement to describe the relationship between the shortest leg and hypotenuse of $\triangle ABC$ and $\triangle ACD$. Then, rewrite the proportional statement as a product.
- d. Add the statement in part (b) to the statement in part (c).

Ask Yourself ...

What can you conclude about the triangles you created?

e. Rewrite the statement in part (d) using the distributive property of equality.

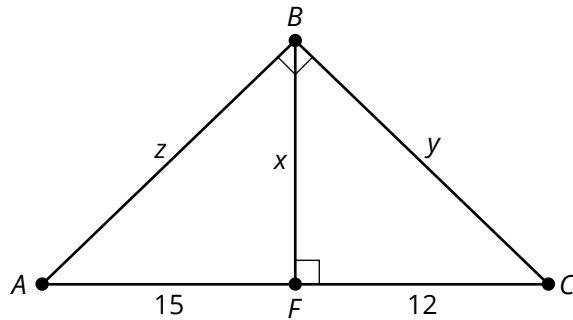
f. What is equivalent to $AD + DB$?

g. Substitute the answer to part (f) into the answer to part (e) to prove the Pythagorean Theorem.

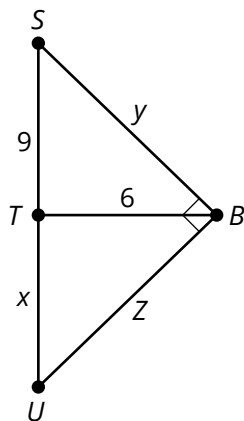
Talk the Talk

By Any Means

1. In the triangle, solve for x , y , and z .



2. In the triangle, solve for x , y , and z .



Lesson 4 Assignment

Write

Write the term from the box that best completes each statement.

right triangle/altitude similarity theorem	geometric mean
right triangle altitude/hypotenuse theorem	right triangle altitude/leg theorem

1. The _____ states that if an altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other.
2. The _____ states that if the altitude is drawn to the hypotenuse of a right triangle, each leg of the right triangle is the geometric mean of the measure of the hypotenuse and the measure of the segment of the hypotenuse adjacent to the leg.
3. The _____ of two positive numbers a and b is the positive number x such that $\frac{a}{x} = \frac{x}{b}$.
4. The _____ states that the measure of the altitude drawn from the vertex of the right angle of a right triangle to its hypotenuse is the geometric mean between the measures of the two segments of the hypotenuse.

Remember

An altitude drawn to the hypotenuse of a right triangle divides the right triangle into triangles similar to the original triangle and to each other. The equivalent proportions of the sides of the triangles show that the altitude is the geometric mean of the measures of the two segments of the hypotenuse.

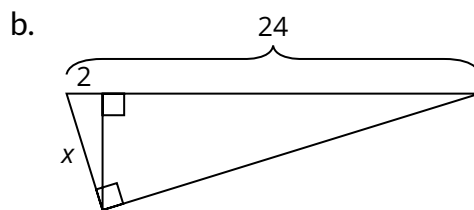
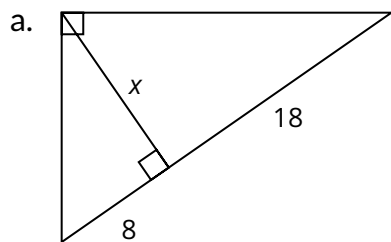
Practice

1. The geometric mean of two numbers is 20. One of the numbers is 50.
What is the other number?

Lesson 4 Assignment

2. The geometric mean of two numbers is $5\sqrt{3}$. One of the numbers is 3. What is the other number?

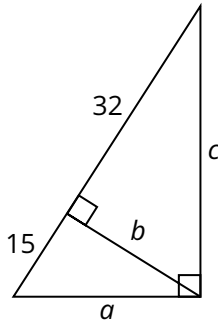
3. Determine each unknown side length.



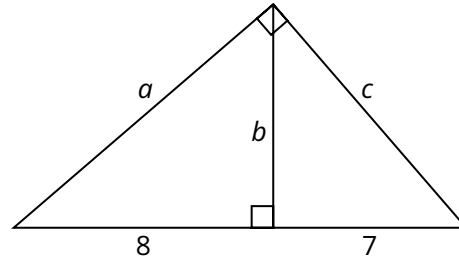
Lesson 4 Assignment

4. Determine the unknown side lengths in each triangle.

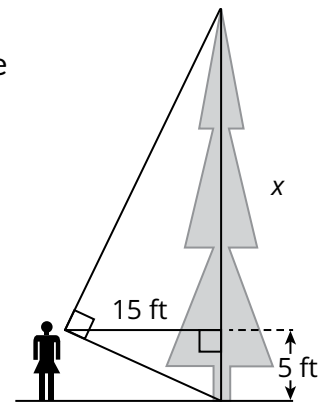
a.



b.



5. You are standing 15 feet from a tree. Your line of sight to the top of the tree and to the bottom of the tree forms a 90-degree angle as shown in the diagram. The distance between your line of sight and the ground is 5 feet. Estimate the height of the tree.

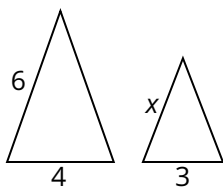


Lesson 4 Assignment

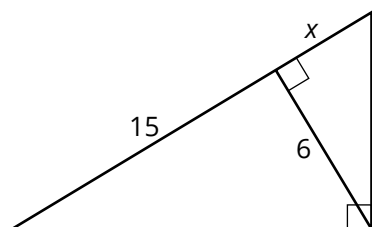
Prepare

Solve for the unknown in each pair of similar triangles.

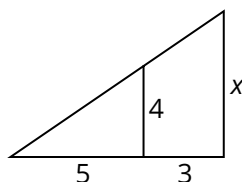
1.



2.



3.



5

Application of Similar Triangles

OBJECTIVES

- Identify similar triangles to calculate indirect measurements.
- Use similarity to solve for unknown measurements and describe them verbally in real-world contexts.

KEY TERMS

- indirect measurement
- angle of incidence
- angle of reflection
- law of reflection

.....

You have used proportions that equate the corresponding side lengths of similar triangles to solve for unknown measurements.

How can you use similar triangles to determine measurements in the real world that are difficult to measure directly?

Getting Started

Me and My Shadow

1. How would you measure the height of a flagpole?

2. How would you measure the height of a very tall building?

3. You are standing outside on a beautiful day. The sky is clear and the sun is blazing. You are looking at your shadow and thinking about geometry.
 - a. Draw a picture of yourself and your shadow.

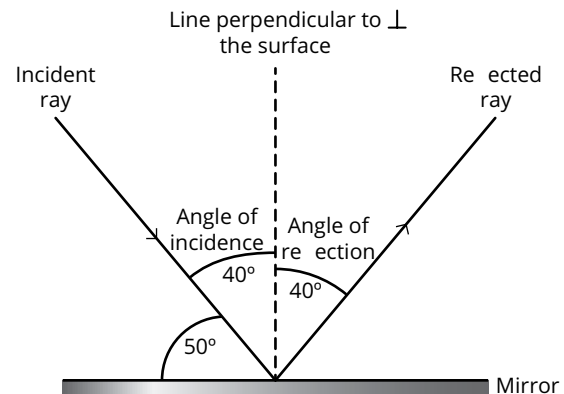
 - b. What is the measure of the angle formed by you and your shadow? Explain your reasoning.

- c. Draw a line segment connecting the top of your head with the top of your shadow's head. What geometric figure is formed?
4. Suppose you are standing outside at different times of the day and the length of your shadow changes due to the sun's position. Does the geometric figure formed by the line segment connecting your head with the top of your shadow's head change? Explain your reasoning.

.....
 Euclid is often credited with developing methods related to indirect measurement around 3000 BC.

At times, measuring something directly is impossible, or physically undesirable. When these situations arise, **indirect measurement**, the technique that uses proportions to calculate measurement, can be implemented. In this activity, you will use a mirror and your knowledge of similar triangles to measure objects indirectly.

Before you get started, you need to know a law of physics that describes the predictable manner in which light travels. An incident ray is a beam of light that reaches and reflects off a mirror, and a reflected ray is the beam of light that bounces off the mirror. Each ray forms an angle with a line perpendicular to the surface. These angles are called the **angle of incidence** and the **angle of reflection**. The **law of reflection** states that the measure of the angle of incidence equals the measure of the angle of reflection.



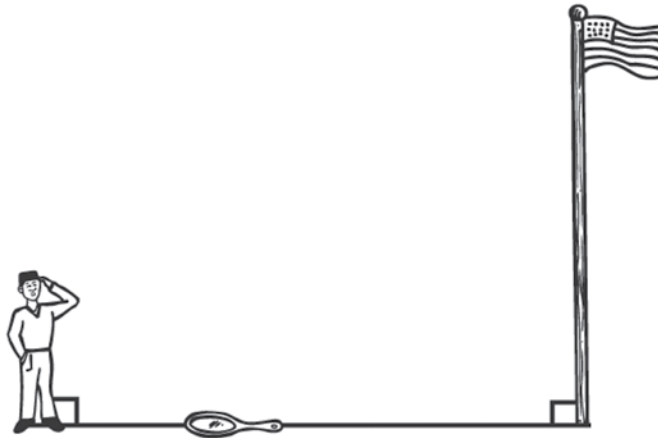
Consider how this law might be applied as you solve indirect measurement problems.

1. Use the following steps to measure the height of the school flagpole or any other tall object outside. You will need a partner, a tape measure, a marker, and a flat mirror.

Step 1: Use a marker to create a dot near the center of the mirror.

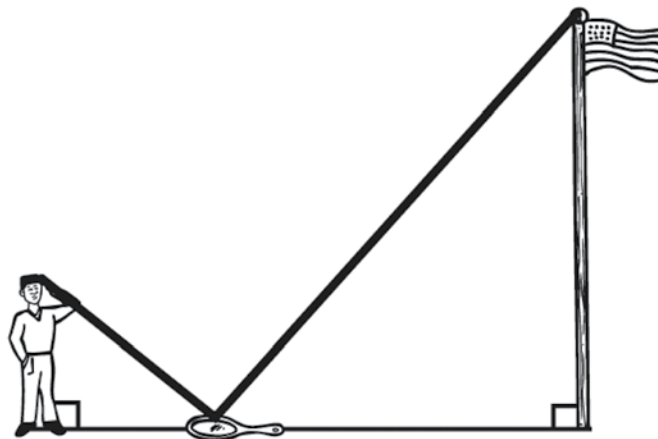
Step 2: Face the object you would like to measure and place the mirror between yourself and the object. Your feet, the mirror, and the base of the object should be collinear.

Step 3: Focus your eyes on the dot on the mirror and walk backward until you can see the top of the object on the dot, as shown.



Step 4: Ask your partner to sketch a picture of you, the mirror, and the object.

Step 5: Review the sketch with your partner. Decide where to place right angles, and where to locate the sides of the two triangles.



Step 6: Determine which segments in your sketch can easily be measured using the tape measure. Describe their locations and record the measurements on your sketch.

2. How do you know that the triangles in your sketch are similar?

.....

Review the steps and questions from the problem-solving model. There are several options for a question to go with the problem-solving model. Select one that fits with what you are trying to highlight.

.....

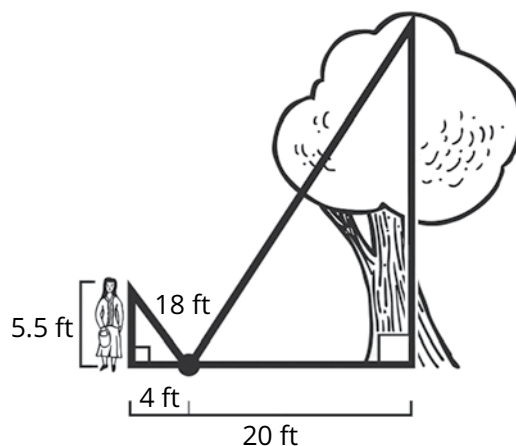
3. Use your sketch to write a proportion to calculate the height of the object and solve the proportion.
4. Compare your answer with others measuring the same object. How do the answers compare?
5. What are some possible sources of error that could result when using this method?
6. Switch places with your partner and identify a second object to measure. Repeat this method of indirect measurement to solve for the height of the new object.

ACTIVITY
5.2

Determining Height and Width Indirectly

When it is not reasonable for you to directly measure something in the real world, you can use indirect measurement and similarities.

1. Linh goes to the Inks Lake State Park and uses the mirror method to gather enough information to calculate the height of one of the trees. The figure shows her measurements. Calculate the height of the tree.



.....

Inks Lake State Park is a state park in Burnet, TX about one hour northwest of Austin. The park is a popular destination for outdoor activities such as fishing, boating, hiking, and camping.

.....

.....

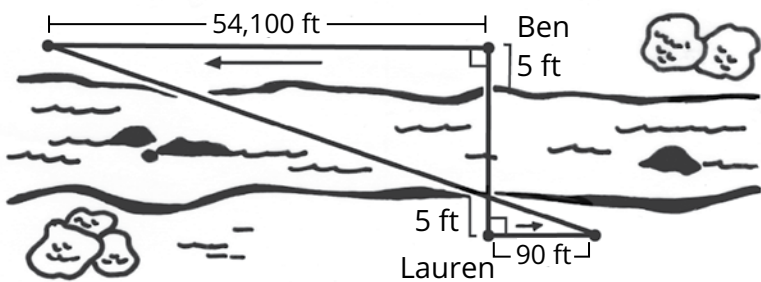
Remember . . .

Whenever you are solving a problem that involves measurements like length (or weight), you may have to rewrite units so they are the same.

.....

2. Alejandra wants to try the mirror method to measure the height of one of her trees. She calculates that the distance between her and the mirror is 3 feet and the distance between the mirror and the tree is 18 feet. Alejandra’s eye height is 60 inches. Draw a diagram of this situation. Then, calculate the height of this tree.

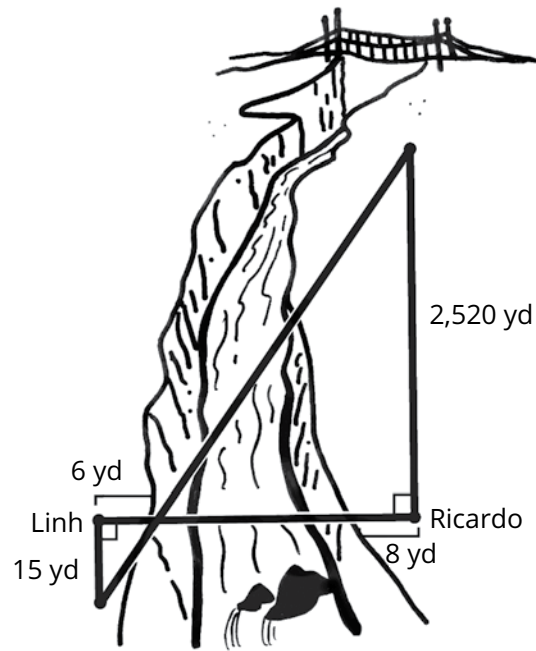
3. At Inks Lake State Park a group of friends visit the lake. Lauren stands on one end of the lake and Ben stands directly across the lake, as shown in the figure.



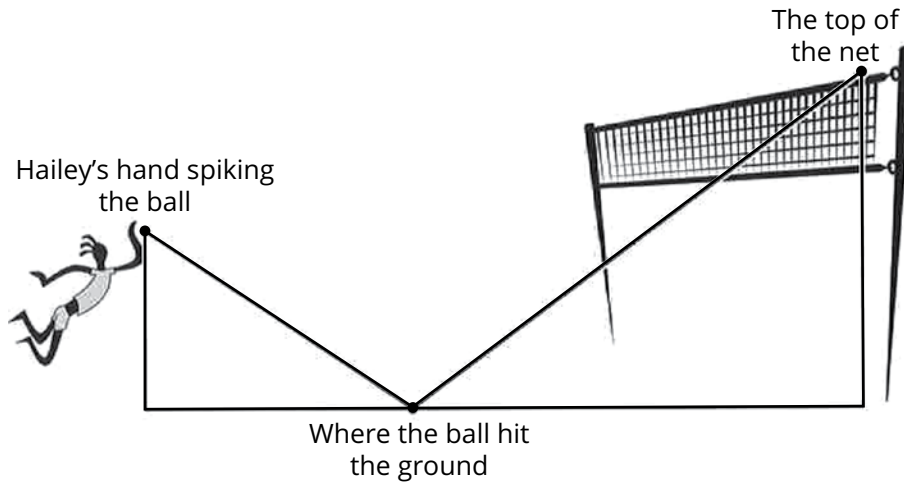
Ben is standing 5 feet from the lake and Lauren is standing 5 feet from the lake. Each friend walks away from the other in opposite parallel directions. Caleb walks 54,100 feet and Lauren walks 90 feet.

- a. How do you know that the triangles formed by the lines are similar?
- b. Calculate the distance from Ben's starting point to Lauren's side of the lake. Round your answer to the nearest tenth, if necessary.
- c. What is the width of the lake? Explain your reasoning.

4. There is also a deep, rocky sinkhole at Inks Lake State Park. Linh and Ricardo take measurements to indirectly calculate the width of the sinkhole. The figure shows their measurements. Calculate the width of the sinkhole.



5. While visiting Inks Lake State Park, Hailey decides to play volleyball. She spikes a volleyball so that it bounces off the ground 4 feet away from her. Her hand was 5 feet off the ground when she hit the ball, and the net is 6 feet beyond the position where the ball hit the ground. The ball bounced over the net, just grazing the top of it. How high is the top of the net?

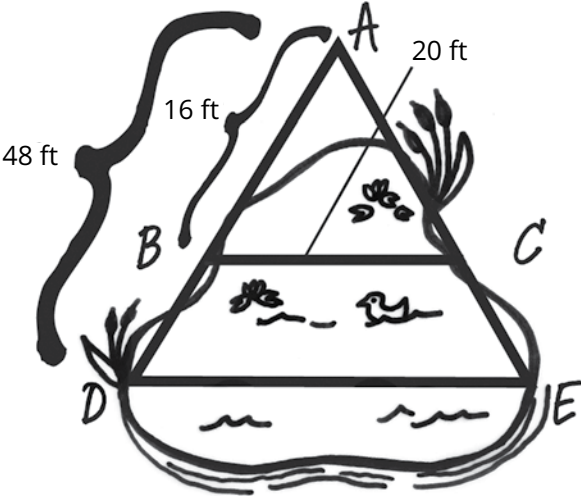


.....

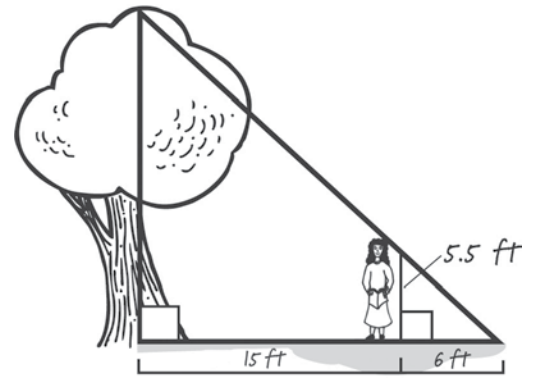
Pedernales Falls State Park is a state park 30 miles west of Austin. Common activities at the Pedernales Falls State Park include hiking, mountain biking, and riding horses. At the river you can swim or fish.

.....

6. There is a large river in Pedernales Falls State Park. Joey wants to calculate the distance across the widest part of the river, labeled as $\overline{DE}$. To indirectly calculate this distance, he first placed a stake at point A. From point A he could see the edge of the river on both sides at points D and E, where he also placed stakes. Then, he measured the distance from point A to point D and from point A to point E.
- At a narrow portion of the river, he placed stakes at points B and C along the string so that $\overline{BC}$ is parallel to $\overline{DE}$. The approximate measurements are shown on the diagram. Calculate the distance across the widest part of the river.



7. While at Pedernales Falls State Park, Gracie notices that a tree casts a shadow and suggests using shadows to calculate the height of the tree. She lines herself up with the tree's shadow so that the tip of her shadow and the tip of the tree's shadow meet. She measures the distance from herself to the tip of the shadows and then measures the distance from herself to the tree. A diagram of this situation is shown. Calculate the height of the tree. Explain your reasoning.





Talk the Talk

A Monumental Measurement

The Washington Monument in Washington, D.C., is the tallest freestanding masonry structure in the world. It is possible to determine the height of the Washington Monument using only a simple tape measure and a few known facts:

- Your eyes are 6 feet above ground level.
 - The reflecting pool is located between the Washington Monument and the Lincoln Memorial.
 - You are standing between the Washington Monument and the Lincoln Memorial facing the Monument with your back to the Lincoln Memorial while gazing into the reflecting pool at the reflection of the Washington Monument.
 - You can see the top of the monument in the reflecting pool.
 - You measured the distance from the spot where you are standing to the spot where you see the top of the monument in the reflecting pool to be 12 feet.
 - You measured the distance from the location in the reflecting pool where you see the top of the monument to the base of the monument to be 1110 feet.
1. With this information, calculate the height of the Washington Monument. Begin by drawing a diagram of the problem situation.

PROBLEM SOLVING



Lesson 5 Assignment

Write

Provide an example of the term *indirect measurement*.

Remember

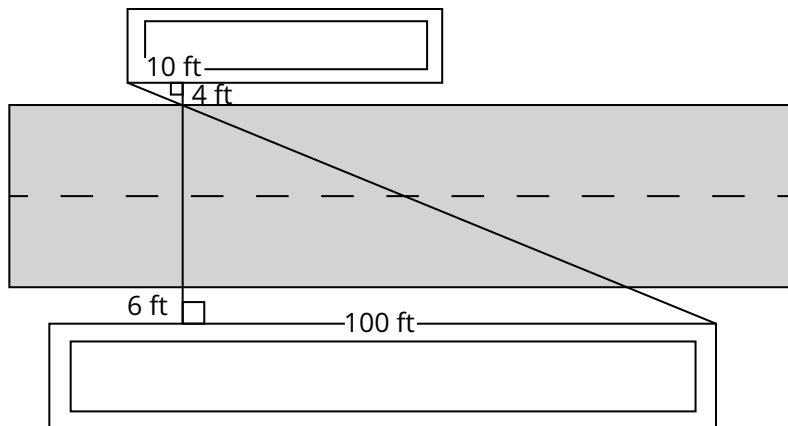
Indirect measurement is the process of using proportions related to similar triangles to determine a measurement when direct measurement is undesirable or impossible.

Practice

1. You want to measure the height of a tree at the community park. You stand in the tree's shadow so that the tip of your shadow meets the tip of the tree's shadow on the ground, 2 meters from where you are standing. The distance from the tree to the tip of the tree's shadow is 5.4 meters. You are 1.25 meters tall. Draw a diagram to represent the situation. Then, determine the height of the tree.

Lesson 5 Assignment

2. You and a friend are each on the 10th floor of apartment buildings that are directly across the street from each other and have balconies. The two of you are making a banner to hang between the apartment buildings. The banner must cross the street. To hang the banner, you and your friend need to attach it to hooks on the wall of each balcony. The wall of your balcony is 6 feet away from the street and the wall of your friend's balcony is 4 feet away from the street. You also know that your friend's balcony is 10 feet away from the end of his building, and your balcony is 100 feet away from the edge of your building. How wide is the street between you and your friend's apartment buildings? How long does the banner need to be? Show all your work and use complete sentences in your answer.



6

Partitioning Segments in Given Ratios

OBJECTIVES

- Use the distance formula to partition a directed line segment into a given ratio, determine the coordinates of the point which divides the line segment into that ratio, and verbally describe the process used.
- Use geometric reasoning and proportionality to partition a directed line segment into a given ratio, determine the coordinates of the point which divides the line segment into that ratio, and verbally describe the process used.

KEY TERM

- directed line segment

.....

You have used similarity and proportionality to solve problems on the plane.

How can you use these concepts to solve problems on a coordinate plane?

Getting Started

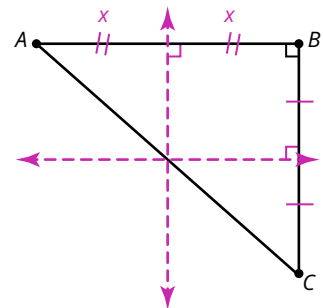
X Marks the Spot

Consider the simple map shown. A treasure is located halfway between points A and C. Use the proportionality relationships you have studied in previous lessons to answer each question.

Remember ...

The midpoint formula states that if (x_1, y_1) and (x_2, y_2) are two points on the coordinate plane, then the midpoint of the line segment that joins these two points is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

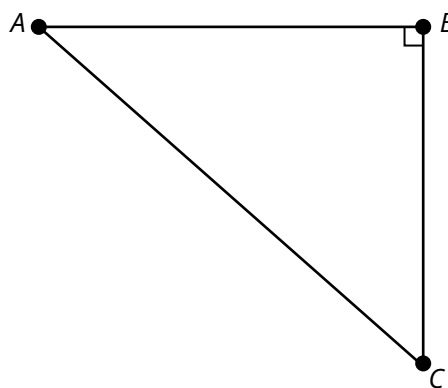
1. Describe how you could construct a line parallel to $\overline{BC}$ that intersects $\overline{AC}$ at its midpoint without measuring $\overline{AC}$.
2. Describe how you could construct a line parallel to $\overline{AB}$ that intersects $\overline{AC}$ at its midpoint without measuring $\overline{AC}$.
3. Complete the constructions to locate the midpoint of $\overline{AC}$. Explain how the midpoint formula is related to the vertical and horizontal lines you drew.



Trisecting a Line Segment

You determined the midpoint of a line segment, so you divided the line segment in a 1:1 ratio. The lengths of the two halves of the line segment are equal.

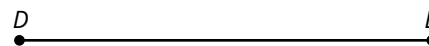
1. Set up a proportion to show that the line drawn parallel to $\overline{BC}$ divides $\overline{AC}$ in a 1:1 ratio. Identify the proportionality theorem you use.



Let's use a similar method to divide segments into other ratios.

WORKED EXAMPLE

Trisect $\overline{DE}$ by dividing its length into thirds.



Draw an auxiliary ray using one of the endpoints of the segment as the endpoint of the ray.



Using a compass, select a length and duplicate that length three times starting at the endpoint of the ray.



Label each equal distance between two consecutive points on the ray as x . Connect the third point, F , to the opposite end of the original line segment to create $\triangle DFE$.



Draw lines parallel to side $\overline{EF}$ through the points on side $\overline{DF}$. Label the distances between parallel lines along $\overline{DE}$ as a , b , and c .



Using the Triangle Proportionality theorem, $\frac{a}{b} = \frac{x}{x}$ and $\frac{b}{c} = \frac{x}{x}$.

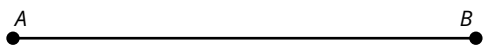
Thus, $a = b$ and $b = c$, and using the transitive property of equality, $a = c$.

Because distances a , b , and c are equal, $\overline{DE}$ has been trisected, or divided into thirds.

2. What change could you make to the diagram to divide $\overline{DE}$ into the ratio 2:1? Use proportions to explain your reasoning.

3. Consider line segment AB .

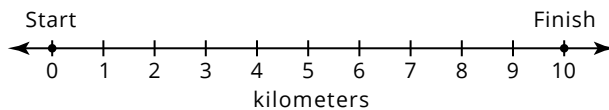
- a. Use construction tools to divide $\overline{AB}$ into five congruent segments.



- b. Explain how the Triangle Proportionality theorem justifies your process.
- c. What change could you make to your diagram to divide $\overline{AB}$ into a 4 : 1 ratio?
- d. How do you know whether you have divided a line segment into a 4 : 1 ratio instead of a 1 : 4 ratio?

Consider a situation where the line segment is located on a number line.

4. An organization is hosting a 10 kilometer race for the community. They are required to locate different stations along the race's path. So, they are using this number line as a model of the race's path. Then, they can use it to determine the locations for the stations along the race's path.



The race's path is represented using a line segment with endpoints at 0 and 10 on the number line. The start of the race is at 0 and the finish line is at 10 on the number line.

- a. Restrooms must be located on the race's path halfway between the start and finish of the race. Describe how to locate and then plot point R on the number line which represents the location of the restrooms.

- b. Water stations must be located every 1 kilometer. Describe how to divide the line segment on the number line into equal parts so that there is a water station every 1 kilometer.
- c. Cooling stations must be located along the race's path $\frac{1}{5}$ of the way and $\frac{3}{5}$ of the way between the race's start and the race's finish. Describe how to locate and then plot points D and E on the number line. Point D is the location of the first cooling station. Point E is the location of the second cooling station.
- d. What ratios describe the locations of the cooling stations? Explain how you know.

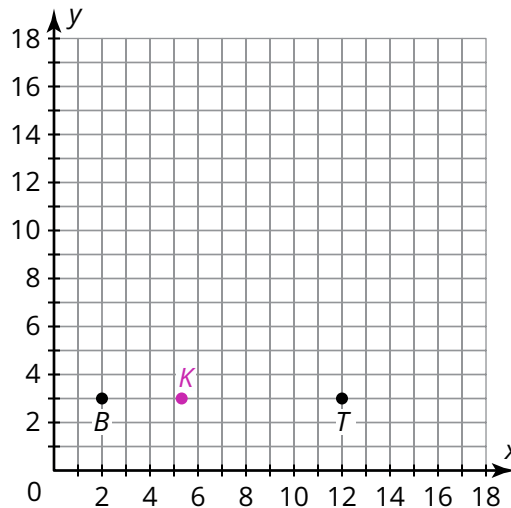
- e. The ratio 1 : 3 describes the location of a fruit station. Describe how to locate and then plot point H on the number line which represents the location of the fruit station.

- f. A photographer is going to stand at point T which is located at 7.5 on the number line. Represent the location of point T as a fraction of the way between the start and finish of the race and also as a ratio. Explain your reasoning and then interpret what the location of point T means in the context of the situation.

Partitioning Directed Line Segments

The midpoint of a line segment divides the line segment in a 1 : 1 ratio. Each half of the line segment is the same length. Let's explore dividing distances into different ratios.

1. Joey buried a spare key to his house in the backyard in case of an emergency. The location of the back door is point $B(2, 3)$, and the location of the oak tree is point $T(12, 3)$.



- a. Suppose Joey buried his spare key $\frac{1}{3}$ of the way between the back door and the oak tree. Determine the location of the key. Plot and label the location as point K .

Point K divides the distance $\overline{BT}$ into different ratios, depending on whether you start at B and move to T or start at T and move to B .

- b. In what ratio does point K divide the distance BT , starting at the back door and moving to the tree? Explain your reasoning.

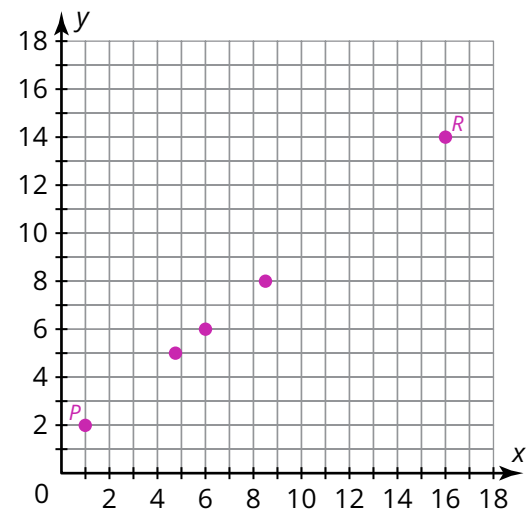
- c. In what ratio does point K divide the distance $\overline{TB}$, starting at the tree and moving to the back door? Explain your reasoning.

When thinking about a line segment and direction is important, you can think about the line segment as a **directed line segment**. The path between the back door and the tree can represent a directed line segment, $\overrightarrow{BT}$. Mathematically, this is different from the directed line segment $\overrightarrow{TB}$, the path from the tree to the back door.

Brianna also buried her house key. She remembers burying the key between the front porch and a rose bush. The location of the front porch is point $P(1, 2)$ and the location of the rose bush is point $R(16, 14)$.

2. Determine the location of Brianna's key in each situation. In what ratio does the location of the key divide the directed line segment $\overrightarrow{PR}$? In what ratio does it divide the directed line segment $\overrightarrow{RP}$?

- a. Suppose Brianna buried her key $\frac{1}{2}$ of the way between the front porch and the rose bush.



- b. Suppose Brianna buried her key $\frac{1}{4}$ of the way between the front porch and the rose bush.

- c. Suppose Brianna buried her key $\frac{1}{3}$ of the way between the front porch and the rose bush.

James and Alejandra used different methods to determine the location that is $\frac{1}{3}$ of the way between the front porch and the rose bush.

James



I drew a vertical line from point R and a horizontal line from point P and determined the coordinates of point Z where they intersected. I divided the horizontal line and the vertical line into 3 equal parts and wrote the coordinates. I used those coordinates to divide segment RP into three parts. Then, I calculated the coordinates along $\overline{RP}$.

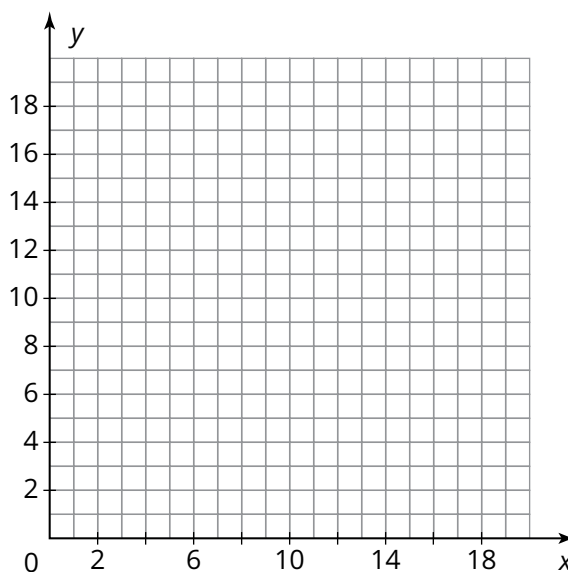
Alejandra



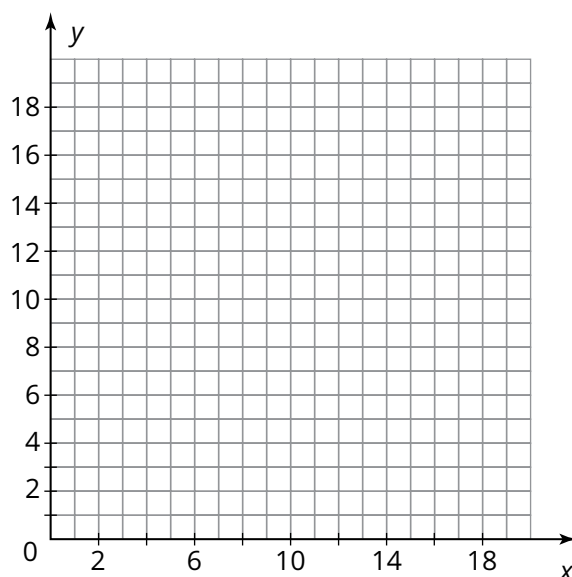
I thought about the slope of segment RP. When I want to divide segment RP into three parts then I need to start at point P and perform three vertical and horizontal shifts to get to point R. So, I can just divide the total horizontal and vertical shifts by three to calculate each vertical and horizontal shift. After that, determining the coordinates is a snap!

3. Calculate the location of the point that is $\frac{1}{3}$ of the way between the front porch and the rose bush using James' method and Alejandra's method. Use a graph to support your work.

a. James' Method



b. Alejandra's Method



4. Suppose Brianna buried her key $\frac{1}{5}$ of the way between the front porch and the rose bush. Determine the location of the key. Identify the two different ratios that the directed segment is divided into by the location of the key.

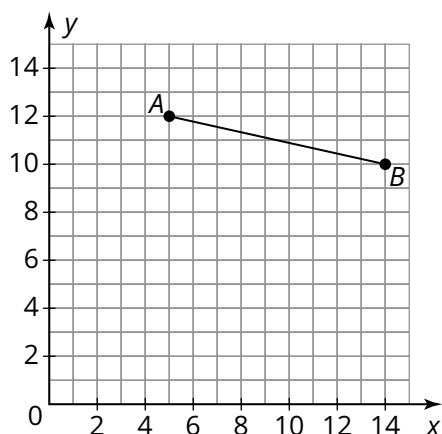
ACTIVITY
6.3

Using Proportionality on the Coordinate Plane

You can combine what you know about algebra and geometry to partition directed line segments into different ratios on the coordinate plane.

Consider, for example, $\overline{AB}$. Let's divide the directed $\overline{AB}$ into a 1 : 2 ratio using algebraic and geometric reasoning.

1. Draw $\overline{AC}$ such that point C is at (5, 0). Then, draw $\overline{BC}$ to form $\triangle ABC$.



2. Determine the slope of $\overline{BC}$.
3. Plot points on $\overline{AC}$ to divide the segment into thirds. List the coordinates of the points.

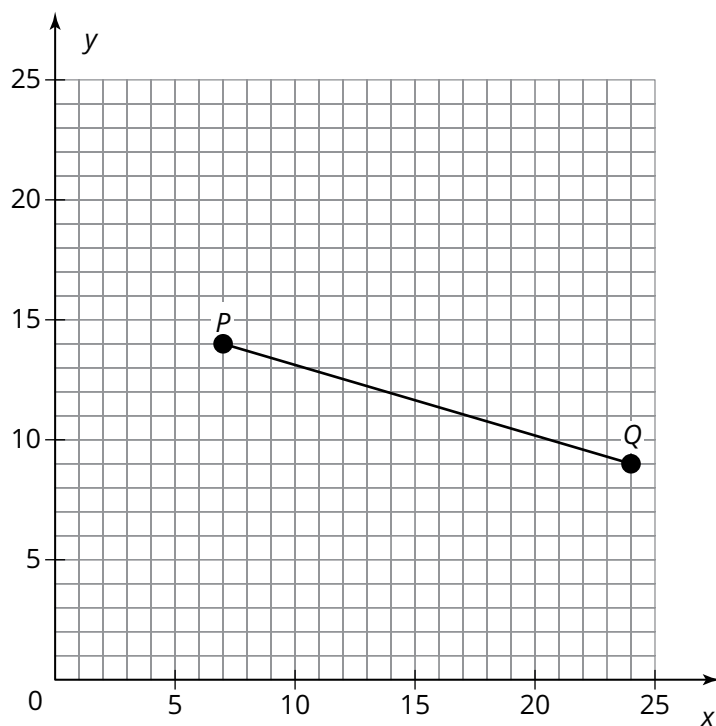
4. Determine and draw lines that are parallel to $\overline{BC}$ and pass through each point you plotted on $\overline{AC}$. Write the equations of the lines and explain your reasoning.

5. Determine at what points the parallel lines you drew intersect $\overline{AB}$. Show your work and explain your reasoning.

6. What point divides the directed line segment AB into a 1 : 2 ratio? What point divides the directed line segment BA into a 1 : 2 ratio? Explain your reasoning. Use a geometric theorem to explain how you know that the lines parallel to $\overline{BC}$ divide $\overline{AB}$ into thirds.

.....
Remember ...
The point-slope form of a linear equation is $y - y_1 = m(x - x_1)$.
.....

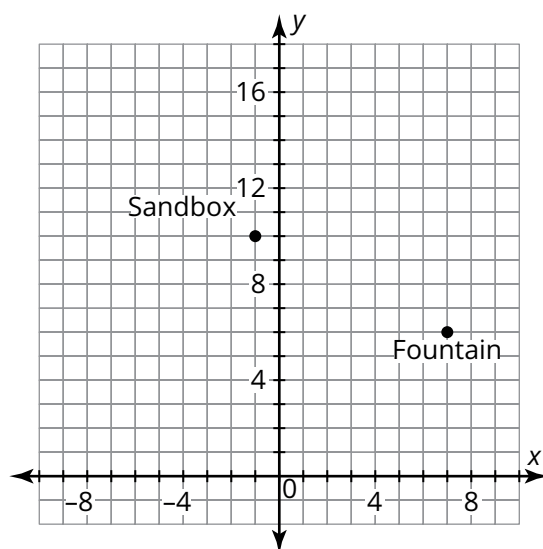
7. Use proportionality to partition the directed $\overline{PQ}$ into a 1 : 2 ratio. Show your work and explain your reasoning.



Talk the Talk

We're Gonna Catch a Big One

The grid shows the locations of a sandbox and a fountain in a park.



1. A scavenger hunt clue is buried $\frac{1}{3}$ of the way between the sandbox and the fountain. Determine the location of the clue. Show your work.

2. The swings are located $\frac{3}{4}$ of the way between the sandbox and the fountain. Determine the location of the swings. Show your work.

Lesson 6 Assignment

Write

In your own words, describe what a directed line segment is.

Remember

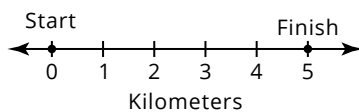
The midpoint of a line segment is the point on the segment that is equidistant from the endpoints of the line segment.

The midpoint formula states that the midpoint between any two points on a coordinate plane, (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

The Triangle Proportionality theorem states: "If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally."

Practice

1. The line segment on the number line represents the path of a 5 kilometer race.



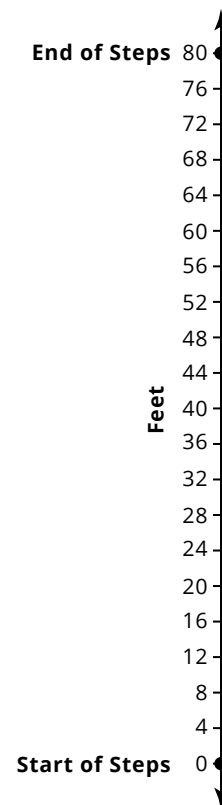
- a. Cooling stations must be located along the race's path $\frac{3}{10}$ of the way and $\frac{7}{10}$ of the way between the race's start and the race's finish. Describe how to locate and then plot points A and B on the number line. Point A is the location of the first cooling station. Point B is the location of the second cooling station.

Lesson 6 Assignment

- b. What ratios describe the locations of the cooling stations?
- c. The ratio 3 : 1 describes the location of a fruit station along the race's path. Describe how to locate and then plot point C on the number line, which represents the location of the fruit station.

2. The line segment on the number line represents the height, in feet, of the vertical steps that follow the side of a waterfall. The steps begin at 0 on the number line which is also where the waterfall hits the pool of water at 0 feet or ground level. The steps end at 80 on the number line which is also where the waterfall begins to fall at 80 feet above ground.

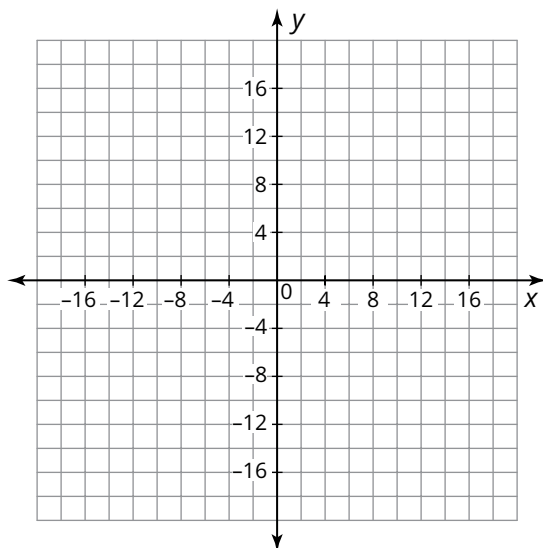
- a. The ratio 2 : 3 describes the location of a platform surrounded by a fence where you can stop and look at the waterfall. Describe how to locate and then plot point G on the number line, which represents the location of the platform.



Lesson 6 Assignment

- b. A hiker is standing at point H on the steps which is located at 56 on the number line. Represent the location of point H as a fraction of the way between the start and the end of the steps and also as a ratio. Explain your reasoning and then interpret what the location of point H means in the context of the situation.

3. Graph the three points on the coordinate plane.
 $A(-10, 3)$, $B(-4, 3)$, $C(-7, 11)$



Lesson 6 Assignment

4. Connect the three points to form triangle ABC .
5. Solve for the coordinates of M_1 , the midpoint of side $\overline{AC}$.
6. Solve for the coordinates of M_2 , the midpoint of side $\overline{BC}$.
7. Connect the two midpoints M_1 and M_2 .
8. Calculate the distance between points M_1 and M_2 .

Lesson 6 Assignment

9. Calculate the distance between points A and B .

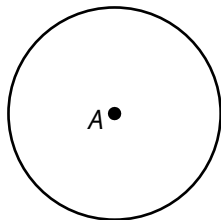
10. Compare the length of the midsegment (line segment M_1M_2) of the triangle to the length of the base of the triangle (line segment AB).

Lesson 6 Assignment

Prepare

Given circle A , draw each part.

1. A chord using points B and C .
2. A tangent at point B .
3. A secant at points D and E .



Similarity

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

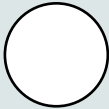
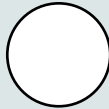
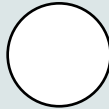
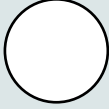
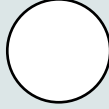
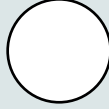
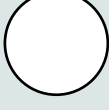
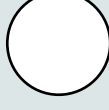
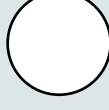
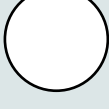
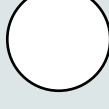
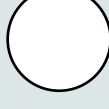
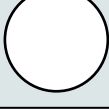
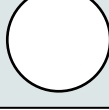
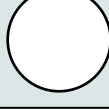
Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Similarity* topic by:

TOPIC 1: <i>Similarity</i>	Beginning of Topic	Middle of Topic	End of Topic
defining <i>dilation</i> and perform a dilation using a given center of dilation and scale factor.			
verifying that the dilation of a line segment is longer or shorter in the ratio given by the scale factor.			
defining <i>similarity</i> as a composition of transformations that preserves angle measure and maintains proportionality of side lengths.			
demonstrating in similar figures that pairs of corresponding angles are congruent and pairs of corresponding sides are proportional.			
verifying through a sequence of similarity transformations that two figures are similar.			
proving the Angle-Angle similarity theorem and using it to explain why two triangles are similar.			
verifying that when the center of dilation does not lie on the given line, the dilation of that line produces a parallel line.			
verifying that when the center of dilation lies on the given line, the dilation of that line leaves it unchanged.			

continued on the next page

TOPIC 1 SELF-REFLECTION
continued

TOPIC 1: <i>Similarity</i>	Beginning of Topic	Middle of Topic	End of Topic
proving theorems about the proportionality of triangle sides and angles and using them to solve real-world and mathematical problems.			
proving the Pythagorean Theorem using triangle similarity.			
using triangle similarity to indirectly measure objects.			
using triangle congruence and similarity to prove relationships in geometric figures.			
calculating the point on a directed line segment that partitions the line segment into a given ratio.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

- Describe a new strategy you learned in the *Similarity* topic.

- What mathematical understandings from the topic do you feel you are making the most progress with?

- Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

Similarity Summary

LESSON

1

Dilating Figures to Create Similar Figures

A **dilation** is a transformation of a figure in which the figure stretches or shrinks with respect to a fixed point, or center of dilation. In a dilation, the coordinates of a point (x, y) transform into the coordinates (kx, ky) where k is the scale factor. If $0 < k < 1$, the dilation is a reduction. If $k > 1$, the dilation is an enlargement.

A dilation produces similar figures. **Similar figures** are geometric figures in which all pairs of corresponding angles are congruent and the lengths of all corresponding sides are proportional.

Two or more geometric figures are similar if a sequence of dilations and/or rigid motions maps one figure onto the other.

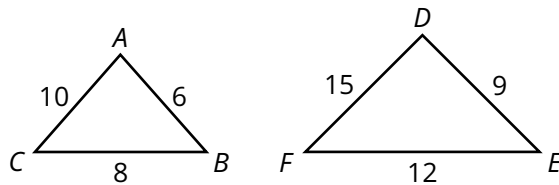
Similar triangles are triangles in which all pairs of corresponding angles are congruent and the lengths of all corresponding sides are proportional. The ratios of corresponding side lengths are equal to the same value. This value is the scale factor of the dilation. Similar triangles have the same shape but not always the same size.

For example, consider $\triangle ABC$ and $\triangle DEF$, which are similar triangles.

$$\triangle ABC \sim \triangle DEF$$

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

$$\frac{6}{9} = \frac{8}{12} = \frac{10}{15}$$



The scale factor of the dilation of $\triangle DEF$ that produces $\triangle ABC$ is $\frac{2}{3}$.

KEY TERMS

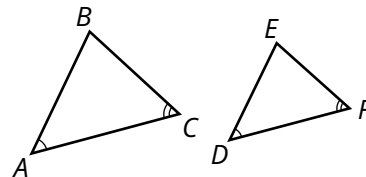
- dilation [dilatación]
- similar figures [figuras similares]
- similar triangles [triángulos similares]
- Angle-Angle similarity theorem [teorema de similitud ángulo-ángulo (AA)]
- Side-Side-Side similarity theorem
- Side-Angle-Side similarity theorem
- angle bisector/proportional side theorem [teorema de la bisectriz del ángulo/lado proporcional]
- Triangle Proportionality theorem [teorema de proporcionalidad del triángulo]
- converse of the Triangle Proportionality theorem
- proportional segments theorem [teorema de segmentos proporcionales]
- triangle midsegment theorem [teorema del segmento medio del triángulo]
- right triangle/altitude similarity theorem

- geometric mean
- right triangle altitude/hypotenuse theorem
- right triangle altitude/leg theorem
- indirect measurement
- angle of incidence [ángulo de incidencia]
- angle of reflection [ángulo de reflexión]
- law of reflection [ley de la reflexión]
- directed line segment

You can conclude that two triangles are similar if you are able to prove that all of their corresponding angles are congruent and all of their corresponding sides are proportional.

The **Angle-Angle similarity theorem** states: "If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar."

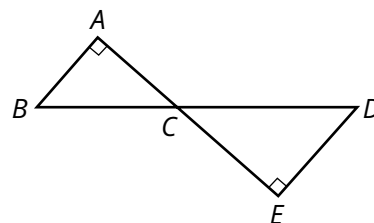
For example, if $m\angle A = m\angle D$ and $m\angle C = m\angle F$, then $\triangle ABC \sim \triangle DEF$.



You can apply the Angle-Angle similarity theorem to verify that two triangles are similar.

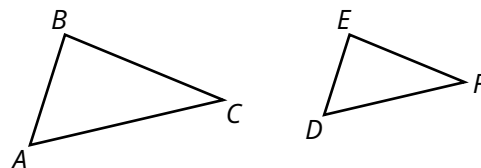
For triangles ABC and EDC , you know that:

- $\angle A \cong \angle E$ because the two angles are right angles.
- $\angle ACB \cong \angle ECD$ because the two angles are vertical angles.



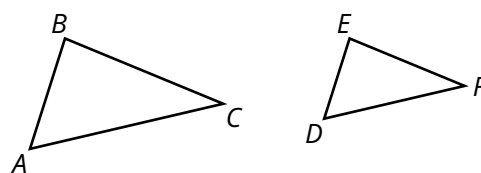
The Angle-Angle similarity theorem tells you that triangles ABC and EDC are similar because they have two pairs of congruent angles.

The **Side-Side-Side similarity theorem** states: "If all three corresponding sides of two triangles are proportional, then the triangles are similar."



For example, if $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$, then $\triangle ABC \sim \triangle DEF$.

The **Side-Angle-Side similarity theorem** states: "If two of the corresponding sides of two triangles are proportional and the included angles are congruent, then the triangles are similar."



For example, if $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle A \cong \angle D$, then $\triangle ABC \sim \triangle DEF$.

Theorems About Proportionality

The **angle bisector/proportional side theorem** states: “A bisector of an angle in a triangle divides the opposite side into two segments whose lengths are in the same ratio as the lengths of the sides adjacent to the angle.”

For example, consider $\triangle ABC$. If $\overline{CD}$ bisects $\angle C$, we can use the angle bisector/proportional side theorem to determine $\overline{BD}$.

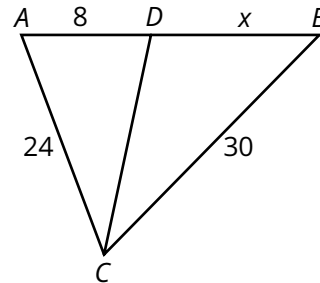
$$\frac{AC}{BC} = \frac{AD}{BD}$$

$$\frac{24}{30} = \frac{8}{x}$$

$$24x = 240$$

$$x = 10$$

$$\overline{BD} = 10$$



The **Triangle Proportionality theorem** states: “If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally.”

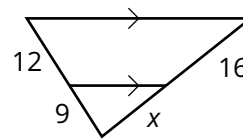
For example, in the diagram shown, four segments are formed by the parallel line that passes through the interior of the triangle.

To determine x , you can use what you know about the Triangle Proportionality theorem to set up and solve a proportion.

$$\frac{9}{12} = \frac{x}{16}$$

$$12x = 144$$

$$x = 12$$



The unknown length, x , is 12 units.

The **converse of the Triangle Proportionality theorem** states: “If a line divides two sides of a triangle proportionally, then it is parallel to the third side.”

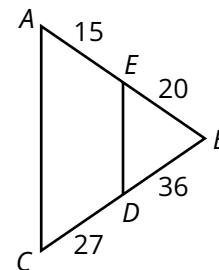
For example, consider $\triangle ABC$.

$$\frac{AE}{EB} = \frac{CD}{DB}$$

$$\frac{15}{20} = \frac{27}{36}$$

$$540 = 540$$

$$\overline{AC} \parallel \overline{ED}$$



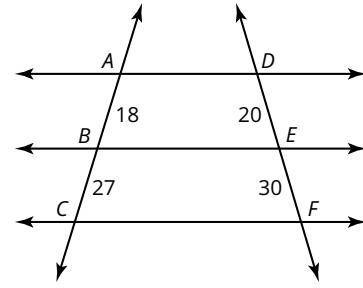
The **proportional segments theorem** states: “If three parallel lines intersect two transversals, then they divide the transversals proportionally.”

For example, $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ are parallel lines intersected by transversals $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$.

$$\frac{AB}{BC} = \frac{DE}{EF}$$

$$\frac{18}{27} = \frac{20}{30}$$

$$540 = 540$$



The **triangle midsegment theorem** states: “The midsegment of a triangle is parallel to the third side of the triangle and is half the measure of the third side of the triangle.”

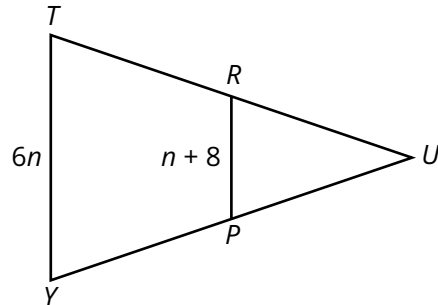
For example, consider $\triangle TUY$ with midsegment RP . You can use the triangle midsegment theorem to solve for n .

$$\frac{1}{2}(6n) = n + 8$$

$$3n = n + 8$$

$$2n = 8$$

$$n = 4$$



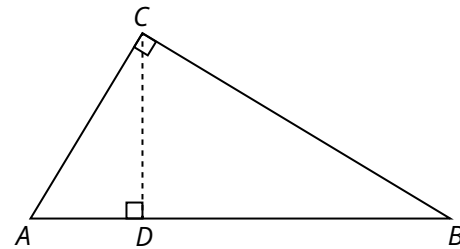
This theorem can also be used in the coordinate plane to find the value of the vertices of a triangle. If you are given the coordinates of the midpoints of the sides of a triangle, you can use the triangle midsegment theorem to determine the slope of the parallel lines and use the slope to determine the coordinates of each vertex.

LESSON 4

More Similar Triangles

The **right triangle/altitude similarity theorem** states: “If an altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other.”

For example, in right triangle ABC , altitude $\overline{CD}$ is drawn to hypotenuse $\overline{AB}$.



$$\triangle ABC \sim \triangle ACD \sim \triangle CBD.$$

The **geometric mean** of two positive numbers a and b is the positive number x such that $\frac{a}{x} = \frac{x}{b}$.

The **right triangle altitude/hypotenuse theorem** states: “The measure of the altitude drawn from the vertex of the right angle of a right triangle to its hypotenuse is the geometric mean between the measures of the two segments of the hypotenuse.”

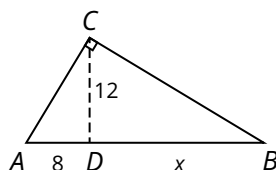
The **right triangle altitude/leg theorem** states: “If the altitude is drawn to the hypotenuse of a right triangle, each leg of the right triangle is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.”

You can identify the geometric mean to help you solve problems. For example, consider $\triangle ABC$. The geometric mean is the altitude length of 12.

$$\frac{8}{12} = \frac{12}{x}$$

$$\frac{2}{3} = \frac{12}{x}$$

$$x = 18$$



The length of $\overline{DB}$ is 18 units.

LESSON

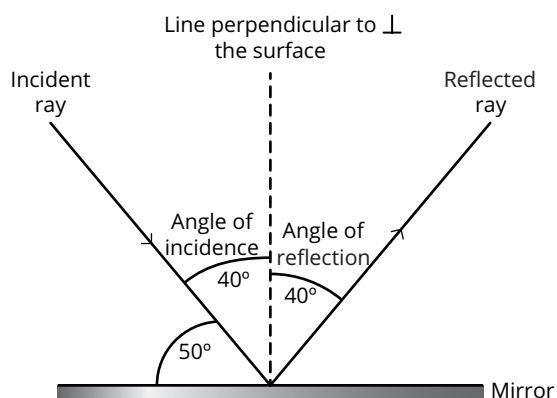
5

Application of Similar Triangles

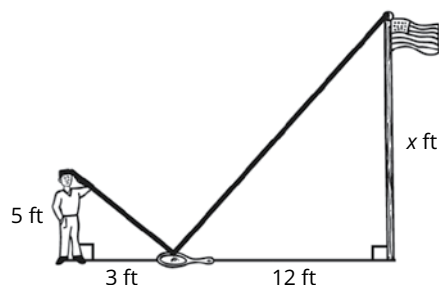
At times, measuring something directly is impossible or physically undesirable. When these situations arise, **indirect measurement**, the technique that uses proportions to calculate measurement, can be implemented.

If you know the direct measurements of parts of similar right triangles, you can use indirect measurement to determine the measure of something in the real world that is not reasonable for you to measure directly.

When using a mirror and your knowledge of similar triangles to measure objects indirectly, you need to know a law of physics that describes the predictable manner in which light travels. An incident ray is a beam of light that reaches and reflects off a mirror, and a reflected ray is the beam of light that bounces off the mirror. Each ray forms an angle with a line perpendicular to the surface. These angles are called the **angle of incidence** and the **angle of reflection**. The **law of reflection** states that the measure of the angle of incidence equals the measure of the angle of reflection.



For example, you can place a mirror on the ground between yourself and a tall object and then stand at a point where you can see the top of the object in the mirror. The triangle created by the object's height, the distance from the object to the mirror, and the incident ray is similar to the triangle created by your height, the distance from yourself to the mirror, and the reflected ray by the Angle-Angle similarity theorem.



$$\begin{aligned}\frac{x}{12} &= \frac{5}{3} \\ 3x &= 60 \\ x &= 20\end{aligned}$$

The height of the flagpole is 20 feet.

LESSON

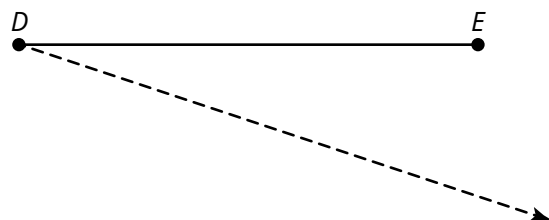
6

Partitioning Segments in Given Ratios

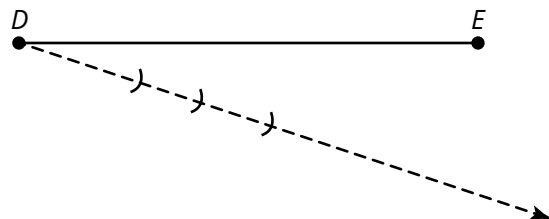
The midpoint of a line segment divides the line segment in a 1 : 1 ratio. You can divide segments into other ratios using constructions.

For example, trisect $\overline{DE}$ by dividing its length into thirds.

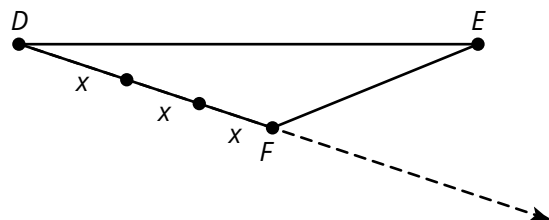
Draw an auxiliary ray using one of the endpoints of the segment as the endpoint of the ray.



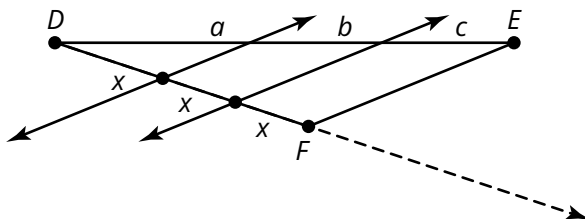
Using a compass, select a length and duplicate that length three times starting at the endpoint of the ray.



Label each equal distance between two consecutive points on the ray as x . Connect the third point, F , to the opposite end of the original line segment to create $\triangle DFE$.



Draw lines parallel to side $\overline{EF}$ through the points on side $\overline{DF}$. Label the distances between parallel lines along $\overline{DE}$ as a , b , and c .



Using the Triangle Proportionality theorem, $\frac{a}{b} = \frac{x}{x}$ and $\frac{b}{c} = \frac{x}{x}$.

Thus, $a = b$ and $b = c$, and using the transitive property of equality, $a = c$.

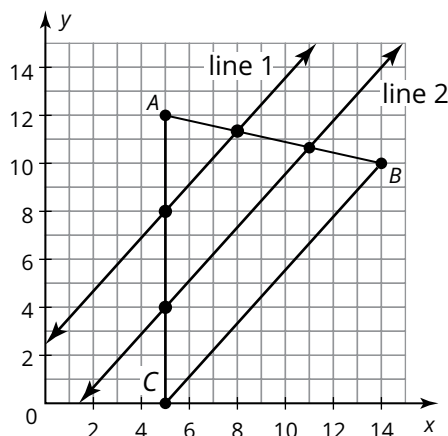
Because distances a , b , and c are equal, $\overline{DE}$ has been trisected, or divided into thirds.

When thinking about a line segment and direction is important, you can think about the line segment as a **directed line segment**. In the previous example, the directed line segment $\overrightarrow{DE}$ is mathematically different from directed line segment $\overrightarrow{ED}$.

You can combine what you know about algebra and geometry to partition directed line segments into different ratios on the coordinate plane.

For example, to divide the directed segment $\overrightarrow{AB}$ into a 1 : 2 ratio, draw $\overline{AC}$ such that point C is at $(5, 0)$ and draw $\overline{BC}$ to form $\triangle ABC$.

Plot points on $\overline{AC}$ to divide the segment into thirds. Draw lines that are parallel to $\overline{BC}$ and pass through each point you plotted. Line 1 divides directed segment $\overrightarrow{AB}$ into a 1 : 2 ratio. Use your knowledge of writing equations of parallel lines given a slope and a point to determine the equation of the line.



$$\text{line 1: } y = \frac{10}{9}x + \frac{22}{9}$$

Write and solve a system of equations to determine the coordinates of the point on directed segment $\overrightarrow{AB}$ that divides it into a 1 : 2 ratio.

$$\begin{cases} y = \frac{10}{9}x + \frac{22}{9} & \leftarrow \text{equation for line 1} \\ y = -\frac{2}{9}x + \frac{118}{9} & \leftarrow \text{equation for the line containing } \overline{AB} \end{cases}$$

The point $\left(8, \frac{102}{9}\right)$ divides directed segment $\overrightarrow{AB}$ into a 1 : 2 ratio.

The location of a point from one end of a line segment to the other on a number line can also be located using fractional distances and ratios.

For example, partition a number line segment from 0 to 12 into the ratio 2 : 3.

The total number of parts is 5 and the total distance is 12. So each segment will have a length of $\frac{12}{5}$ or 2.4 units.

The first point is at zero. Determine the location of each of the remaining points on the number line.

$$0 + 2.4 = 2.4$$

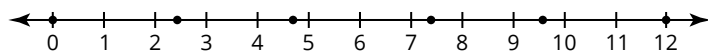
$$2.4 + 2.4 = 4.8$$

$$4.8 + 2.4 = 7.2$$

$$7.2 + 2.4 = 9.6$$

$$9.6 + 2.4 = 12.$$

Finally, plot the points on a number line.





Sundials are mostly decorative, but in the past they were functional timekeepers, in some cases with to-the-minute accuracy. Sundials work by applying trigonometry to the angle of the sun.

Trigonometry

LESSON 1	Introduction to Trigonometry	889
LESSON 2	Tangent Ratio and Inverse Tangent	905
LESSON 3	Exploring Sine and Cosine Ratios	925
LESSON 4	Complement Angle Relationships	953



1

Introduction to Trigonometry

OBJECTIVES

- Explore trigonometric ratios as measurement conversions.
- Analyze the properties of similar right triangles and write a description of the side length proportionality.

KEY TERMS

- reference angle
- opposite side
- adjacent side

.....

You know about the proportionality between the side lengths of similar figures.

What is the relationship between the ratios of the side lengths of similar figures?

Getting Started

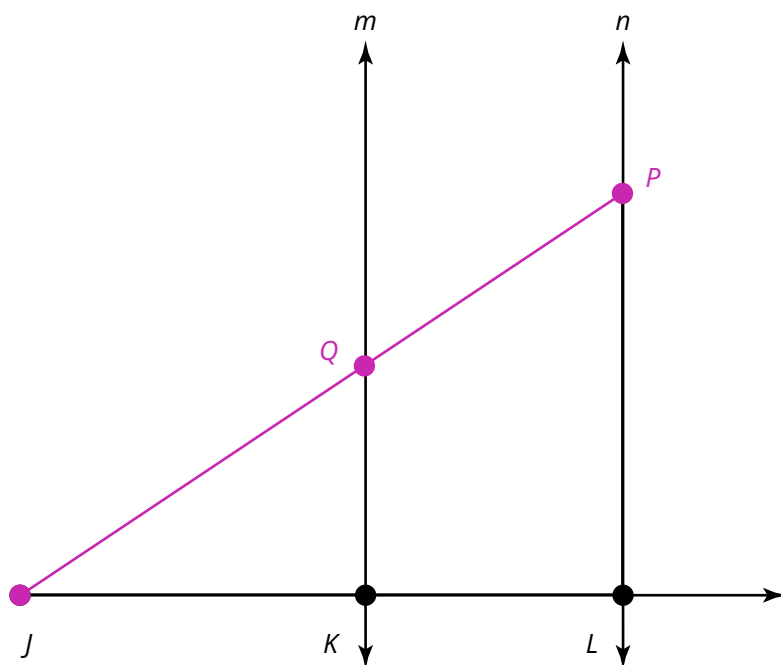
Remember...

You cannot make assumptions about a diagram by sight.

You're Acute Angle

Lines m and n are parallel.

1. Locate a point on line n and label it point P to create $\triangle JLP$. Then label the intersection of $\overline{JP}$ and line m as point Q to create $\triangle JKQ$.

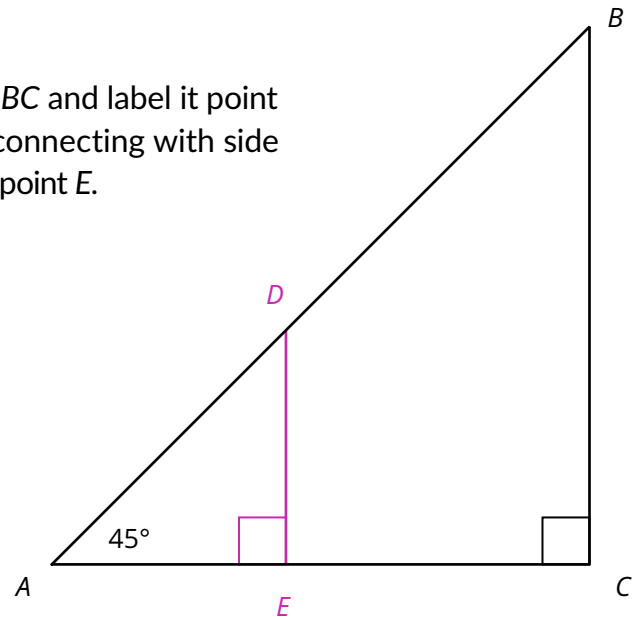


2. Verify that $\triangle JLP \sim \triangle JKQ$. Explain your reasoning.
3. Measure and analyze the side length ratios of the triangles:
 $\frac{KQ}{JK}$ to $\frac{LP}{JL}$, $\frac{KQ}{JQ}$ to $\frac{LP}{JP}$, $\frac{JK}{JQ}$ to $\frac{JL}{JP}$.

Constant Ratios in Right Triangles

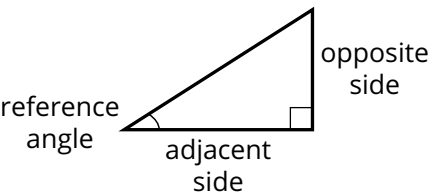
Triangle ABC shown is a 45° - 45° - 90° triangle.

1. Choose any point along the hypotenuse of $\triangle ABC$ and label it point D . Then construct a vertical line segment, $\overline{DE}$, connecting with side $\overline{AC}$ so that $\overline{DE} \perp \overline{AC}$. Label the other endpoint as point E .



2. Determine all of the side length ratios and corresponding angle measures for the two triangles. Explain how you know that $\triangle ABC$ is similar to $\triangle ADE$.

You know that the hypotenuse of a right triangle is the opposite side of the right angle. In trigonometry, the legs of a right triangle are often referred to as the *opposite side* and the *adjacent side*. These references are based on the angle of the triangle that you are considering, which is called the **reference angle**. The **opposite side** is the opposite side of the reference angle. The **adjacent side** is the adjacent side of the reference angle that is *not* the hypotenuse.



3. For $\triangle ABC$ and $\triangle ADE$, identify the opposite side, adjacent side, and hypotenuse, using $\angle A$ as the reference angle.

4. Record the side length measurements for both triangles in the table.

Triangle Name	Length of Opposite Side of $\angle A$	Length of Adjacent Side of $\angle A$	Length of Hypotenuse
$\triangle ABC$			
$\triangle ADE$			

5. Determine each side length ratio using $\angle A$ as the reference angle.

Triangle Name	$\frac{\text{Opposite Side of } \angle A}{\text{Hypotenuse}}$	$\frac{\text{Adjacent Side of } \angle A}{\text{Hypotenuse}}$	$\frac{\text{Opposite Side of } \angle A}{\text{Adjacent Side of } \angle A}$
$\triangle ABC$			
$\triangle ADE$			

6. Explain why Nakota is correct.

Nakota



The side length ratios of the opposite side to the hypotenuse or the adjacent side to the hypotenuse is a percent. If the ratio is approximately 0.70, that means the length of the side is about 70% the length of the hypotenuse.

Let's analyze the recorded measurements.

7. Compare the side length ratios of the triangles from your table. What do you notice?

Ask Yourself ...

What patterns do you see in the measurements you recorded?

8. Compare your measurements and ratios with those of your classmates. What do you notice?

Given the same reference angle measure, are each of the ratios you studied constant in similar right triangles? You can investigate this question by analyzing similar right triangles without side measurements.

WORKED EXAMPLE

Consider $\triangle ABC$ and $\triangle ADE$ shown. They are both 45° - 45° - 90° triangles.

$$\frac{\text{leg length of } \triangle ADE}{\text{hypotenuse length of } \triangle ABC}$$

Triangle ABC is similar to $\triangle ADE$ by the AA similarity theorem.

Therefore, the lengths of the corresponding sides are proportional.

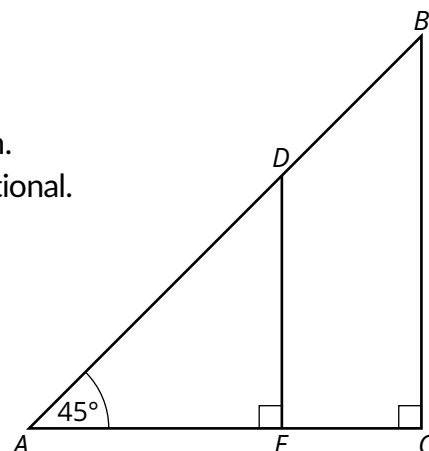
$$\frac{AE}{AC} = \frac{AD}{AB}$$

You can rewrite the proportion.

length of adjacent side of $\angle A$

$$\frac{AE}{AD} = \frac{AC}{AB}$$

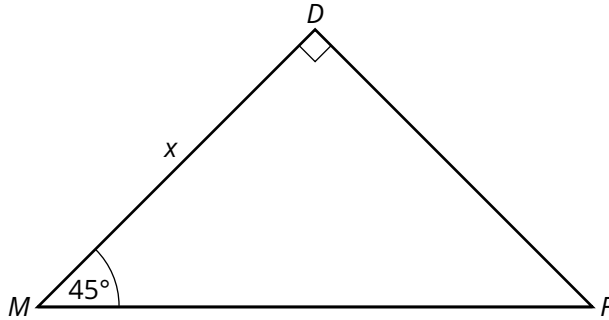
length of hypotenuse



So, given the same reference angle measure, the ratio $\frac{\text{length of adjacent side of reference angle}}{\text{length of hypotenuse}}$ is constant in similar right triangles.

9. Use $\triangle ABC$ in the Worked Example with reference $\angle A$ to verify that the side length ratios $\frac{\text{opposite side of reference angle}}{\text{hypotenuse}}$ and $\frac{\text{opposite side of reference angle}}{\text{adjacent side of reference angle}}$ are constant in similar right triangles. Show your work.

You have estimated certain constant ratios of side lengths in 45° - 45° - 90° triangles. You can use what you know about the 45° - 45° - 90° triangle theorem to determine the exact ratios.



10. Given a leg length x , determine the lengths of the other sides of the 45° - 45° - 90° triangle. Label the triangle.

11. Determine the ratio of the:

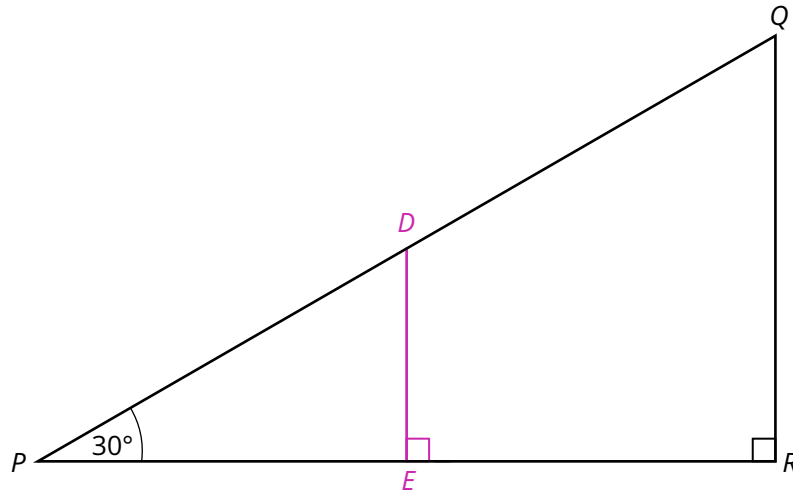
a. opposite side length to the length of the hypotenuse.

b. adjacent side length to the length of the hypotenuse.

c. opposite side length to the adjacent side length.

Ratios in 30°-60°-90° Triangles

Triangle PQR shown is a 30°-60°-90° triangle.



1. Choose any point along the hypotenuse of $\triangle PQR$ and label it point D . Then, construct a vertical line segment, $\overline{DE}$, connecting with side $\overline{PR}$ so that $\overline{DE} \perp \overline{PR}$. Label the other endpoint as point E .
2. Measure each of the sides of the similar right triangles in millimeters. Record the side length measurements in the table.

Triangle Name	Length of Opposite Side of $\angle P$	Length of Adjacent Side of $\angle P$	Length of Hypotenuse
$\triangle PQR$			
$\triangle PDE$			

3. Determine each side length ratio for the triangles using $\angle P$ as the reference angle.

Triangle Name	$\frac{\text{Opposite Side of } \angle P}{\text{Hypotenuse}}$	$\frac{\text{Adjacent Side of } \angle P}{\text{Hypotenuse}}$	$\frac{\text{Opposite Side of } \angle P}{\text{Adjacent side of } \angle P}$
$\triangle PQR$			
$\triangle PDE$			

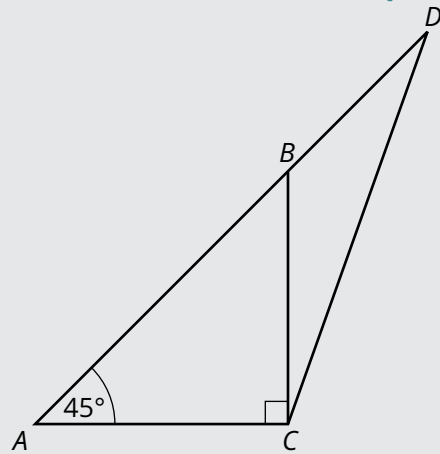
Let's analyze the recorded measurements.

4. Compare the side length ratios of the triangles in your table.
What do you notice?
5. What conclusions can you draw about the three ratios you studied in 45° - 45° - 90° triangles and 30° - 60° - 90° triangles?
6. What conclusions can you draw about these three ratios for any right triangles with congruent reference angles? Explain your reasoning.

7. Explain why Alicia is incorrect.

Juliana

The ratio $\frac{BC}{AB}$ is equal to the ratio $\frac{DC}{AD}$, because the ratio $\frac{\text{opposite side of } \angle A}{\text{hypotenuse}}$ is the same for both $\triangle ABC$ and $\triangle ADC$, given the reference angle A , which is 45° .

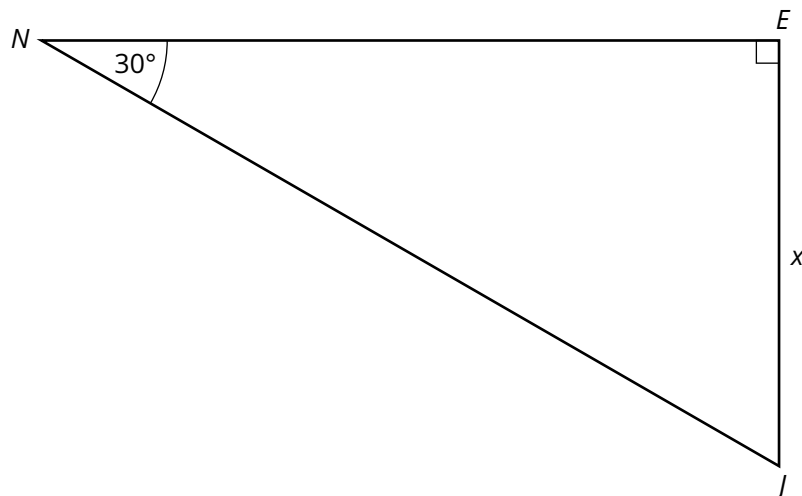


Ask Yourself . . .

Did you justify your mathematical reasoning?

8. Is each of the three ratios you studied in this lesson the same for any triangles with congruent reference angles? Explain your reasoning.

You have estimated certain constant ratios of side lengths in 30° - 60° - 90° triangles. You can use what you know about the 30° - 60° - 90° triangle theorem to determine the exact ratios.



9. Given a length of the shortest side, x , determine the lengths of the other sides of the 30° - 60° - 90° triangle. Label the triangle.

.....
Think About...
Don't forget to
rationalize the
denominators.
.....

10. Determine the ratio of the:
- a. opposite side length to the length of the hypotenuse.
 - b. adjacent side length to the length of the hypotenuse.
 - c. opposite side length to the adjacent side length.

The three ratios you worked with in this lesson are very important to trigonometry and have special names and properties. You will learn more about these ratios in the next several lessons.



Talk the Talk

To Each Reference Angle, a Ratio

You have investigated and verified algebraically the three different side length ratios for 45° - 45° - 90° and 30° - 60° - 90° triangles. Consider how those ratios will change when the reference angle changes.

1. As the reference angle measure increases, what happens to each side length ratio? Explain your reasoning.

.....

In each ratio, the terms *opposite side*, *adjacent side*, and *hypotenuse* refer to the lengths of sides.

.....

a. $\frac{\text{opposite side}}{\text{hypotenuse}}$

b. $\frac{\text{adjacent side}}{\text{hypotenuse}}$

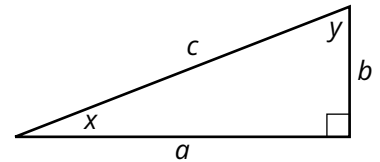
c. $\frac{\text{opposite side}}{\text{adjacent side}}$

Lesson 1 Assignment

Write

Use the diagram to complete each sentence.

1. When b is the opposite side, then x is the _____.
2. When y is the reference angle, then b is the _____.
3. When x is the reference angle, then b is the _____.



Remember

Given the same reference angle for similar right triangles, the side length ratios $\frac{\text{opposite side}}{\text{hypotenuse}}$, $\frac{\text{adjacent side}}{\text{hypotenuse}}$, and $\frac{\text{opposite side}}{\text{adjacent side}}$ are constant.

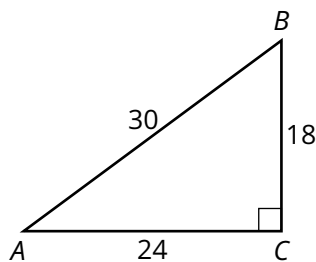
The side length ratios in right triangles with congruent reference angles are equal.

Practice

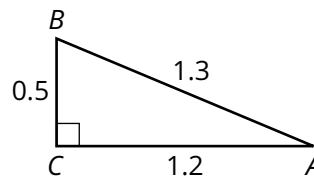
1. Determine the side length ratios $\frac{\text{opposite side}}{\text{hypotenuse}}$, $\frac{\text{adjacent side}}{\text{hypotenuse}}$, and $\frac{\text{opposite side}}{\text{adjacent side}}$ using $\angle A$ as the reference angle in each triangle.

Write your answers as fractions in simplest form.

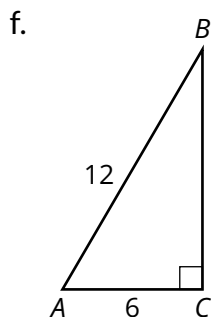
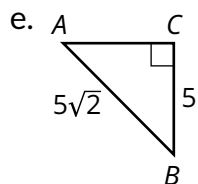
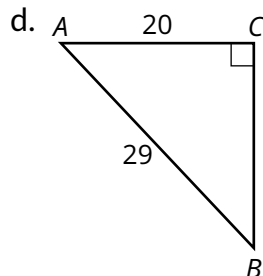
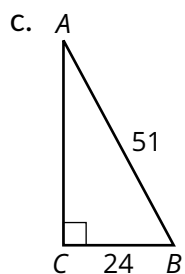
a.



b.

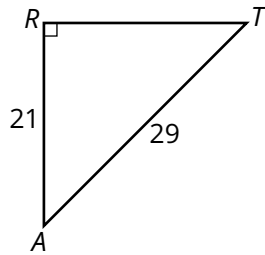


Lesson 1 Assignment



Lesson 1 Assignment

Prepare



1. Determine the length of $\overline{RT}$.
2. Write a ratio to compare the length of the opposite side of $\angle A$ to the length of the adjacent side of $\angle A$.
3. Write a ratio to compare the length of the adjacent side of $\angle A$ to the length of the opposite side of $\angle A$.
4. What is the difference between the ratios you wrote in Question 2 and Question 3?

2

Tangent Ratio and Inverse Tangent

OBJECTIVES

- Define the relationship between the lengths of the opposite side and the adjacent side in a right triangle as the tangent ratio.
- Use the inverse tangent in a right triangle to solve for unknown angle measures.

KEY TERMS

- tangent ($\tan$)
- inverse tangent (arctangent)
- angle of elevation

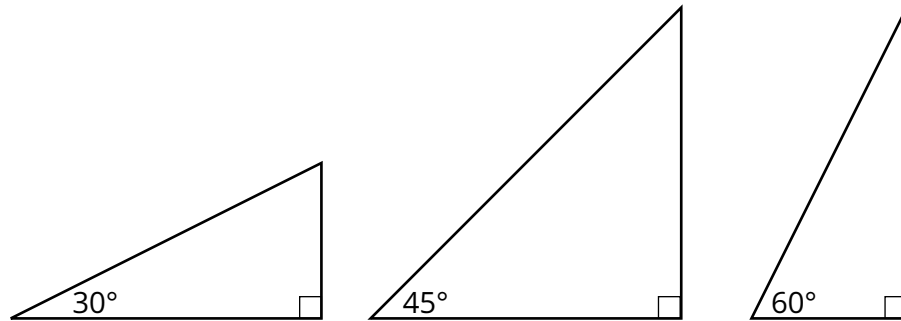
You have learned about constant ratios in similar triangles, given a specific reference angle.

How can you use the ratio of the length of the opposite side of the reference angle to the length of the adjacent side of the reference angle to solve for unknown measurements?

Getting Started

Steepness

Consider the right triangles shown.



1. Describe the steepness of a right triangle in terms of its side lengths.

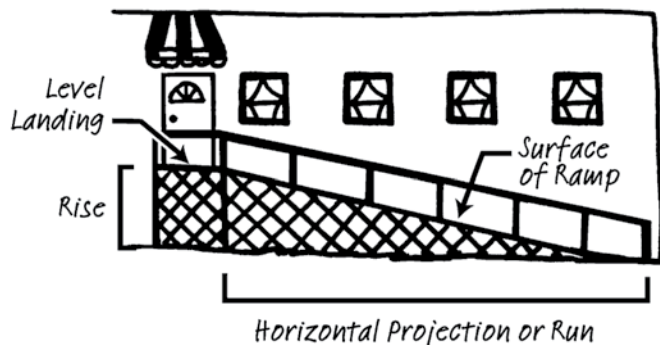
You can also use angle measures to describe a right triangle's steepness.

2. Compare the steepness of the triangles shown in terms of their given reference angle. What do you notice?

ACTIVITY 2.1

Connecting Slope and Tangent

The maximum incline for a safe wheelchair ramp should not exceed a ratio of 1 : 12. This means that every 1 unit of vertical rise requires 12 units of horizontal run. The maximum rise for any run is 30 inches. The ability to manage the incline of the ramp is related to both its steepness and its length.



.....

The Americans with Disabilities Act (ADA) specifies the maximum incline for a safe wheelchair ramp. President George H.W. Bush signed the Americans with Disabilities Act (ADA) into law on July 26, 1990.

.....

Omar decides to build 2 ramps, each with the ratio 1 : 12.

1. The first ramp extends from the front yard to the front porch. The vertical rise from the yard to the porch is 2.5 feet.
 - a. Draw a diagram of the ramp. Include the measurements for the vertical rise and horizontal run of the ramp.
 - b. Calculate the length of the surface of the ramp.

2. The second ramp extends from the deck on the back of the house to the backyard. The vertical rise from the yard to the deck is 18 inches.
 - a. Draw a diagram of the ramp. Include the measurements for the vertical rise and horizontal run of the ramp.
 - b. Calculate the length of the surface of the ramp.
3. Compare the two ramps. Are the triangles similar? Explain your reasoning.
4. What does the ratio of the rise of the ramp to the run of the ramp represent?

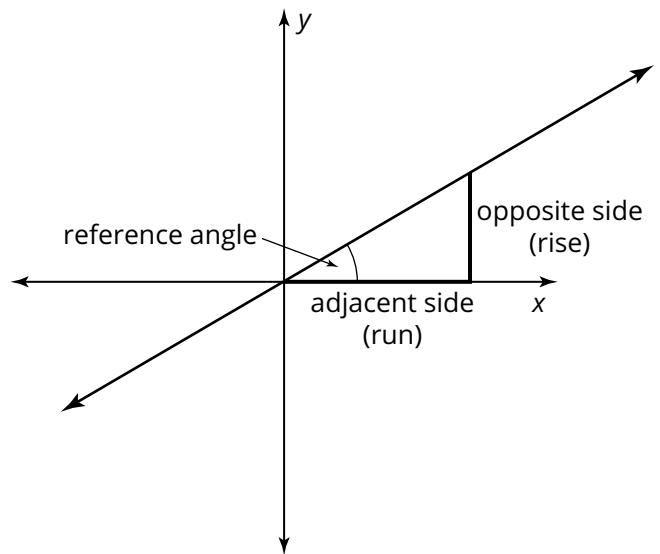
.....

An **angle of elevation** is the angle above the horizontal. In this case, you can think about it as the angle from the ground up to the top of the ramp.

.....

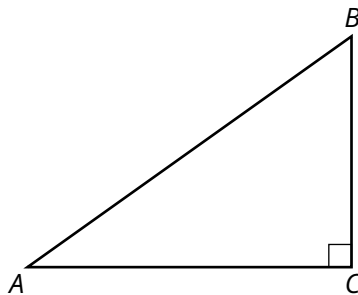
5. Compare and describe the angles of elevation of the two ramps.

The **tangent (tan)** of an acute angle in a right triangle is the ratio of the length of the opposite side of the reference angle to the length of the adjacent side of the reference angle. The expression “tan A” means “the tangent of $\angle A$.” The tangent of an angle can be used to describe the slope of a line.



WORKED EXAMPLE

Consider $\angle A$ in the right triangle shown.



The tangent ratio describes the relationship between $\angle A$, the opposite side of $\angle A$, and the adjacent side of $\angle A$.

$$\tan A = \frac{\text{length of the opposite side of } \angle A}{\text{length of the adjacent side of } \angle A} = \frac{BC}{AC}$$

Remember, the ratios of corresponding side lengths between two similar triangles are congruent. The tangent values of congruent angles in similar triangles are always the same.

.....

The tangent ratio is different from the tangent of a circle that you previously studied. The tangent of a circle is a line that touches a circle at one point. The tangent of an acute angle in a right triangle is a ratio. The tangent ratio in a right triangle uses the lengths of the opposite side and the adjacent side of an angle. The two sides come together, or touch, at the vertex of the right angle.

.....

6. Use the Worked Example to complete the ratio that represents the tangent of $\angle B$.

$$\tan B = \frac{\text{length of the opposite side of } \angle B}{\text{length of the adjacent side of } \angle B} = \frac{\boxed{}}{\boxed{}}$$

.....

When completing Question 6, first identify the reference angle, opposite side, adjacent side, and hypotenuse.

.....

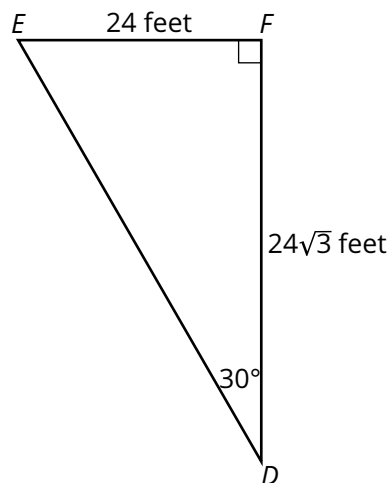
.....

Think About . . .

What happens to the tangent value of an angle as the measure of the angle increases up to 90° ?

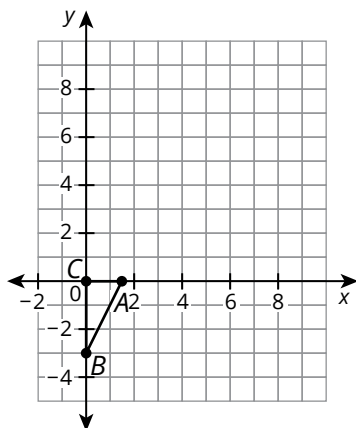
.....

7. Determine the tangent values of both acute angles in $\triangle DEF$. Leave your answers as exact values and rationalize the denominator.



The slope of a line is the tangent of the angle the line makes with the x-axis.

8. Consider the graph of the line $y = 2x - 3$. The x-intercept of the line is labeled as point A. The y-intercept of the line is labeled as point B. Point C is at the origin, creating a slope triangle.



- a. Determine the length of BC .
- b. Determine the length of AC .

- c. Use the slope triangle to determine the ratio of the vertical change (rise) to the horizontal change (run) between the two points on the line.
 - d. Which side of the slope triangle is the opposite side of $\angle A$?
 - e. Which side of the slope triangle is the adjacent side of $\angle A$?
 - f. Determine the tangent of $\angle A$.
 - g. Why are the slope of a line and the tangent of the angle the line makes with the x-axis the same?
9. Create another slope triangle on the graph by choosing any two points on the line and labeling the points as D and E . Label the point of the horizontal and vertical line you use to create the slope triangle as point F .

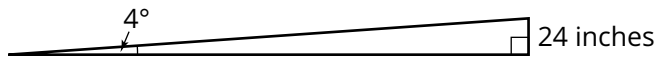
10. Is $\triangle ABC$ similar to $\triangle DEF$? Explain your reasoning.
11. Describe the relationship between $\angle BAC$ and $\angle EDF$. Explain your reasoning.
12. Determine the tangent of $\angle D$.
13. Explain why the tangent of an angle is the same no matter the size of the triangle.

ACTIVITY
2.2

Problem Solving with Tangents

You have used the Pythagorean Theorem to solve for an unknown side length in right triangles given two of the sides. In this activity, you will explore how you can use the tangent ratio to solve for unknown sides given an angle and a side length.

1. A proposed wheelchair ramp is shown.



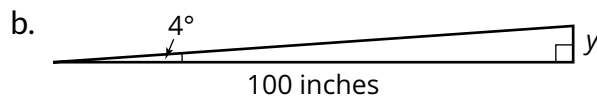
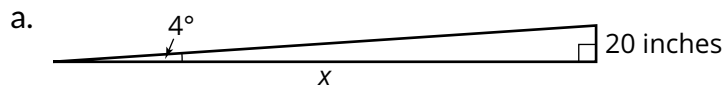
- What information about the ramp is required to show that the ramp meets the safety rules?
- Write a decimal that represents the greatest value of the slope of a safe ramp.
- Use technology to calculate the value of $\tan 4^\circ$ and use it to determine whether the ramp meets the safety rules. Round your answer to the nearest hundredth.
- Compute the run of the proposed ramp.

.....

Make sure that your calculator is calculating the tangent in *degrees* and not *radians*. Radians are a different unit of measure for angles that you will learn about in a future topic.

.....

2. Two proposed wheelchair ramps are shown. Determine the run of each ramp to meet safety regulations. When necessary, round your answer to the nearest inch.



3. When other ramps that have a 4° angle with different side measurements are drawn by extending or shortening the rise and run, will the tangent ratios always be equivalent? Explain your reasoning.

ACTIVITY 2.3

Inverse Tangent

The **inverse tangent** (or arctangent) of x is the measure of an acute angle whose tangent is x . If you know the length of any two sides of a right triangle, it is possible to compute the measure of either acute angle by using the inverse tangent, or the $\tan^{-1}$ button on a calculator.

In right triangle ABC , if $\tan A = x$, then $\tan^{-1} x = m\angle A$.

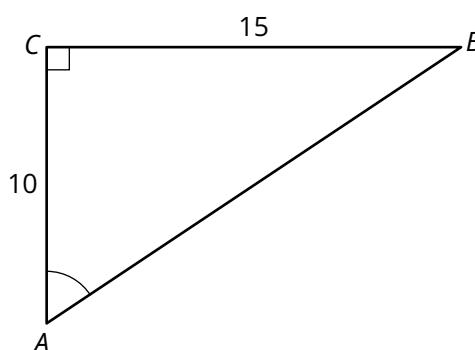
WORKED EXAMPLE

Consider $\triangle ABC$.

$$\tan A = \frac{15}{10}$$

You can use the inverse tangent to determine $m\angle A$.

$$\begin{aligned}\tan^{-1}(\tan A) &= \tan^{-1}\left(\frac{15}{10}\right) \\ A &\approx 56.31^\circ\end{aligned}$$



1. Consider $\triangle ABC$ from the Worked Example.
 - a. Determine the ratio for $\tan B$, and then use the inverse tangent to calculate $m\angle B$.
 - b. Add $m\angle A$ and $m\angle B$. Does your sum make sense in terms of the angle measures of a triangle?

PROBLEM SOLVING



For any operation, the inverse is the operation that undoes it. For example, division is the inverse operation of multiplication.

The inverse tangent undoes the tangent and takes you back to the angle.

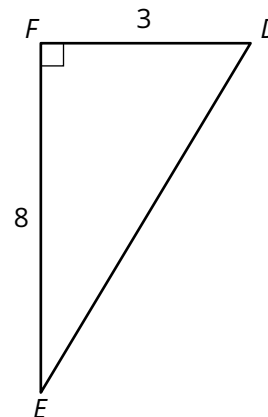
Arctangent is the function representation while $\tan^{-1}$ is the trigonometric representation of the inverse tangent.

Think About...

You are calculating the measure of $\angle B$ using the given information instead of using $\angle A$. Why is this important?

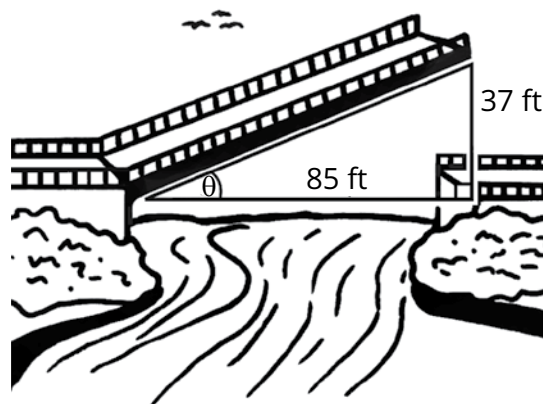
.....
Remember ...
 you must use the
 degree symbol
 when solving
 for an unknown
 angle measure.

2. Calculate $m\angle E$.

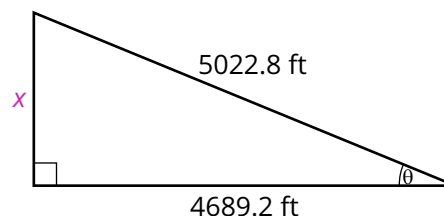


3. Movable bridges are designed to open water ways for large boats and barges. When the bridge moves, all vehicle traffic stops. The maximum height of the open bridge deck of the movable bridge shown is 37 feet above the water surface. The waterway width is 85 feet. Calculate the angle measure formed by the movement of the bridge.

.....
 The Greek letter
 "theta" (θ) is
 commonly used to
 represent an angle
 in a triangle. It
 acts as a variable
 and represents an
 unknown angle.



4. A ski slope at a ski resort has these measurements. Determine the angle of elevation of the ski slope. Show your work.



Talk the Talk

Road Grades

Many mountainous areas have road signs like the sign shown that refer to the percentage grade for the road. An 8% grade, for example, means that the altitude changes by 8 feet for each 100 feet of horizontal distance.



1. What function would be most helpful to determine the percentage grade that should be put on a road sign where the angle of elevation of the road is 9° ?
2. What function would be most helpful to determine the angle of elevation of a road with a percentage grade of 7%?
3. Determine the angle of elevation of a road with a percentage grade of 6%.

Ask Yourself . . .

How can you use inverse tangents in everyday life?

4. Determine the percentage road grade that should be put on a road sign where the angle of elevation is 10° .

.....

Review the definition of **represent** in the Academic Glossary.

.....

5. Does the image in the sign accurately represent an 8% grade? Explain how you can determine the answer.

6. What is the approximate angle of elevation that is actually shown in the sign?

Lesson 2 Assignment

Write

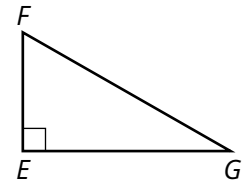
Match each description to its corresponding term for $\triangle EFG$.

1. $\frac{EF}{EG}$ relation to $\angle G$

a. tangent

2. $\tan^{-1}\left(\frac{EF}{EG}\right)$ in relation to $\angle G$

b. inverse tangent



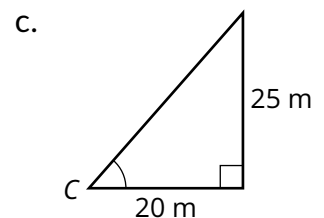
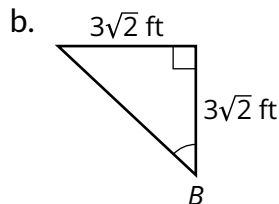
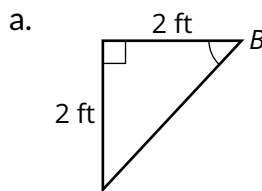
Remember

The tangent (tan) of an acute angle in a right triangle is the ratio of the length of the opposite side of the angle to the length of the adjacent side of the angle.

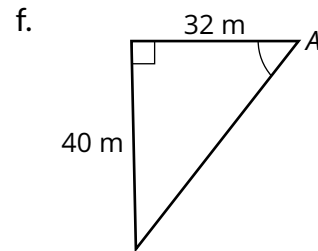
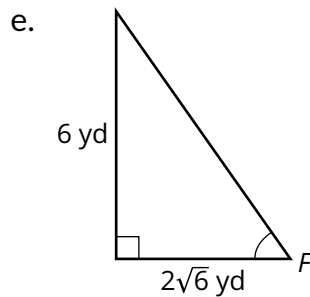
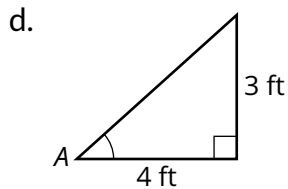
The inverse tangent (or arctangent) of x is the measure of an acute angle whose tangent is x .

Practice

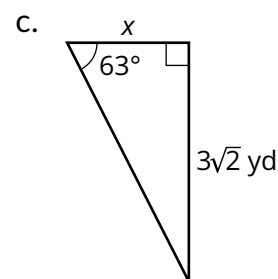
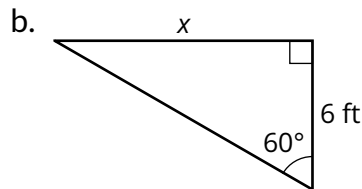
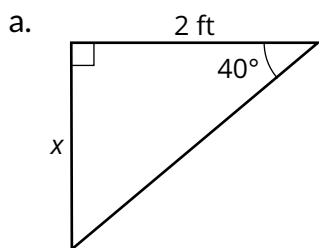
1. Calculate the tangent of the indicated angle in each triangle. Write your answers in simplest form.



Lesson 2 Assignment



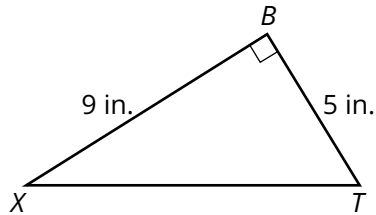
2. Use a tangent ratio to calculate the unknown length of each triangle. Round your answers to the nearest hundredth.



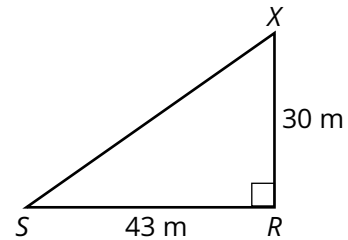
Lesson 2 Assignment

3. Calculate the measure of $\angle X$ in each triangle. Round your answers to the nearest hundredth.

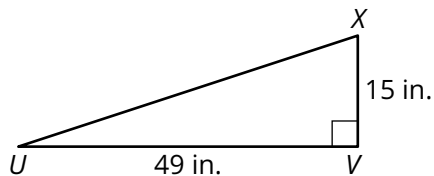
a.



b.

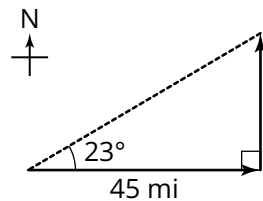


c.



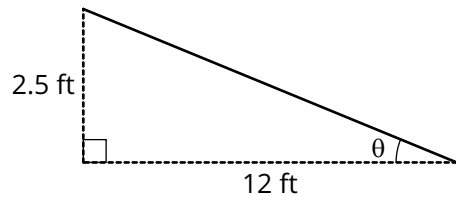
4. Solve each problem. Round your answers to the nearest hundredth.

- a. A boat travels in the following path. How far north did it travel?



Lesson 2 Assignment

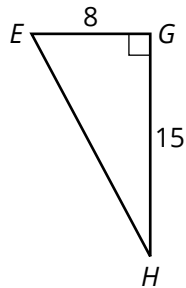
- b. A moving truck is equipped with a ramp that extends from the back of the truck to the ground. When the ramp is fully extended, it touches the ground 12 feet from the back of the truck. The height of the ramp is 2.5 feet. Calculate the measure of the angle formed by the ramp and the ground.



Lesson 2 Assignment

Prepare

Consider $\triangle EGH$.



1. What is the length of the hypotenuse?

The hypotenuse is 17.

2. Determine each length.
 - a. What is the length of the opposite side of $\angle E$?
 - b. What is the length of the adjacent side of $\angle E$?
 - c. What is the length of the opposite side of $\angle H$?
 - d. What is the length of the adjacent side of $\angle H$?

3

Exploring Sine and Cosine Ratios

OBJECTIVES

- Define the relationship between the length of the opposite side and the length of the hypotenuse in a right triangle as the sine ratio.
- Use the sine ratio in a right triangle to solve for unknown side lengths by reading and analyzing real-world problems.
- Define the relationship between the length of the adjacent side and the length of the hypotenuse in a right triangle as the cosine ratio.
- Use the inverse sine and inverse cosine in a right triangle to solve for unknown angle measures by reading and analyzing real-world problems.

KEY TERMS

- sine (sin)
- inverse sine (arcsine)
- cosine (cos)
- inverse cosine (arccosine)

.....

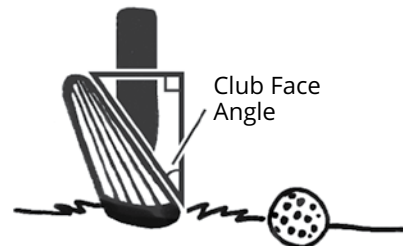
You have used the ratio between the lengths of the opposite side and adjacent side of an acute angle in a right triangle to solve for unknown lengths and angles.

How can you use the ratio of the length of the opposite side of the acute angle to the hypotenuse length and the ratio of the adjacent side of the acute angle to the hypotenuse length of a right triangle to solve for unknown measurements?

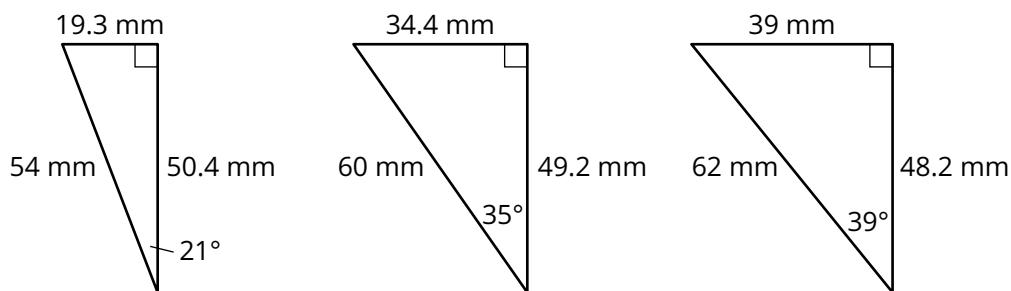
Getting Started

Fore!

Each golf club in a set of clubs is designed to cause the ball to travel different distances and different heights. One design element of a golf club is the angle of the club face.



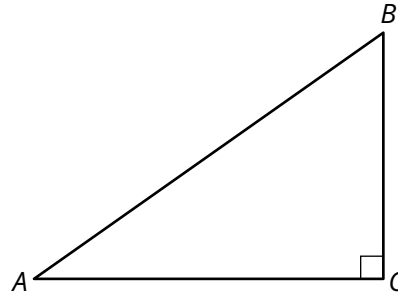
You can draw a right triangle that is formed by the club face angle. The right triangles formed by different club face angles are shown.



1. How do you think the club face angle affects the path of the ball?
2. For each club face angle, write the ratio of the side length opposite the given acute angle to the length of the hypotenuse. Write your answers as decimals rounded to the nearest hundredth.
3. What happens to this ratio as the angle measure gets larger?

The Sine Ratio

The **sine (sin)** of an acute angle in a right triangle is the ratio of the length of the opposite side of the angle to the length of the hypotenuse.



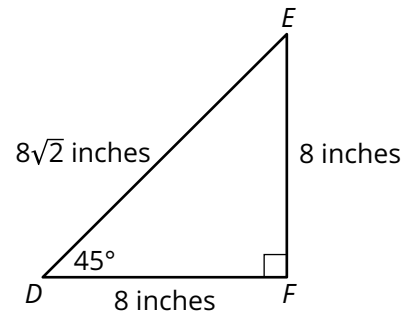
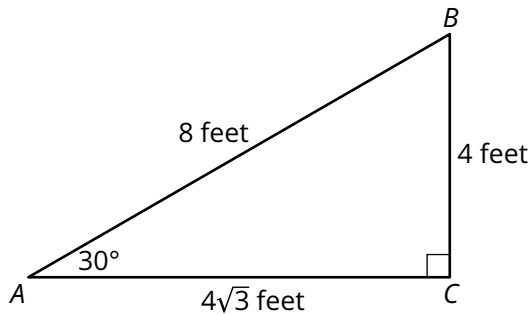
.....
The expression
"sin A" means
"the sine of $\angle A$."
.....

1. Complete the ratio that represents the sine of $\angle A$.

$$\sin A = \frac{\text{length of opposite side of } \angle A}{\text{length of hypotenuse}} = \frac{\boxed{}}{\boxed{}}$$

2. For each triangle in the Getting Started, you calculated the sine value of the club face angle. Calculate the sine value of the other acute angle in each triangle. Round your answers to the nearest hundredth.

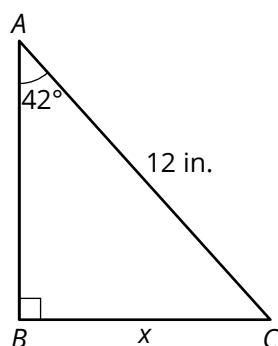
3. What do the sine values of the angles in Question 2 all have in common?
4. What happens to the sine values of an angle as the measure of the angle increases?
5. You calculated the sine ratios for different club faces in the Getting Started and in Question 2. Consider a golf club with a club face angle $\angle A$ for which $\sin A \approx 0.45$. Estimate the measure of the club face angle based on the decimal value of the ratio. Use a calculator to verify your answer.
6. Use the right triangles shown to calculate the values of $\sin 30^\circ$, $\sin 45^\circ$, and $\sin 60^\circ$. Leave your answers as exact values and rationalize the denominator.



WORKED EXAMPLE

You can use the sine ratio to determine a missing side length.

Determine the value of x .



Step 1: Use the acute angle of the triangle and write a sine ratio that includes the missing side length x .

$$\sin 42^\circ = \frac{x}{12}$$

Step 2: Use the multiplication property of equality to solve for x .

$$(\sin 42^\circ)(12) = x$$

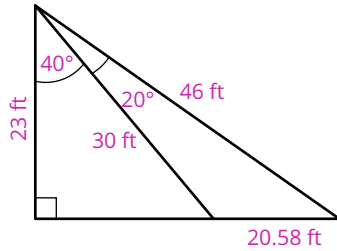
Step 3: Use technology to determine the value of $\sin 42^\circ$.

$$(0.6691)(12) = x$$

$$x \approx 8.03 \text{ in.}$$

7. Use the Pythagorean Theorem to determine AB .

8. Two cables supporting the center pole of a circus tent are both connected at the top of the pole and are staked into the ground several feet apart. The length of the first cable is 30 feet and the length of the second cable is 46 feet. The angle formed by the pole and the first cable is 40° . The angle formed by the pole and the second cable is 60° . Label the diagram. Then, calculate the height of the center pole and the distance between the two stakes.

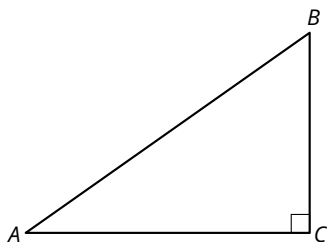


Inverse Sine

The **inverse sine** (or arcsine) of x is the measure of an acute angle whose sine is x . When you know the length of any two sides of a right triangle, it is possible to calculate the measure of either acute angle by using the inverse sine, or $\sin^{-1}$ button on a graphing calculator.

In right triangle ABC , if $\sin A = x$, then $\sin^{-1} x = m\angle A$.

1. In $\triangle ABC$, $\sin A = \frac{2}{5}$. Use the inverse sine to determine $m\angle A$.



2. Ben calculated the inverse sine of $\angle A$. Explain what Ben did incorrectly when calculating the inverse sine.

Ben

$$\sin A = \frac{2}{5}$$

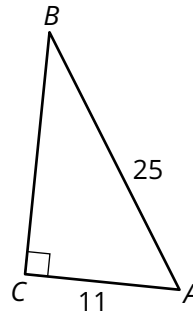
I can use the sin button on my calculator to determine $\sin\left(\frac{2}{5}\right) \approx 0.007$.

I can then take the inverse of 0.007 to determine $m\angle A$.

$$m\angle A \approx 143^\circ$$

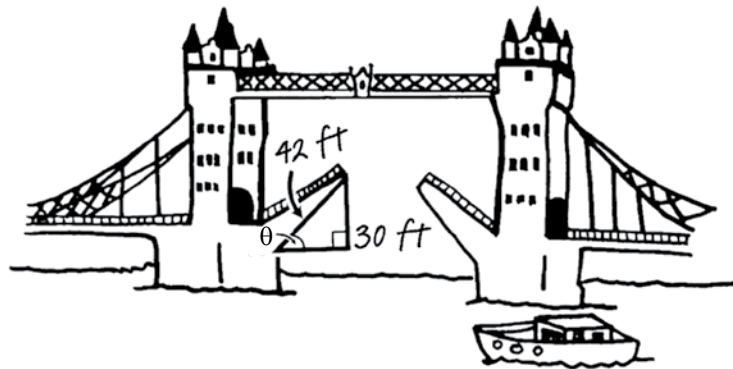


3. Determine the ratio for $\sin B$, and then use the inverse sine to calculate $m\angle B$.



4. The movable bridge below is called a double-leaf Bascule bridge. It has a counterweight that continuously balances the bridge deck, or “leaf,” throughout the entire upward swing, providing an open waterway for boat traffic. The counterweights on double-leaf bridges are usually located below the bridge decks.

The length of one leaf, or deck, is 42 feet. The maximum height of an open leaf is 30 feet. Calculate the measure of the angle formed by the movement of the bridge.



.....

The first movable bridge in the United States was built in 1834 in Chicago, Illinois.

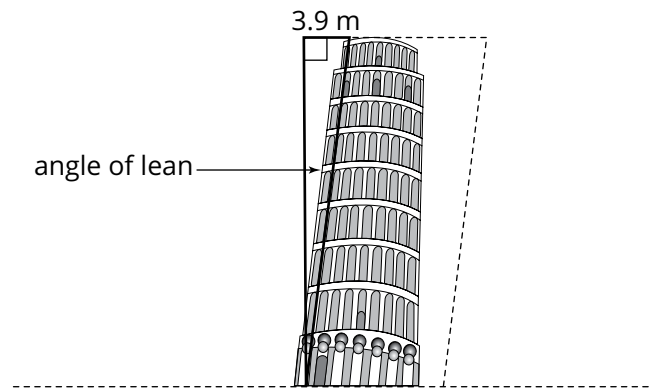
.....

5. The Leaning Tower of Pisa is a tourist attraction in Italy. It was built on unstable land, and as a result, it really does lean!

The height of the tower is approximately 55.86 meters from the ground on the low side and 56.7 meters from the ground on the high side. The top of the tower is displaced horizontally 3.9 meters as shown. Determine the angle at which the tower leans.

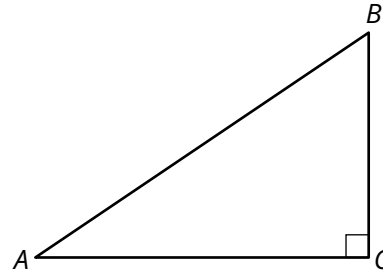
Ask Yourself . . .

Did you make a plan to solve the problem?



.....
The expression “cos A”
means “the cosine
of ∠A.”
.....

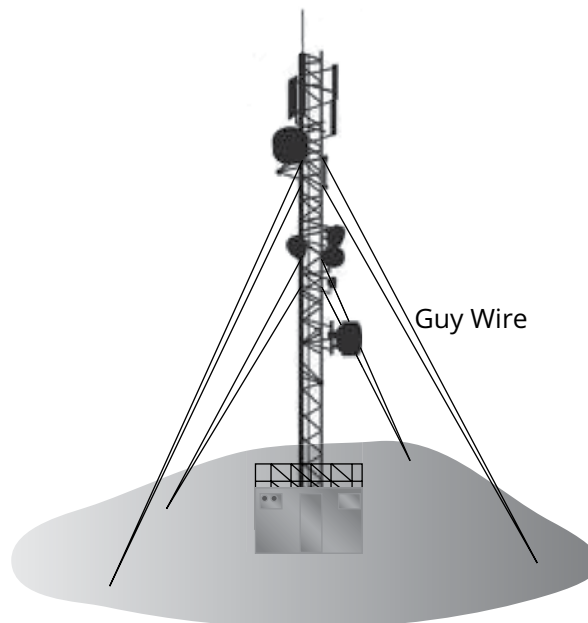
The **cosine (cos)** of an acute angle in a right triangle is the ratio of the length of the adjacent side of the angle to the length of the hypotenuse.



1. Complete the ratio that represents the sine of ∠A.

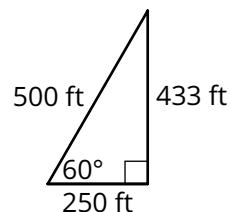
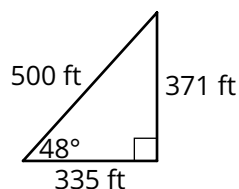
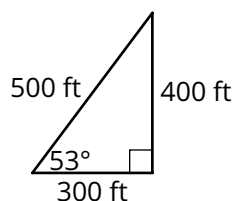
$$\cos A = \frac{\text{length of adjacent side of } \angle A}{\text{length of hypotenuse}} = \frac{\boxed{}}{\boxed{}}$$

Radio masts and radio towers are both tall human-made structures. The difference between a mast and a tower is that a mast has guy wires attached to it to provide stability. Guy wires are attached near the top of a mast and are attached to the ground. Radio masts are usually found in more open areas where there is room for the guy wires.



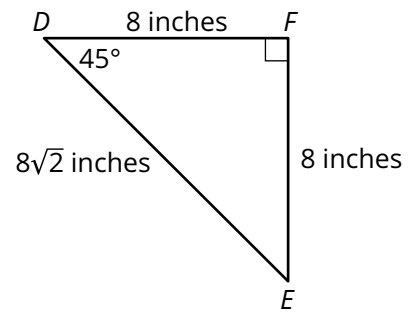
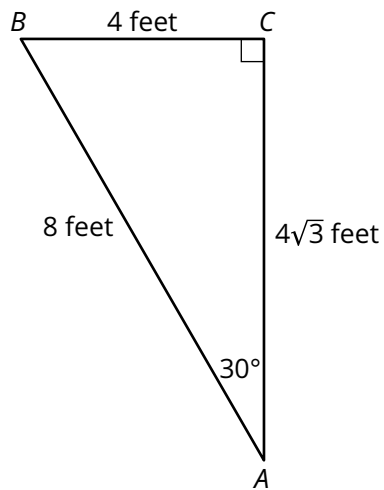
A guy wire and its mast form a right triangle. It is important that all pairs of guy wires form congruent triangles so that the tension on each wire is the same.

Each triangle shown represents the triangle formed by a mast and guy wire. The angle formed by the wire and the ground is given in each triangle.

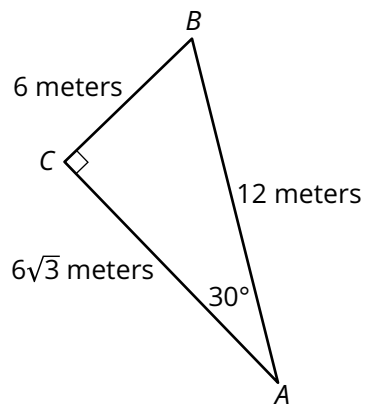


2. For each acute angle formed by the wire and the ground, write the ratio of the adjacent side length of the angle to the hypotenuse length. Write your answers as decimals rounded to the nearest hundredth, when necessary.
3. What do the cosine values of the angles in Question 2 all have in common?
4. Is the cosine value of every acute angle less than 1? Explain your reasoning.
5. What happens to the cosine value of an angle as the measure of the angle increases?

6. Use the right triangles below to calculate the values of $\cos 30^\circ$, $\cos 45^\circ$, and $\cos 60^\circ$. Leave your answers as exact values and rationalize the denominator.



7. Consider the triangle shown. Leave your answers as exact values and rationalize the denominator.



- a. Calculate the values of $\sin 30^\circ$, $\cos 30^\circ$, and $\tan 30^\circ$.

b. Calculate the value of $\frac{\sin 30^\circ}{\cos 30^\circ}$.

c. What do you notice about the value of $\frac{\sin 30^\circ}{\cos 30^\circ}$?

8. Do you think that the relationship between the sine, cosine, and tangent values of an angle is true for any angle? Explain your reasoning.

WORKED EXAMPLE

You can also use the cosine ratio to determine a missing side length.

Determine the value of x .

Step 1: Use the acute angle to write a cosine ratio that includes the missing side length x .

$$\cos 23^\circ = \frac{18}{x}$$

Step 2: Use the multiplication property of equality multiplying each side by x .

$$(\cos 23^\circ)(x) = 18$$

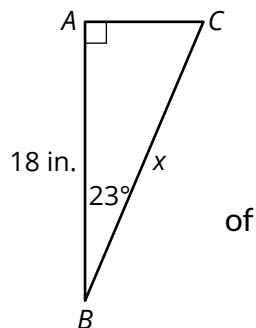
Step 3: Use the division property of equality to solve for x .

$$x = \frac{18}{\cos 23^\circ}$$

Use technology to determine the value of $\cos 23^\circ$.

$$x = \frac{18}{0.9205}$$

$$x \approx 19.6 \text{ in.}$$



9. Use the Pythagorean Theorem to determine AC .
10. A guy wire is 600 feet long and forms a 55° angle with the ground. First, draw a diagram of this situation. Then, calculate the number of feet from the tower's base to where the wire is attached to the ground.

Inverse Cosine

The **inverse cosine** (or arccosine) of x is the measure of an acute angle whose cosine is x . When you know the length of any two sides of a right triangle, it is possible to compute the measure of either acute angle by using the inverse cosine, or $\cos^{-1}$ button on a graphing calculator.

In right triangle ABC , if $\cos A = x$, then $\cos^{-1} x = m\angle A$.

1. In right triangle ABC , if $\cos A = \frac{2}{7}$, use the inverse cosine to determine $m\angle A$.
2. Lucia draws acute angle QRS . She measures RS to be 3 inches and RQ to be 6 inches. She determines the measure of $\angle R$. What is the mistake in Lucia reasoning?

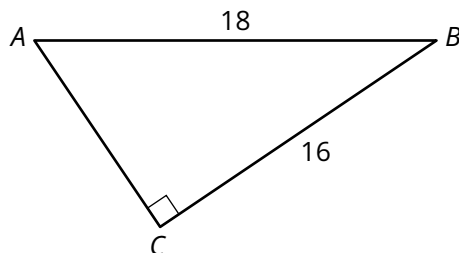
Lucia

$$\cos R = \frac{3}{6} = \frac{1}{2}$$

$$m\angle R = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$



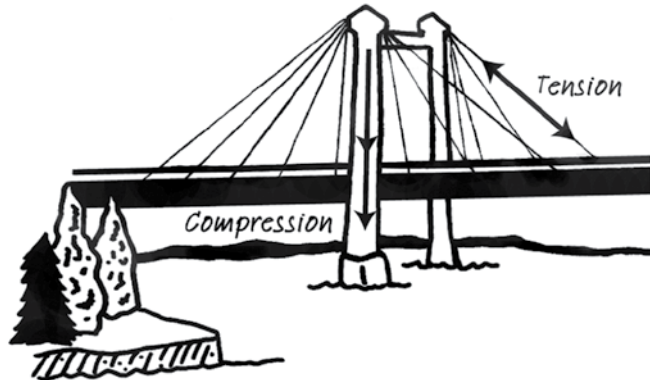
3. Use the inverse cosine to calculate $m\angle B$.



PROBLEM SOLVING



4. A typical cable-stayed bridge is a continuous girder with one or more towers erected above piers in the middle of the span. From these towers, cables stretch down diagonally (usually to both sides) and support the girder. Tension and compression are calculated into the design of this type of suspension bridge.



.....

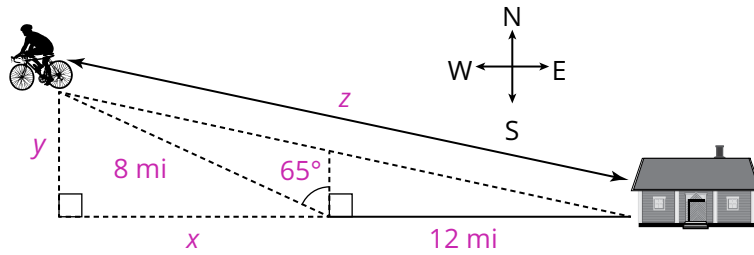
The Bluff Dale Bridge and the Barton Creek Bridge in Texas are early examples of cable-stayed bridges.

.....

One cable is 95 feet. The span on the deck of the bridge from that cable to the girder is 80 feet. Calculate the angle formed by the deck and the cable.



5. Nia is training for a charity bicycle marathon. She leaves her house at noon and heads due west, biking at an average rate of 4 miles per hour. At 3 PM she changes course to N 65° W as shown. Label the diagram and determine the bike's distance from Nia's home at 5 PM.





Cut out each diagram at the end of the lesson and glue it below the appropriate situation described. Then, calculate the unknown measurement.

1. An observer stands an unknown distance away from a building that is 80 feet tall. Looking up to the top of the building, he notes that the angle of elevation is 52° . Determine the distance from the base of the building to the observer.
2. An observer stands 62.5 feet away from a building that is 80 feet tall. Looking up to the top of the building, he ponders the measure of the angle of elevation. Determine the measure of the angle of elevation.

3. An observer stands an unknown distance away from a building. Looking up to the top of the building, he notes that the angle of elevation is 52° . He also knows the distance from where he is standing to the top of the building is 101.52 feet. Determine the height of the building.

4. An observer stands an unknown distance away from a building that is 80 feet tall. Looking up to the top of the building, he ponders the measure of the angle of elevation. He knows the distance from where he is standing to the top of the building is 101.52 feet. Determine the measure of the angle of elevation.

5. An observer stands 100 feet away from a building that is an unknown height. Looking up to the top of the building, he notes that the angle of elevation is 52° . Determine the distance from the observer to the top of the building.

Diagrams for Talk the Talk



Diagram A

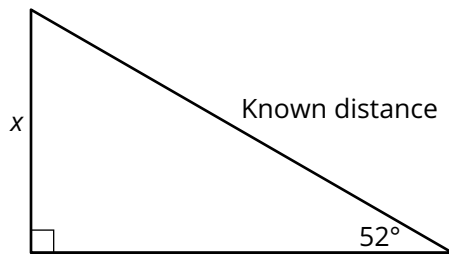


Diagram B

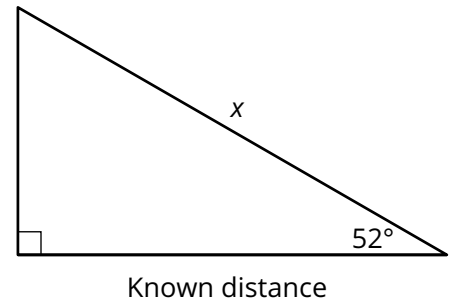


Diagram C

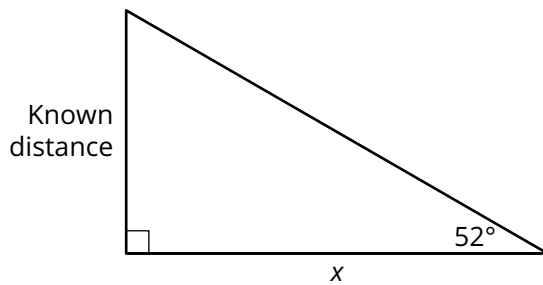


Diagram D

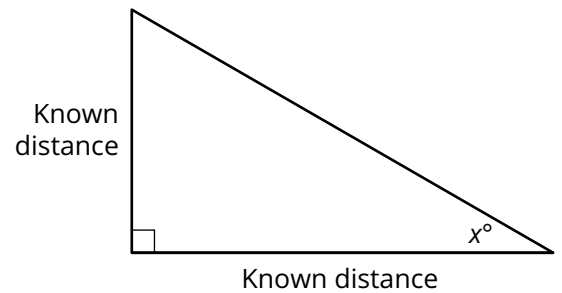
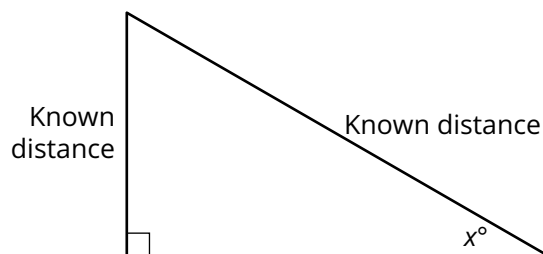


Diagram E



Why is this page blank?

So you can cut out the statements and reasons on the other side.

Lesson 3 Assignment

Write

1. Describe the similarities and differences between the *cosine ratio* and *sine ratio*.
2. Define the terms *inverse sine* and *inverse cosine* in your own words.

Remember

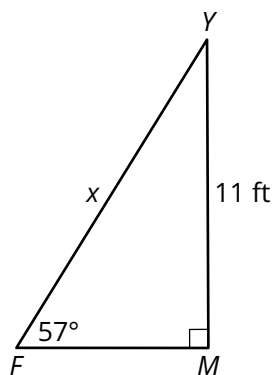
The sine of an acute angle in a right triangle is the ratio of the length of the opposite side of the angle to the length of the hypotenuse. The inverse sine of x , or $\sin^{-1} x$, can be used to determine the measure of an acute angle whose sine is x .

The cosine of an acute angle in a right triangle is the ratio of the length of the adjacent side of the angle to the length of the hypotenuse. The inverse cosine of x , or $\cos^{-1} x$, can be used to determine the measure of an acute angle whose cosine is x .

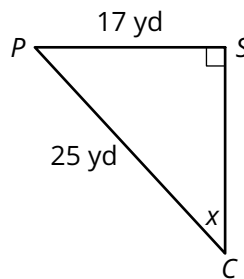
Practice

1. Use the sine ratio, or the inverse sine, to solve for x . Round each answer to the nearest tenth.

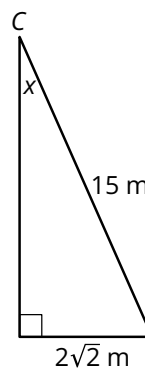
a.



b.

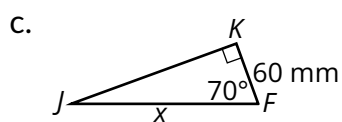
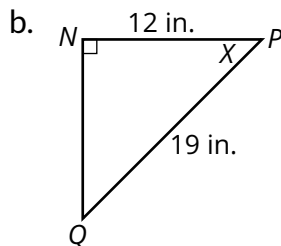
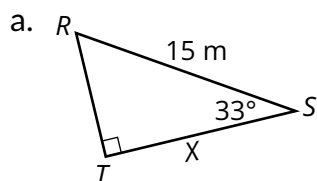


c.



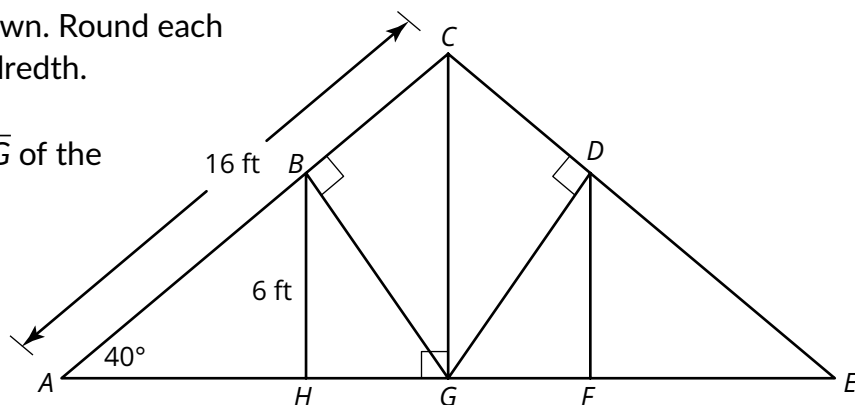
Lesson 3 Assignment

2. Use the cosine ratio or the inverse cosine to solve for x .
Round each answer to the nearest tenth.



3. Consider the roof truss shown. Round each answer to the nearest hundredth.

- a. Determine the height $\overline{CG}$ of the roof truss.



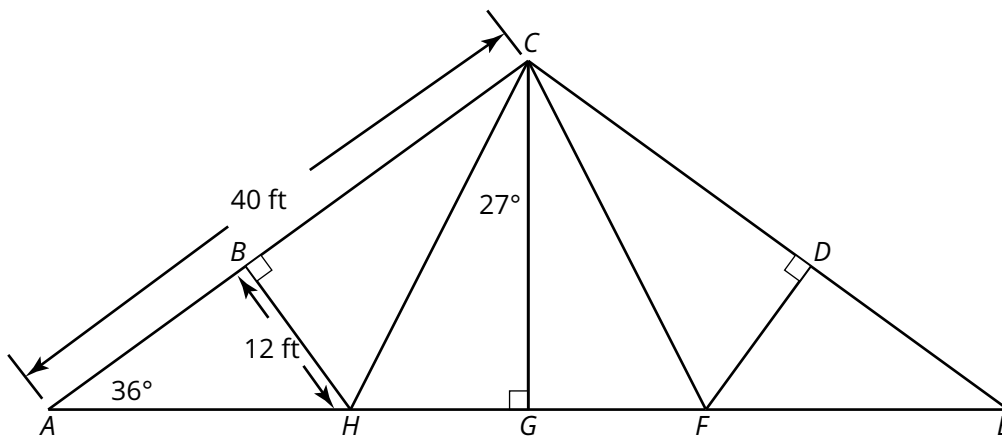
- b. Determine length $\overline{AB}$.

Lesson 3 Assignment

c. Determine the measure of $\angle BGC$.

d. Determine length $\overline{BG}$ of the support beam.

4. Consider the bridge shown. Use the figure and the fact that $\triangle AGC$ is congruent to $\triangle EGC$ to complete parts (a) through (e). Round each answer to the nearest tenth.



a. Determine the width $\overline{AE}$ of the bridge.

Lesson 3 Assignment

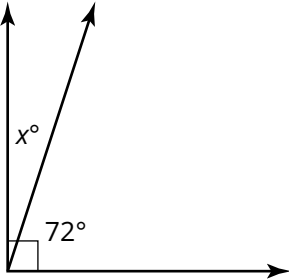
- b. Determine the height $\overline{CG}$ of the bridge.
- c. Determine the length of $\overline{CH}$.
- d. Determine the measure of $\angle BHC$.
- e. Does $\overline{CH}$ bisect $\angle ACG$? Explain your reasoning.

Lesson 3 Assignment

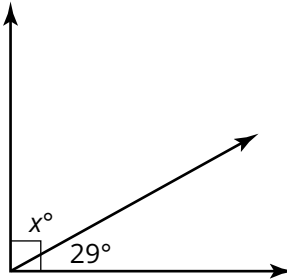
Prepare

Solve for each unknown angle measure.

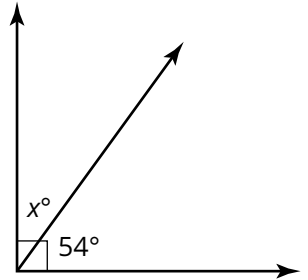
1.



2.



3.



4

Complement Angle Relationships

OBJECTIVES

- Explore complement angle relationships in a right triangle.
- Solve problems using complement angle relationships.
- Select strategies for determining an unknown side length or angle measure in a right triangle.

KEY TERM

- angle of depression

.....

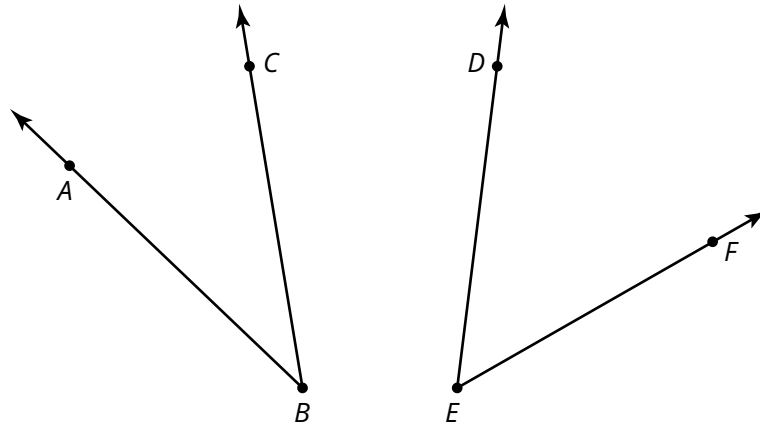
You have explored the ratios formed between two side lengths of a right triangle and how they relate to the acute angles within the triangle.

How do the trigonometric ratios of the two acute angles in a right triangle compare to one another?

Getting Started

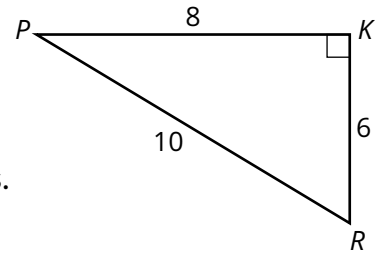
Angles Can Be Very Complementary!

Consider $\angle ABC$ and $\angle DEF$.



1. Use a protractor to determine the measure of each angle.
2. Trace $\angle ABC$ on a sheet of patty paper. Align $\overline{BC}$ and $\overline{ED}$, and then trace the ray from point B to point F on your patty paper to create $\angle ABF$.
3. What type of angle is $\angle ABF$? Justify your answer.

Angle ABC and angle DEF are complementary angles. The two acute angles in a right triangle are always complementary angles. Consider $\triangle PKR$.



4. Explain how you know that $\angle P$ and $\angle R$ are complementary angles.

5. Write a ratio that represents $\sin \angle P$.

.....
Remember ...

Two angles are complementary angles when the sum of the angle measures is equal to 90° .

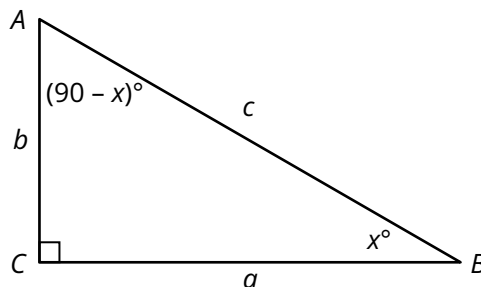
6. Write a ratio that represents $\cos \angle R$.

7. What do you notice about the ratios representing $\sin \angle P$ and $\cos \angle R$?

8. How does the ratio representing $\cos \angle P$ compare to the ratio representing the $\sin \angle R$?

Complementary Relationships

Consider $\triangle ABC$ with right angle C . Angles A and B are complementary angles because the sum of their measures is equal to 90° . The trigonometric ratios also have complementary relationships.



1. Use $\triangle ABC$ to complete the table.

Reference Angle	sin	cos
A		
B		

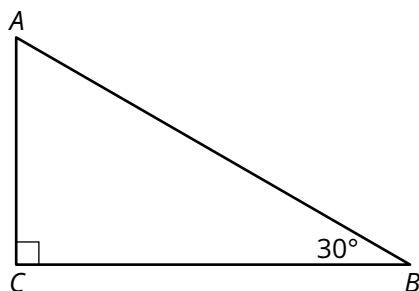
2. Summarize the relationship between the trigonometric functions of complementary angles.

ACTIVITY 4.2

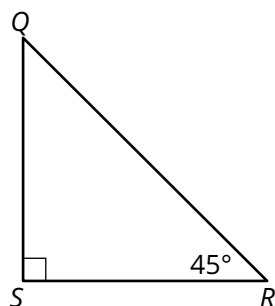
Using Complements

In this activity, you will investigate the relationships between complementary trigonometric functions for 30° - 60° - 90° and 45° - 45° - 90° triangles.

1. Use the complementary angle relationships, your knowledge of the side relationships of special right triangles, and the Pythagorean Theorem to complete the chart with the numeric ratios for each triangle.



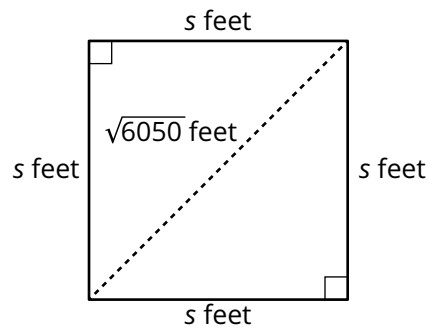
Reference Angle	sin	cos
30°		
60°		



Reference Angle	sin	cos
45°		

Ask Yourself ...
What patterns do you notice?

2. The base of the Washington Monument located in Washington, D.C. is a square. The monument was built to honor George Washington, the first U.S. president.



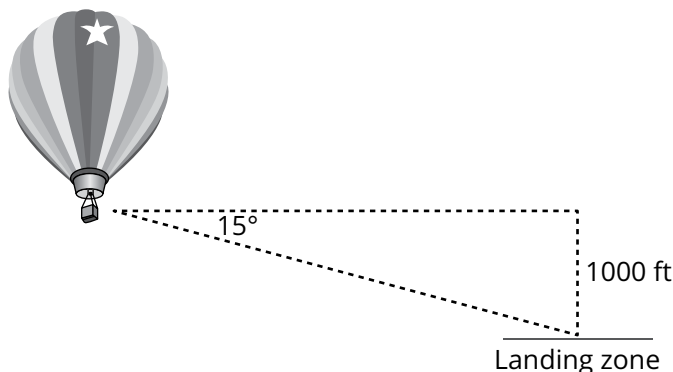
- a. Use a trigonometric function to solve for the dimensions of the base of the Washington Monument.
- b. Use the Pythagorean Theorem to verify the dimensions of the base of the Washington Monument.

ACTIVITY
4.3

Applying Trigonometric Ratios

In this activity, you will use your knowledge of trigonometric ratios to solve problems.

1. At an altitude of 1000 feet, a balloonist measures the angle of depression from the balloon to the landing zone. The measure of that angle is 15° . How far is the balloon from the landing zone?



2. An aircraft uses radar to spot another aircraft 8000 feet away at a 12° angle of depression. Sketch the situation, and determine the vertical and horizontal separation between the two aircraft.

.....

There is a yearly hot air balloon festival that takes place in Albuquerque, New Mexico.

.....

.....

An angle of depression is an angle below horizontal. In this case, you can think about it as the angle from skyline to the airplane with the lower altitude (height).

.....

.....

The Wright Brothers from Dayton, Ohio flew an airplane they built for 12 seconds above sand dunes in Kitty Hawk, North Carolina.

.....

Ask Yourself . . .

How can you organize and record your mathematical ideas?

.....
Lieutenant Joseph Christmas Ives and geologist John Strong Newberry conducted the first survey of the Grand Canyon. The survey, commissioned by the U.S. government, was intended to help the U.S. learn about the land they acquired at the end of the Mexican-American War in 1848. The Grand Canyon was established as a national park on February 26, 1919 by President Woodrow Wilson—two and a half years after he signed the National Park Service Act on August 25, 1916.

.....

Remember ...

The altitude of a right triangle drawn to its hypotenuse is the geometric mean between the measures of the lengths of the segments of the hypotenuse.

.....

The first national park ranger was Harry Yount. He worked as a gamekeeper at Yellowstone National Park in 1880.

.....

3. To measure the width of the Grand Canyon, a surveyor stands at a point on the North Rim of the canyon and measures the angle of depression to a point directly across on the South Rim of the canyon. The angle of depression is 19° .

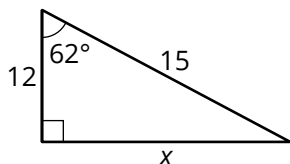
At the surveyor's position on the North Rim, the Grand Canyon is 7256 feet above sea level. The point on the South Rim, directly across, is 6159 feet above sea level. Sketch a diagram of the situation and determine the width of the Grand Canyon at the surveyor's position.

4. Antonia is camping in the woods. He notices on a map that his campsite is 50 yards from the ranger station. He walks 18 yards towards the ranger station before stopping to climb to the top of a tree to locate the station. The tree forms the altitude of a right triangle that connects the top of the tree, Antonia's campsite, and the ranger station. Sketch a diagram of the situation and determine the angle of depression formed by Antonia's sightline to the ranger station.

Making Choices



Jackson, Linh, and Nakota are each trying to determine the value of x .



Jackson

Jackson says he can use the cosine ratio to determine the value of x .

$$\cos 62^\circ = \frac{x}{15}$$

Linh

Linh says he can use the tangent ratio to determine the value of x .

$$\tan 62^\circ = \frac{x}{12}$$

Nakota

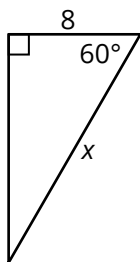
Nakota says she can use the Pythagorean Theorem to determine the value of x .

1. Who's correct? Explain your reasoning.

2. Determine the value of x .



Alejandra and Hailey are trying to determine the value of x .



Alejandra

Alejandra says there is only one way to determine the value of x since you are only given one side of the triangle.

Hailey

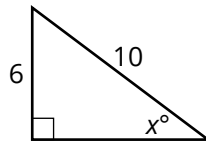
Hailey says there is more than one way to determine the value of x .

3. Who's correct? Explain your reasoning.

4. Determine the value of x .



Miguel and Briana are each trying to determine the value of x .



Miguel

Miguel says you can use the Pythagorean triple 6, 8, 10 to determine the value of x .

Briana

Briana says you can not use the Pythagorean Theorem or a Pythagorean triple to determine the value of x .

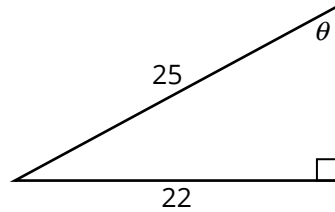
5. Who's correct? Explain your reasoning.

6. Determine the value of x .

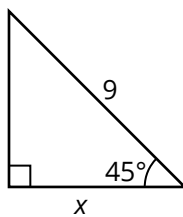
While there is often more than one way to determine an unknown side length or angle in a right triangle, one way is generally more efficient than another. You can use special right triangle theorems, the Pythagorean Theorem, Pythagorean triples and trigonometric ratios to determine unknown values.

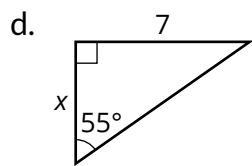
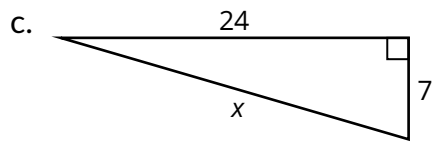
7. Choose a strategy to determine the unknown value in each right triangle. Explain your reasoning.

a.



b.



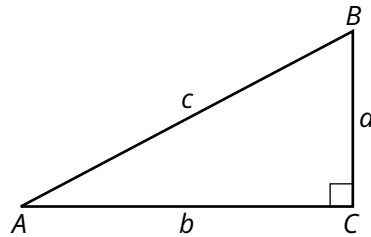




Talk the Talk

On a Need-to-Know Basis

Consider $\triangle ABC$.



1. Can you determine all of the unknown measures of any right triangle when you know just one of the measures? Explain your reasoning.
2. Xavier made the given statement. Give a counterexample to explain why Felix is incorrect.

Xavier



You need to know any two measures of a right triangle to determine all the unknown measures of the triangle.

3. When $c = 8$ and $m\angle A = 32^\circ$, determine the other measures of $\triangle ABC$.

Lesson 4 Assignment

Write

Describe the minimum side and angle measures you need to determine all the unknown measurements in a right triangle.

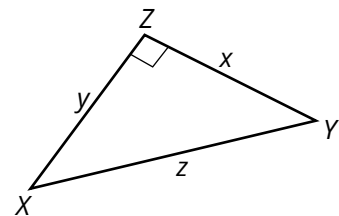
Remember

The sine of an angle is equal to the cosine of its complement.

Practice

1. Use triangle XYZ to complete the table of ratios.

Reference Angle	sin	cos
X		
Y		



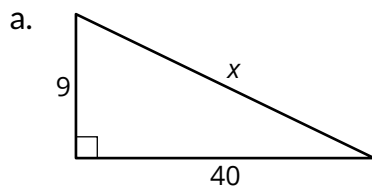
2. A pilot and co-pilot are performing a test run in a new airplane. The pilot is required to take off and fly in a straight path at an angle of elevation that is between 33° and 35° until the plane reaches an altitude of 10,000 feet. When the plane reaches 10,000 feet, the co-pilot will take over. Round each distance to the nearest tenth.
- a. Draw a figure to model this situation. Label the angle of elevation and the opposite side of the angle of elevation. Label the adjacent side of the angle of elevation as x and the hypotenuse as y .

Lesson 4 Assignment

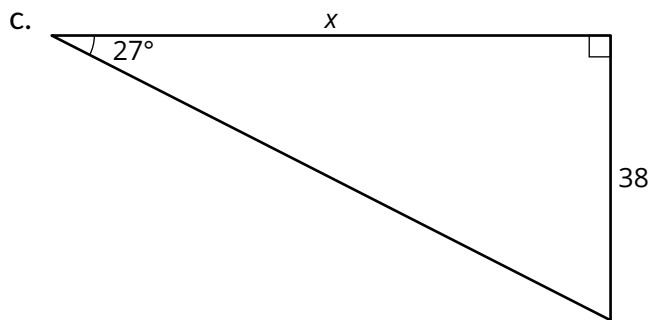
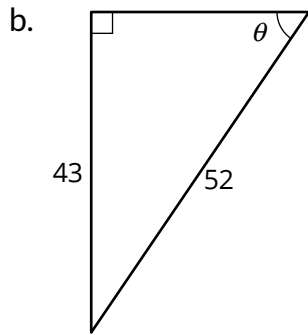
- b. Determine the minimum and maximum horizontal distance between the point of takeoff and the point at which the co-pilot takes over.

- c. What is the minimum distance that the pilot flies the plane? What is the maximum distance that the pilot flies the plane?

3. Choose a strategy to determine the unknown value in each right triangle. Explain your reasoning.

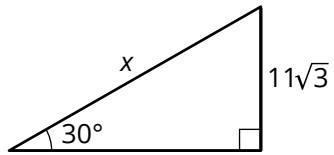


Lesson 4 Assignment



Lesson 4 Assignment

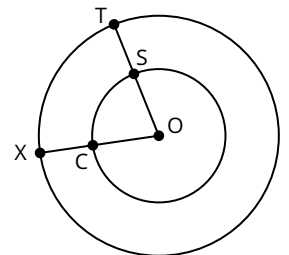
d.



Lesson 4 Assignment

Prepare

1. When the measures of two angles are equal, are the angles congruent?
2. When the measures of line segments are equal, are the line segments congruent?
3. When the measures of two arcs are equal, are the arcs congruent?
4. Alicia explains to her geometry partner that $\widehat{SC}$ is congruent to $\widehat{TX}$. How did Alicia arrive at this conclusion? Is Alicia correct? Explain.



TOPIC 2 SELF-REFLECTION

Name: _____

Trigonometry

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

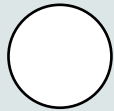
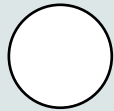
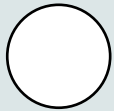
Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Trigonometry* topic by:

TOPIC 2: <i>Trigonometry</i>	Beginning of Topic	Middle of Topic	End of Topic
naming the sides of right triangles in terms of a reference angle.			
comparing common ratios for similar right triangles.			
developing the relationship between the side length ratios and the reference angle.			
defining the tangent, sine, and cosine ratios.			
using the trigonometric ratios to solve for unknown side lengths in right triangles.			
using the inverse tangent, inverse sine, and inverse cosine to solve for unknown angle measures in right triangles.			
applying the trigonometric ratios and the Pythagorean Theorem to solve real-world problems involving right triangles.			
identifying the relationship between the sine and cosine of complementary angles.			

continued on the next page

TOPIC 2 SELF-REFLECTION *continued*

TOPIC 2 <i>Trigonometry</i>	Beginning of Topic	Middle of Topic	End of Topic
using the relationship between the sine and cosine of complementary angles to solve problems.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Trigonometry* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 2 SUMMARY

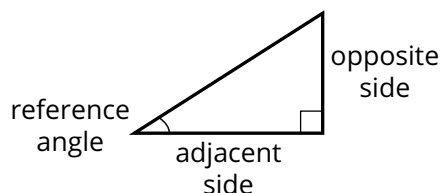
Trigonometry

LESSON

1

Introduction to Trigonometry

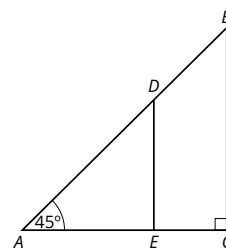
You know that the hypotenuse of a right triangle is the opposite side of the right angle. In trigonometry, the legs of a right triangle are often referred to as the *opposite side* and the *adjacent side*. These references are based on the angle of the triangle that you are considering, which is called the **reference angle**. The **opposite side** is the opposite side of the reference angle. The **adjacent side** is the adjacent side of the reference angle that is *not* the hypotenuse.



Consider $\triangle ABC$ and $\triangle ADE$. They are both 45° - 45° - 90° triangles.

Triangle ABC is similar to $\triangle ADE$ by the AA similarity theorem. Therefore, the lengths of the corresponding sides are proportional.

$$\frac{AE}{AC} = \frac{AD}{AB}$$

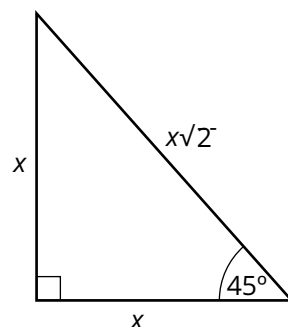


So, given the same reference angle measure, the ratio $\frac{\text{length of the side adjacent of reference angle}}{\text{length of hypotenuse}}$ is constant in similar right triangles.

You can use what you know about the Pythagorean Theorem to determine the exact ratios of side lengths in 45° - 45° - 90° triangles and use these ratios to determine the lengths of the sides.

The ratio of the opposite side length to the length of the hypotenuse is $\frac{x}{x\sqrt{2}} = \frac{\sqrt{2}}{2}$. This is the same as the ratio of the adjacent side length to the length of the hypotenuse.

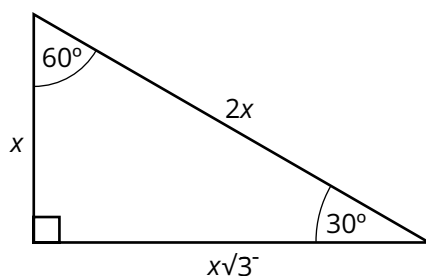
The ratio of the opposite side length to the adjacent side length is $\frac{x}{x}$, or 1.



KEY TERMS

- reference angle [ángulo de referencia]
- opposite side
- adjacent side
- tangent (tan) [tangente]
- inverse tangent (arctangent) [tangente inversa / arco tangente]
- angle of elevation
- sine (sin) [seno]
- inverse sine (arcsine) [seno inverso / arco seno]
- cosine (cos) [coseno]
- inverse cosine (arccosine) [coseno inverso / arco coseno]
- angle of depression

The same properties of ratios apply to 30° - 60° - 90° triangles as well.



For the 30° reference angle, the ratio of the opposite side length to the length of the hypotenuse is $\frac{x}{2x} = \frac{1}{2}$. The ratio of the adjacent side length to the length of the hypotenuse is $\frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$. The ratio of the opposite side length to the adjacent side length is $\frac{x}{x\sqrt{3}} = \frac{\sqrt{3}}{3}$.

For the 60° reference angle, the ratio of the opposite side length to the length of the hypotenuse is $\frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$. The ratio of the adjacent side length to the length of the hypotenuse is $\frac{x}{2x} = \frac{1}{2}$. The ratio of the opposite side length to the adjacent side length is $\frac{x\sqrt{3}}{x} = \sqrt{3}$.

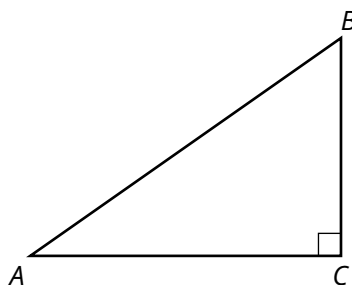
LESSON

2

Tangent Ratio and Inverse Tangent

The **tangent (tan)** of an acute angle in a right triangle is the ratio of the length of the opposite side of the reference angle to the length of the adjacent side of the reference angle. The expression “ $\tan A$ ” means “the tangent of $\angle A$.” The tangent of an angle can be used to describe the slope of a line.

Consider $\angle A$ in the right triangle.

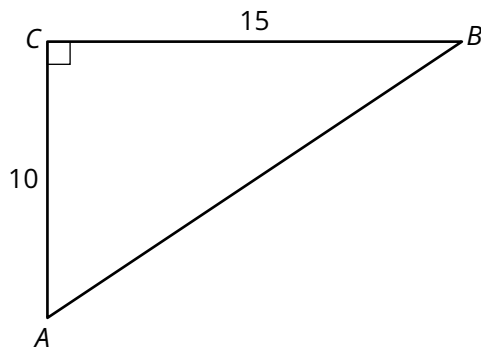


The tangent ratio describes the relationship between $\angle A$, the opposite side of $\angle A$, and the adjacent side of $\angle A$.

$$\tan A = \frac{\text{length of opposite side of } \angle A}{\text{length of adjacent side to } \angle A} = \frac{BC}{AC}$$

Remember, the ratios of corresponding side lengths between two similar triangles are congruent. The tangent values of congruent angles in similar triangles are always the same.

The **inverse tangent** (or arctangent) of x is defined as the measure of an acute angle whose tangent is x . If you know the length of any two sides of a right triangle, it is possible to compute the measure of either acute angle using the inverse tangent, or the $\tan^{-1}$ button on a graphing calculator. In right triangle ABC , if $\tan A = x$, then $\tan^{-1}x = m\angle A$.



For example, consider $\triangle ABC$. Determine $m\angle A$.

$$\begin{aligned}\tan A &= \frac{15}{10} \\ \tan^{-1}(\tan A) &= \tan^{-1}\left(\frac{15}{10}\right) \\ A &\approx 56.31^\circ\end{aligned}$$

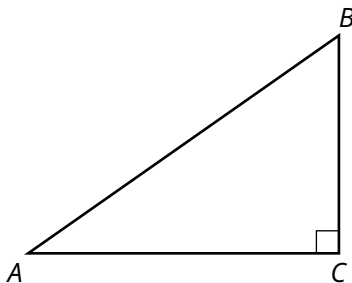
LESSON

3

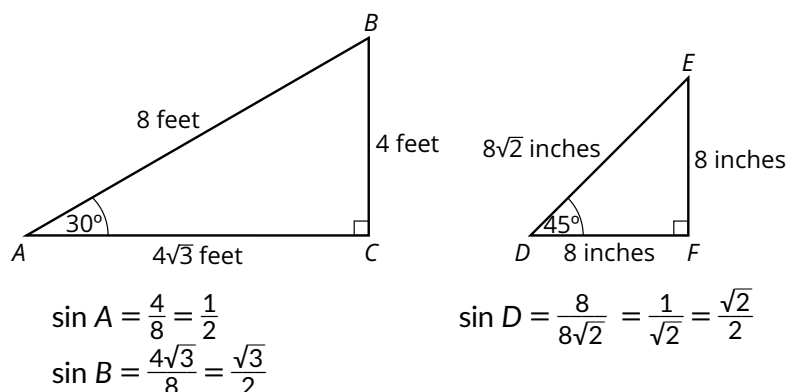
Exploring Sine and Cosine Ratios

The **sine (sin)** of an acute angle in a right triangle is the ratio of the length of the opposite side of the angle to the length of the hypotenuse. The expression “sin A ” means “the sine of $\angle A$.”

In right triangle ABC , $\sin A = \frac{\text{length of opposite side of } \angle A}{\text{length of hypotenuse}} = \frac{BC}{AB}$.



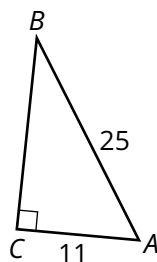
You can use these ratios to calculate the values of $\sin 30^\circ$, $\sin 45^\circ$, and $\sin 60^\circ$. For example, consider $\triangle ABC$ and $\triangle DEF$.



The **inverse sine** (or arcsine) of x is defined as the measure of an acute angle whose sine is x . If you know the length of any two sides of a right triangle, it is possible to calculate the measure of either acute angle by using the inverse sine, or $\sin^{-1}$ button on a graphing calculator.

In right triangle ABC , if $\sin A = x$, then $\sin^{-1} x = m\angle A$.

For example, consider $\triangle ABC$. Determine $m\angle B$.

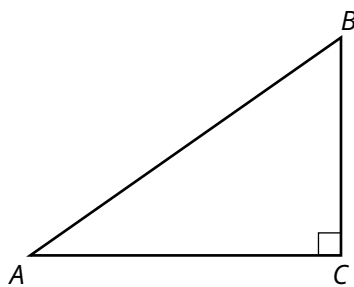


$$\sin B = \frac{11}{25}$$

$$\sin^{-1}(\sin B) = \sin^{-1}\left(\frac{11}{25}\right)$$

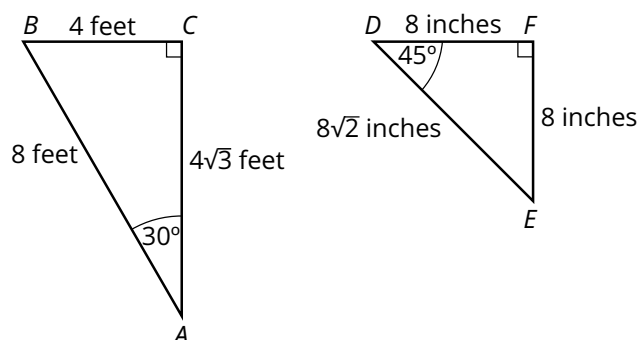
$$B \approx 26.1^\circ$$

The **cosine (cos)** of an acute angle in a right triangle is the ratio of the length of the adjacent side of the angle to the length of the hypotenuse. The expression “ $\cos A$ ” means “the cosine of $\angle A$.”



In right triangle ABC , $\cos A = \frac{\text{length of adjacent side of } \angle A}{\text{hypotenuse}} = \frac{AC}{AB}$.

You can use these ratios to calculate the values of $\cos 30^\circ$, $\cos 45^\circ$ and $\cos 60^\circ$. For example, consider $\triangle ABC$ and $\triangle EDF$.



$$\cos A = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

$$\cos B = \frac{4}{8} = \frac{1}{2}$$

$$\cos D = \frac{8}{8\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

The **inverse cosine** (or arccosine) of x is defined as the measure of an acute angle whose cosine is x . If you know the length of any two sides of a right triangle, it is possible to compute the measure of either acute angle by using the inverse cosine, or $\cos^{-1}$ button on a graphing calculator.

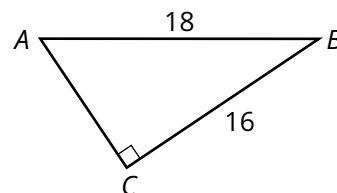
In right triangle ABC , if $\cos A = x$, then $\cos^{-1} x = m\angle A$.

For example, consider $\triangle ABC$. Determine $m\angle B$.

$$\cos B = \frac{16}{18}$$

$$B = \cos^{-1}\left(\frac{16}{18}\right)$$

$$B \approx 27.3^\circ$$



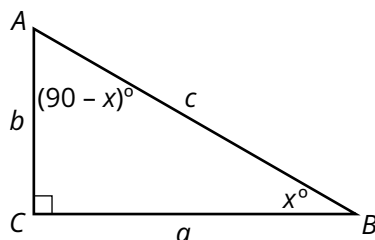
For any acute angle in a right triangle, $\tan A = \frac{\sin A}{\cos A}$.

LESSON

4

Complement Angle Relationships

Consider $\triangle ABC$ with right angle C . Angles A and B are complementary angles because the sum of their measures is equal to 90 degrees. The trigonometric ratios also have complementary relationships.



The table displays the trigonometric ratios for a 30° , 60° , and 45° angle based on the Pythagorean Theorem and complementary relationships.

Reference Angle	sin	cos
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

ISBN: 978-1-970198-54-6

© 2026 Texas Education Agency. Portions of this work are adapted,
with permission, from the originals created by and copyright
© 2021 Carnegie Learning, Inc.

This work is licensed under a Creative Commons Attribution-Non-Commercial-
4.0 International License.

You are free:

to Share—to copy, distribute, and transmit the work

to Remix—to adapt the work

Under the following conditions:

Attribution—You must attribute any adaptations of the work in the following manner:

This work is based on original works of the Texas Education Agency and
Carnegie Learning, Inc. This work is made available under a Creative
Commons Attribution-Non-Commercial-4.0 International License. This does
not in any way imply endorsement by those authors of this work.

NonCommercial—You may not use this work for commercial purposes.

With the understanding that:

For any reuse or distribution, you must make clear to others the license terms of this
work. The best way to do this is with a link to this web page:

<https://creativecommons.org/licenses/by-nc/4.0/>

Trademarks and trade names are shown in this book strictly for illustrative
and educational purposes and are the property of their respective owners.
References herein should not be regarded as affecting the validity of said
trademarks and trade names.

Printed in the USA