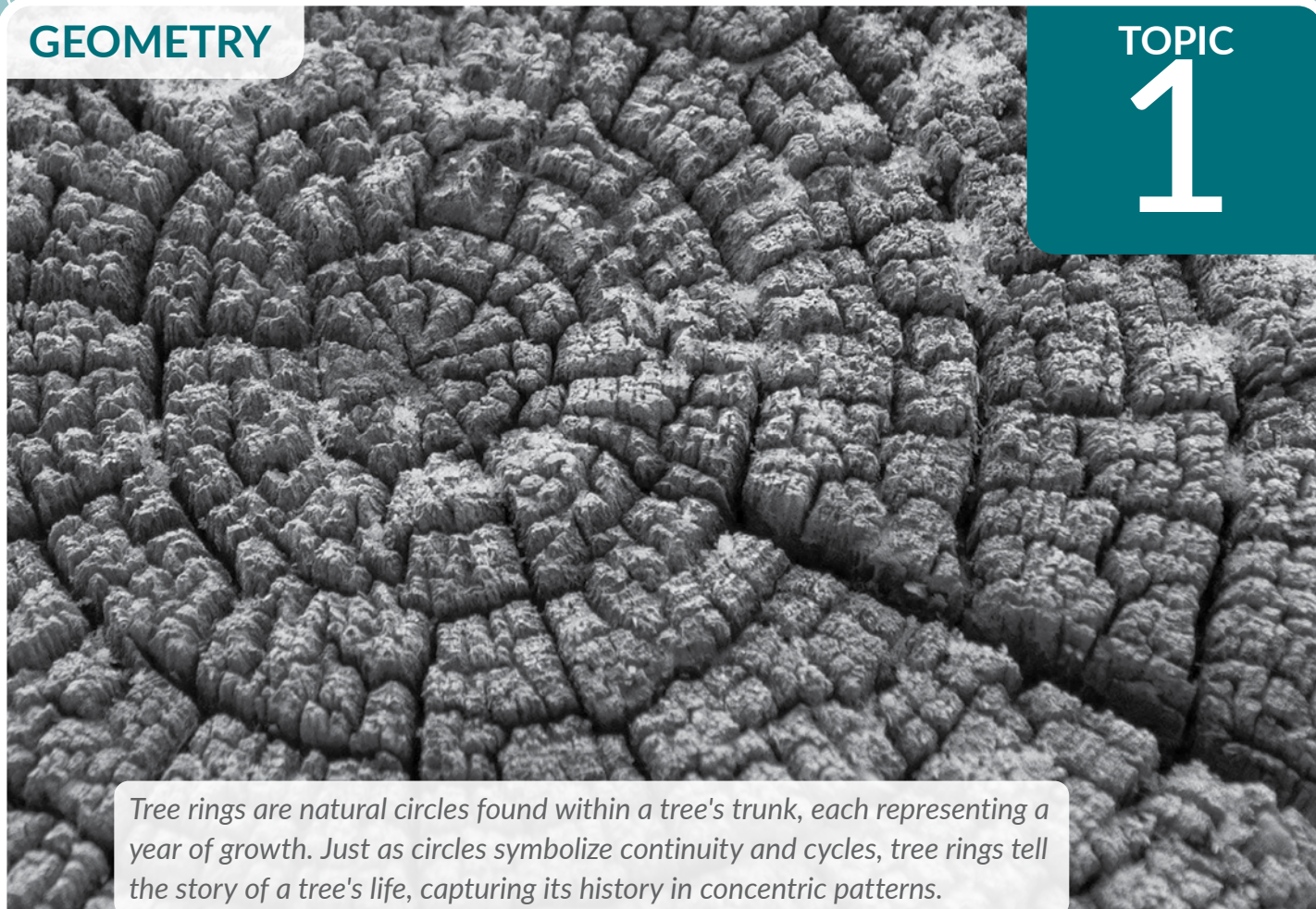


Connecting Geometric and Algebraic Descriptions

TOPIC 1	Circles	985
TOPIC 2	Building Three-Dimensional Shapes	1087





Tree rings are natural circles found within a tree's trunk, each representing a year of growth. Just as circles symbolize continuity and cycles, tree rings tell the story of a tree's life, capturing its history in concentric patterns.

Circle

LESSON 1	Tangents, Secants, and More Chords.....	987
LESSON 2	Similarity Relationships in Circles.....	1003
LESSON 3	Sectors and Segments of a Circle.....	1023
LESSON 4	Deriving the Equation of a Circle.....	1047
LESSON 5	Determining Points on a Circle.....	1065



1

Tangents, Secants, and More Chords

OBJECTIVES

- Prove the segment chord theorem.
- Explain the relationship between a tangent line and a radius orally and in writing.
- Prove the secant segment theorem.
- Prove the secant tangent theorem.
- Solve problems using theorems about circles.

KEY TERMS

- segments of a chord
- segment chord theorem
- external secant segment
- secant segment theorem
- secant tangent theorem

.....

You know the criteria for proving that two triangles are similar and the relationship between the corresponding sides and angles of similar triangles.

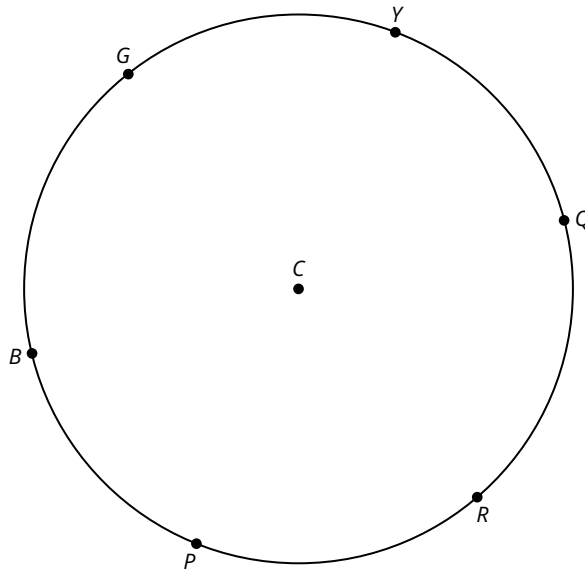
How can you use what you know about similar triangles to prove theorems about tangents, secants, and chords?

Getting Started

Segments on Chords

Segments of a chord are the segments formed on a chord when two chords of a circle intersect.

1. Consider circle C .

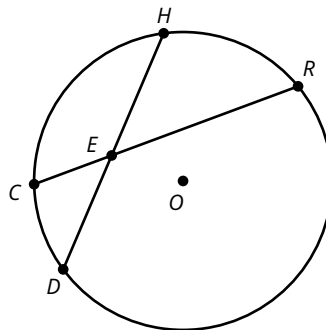


- a. Draw chords PQ and RY .
 - b. Label the point at which the two chords intersect point E .
 - c. Use a ruler to measure the length of each segment on the two chords.
2. Make a conjecture about the two segment lengths that make up the first chord and the two segment lengths that make up the second chord.

ACTIVITY
1.1

Chord Segments

1. Prove the segment chord conjecture.
Given: Chords HD and RC intersect at point E in circle O .
Prove: $EH \cdot ED = ER \cdot EC$

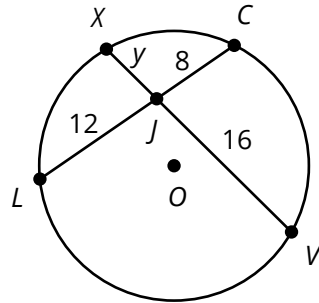


.....
Connect points C and D , and points H and R .
Show the triangles are similar.
.....

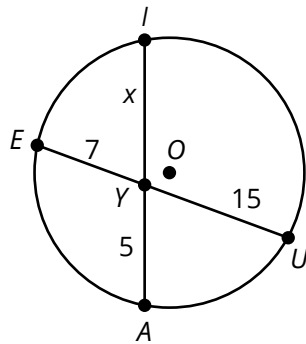
Since you have proven your conjecture you can use this theorem as a valid reason in proofs.

The **segment chord theorem** states that “If two chords in a circle intersect, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the second chord.”

2. Use the segment chord theorem to determine the value of y .



3. Use the segment chord theorem to determine the value of x .

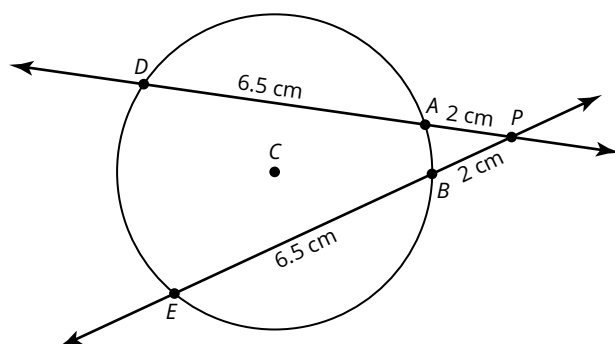


Secants and Tangents

Recall that a secant segment is a line segment formed when two secants intersect outside a circle. A secant segment begins at the point at which the two secants intersect, continues into the circle, and ends at the point at which the secant exits the circle.

An **external secant segment** is the portion of each secant segment that lies on the outside of the circle. It begins at the point at which the two secants intersect and ends at the point where the secant enters the circle.

1. Consider circle C with the measurements as shown.



The vertex of $\angle DPE$ is located outside of circle C . Because this angle is formed by the intersection of two secants, each secant line contains a secant segment and an external secant segment.

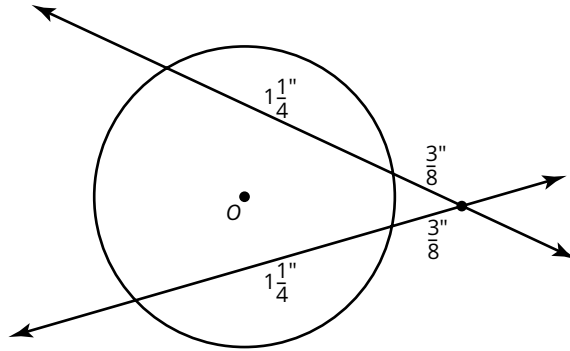
- a. Identify the two secant segments.
- b. Identify the two external secant segments.

It appears that the product of the lengths of the segment and its external secant segment is equal to the product of the lengths of the second secant segment and its external secant segment.

This observation can be proved and stated as a theorem.

.....
 It may be helpful to
 connect points A and
 S, and points E and N.

2. Prove the secant segment conjecture.
 Given: Secants CS and CN intersect at point C in the exterior
 of circle O.
 Prove: $CS \cdot CE = CN \cdot CA$



Since you have proven your conjecture you can use this theorem as a
 valid reason in proofs.

The **secant segment theorem** states: “If two secants intersect in the exterior of a circle, then the product of the lengths of the secant segment and its external secant segment is equal to the product of the lengths of the second secant segment and its external secant segment”

You have previously proven the tangent segment theorem. Recall that the tangent segment theorem states that if two tangent segments are drawn from the same point on the exterior of a circle, then the tangent segments are congruent.

3. Follow these steps to construct a tangent line to a circle through a point outside of the circle

Step 1: Draw a circle with center point C and locate point P outside of the circle.

Step 2: Draw segment PC .

Step 3: Construct the perpendicular bisector of segment PC .

Step 4: Label the midpoint of the perpendicular bisector of segment PC point M .

Step 5: Adjust the radius of your compass to the distance from point M to point C .

Step 6: Place the compass point on point M , and construct one arc that intersects circle C .

Step 7: Label the point at which the arc cuts through circle C as point A .

Step 8: Connect point P and A to form tangent line PA .

Step 9: Draw a secant that passes through the same external point P and intersects the circle at two points. Label the two points T and U .

Use a ruler to measure the length of the tangent segment PA and the length of the secant segment UP and the external secant segment TP created by your construction.

.....
Remember ...

A tangent segment is a line segment formed by connecting a point outside a circle to a point of tangency.

The point of tangency is where the tangent intersects the circle.
.....

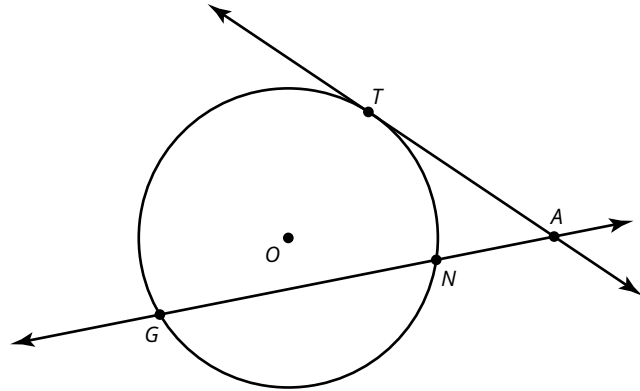
4. Make a conjecture about the relationship between the secant segment and its external segment and the tangent segment.

You can prove your conjecture and state it as a theorem.

.....
Try connecting points
N and T, and points G
and T.
.....

5. Prove the secant
tangent conjecture.
Given: Tangent AT
and secant
 AG intersect
at point A in
the exterior of
circle O .

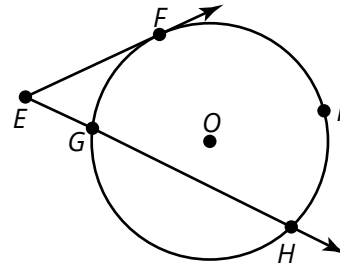
Prove: $(AT)^2 = AG \cdot AN$



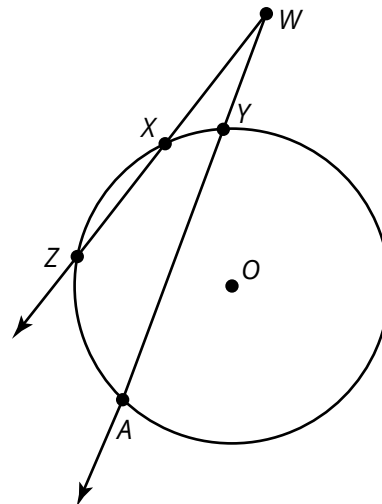
Since you have proven your conjecture you can use this theorem as a valid reason in proofs.

The **secant tangent theorem** states: "If a tangent and a secant intersect in the exterior of a circle, then the product of the lengths of the secant segment and its external secant segment is equal to the square of the length of the tangent segment."

6. The length of $\overline{EF}$ is 8 inches and the length of $\overline{EG}$ is 4 inches. Determine the length of $\overline{GH}$.



7. The length of $\overline{WX} = 12$ centimeters. The length of $\overline{WY}$ is 9 centimeters and the length of $\overline{YA}$ is 21 centimeters. Determine the length of $\overline{XZ}$.

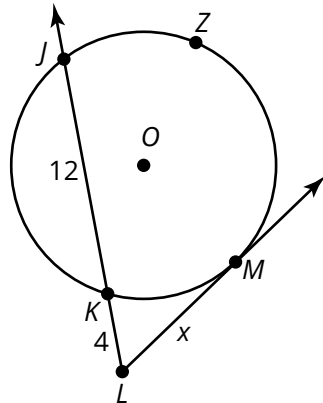


Talk the Talk

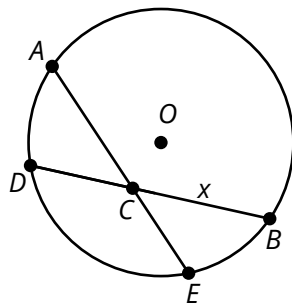
All the Pieces

Determine the unknown length in each figure.

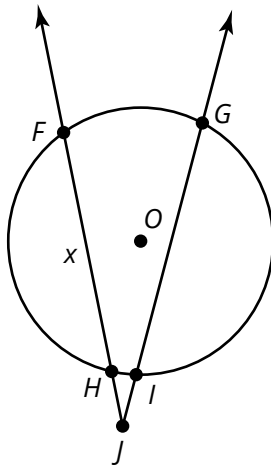
1.



2. $DC = 4$, $CE = 3$, and $AE = 11$.



3. $JH = 3$, $GI = 13$, and $IJ = 2$



Lesson 1 Assignment

Write

What is the difference between a segment on a chord, a segment on a tangent, and a segment on a secant?

Remember

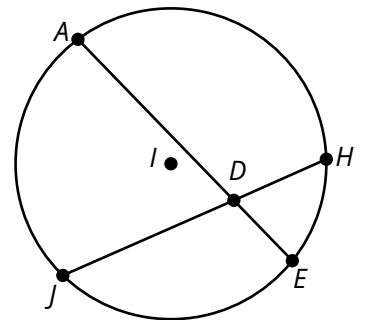
The segment chord theorem states that “If two chords in a circle intersect, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the second chord.”

The secant segment theorem states: “If two secants intersect in the exterior of a circle, then the product of the lengths of the secant segment and its external secant segment is equal to the product of the lengths of the second secant segment and its external secant segment.”

The secant tangent theorem states: “If a tangent and a secant intersect in the exterior of a circle, then the product of the lengths of the secant segment and its external secant segment is equal to the square of the length of the tangent segment.”

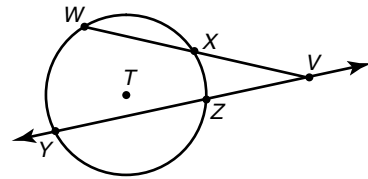
Practice

1. In the figure shown, chords AE and JH intersect at point D inside circle I . $AD = 12$ meters, $JD = 10$ meters, and $DE = 4.5$ meters. What is DH ? Explain your reasoning.

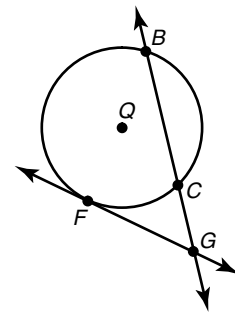


Lesson 1 Assignment

2. In the figure shown, $WV = 36$ inches, point X is a midpoint of segment WV , and $YV = 40$ inches. What is YZ ? Explain your reasoning.



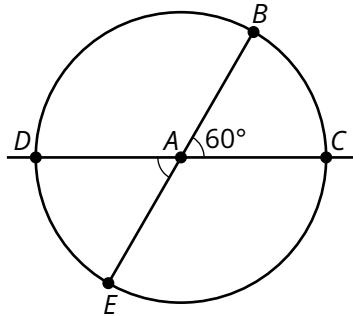
3. In the figure shown, line FG is tangent to circle Q , $BC = 10$ feet, and $CG = 4$ feet. What is FG ? Explain your reasoning.



Lesson 1 Assignment

Prepare

Determine each arc measure.



1. $m\widehat{BD}$

2. $m\widehat{DE}$

3. $m\widehat{CE}$

4. $m\widehat{CD}$

2

Similarity Relationships in Circles

OBJECTIVES

- Demonstrate using informal arguments that all circles are similar.
- Calculate the length of an arc of a circle given a central angle measure.
- Solve real-world and mathematical problems using arc length measurements.
- Use similarity to show that an arc length of a circle is proportional to the radius.
- Understand and use radians to measure arc lengths and angles.

KEY TERMS

- arc length
- radian

.....

You have learned how to measure an arc of a circle in degrees called arc measure. In this lesson, you will explore arc lengths as linear measures and use different units to represent their lengths.

What other unit of measure can be used for arc measures and what is another way to measure arcs of circles?

Getting Started

Remember ...

A circle is the set of points equidistant from a given point.

Two Pi Radii

Let's explore a different way of measuring around a circle.

1. Construct a circle in the space provided and label it O . Then, draw a radius and label it as $\overline{OA}$.

Think About ...

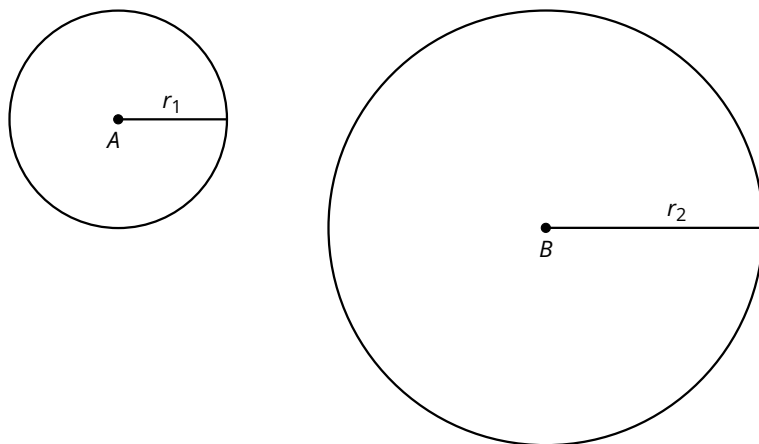
How can you and your classmates measure radius lengths from the same starting position?

2. Cut out a length of string equal to the radius length of your circle. Then, answer each question and compare your answers with your classmates' answers.
 - a. How many radius lengths does it take to go all the way around the circle? Explain why.
 - b. How many radius lengths does it take to go halfway around the circle? Explain why.
 - c. Estimate where on the circle the point $\frac{3\pi}{2}$ radius lengths falls in relation to point A .

ACTIVITY
2.1**All Circles Are Similar**

Are all circles similar figures? You may guess that the answer is yes, but in geometry, you need to build an argument to justify this answer. Recall that two figures are similar if there is a set of transformations that will move one figure onto the other.

Consider the two circles shown to prove that all circles are similar.



1. Describe a sequence of transformations you can use to map circle A to circle B.

Since you can map circle A onto circle B, you can show the two circles are similar.

2. Use patty paper to translate the center of circle A along $\overline{AB}$ onto the center of circle B.

For the circles to be similar, there must be some scale factor, k , that maps circle A onto circle B.

3. Identify the scale factor in terms of r_1 and r_2 .
4. Summarize the proof to show that circles are similar.

Draw and Deduce

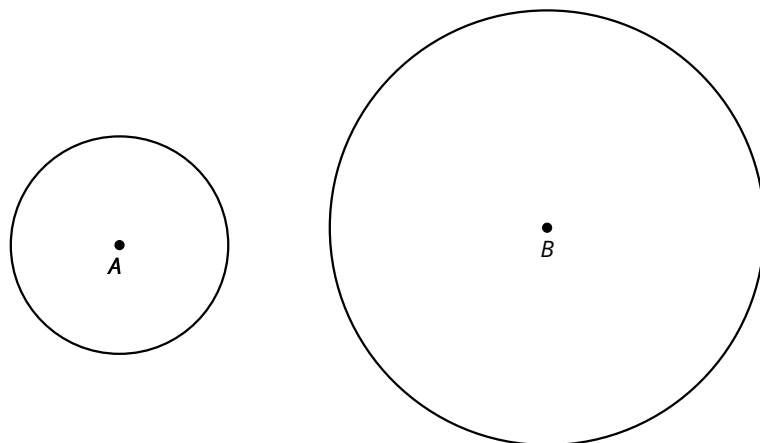
If the circles are similar, then the ratios of the measures of their corresponding parts should be equal.

Remember ...

Two circles with the same center are concentric circles.

You learned that the circumference of a circle with radius r is $2\pi r$.
Let's use similarity to understand why this formula is true.

Consider circle A with a diameter of 1 and circle B with a diameter of $2r$.



5. Draw and label the diameter in each circle.

There are proportional relationships that exist within and between circles that are necessary to explain the circumference formula. You know that the ratio of the circumference to the diameter of all circles is π , and you know that all circles are similar.

.....
This scale factor, k ,
can also be referred
to as the *constant
of proportionality*.
.....

6. Because all circles are similar, there exists a scale factor k such that $C_B = kC_A$. Identify the scale factor k that shows circle B is similar to circle A .

Because the ratio of the circumference to the diameter is the same for all circles, then $\frac{C_A}{d_A} = \frac{C_B}{d_B}$. You can use substitution and properties of equality to show that the $C_B = 2\pi r$.

WORKED EXAMPLE

You can use the proportional relationship between two similar circles to verify the circumference formula.

Given $\frac{C_A}{d_A} = \frac{C_B}{d_B}$

Step 1 $\frac{C_A}{1} = \frac{C_B}{2r}$

Step 2 $(2r)C_A = C_B$

Step 3 $(2r)\pi = C_B$

Step 4 $C_B = 2\pi r$

The circumference of any circle with radius r is $2\pi r$.

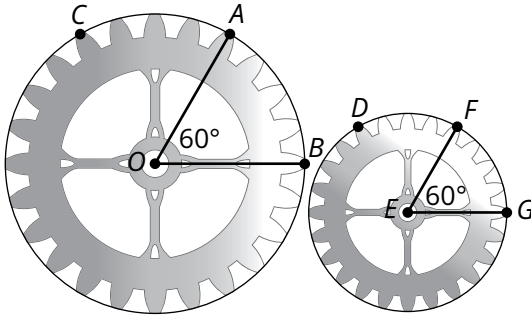
7. Identify the constant of proportionality in Step 2 of the Worked Example. Is this the same k -value you determined to show circle A and circle B are similar?

8. Justify why π is substituted for C_A in Step 3.

9. The final equation in the Worked Example represents a proportional relationship. Identify the proportional relationship and state its constant of proportionality.

Determining Arc Length

Consider the large gear represented by circle O , containing a central angle, $\angle AOB$, whose measure is equal to 60° ; a minor arc, $\widehat{AB}$; and a major arc, $\widehat{ACB}$, as shown. Consider the small gear represented by circle E , containing a central angle, $\angle FEG$, whose measure is equal to 60° ; a minor arc, $\widehat{FG}$; and a major arc, $\widehat{FDG}$, as shown.



1. Compare the two gears.

- Is the length of the radii in the large gear proportional to the length of the radii in the small gear? Explain your reasoning.

Remember ...

Two points on a circle determine a major arc and a minor arc. The arc with the greater measure is the major arc. The other arc is the minor arc.

- Determine the degree measure of the minor arc in each circle.

- What is the ratio of the degree measure of the minor arc to the degree measure of the entire circle for each of the two gears?

2. Explain why Trung is incorrect.

Trung

The two minor arcs, $\widehat{AB}$ and $\widehat{FG}$, on the gears have the same measure, which is 60° . So, the two arcs are the same length.



In a circle, the degree measure of an arc is equal to the measure of the central angle. **Arc length** is a linear measurement that represents a portion of the circumference of a circle. To determine the arc length of the minor arc, you need to know the circumference of the circle, which requires knowing the radius of the circle.

3. When the length of the radius of the large gear, or $\overline{OB}$, is equal to 4 centimeters, determine the circumference of circle O . Leave your answer as an exact value in terms of π .
4. Use the circumference of circle O determined in Question 3 and the ratio determined in Question 1, part (c) to solve for the length of the minor arc. Leave your answer as an exact value in terms of π .
5. When the length of the radius of the small gear, or $\overline{EF}$, is equal to 2 centimeters, determine the circumference of circle E . Leave your answer as an exact value in terms of π .
6. Use the circumference of circle E determined in Question 5 and the ratio determined in Question 1, part (c) to solve for the length of the minor arc. Leave your answer as an exact value in terms of π .

.....

The length of an arc is different from the degree measure of the arc. The degree measure of an arc tells you how much of the circle the arc covers in terms of the angle and is measured in degrees. The length of an arc is the distance traveled between two points on the circle and is measured in a unit of length such as centimeters or inches.

.....

.....

Arc length is directly proportional to the circumference.

.....

.....

Remember ...

Pi is a non-repeating, non-terminating decimal so it continues on indefinitely. To determine the exact arc length, you can leave your answer in terms of pi as $\frac{40}{9}\pi$ inches, since in this case you are not rounding or approximating the decimal pi. To provide an approximate value for the arc length you can round π to 3.14 and multiply 3.14 times $\frac{40}{9}$ in your calculator. $\frac{40}{9}(3.14) = 13.95$, which means that the arc length is approximately 14 inches.

.....

.....

$$s = \frac{m}{360^\circ} \cdot 2\pi r$$

$$s = \frac{m}{180^\circ} \cdot \pi r$$

$$\frac{s}{r} = \frac{\pi}{180^\circ} \cdot m$$

.....

To determine arc length, s , you multiplied the circumference of the circle by a fraction representing the portion of the circumference determined by the central angle measure, m .

$$\text{arc length} = \frac{\text{central angle measure}}{360^\circ} (\text{circumference})$$

$$s = \frac{m}{360^\circ} (2\pi r)$$

There is a proportional relationship between the measure of an arc length and the circumference of the circle.

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\text{central angle measure}}{360^\circ}$$

$$\frac{s}{2\pi r} = \frac{m}{360^\circ}$$

WORKED EXAMPLE

You can apply the formula $s = \frac{m}{360^\circ} (2\pi r)$ to determine the length of an arc for a circle with a radius of 10 inches and central angle of 80° .

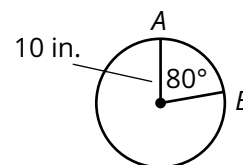
$$s = \frac{m}{360^\circ} (2\pi r)$$

$$m\widehat{AB} = \frac{80^\circ}{360^\circ} (2\pi(10))$$

$$m\widehat{AB} = \frac{2}{9} (20\pi)$$

$$m\widehat{AB} = \frac{40}{9}\pi$$

The measure of $\widehat{AB}$ is $\frac{40}{9}\pi$, or approximately 14 inches.



7. Determine the arc length of an entire circle with each given radius. Leave your answers as exact values.

a. 1 inch

.....

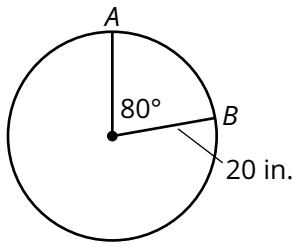
b. r units

The relationship also implies that the ratio of the arc length to the radius, $\frac{s}{r}$, is directly proportional to m , the measure of the central angle.

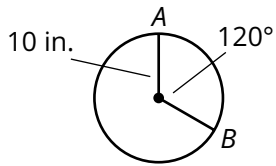
8. What is the constant of proportionality in this proportional relationship?

9. Use the formula that compares the ratio $\frac{s}{r}$ to calculate the length of the minor arc AB in each circle. Express your answer in terms of π .

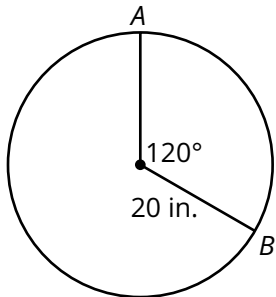
a.



b.

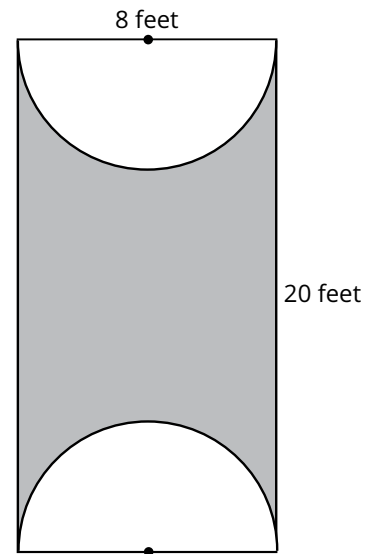


c.



- d. Compare the Worked Example and part (a), as well as parts (b) and (c). In each pair, the central angle is the same but the radius has been doubled. What effect does doubling the radius have on the length of the arc? Justify why this relationship exists.

10. Two semicircular cuts were taken from the rectangular region shown. Determine the perimeter of the shaded region in feet to the nearest hundredth.



.....

The Treaty Oak tree in Austin, TX is believed to be more than 500 years old. It was held sacred by local Native American tribes as a meeting place and is reputed to be where Stephen F. Austin established a boundary agreement.

.....

Tree rings can help determine the age of a tree, as well as the circumferences at different diameters of the tree. Consider the diagram of tree rings shown. Let the radius of the small tree ring (small circle) be r and the radius of the larger tree ring (large circle) be $10r$.



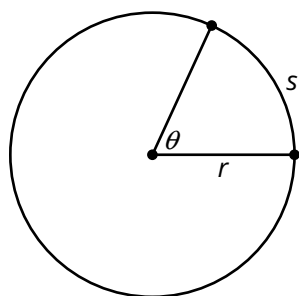
11. Compare the tree rings.

- How does the length of the minor arc in the small tree ring compare to the length of the minor arc in the large tree ring?
- When the length of the minor arc in the small tree ring is equal to 3 inches, what is the length of the minor arc in the large tree ring?
- When $m\angle A = 20^\circ$ and the length of the minor arc of the small tree ring is 3 inches, determine the circumference of the large tree ring.

ACTIVITY 2.3

Measuring Angles and Arcs Using Radians

You read the symbol θ as “theta.”



The central angle of one radian of a unit circle produces an arc length of 1.

So far you have described the measures of angles and arcs using degrees and determined the arc length intercepted by a central angle as a linear measurement.

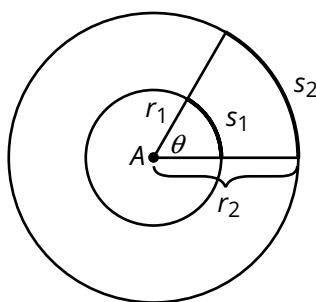
Let's investigate another unit of angular measure called a *radian*. A **radian** is defined as the measure of a central angle whose arc length is the same as the radius of the circle. The radian measure, θ , of a central angle is the ratio of the length of the intercepted arc to the radius of a circle.

$$\theta = \frac{s}{r}$$

The length of the intercepted arc of the central angle is proportional to the radius. The radian measure of the angle is the constant of proportionality.

$$s = \theta r$$

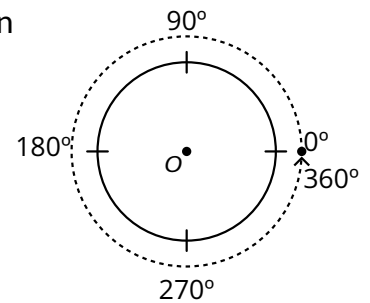
1. Use the fact that all circles are similar to show that the length of the arc intercepted by an angle is proportional to the radius.



You know that the arc measure of an entire circle is 360° and has an arc length of $2\pi r$ units.

2. What is the radian measure of an entire circle? Leave your answer as an exact value.

3. Write the equivalent radian measure for each degree measure as an exact value.



The degree measure and radian measure are related by the equations $360^\circ = 2\pi$ radians and $180^\circ = \pi$ radians.

4. Write an equivalent equation to show each relationship. Write your answer as a decimal approximation.

a. 1° and radians

b. degrees and 1 radian

.....
The decimal approximation is not used to convert between radians and degrees.
.....

WORKED EXAMPLE

You can convert from degrees to radians and from radians to degrees.

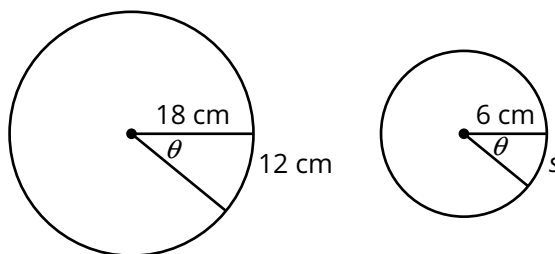
Convert 60° to radians.

$$60^\circ \left(\frac{\pi}{180^\circ} \right) = \frac{\pi}{3}$$

Convert $\frac{\pi}{3}$ to degrees.

$$\frac{\pi}{3} \left(\frac{180^\circ}{\pi} \right) = 60^\circ$$

5. Determine the unknown arc length. Explain your reasoning.



6. The measure of a central angle is 120° . The length of the radius is 20 centimeters. Determine the arc length using each equation. Leave your answers as exact values.

a. $s = \frac{\text{measure of angle}}{360^\circ} \cdot 2\pi r$

b. $s = \theta r$

- c. Compare your answers in part (a) and part (b).

7. When $\theta = \frac{\pi}{2}$ and $r = 4$, solve for the length of the intercepted arc. Leave your answer as an exact value.

8. When $r = 2$ and the intercepted arc length is 5, what is the measure of the central angle in radians?

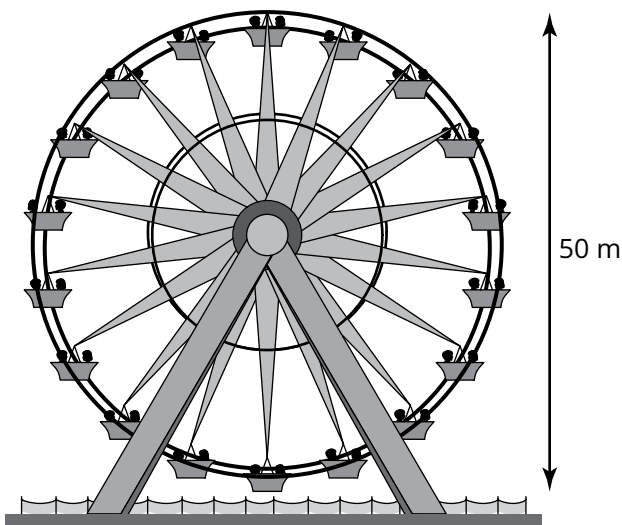
9. Describe the similarities and differences between radians and degrees.

10. Which unit do you prefer to use? Explain your reasoning.

Talk the Talk

Ferris Wheel

A Ferris wheel is 50 meters tall. The actual wheel is constructed with steel arcs that connect one passenger car to the next passenger car.



1. Determine the length of each steel arc connecting one passenger car to the next passenger car. Approximate to the nearest tenth.
2. Convert 150° to radians. Leave your answer as an exact answer in terms of π .
3. The diameter of a circle is 24 inches. What is the length of an arc intercepted by an angle of $\frac{\pi}{3}$ radians? Leave your answer in terms of π .

PROBLEM SOLVING



The oldest well-known Ferris wheel in Texas is called the Texas Star. This Ferris wheel is located at the Texas State Fair in Dallas. The Ferris wheel is just over 212 feet tall. The actual wheel is constructed of steel arcs that connect one passenger car to the next passenger car. The Texas Star has 44 passengers cars.

Lesson 2 Assignment

Write

Write a definition for *arc length* and for *radian* in your own words.

Remember

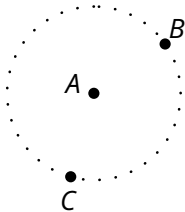
All circles are similar figures.

The formula for arc length can be written as $s = \frac{m}{360^\circ} \cdot 2\pi r$, where s is the arc length and m is the central angle measure.

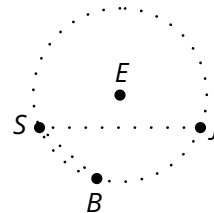
The length of the intercepted arc of the central angle is proportional to the radius.

Practice

1. In circle A , describe the difference between the measure of minor arc BC and the length of major arc BC .



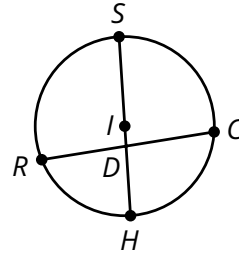
2. In circle E , the radius of the circle is 16 centimeters and $m\angle JSB$ is 40° . Determine the length of $\widehat{JB}$. Approximate your answer to the nearest hundredth.



Lesson 2 Assignment

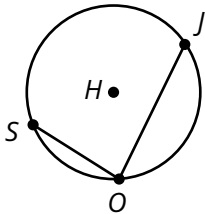
3. In circle I , the radius is 6 millimeters and $m\widehat{HC}$ is 80° .

- Determine the length of $\widehat{SC}$ in millimeters. Approximate your answer to the nearest hundredth.
- Determine the measure of $\widehat{SC}$ in radians. Leave your answer as an exact answer in terms of π .



4. In circle H , the length of $\widehat{SJ}$ is 24π centimeters and $m\angle JOS$ is 80° .

Determine the length of a diameter of circle H .



Lesson 2 Assignment

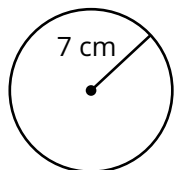
5. When $\theta = \frac{\pi}{3}$ and $r = 3$, what is the length of the intercepted arc in terms of π ?
6. When $r = 8$ and the intercepted arc length is 6π , what is the measure of the central angle?
7. When $r = 6$ and the measure of the central angle is $\frac{2\pi}{3}$, what is the length of the intercepted arc?
8. When $r = 6$ and the intercepted arc length is 4π , what is the measure of the central angle?
9. The measure of a central angle is 80° . The length of the radius is 40 mm. Determine the length of the intercepted arc. Leave your answer as an exact value in terms of π .
10. The measure of a central angle is 110° . The length of the radius is 15 ft. Determine the length of the intercepted arc. Leave your answer as an exact value in terms of π .

Lesson 2 Assignment

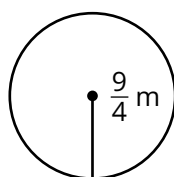
Prepare

Determine the area of each circle.

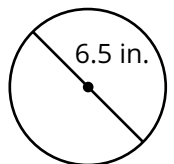
1.



2.



3.



3

Sectors and Segments of a Circle

OBJECTIVES

- Determine the area of sectors of a circle orally and in writing.
- Derive the formula for the area of a sector.
- Determine the area of segments of a circle.

KEY TERMS

- sector of a circle
- segment of a circle

.....

You know how to determine the length of an intercepted arc given its central angle measure and the radius of the circle.

How does this calculation relate to the calculation of the area of a sector of a circle?

Getting Started

Remember ...

Concentric circles are circles that have the same center, but radii of different lengths.

Bull's-Eye!

A standard dartboard is shown at the end of the lesson. The dartboard has six concentric circles of wire on the playable section of the board including the inner and outer bull's-eye.

A **sector of a circle** is a region of the circle bounded by two radii and the included arc. Notice that the dartboard is divided into 20 congruent sectors.

1. Determine the measure of the central angle and the measure of the intercepted arc formed by each sector.

Think About ...

What are the corresponding parts of the sectors that make them congruent?

2. What is the measure of the length of the intercepted arc formed by each sector in terms of π ? Explain your reasoning.

3. Cut out each sector of the dartboard. Arrange all 20 sectors to form a figure that resembles a parallelogram.

- a. How do the base and height of the figure relate to the dartboard?

b. Determine the area of the figure you created in terms of π . Explain your reasoning.

c. How does the area of the figure relate to the area of the dartboard?

d. Determine the area contained by each of the sectors in terms of π . Explain your reasoning.

4. When you divided the dartboard into 40 congruent sectors and arranged them in the same way, how would it affect the shape and area of your figure?

5. When you divided the dartboard into 40 congruent sectors, what would be the area of each sector? Explain your reasoning.

6. How could you determine the area of the dartboard when you knew only the area of one of the sectors?

.....

Think About ...

What happens when you divide the dartboard into 80 congruent sectors?

.....

Area of a Sector

There is a proportional relationship between the measure of the area of a circle sector, A , and the area of the circle.

$$\frac{\text{area of a sector}}{\text{area of circle}} = \frac{\text{central angle measure}}{360^\circ}$$

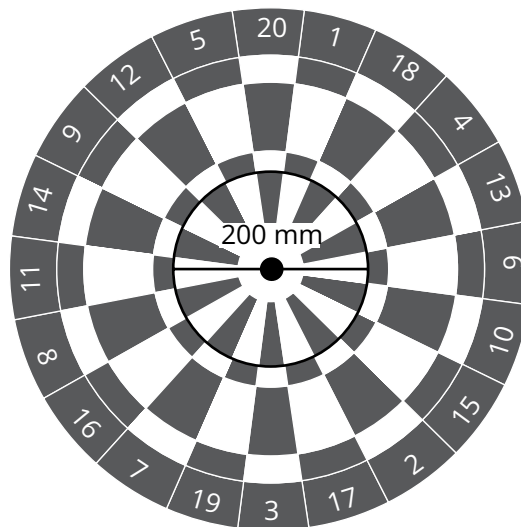
To determine the area of a sector, A , you multiply the area of the circle by a fraction that represents the portion of the area determined by the central angle measure, m .

$$A = \frac{\text{central angle measure}}{360^\circ} \cdot \text{area of circle}$$

$$A = \frac{m}{360^\circ} \pi r^2$$

1. How does the formula to determine the area of a sector compare to the formula to determine the arc length?

The radius of the entire dartboard is approximately 225 millimeters. The innermost circle of the triple ring of the dartboard has a diameter of approximately 200 millimeters, as shown.



2. Josh and Alyssa each determined the area of each sector of the innermost circle of the triple ring. Identify the mistake that each student made.

Josh



Each sector of the innermost circle of the triple ring has the same central angle measure as each sector of the dartboard.

I know the area of the dartboard is $50,625\pi$ square millimeters. I can divide this by 20 to determine the area of each sector of the innermost circle of the triple ring.

Alyssa



The radius of the dartboard is 225 millimeters, and the radius of the innermost circle of the triple ring is 100 millimeters.

$\frac{100}{225} = \frac{4}{9}$, so the area of each sector of the innermost circle of the triple ring is $\frac{4}{9}$ the area of each sector of the dartboard.

3. Calculate the area of each sector of the innermost circle of the triple ring. Express your answer in terms of π .

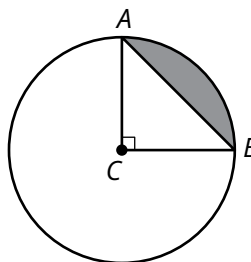


4. Lizzie and Matthew notice that half of the sectors of each circle on the dartboard are the same color. Matthew says that to calculate the total area of all the sectors of the same color, he could apply the proportional relationship between sector area and circle area to calculate the area of half the circle. Lizzie says to, instead, apply this proportional relationship to calculate the area of one sector and multiply that area by 10. Who's correct? Explain your reasoning.

Area of a Segment

A **segment of a circle** is a region of the circle bounded by a chord and the included arc.

1. Name the chord and the arc that bound the shaded segment of the circle.

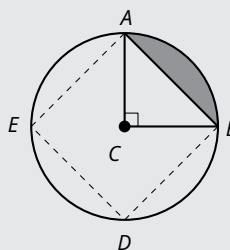


The length of chord $\overline{AB}$ is $8\sqrt{2}$ centimeters.

2. Jamal and Tiara each determined the area of the segment of the circle. Use each student's method to calculate the area of the segment. Express your answers in terms of π .

Jamal

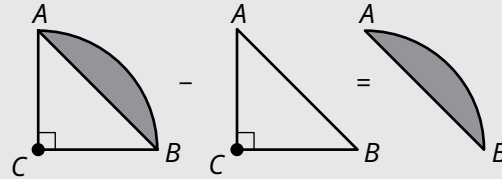
I can draw square $ABDE$ inside circle C . Triangle ABC is a 45° - 45° - 90° triangle, so the measure of the radius of circle C is 8 centimeters. I can determine the area of square $ABDE$ and subtract it from the area of circle C . Then, I can divide this area by 4 to determine the area of the segment of the circle.



Tiara



Triangle ABC and the segment together make up a sector of circle C . Triangle ABC is a 45° - 45° - 90° triangle, so its base and height are each 8 centimeters. The sector of the circle is $\frac{1}{4}$ the area of circle C since the central angle measure is 90° . I can determine the area of the sector and subtract the area of triangle ABC to determine the area of the segment of the circle.



You may not always be able to inscribe a shape inside a circle that you can easily determine the area of, but you can always determine the area of a segment by subtracting the area of the triangle formed by the two radii and the chord from the area of the sector formed by the two radii and the included arc.

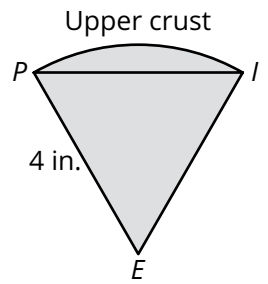
$$\text{Area of segment} = \frac{m}{360^\circ}(\pi r^2) - \frac{1}{2}bh$$

3. Chris covers the entire top of his piece of pie with whipped cream, but he doesn't like to eat the upper portion of the pie crust where it is thickest.

$\overline{EP}$ and $\overline{EI}$ are radii of a circle.

$$m\overline{EP} = 4 \text{ inches}$$

$$m\angle PEI = 60^\circ$$

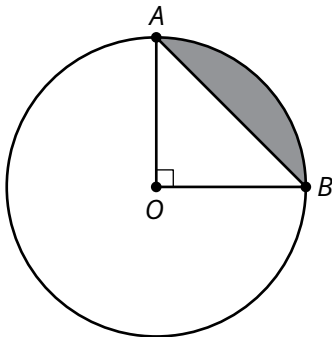


- a. How can you determine the area of $\triangle PIE$?

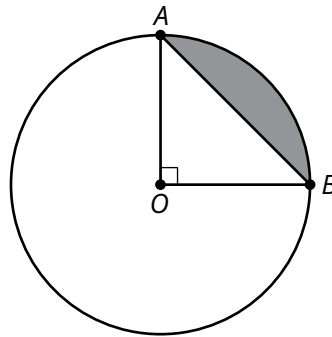
- b. Determine the area of the upper crust. Round your answer to the nearest hundredth.

4. Calculate the radius of each circle with the given area of the shaded segment.

- a. Area of segment = $9\pi - 18$ square feet



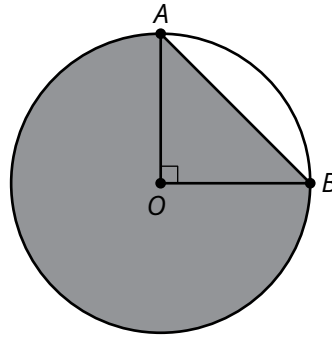
- b. Area of segment = 10.26 square feet



.....
Think About ...

How are the given values for the area of the segment in each figure related?
.....

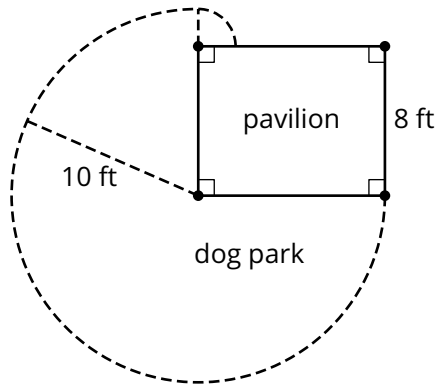
5. The length of the radius is 10 inches. Calculate the area of the shaded region of circle O . Express your answer in terms of π .



ACTIVITY
3.3

Applications of Sectors

A dog park next to a pavilion has 2 sectors. The larger sector has a radius of 10 feet and the smaller sector located at the top left of the pavilion has a radius of 2 feet.

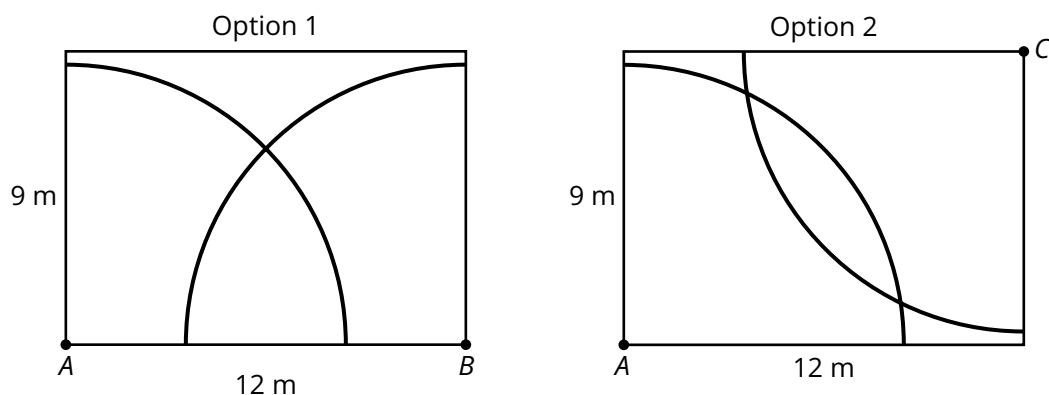


1. Determine the area of the larger sector created by the 10-foot radius. Round your answer to the nearest hundredth.
2. Determine the total area or space the dogs have to play in the dog park. Round your answer to the nearest hundredth.

PROBLEM SOLVING

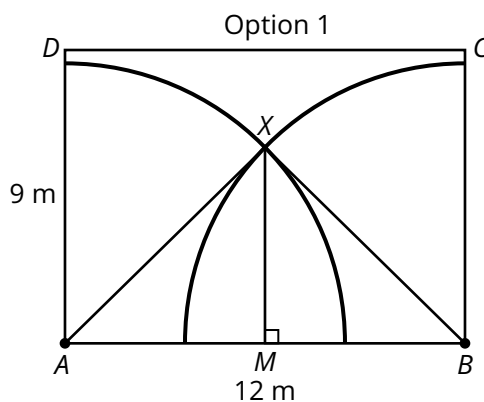


Ashley owns a lawn care business. One client has a rectangular lawn that is 9 meters by 12 meters. He wants to install sprinklers to water the lawn. He is willing to pay for two sprinklers and suggests the following placements for the sprinklers. Each sprinkler rotates in quarter circles and sprays water 8.4 meters.



Ashley must prepare a report for the client that includes her recommendation for how to best water his lawn.

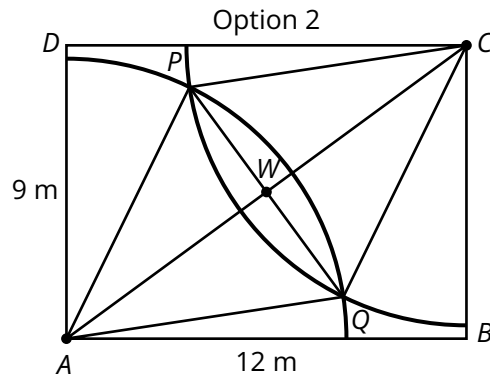
3. Ashley begins by analyzing the suggestions made by the client. She begins by drawing some additional lines to decompose the area of the lawn that is covered by the sprinklers in Option 1.



- a. Describe the shapes that make up the composed area.
- b. List the formulas needed to determine the composite area and describe how to determine the composite area.

4. Use your plan to determine the total area of the lawn covered by the sprinkler in the first option. Round your answer to the nearest hundredth.

5. Ashley draws some additional lines to decompose the area of the lawn that is covered by the second option.



- Describe the shapes that make up the composed area.
- List the formulas needed to determine the composite area and describe how to determine the composite area.

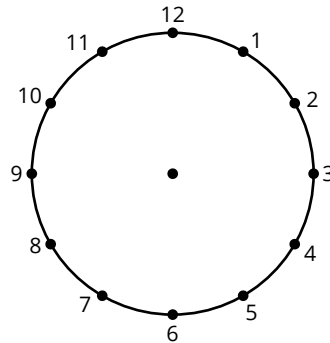
6. Determine the total area of the lawn covered by the sprinklers in the second option. Round your answer to the nearest hundredth.

7. Which option covered a larger area of the lawn?

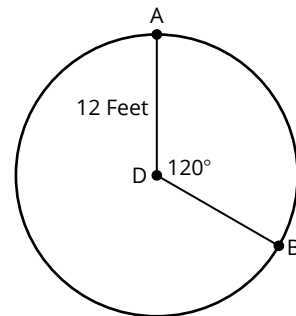
Talk the Talk

What's the Difference?

1. The radius of a clock face is 8 inches.
Determine the area of the sector formed when the hour hand is at 3 and the minute hand is at 1. Round your answer to the nearest tenth.

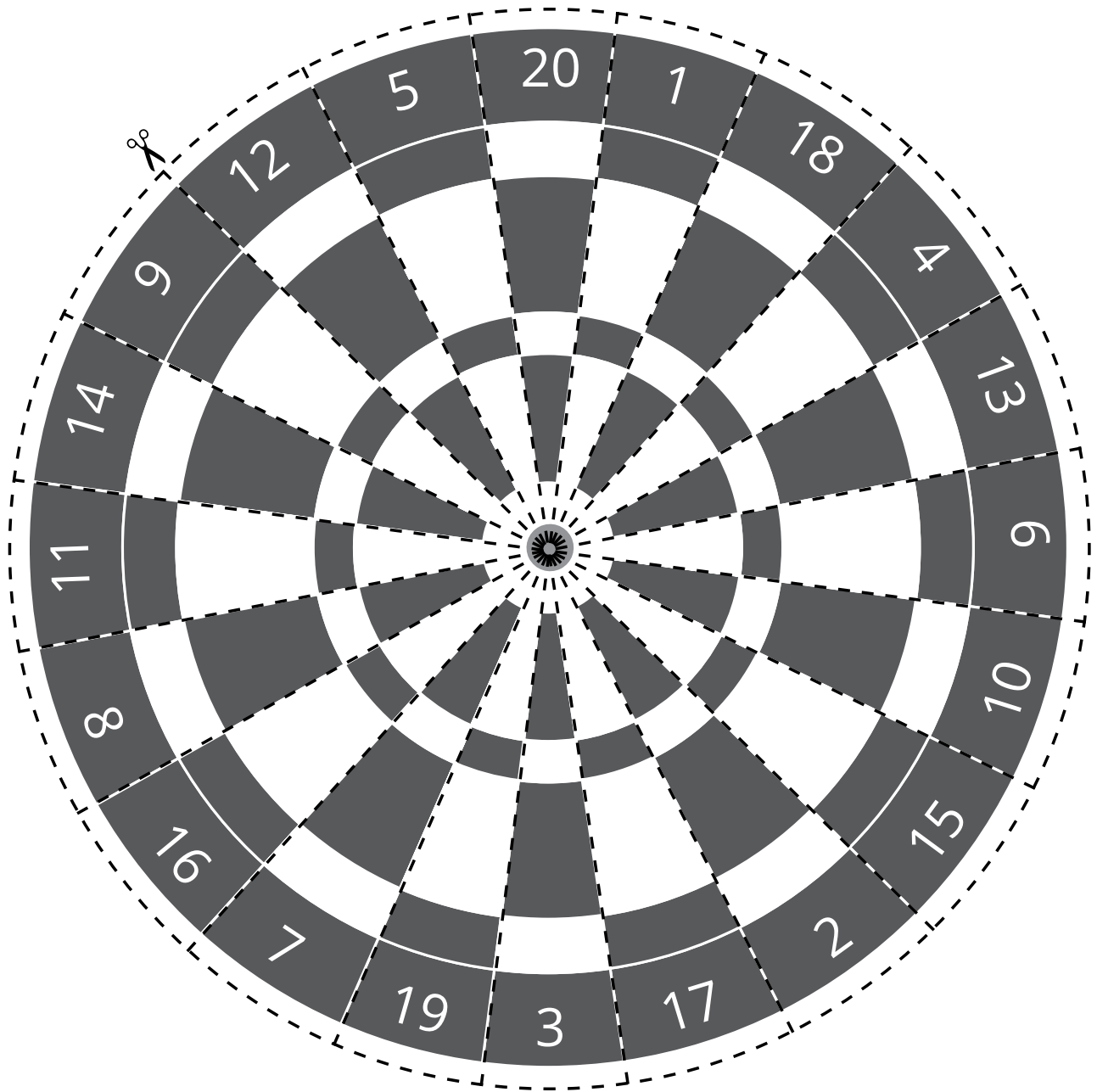


2. Determine the length of the circular track between points A and B. Round your answer to the nearest tenth.



3. Describe the difference between arc length and the measure of the area of a sector.

Dartboard Cut Out



Why is this page blank?

So you can cut out the dartboard on the other side.

Lesson 3 Assignment

Write

Draw an example of each term.

1. sector of a circle
2. segment of a circle

Remember

A sector of a circle is a region of the circle bounded by two radii and the included arc. Its area is a fraction of the area of the circle and can be calculated using the formula $A = (\pi r^2) \frac{m}{360^\circ}$, where m is the measure of the central angle. A segment of a circle is a region of the circle bounded by a chord and the included arc. Its area can be found by subtracting the area of the triangle formed by the two radii and the chord from the area of the sector formed by the two radii and the included arc.

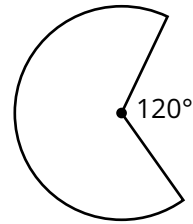
Practice

Use 3.14 for π . Round your answer to the nearest hundredth if necessary.

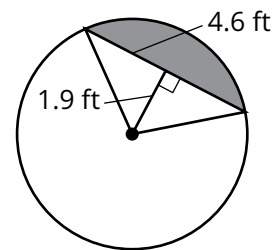
1. Abraj Al Bait Towers clock in Mecca, Saudi Arabia, has a clock face with a diameter of 43 meters.
 - a. Determine the area of the sector formed by the minute hand and the hour hand when the time is 3:00.
 - b. Determine the area of the sector formed by the minute hand and the hour hand when the time is 5:00.

Lesson 3 Assignment

2. A company has a circular card table with a 4-foot diameter. They want to remove a portion to provide a place for the dealer to stand as shown in the diagram. How much surface area of the table will be left for those who are sitting at the table?



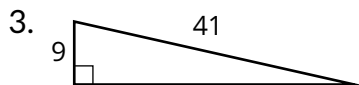
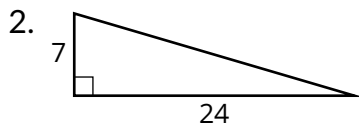
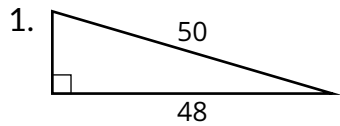
3. Geneva has a circular table with a 6-foot diameter that she would like to put in her new kitchen. In order for it to fit up against the wall, she must cut off the shaded portion of the table as shown in the diagram. The measure of the central angle is 100° . How much surface area will she lose when she removes this part of the table?



Lesson 3 Assignment

Prepare

Determine each unknown side length.



4

Deriving the Equation of a Circle

OBJECTIVES

- Use the Pythagorean Theorem to derive the equation of a circle given the center and radius.
- Distinguish and explain the difference between the equation of a circle written in general form and the equation of a circle written in standard form (center-radius form).
- Complete the square to determine the center and the radius of a circle.

.....

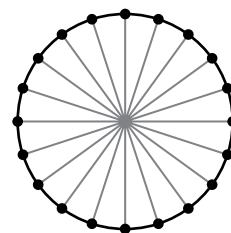
You have graphed linear and quadratic functions and written equations for their graphs.

How can you write the equation for a circle and graph it?

Getting Started

Spinning Right Triangles

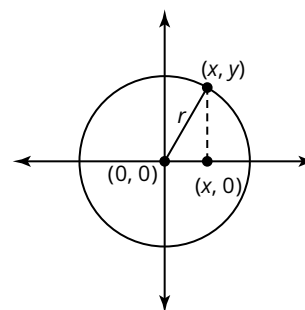
Recall that a *circle* is the set of points on a plane equidistant from a given point. When the circle shown is drawn on a coordinate plane, it is possible to write an algebraic equation for the circle.



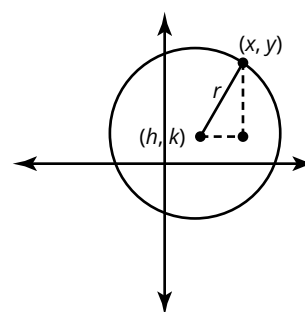
.....
Because your equation is true for every point on the circle, it can be used as the equation of a circle centered at the origin.
.....

1. Follow the steps to write an equation for each circle shown.

- a. Consider a circle drawn with its center located at $(0, 0)$ and a point (x, y) on the circle. Label the sides of the right triangle formed. Then, use the Pythagorean Theorem to solve for r^2 .



- b. Next, consider a circle with its center located at point (h, k) and a point (x, y) on the circle. Label the sides of the right triangle formed. Then, use the Pythagorean Theorem to solve for r^2 .



.....
Think About . . .

The vertical height of the triangle can be expressed in the form $y_2 - y_1$. The horizontal length of the triangle can be expressed in the form $x_2 - x_1$.
.....

2. How does the equation you wrote with a center at (h, k) also describe a circle with its center point at the origin? Explain your reasoning.

The Equation for a Circle

The standard form of the equation of a circle centered at (h, k) with radius of length r can be expressed using the equation shown.

$$(x - h)^2 + (y - k)^2 = r^2$$

1. Write an equation for each description.

a. A circle with center at the origin and $r = 8$

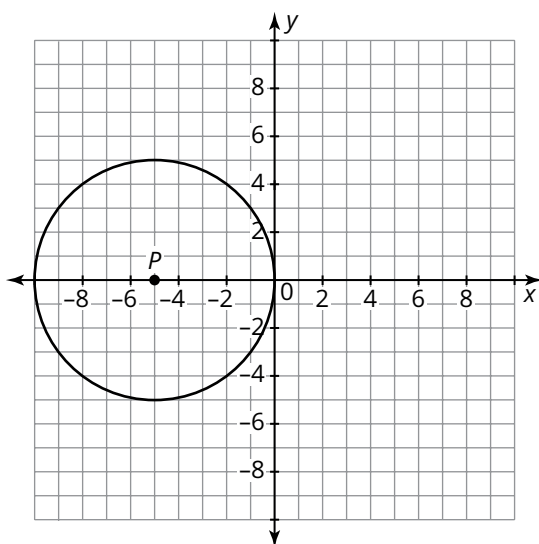
Pay attention to the form of the equation. The coefficients of both x and y are 1.

b. A circle with center $(3, -5)$ and $r = 6$

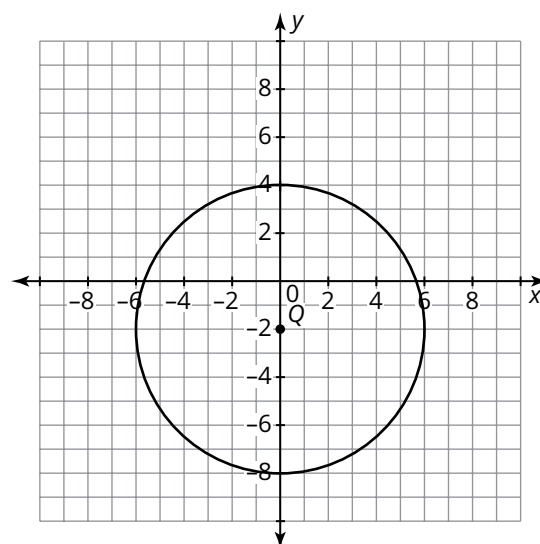
Ask Yourself ...

Is a circle a function?

c. Circle P



d. Circle Q



ACTIVITY 4.2

Completing the Square to Determine the Center

Remember ...

The variables h and k are constants; they represent the center of the circle.

By expanding the binomial terms of the standard form of a circle, $(x - h)^2 + (y - k)^2 = r^2$, the equation can be written as

$$\begin{aligned} x^2 - 2hx + h^2 + y^2 - 2ky + k^2 &= r^2 \\ \text{or} \\ x^2 + y^2 - 2hx - 2ky + h^2 + k^2 &= r^2 \end{aligned}$$

The equation for a circle in general form is $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where A , C , D , E , and F are constants, $A = C$, and $x \neq y$.

To identify the center and radius of a circle written in general form, you need to rewrite the equation in standard form. It is often necessary to use completing the square to rewrite the equation in standard form.

Remember ...

The equation of a circle centered at $(0, 0)$ is $x^2 + y^2 = r^2$.

The equation of a circle centered at (h, k) : $(x - h)^2 + (y - k)^2 = r^2$ is a result of translating a circle with the same radius centered at the origin by h units horizontally and k units vertically.

The composition of translations shifts the entire graph of the circle, so that every point on the circle is moved accordingly. The circle in the worked example is a translation of the circle $x^2 + y^2 = r^2$ to the right 2 and up 3.

WORKED EXAMPLE

You can rewrite $x^2 + y^2 - 4x - 6y + 9 = 0$ in standard form by completing the square.

First, use an algebraic transformation to remove the constant term from the variable expression. Then, write the resulting equation grouping the x -terms together and the y -terms together using sets of parentheses.

$$(x^2 - 4x) + (y^2 - 6y) = -9$$

Next, complete the square within each pair of parentheses. To do this, examine the first two terms of each quadratic expression, and determine the constant term to rewrite each expression as a perfect square trinomial. Add those constant terms to both sides of the equation. Write the resulting equation.

$$(x^2 - 4x + 2^2) + (y^2 - 6y + 3^2) = -9 + 2^2 + 3^2$$

Finally, rewrite the left side of the equation as the sum of the perfect squares.

$$(x - 2)^2 + (y - 3)^2 = 4$$

1. Identify the center point and radius of the circle described by the equation in the Worked Example.

2. Consider the general form of the equation
 $2x^2 + 2y^2 - x + 4y + 2 = 0$.
 - a. How do the coefficients of the squared terms in this equation differ from the equation in the Worked Example? How can you tell this equation represents a circle?

 - b. How does this general form of a circle equation change your strategy to rewrite it in standard form?

 - c. Rewrite the equation in standard form.

3. Determine whether each equation represents a circle. If so, rewrite the equation in standard form and then describe the location of the center and the length of the radius.

a. $x^2 + y^2 - 2x + 4y + 4 = 0$

b. $x^2 + 4y^2 - 24y + 32 = 0$

c. $2x^2 + 2y^2 - 5x + 8y + 10 = 0$

ACTIVITY
4.3

Doubling and Tripling Circumference and Area

Let's consider what happens to the equation of a circle when its circumference or area is changed.

1. Circle P is represented by the equation $(x - 4)^2 + (y + 1)^2 = 36$.
 - a. Determine the equation of a circle that has the same center as circle P , but whose circumference is twice that of circle P .
 - b. Determine the equation of a circle that has the same center as circle P , but whose circumference is three times that of circle P .
 - c. Determine the equation of a circle that has the same center as circle P , but whose area is twice that of circle P .
 - d. Determine the equation of a circle that has the same center as circle P , but whose area is three times that of circle P .

2. Complete the table. Then, identify any pattern(s) you notice.

Equation of Circle P	$r = 6$	$(x - 4)^2 + (y + 1)^2 = 36$
2 Times the Circumference of Circle P		
3 Times the Circumference of Circle P		
4 Times the Circumference of Circle P		
5 Times the Circumference of Circle P		
n Times the Circumference of Circle P		

3. Complete the table. Then, identify any pattern(s) you notice.

Equation of Circle P	$r = 6$	$(x - 4)^2 + (y + 1)^2 = 36$
2 Times the Area of Circle P		
3 Times the Area of Circle P		
4 Times the Area of Circle P		
5 Times the Area of Circle P		
n Times the Area of Circle P		

4. Circle F is represented by the equation $x^2 + (y - 3)^2 = 25$.
- Determine the equation of a circle that has the same center as circle F , but whose circumference is 10 times that of circle F .
 - Determine the equation of a circle that has the same center as circle F , but whose area is 10 times that of circle F .



Talk the Talk

Match Game

Cut out each circle equation located at the end of the lesson. Then, tape each equation into the table with its corresponding center and radius.

Radius Length	Center at (0, 2)	Center at (1, 2)	Center at (1, -2)	Center at (-1, 2)
5				
10				
$\sqrt{5}$				
$\sqrt{10}$				

Circle Equations

**A**

$$x^2 + (y - 2)^2 + 5 = 10$$

B

$$(x + 1)^2 + (y - 2)^2 = 25$$

C

$$(x - 1)^2 + (y + 2)^2 = 10$$

D

$$x^2 + y^2 - 4y + 4 = 100$$

E

$$(x - 1)^2 + (y - 2)^2 = 25$$

F

$$x^2 + y^2 - 2x + 4y = 0$$

G

$$x^2 + y^2 - 2x - 4y - 95 = 0$$

H

$$x^2 + y^2 + 2x - 4y = 5$$

I

$$(x - 1)^2 + (y + 2)^2 = 25$$

J

$$x^2 + (y - 2)^2 = 10$$

K

$$(x + 1)^2 + (y - 2)^2 = 100$$

L

$$(x - 1)^2 + (y - 2)^2 = 5$$

M

$$x^2 + y^2 - 2x - 4y - 5 = 0$$

N

$$(x - 1)^2 + (y + 2)^2 = 100$$

O

$$(x - 1)^2 + (y - 2)^2 = 5$$

P

$$x^2 + y^2 - 4y - 21 = 0$$

Why is this page blank?

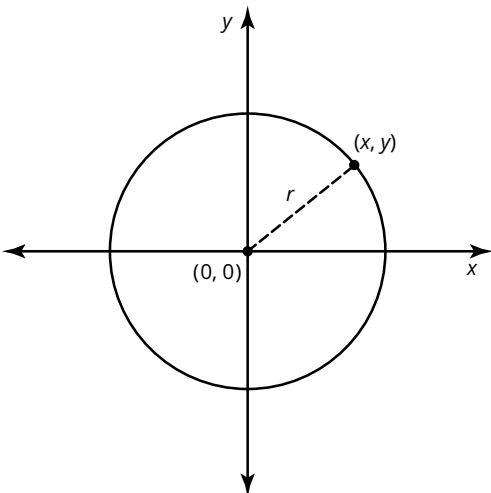
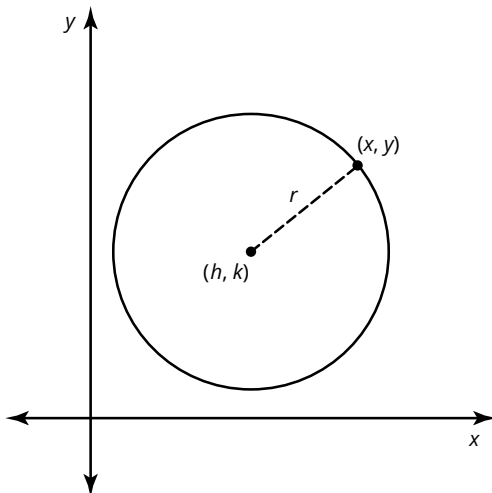
So you can cut out the equations on the other side.

Lesson 4 Assignment

Write

Describe how the graph of $(x - h)^2 + (y - k)^2 = r^2$ is related to the graph of $x^2 + y^2 = r^2$.

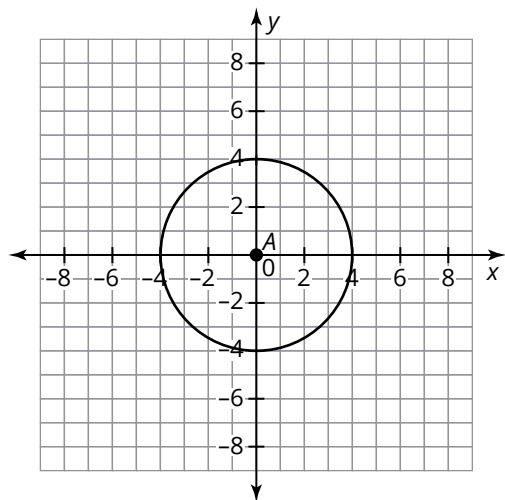
Remember

Circle with center at origin and radius r	Circle with center at (h, k) and radius r
	
Center: $(0, 0)$ Radius: r	Center: (h, k) Radius: r
$x^2 + y^2 = r^2$	$(x - h)^2 + (y - k)^2 = r^2$

Lesson 4 Assignment

Practice

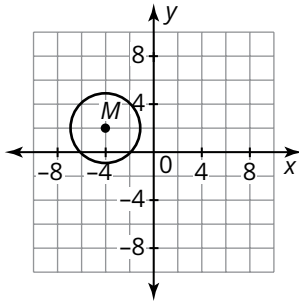
1. Write an equation in standard form given circle A.



- A circle with a center at A and a radius of 4.
- A circle with the same center as circle A, but whose circumference is 4 times that of circle A.
- A circle with the same center as circle A, but whose area is 4 times that of circle A.

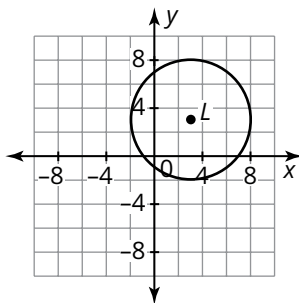
Lesson 4 Assignment

2. Write an equation in standard form given circle M .



- a. A circle with a center at $M(-4, 2)$ and a radius of 3.
- b. A circle with the same center as the circle M , but whose circumference is 20 times that of circle M .

3. Write an equation in standard form given circle L .



- a. A circle with center at $L(3, 3)$ and a radius of 5.
- b. A circle with the same center as the circle L , but whose area is 20 times that of circle L .

Lesson 4 Assignment

4. Determine whether each equation represents a circle. If so, describe the location of the center and radius.

a. $x^2 + y^2 - 4x + 6y + 9 = 0$

b. $4x^2 + 4y^2 - 8x - 20y - 30 = 0$

c. $3x^2 + y^2 + 3x + 9y + 15 = 0$

Lesson 4 Assignment

Prepare

Write the equation of each circle, then describe the composition of transformations that maps a circle with the same radius centered at the origin to the circle.

1. center at $(-2, 6)$, radius of 2
2. center at $(5, -10)$, radius of 10
3. radius of 5, points on the circle at $(3, 2)$ and $(3, -8)$

5

Determining Points on a Circle

OBJECTIVES

- Use the Pythagorean Theorem to determine and explain whether a point lies on a circle on the coordinate plane given the circle's center at the origin or not at the origin, the radius of the circle, and the coordinates of the point.
- Use rigid motion to transform a circle about the coordinate plane to determine whether a point lies on a circle's image given the pre-image's center, radius, and the coordinates of the point.
- Determine the coordinates of a point that lies on a circle given the location of the center point and the radius of the circle.

.....

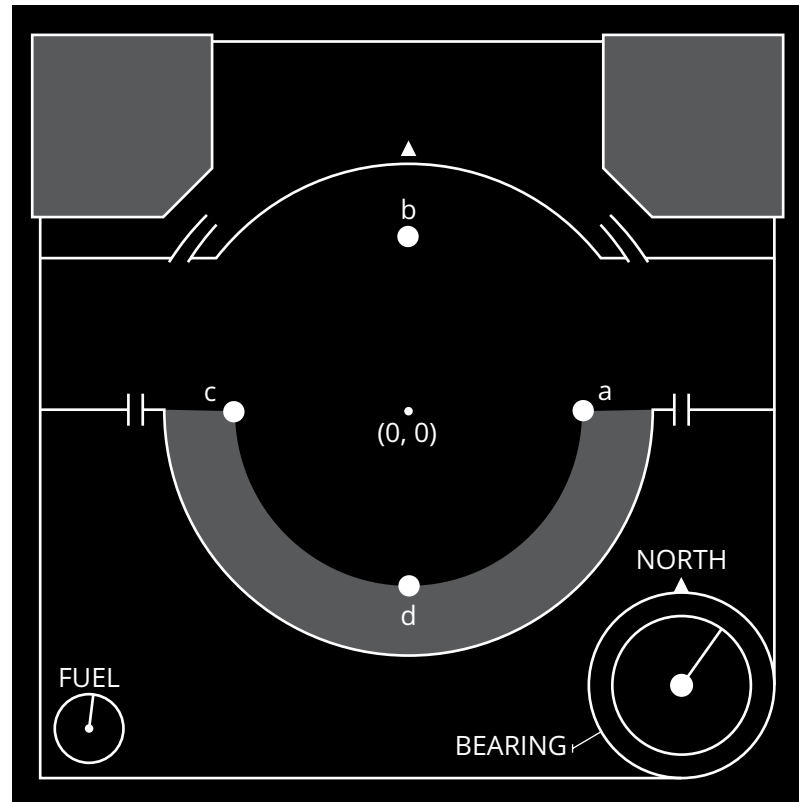
You have determined the equation for a circle centered at the origin and not centered at the origin.

How can you determine the coordinates of points on the circumference of a circle?

Getting Started

Peripheral Vision

In a video game, the locations of objects are tracked by “radar” using coordinates.



1. The points at a, b, c, and d represent objects on the circumference of the circle, which has a radius of 10 units. Label the objects with their coordinates.

2. Consider an object on the circumference of the circle which is not located at one of the four points shown in the diagram. What is the minimum information you need to determine the coordinates of the point? Justify your answer.

Ask Yourself . . .

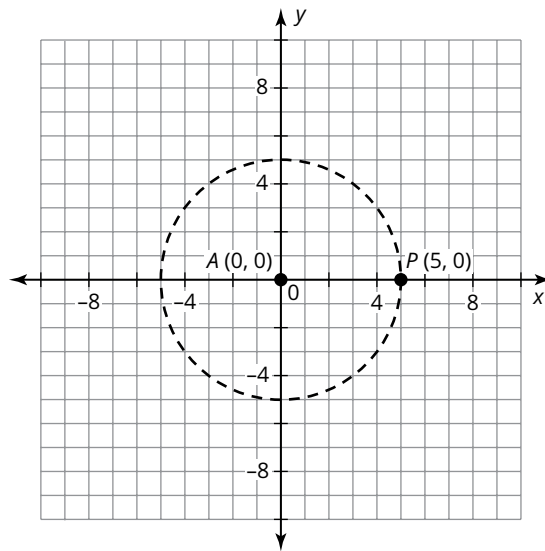
How can you use the ability to determine points on a circle in everyday life?

Identifying Points on a Circle

Let's continue to explore the connection between the Pythagorean Theorem and circles.

Consider circle A with its center point located at the origin and point $P(5, 0)$ on the circle.

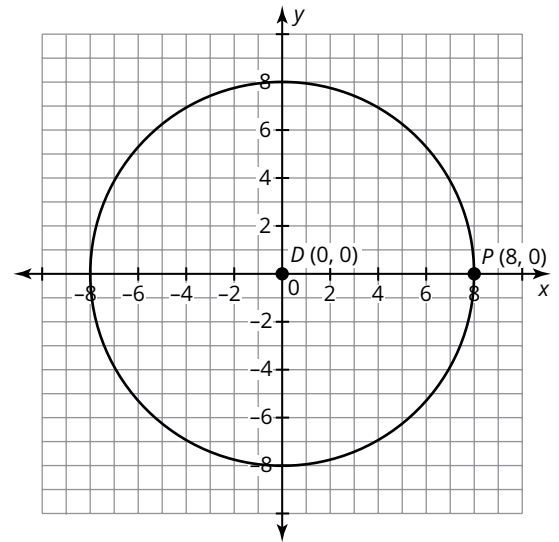
1. Use the axes to plot three additional points on circle A and label the coordinates for each point. Then write the equation of circle A.



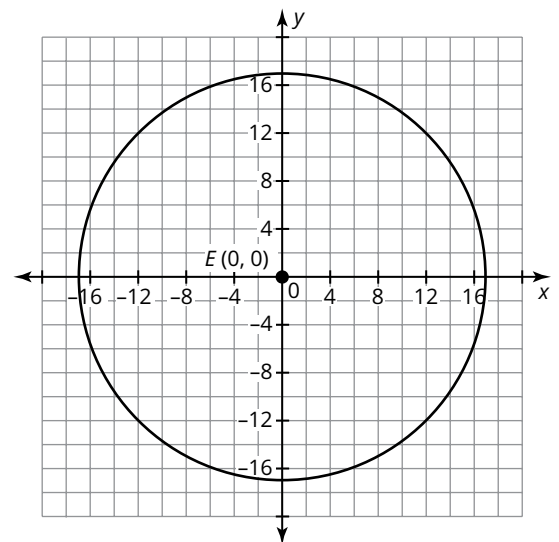
There are an infinite number of points located on circle A. To determine the coordinates of other points located on circle A, you can use the Pythagorean Theorem.

2. Use the Pythagorean Theorem to determine whether point $B(3, 4)$ lies on circle A, then explain your reasoning.

3. Consider circle D centered at the origin with a diameter of 16 units. Write the equation of the circle. Then, determine whether point $H(5, \sqrt{38})$ lies on circle D . If so, use symmetry to determine 3 more points on circle D . When not, explain your reasoning.



4. Consider circle E centered at the origin with a diameter of 34 units. Write the equation of the circle. Then, determine whether point $J(8, 15)$ lies on circle E . If so, use symmetry to determine 3 more points on circle E . When not, explain your reasoning.



ACTIVITY
5.2

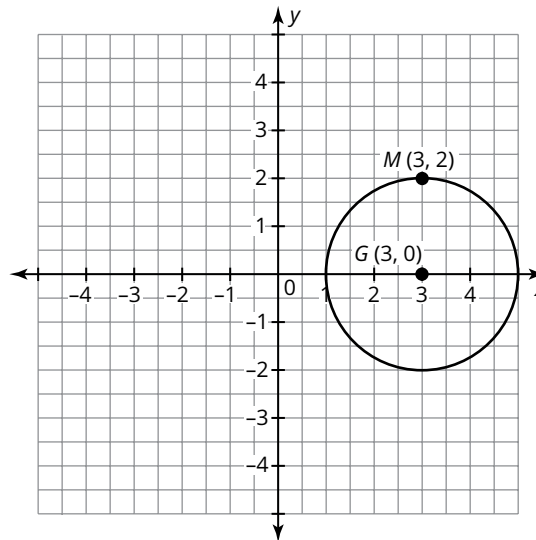
The Center Is Not at the Origin

How can you identify points on a circle when the center is not located at the origin?

.....
Think About...

How does your strategy change when the center of the circle is not on the origin?
.....

1. Consider circle G with its center point located at $(3, 0)$ and point $M(3, 2)$ on the circle.
 - a. Write the equation in standard form of circle G . Explain your reasoning.



- b. Determine whether the point $J\left(4.5, \frac{\sqrt{3}}{2}\right)$ lies on circle G . Explain your reasoning.
 - c. Determine whether the point $P(4, \sqrt{3})$ lies on circle G . Explain your reasoning.

Consider Mariana's statement about additional points on circle G.

Mariana



Only one of those points is located on circle G.
I can use that point and what I know about
symmetry in a circle to identify other points
on circle G.

2. Justify Mariana's reasoning, and identify additional points on circle G.

A figure can be in any of the four quadrants of a coordinate plane—and sometimes a figure is in multiple quadrants. This is also true of circles.

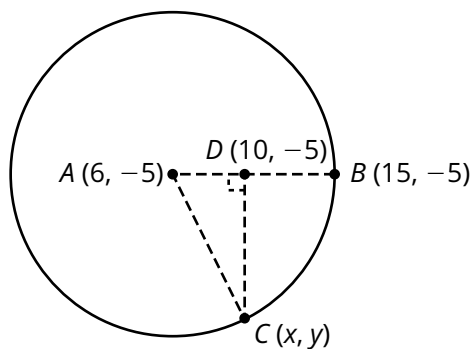
3. Circle K has its center at $(-2, -3)$ and contains point $N(-4, -1.5)$.
Point C is at $(-2, -5.5)$.
 - a. Write the equation in standard form for circle K . Explain your reasoning.

- b. Describe what method(s) you can use to determine whether point C lies on circle K .
- c. Determine whether point C lies on circle K . Describe how you determined your answer.

Talk the Talk

Algebra and Geometry, Together Again

1. Given circle A with radius $\overline{AB}$ and radius $\overline{AC}$, and $\overline{AB} \perp \overline{DC}$ at point D, determine the coordinates of point C.



Lesson 5 Assignment

Write

Explain how to determine the locations of other points on a circle given the coordinates of one point. Draw an example.

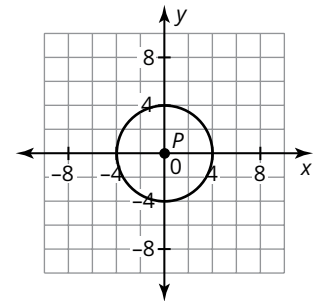
Remember

You can use the Pythagorean Theorem, the distance formula, and symmetry to determine whether a point lies on a circle, given the coordinates of the center point at the origin or not at the origin and the coordinates of a point on the circle.

Practice

1. Consider circle P , centered at the origin with a radius of 4 units.

- a. Write the equation in standard form for circle P .

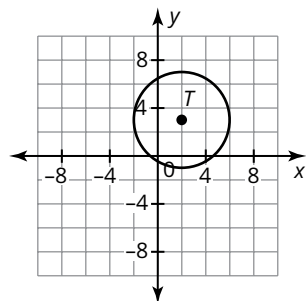


- b. Verify that point $K(1, \sqrt{15})$ lies on circle P .

- c. Use symmetry to determine three more points on circle P .

Lesson 5 Assignment

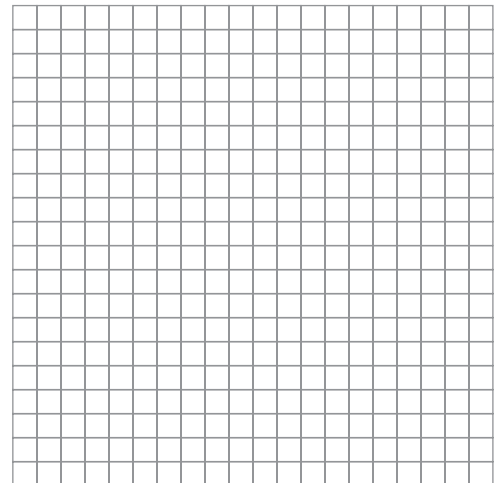
2. Consider circle T , with its center point located at $(2, 3)$ with a radius of $3\sqrt{2}$ units.



- Write the equation in standard form for circle T .
- Verify that point $R(5, 0)$ lies on circle T .
- Use symmetry to determine three more points on circle T .

Lesson 5 Assignment

3. Elena brought home a new puppy who she needs to introduce to her current dog. She temporarily puts her current dog on a runner in the middle of her backyard and her new puppy on a shorter runner. The shorter runner is exactly 4 feet to the east and 4 feet north of where her current dog's runner is located. Based on the graph, the dogs can meet at the point (3, 4).
- a. Graph the range each dog will be able to travel on its runner. Use the origin as the place where the current dog is located. Show your work.

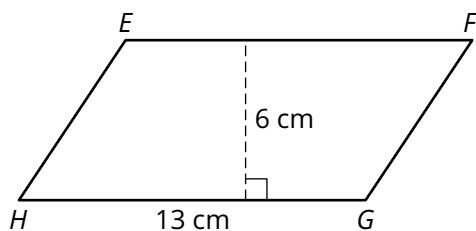
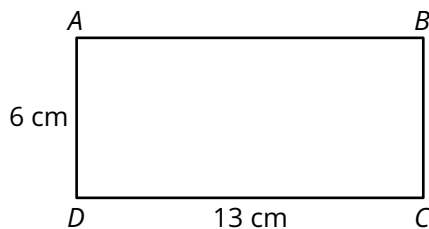


- b. Use symmetry to describe the location of the other point that is on the very edge of both of the dogs' ranges.

Lesson 5 Assignment

Prepare

Consider rectangle $ABCD$ and parallelogram $EFGH$.



1. What is the length of every line segment drawn in the interior of rectangle $ABCD$ and parallel to $\overline{AB}$ and $\overline{CD}$?
2. What is the length of every line segment drawn in the interior of parallelogram $EFGH$ and parallel to $\overline{EF}$ and $\overline{GH}$?
3. Compare the area of rectangle $ABCD$ to the area of parallelogram $EFGH$.

Circles

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Circles* topic by:

TOPIC 1 <i>Circles</i>	Beginning of Topic	Middle of Topic	End of Topic
proving and applying the segment chord theorem, secant segment theorem, and secant tangent theorem.			
proving through similarity transformations that all circles are similar.			
providing an informal argument for the circumference and area formulas for circles.			
using proportional reasoning to determine an arc length.			
defining the radian measure of an angle as the ratio of an arc length to its radius.			
calculating a radian measure given an arc length and its radius.			
using the constant of proportionality to convert between degrees and radians.			
using proportional reasoning to determine the area of a segment or a sector of a circle.			

TOPIC 1 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Circles* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 1 SUMMARY

Circles

LESSON

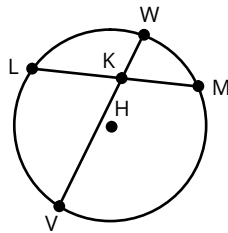
1

Tangents, Segments, and More Chords

Segments of a chord are the segments formed on a chord when two chords of a circle intersect.

The **segment chord theorem** states: “If two chords in a circle intersect, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the second chord.”

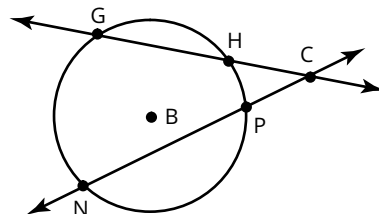
For example, in circle H , chords $\overline{LM}$ and $\overline{VW}$ intersect to form $\overline{LK}$ and $\overline{MK}$ of chord $\overline{LM}$ and $\overline{WK}$ and $\overline{VK}$ of chord $\overline{VW}$. So $LK \cdot MK = WK \cdot VK$.



A secant segment is a segment formed when two secants intersect in the exterior of a circle. An **external secant segment** is the portion of a secant segment that lies on the outside of the circle.

The **secant segment theorem** states: “If two secant segments intersect in the exterior of a circle, then the product of the lengths of one secant segment and its external secant segment is equal to the product of the lengths of the second secant segment and its external secant segment.”

For example, in circle B , secant segments $\overline{GH}$ and $\overline{NP}$ intersect at point C outside the circle. So, $GC \cdot HC = NC \cdot PC$.

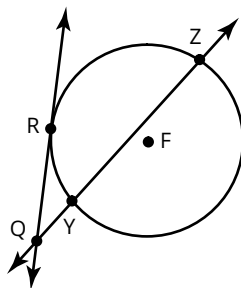


KEY TERMS

- segments of a chord [segmentos de una cuerda]
- segment chord theorem [teorema del segmento-cuerda]
- external secant segment [segmento secante externo]
- secant segment theorem [teorema del segmento secante]
- secant tangent theorem [teorema de la sacante y la tangente]
- arc length
- radian [radián]
- sector of a circle [sector de un círculo]
- segment of a circle [segmento de un círculo]

The **secant tangent theorem** states: “If a tangent and a secant intersect in the exterior of a circle, then the product of the lengths of the secant segment and its external secant segment is equal to the square of the length of the tangent segment.”

For example, in circle F , tangent $\overline{QR}$ and secant $\overline{YZ}$ intersect at point Q outside the circle. So, $QZ \cdot QY = QR^2$.



LESSON

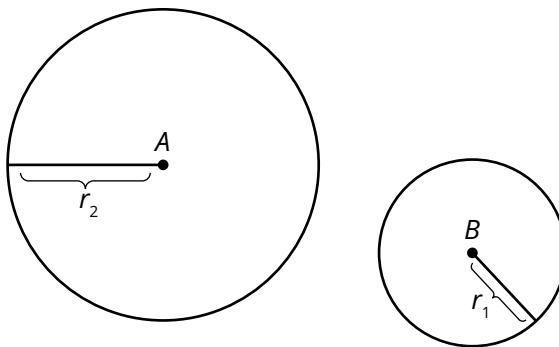
2

Similarity Relationships in Circles

You can use dilations to show that all circles are similar. Recall that two figures are similar if there is a set of transformations that will move one figure onto the other. To show that any two circles are similar, only a translation and a dilation are necessary. To determine their scale factor, find the ratio of the radii.

For example, a dilation is needed to increase circle B to the size of circle A .

The scale factor, k , is $k = \frac{r_2}{r_1}$



Arc length is a portion of the circumference of a circle. The length of an arc is different from the degree measure of the arc. Arcs are measured in degrees, whereas arc lengths are linear measurements.

To determine arc length, s , multiply the circumference of the circle by a fraction that represents the portion of the circumference determined by the central angle measure, m . There is a proportional relationship between the measure of an arc length of a circle, s , and the circumference of the circle.

$$s = \frac{m}{360^\circ} \cdot \text{circumference}$$

The formula for arc length can also be written as follows:

$$s = \frac{m}{360^\circ} \cdot 2\pi r$$

You can apply the formula $s = \frac{m}{360^\circ} \cdot 2\pi r$ to determine the measure of an arc. For example, consider a circle with a radius of 10 inches and a central angle of 80° .

$$\begin{aligned} s &= \frac{m}{360^\circ} \cdot 2\pi r \\ m\widehat{AB} &= \frac{80}{360^\circ} \cdot 2\pi(10) \\ m\widehat{AB} &= \frac{2}{9} \cdot 20\pi \\ m\widehat{AB} &= \frac{40\pi}{9} \end{aligned}$$

The measure of arc length AB is $\frac{40\pi}{9}$, or approximately 14 inches.

The formula implies that the ratio of the arc length to the radius, $\frac{s}{r}$, is directly proportional to m , the measure of the central angle.

$$\begin{aligned} s &= \frac{m}{360^\circ} \cdot 2\pi r \\ s &= \frac{m}{180^\circ} \cdot \pi r \\ \frac{s}{r} &= \frac{\pi}{180^\circ} \cdot m \end{aligned}$$

So far, you have described the measures of angles and arcs using degrees.

A *radian* is another unit that is used to measure angles and arcs. A **radian** is defined as the measure of a central angle whose arc length is the same as the radius of the circle.

Let r represent the length of the radius of the circle, θ represent the measure of the central angle in radians, and s represent the length of the intercepted arc.

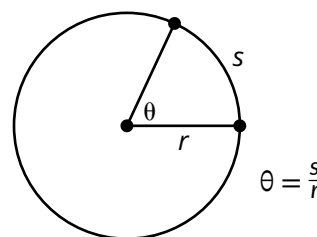
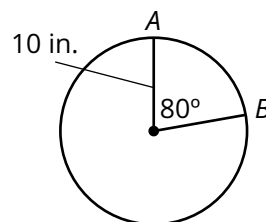
Note that $\frac{s}{r}$ is equal to θ .

The circle has a radius of length r . The circumference of the circle is $2\pi r$. There are 360° in a circle. Because 360° is equivalent to $\frac{2\pi r}{r} = 2\pi$ radians, it follows that 180° is equivalent to π radians.

Therefore, $\frac{\text{radian measure}}{\text{degree measure}} = \frac{\pi}{180^\circ}$ and $\frac{\text{degree measure}}{\text{radian measure}} = \frac{180^\circ}{\pi}$.

When converting degrees to radians, multiply a degree measure by $\frac{\pi}{180^\circ}$.

When converting radians to degrees, multiply a degree measure by $\frac{180^\circ}{\pi}$.



Sectors and Segments of a Circle

A **sector of a circle** is a region of the circle bounded by two radii and the included arc. To determine the area of a sector, A , multiply the area of the circle by a fraction that represents the portion of the area determined by the central angle measure, m . There is a proportional relationship between the measure of the area of a circle sector, A , and the area of the circle.

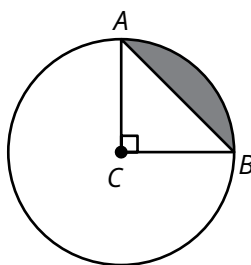
$$A = \frac{m}{360^\circ} \cdot \text{area of a circle}$$

$$\frac{A}{\text{area of a circle}} = \frac{m}{360^\circ}$$

The formula for sector area can also be written as follows:

$$A = \frac{m}{360^\circ} \cdot \pi r^2$$

A **segment of a circle** is a region of the circle bounded by a chord and the included arc.



You can always determine the area of a segment by subtracting the area of the triangle formed by the two radii and the chord from the area of the sector formed by the two radii and the included arc.

$$\text{Area of segment} = \frac{m}{360^\circ}(\pi r^2) - \frac{1}{2}bh$$

Deriving the Equation of a Circle

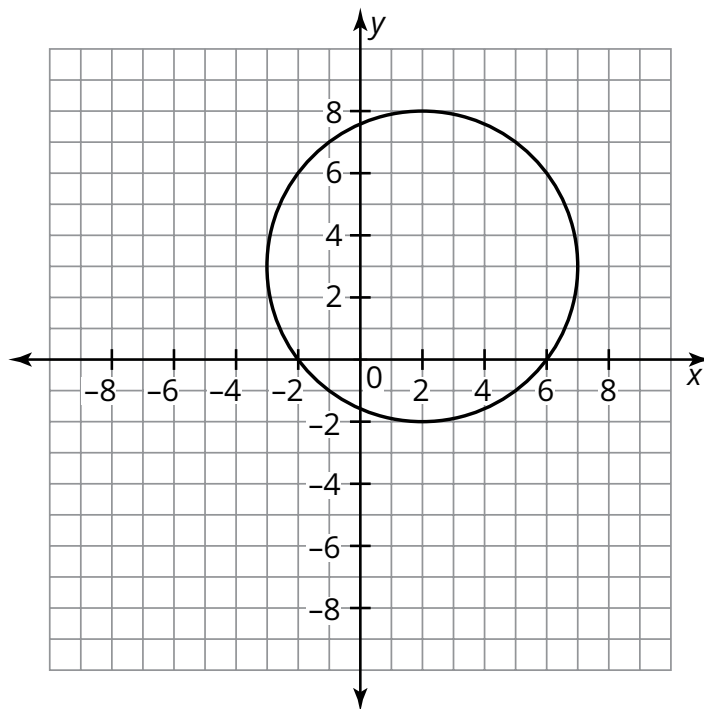
The equation of a circle centered at the origin is $x^2 + y^2 = r^2$, where r is the radius of the circle.

The standard form of the equation of a circle centered at (h, k) with radius r can be expressed as $(x - h)^2 + (y - k)^2 = r^2$.

For example, a circle with a center at $(2, 3)$ and a radius of 5 has the equation $(x - 2)^2 + (y - 3)^2 = 25$.

The equation for a circle in general form is $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where A , C , D , E , and F are constants, $A = C$, and $x \neq y$.

In order to identify the center and the radius of a circle written in general form, it is necessary to rewrite the equation in standard form. Completing the square can be used to rewrite the equation.



For example, rewrite $x^2 + y^2 - 4x - 6y + 9 = 0$ in standard form by completing the square.

First, use an algebraic transformation to remove the constant term from the variable expression. Write the resulting equation by grouping the x -terms together and the y -terms together using sets of parentheses.

$$(x^2 - 4x) + (y^2 - 6y) = -9$$

Next, complete the square within each set of parentheses. To do this, examine the first two terms of each quadratic expression. Determine what the constant term would be if each expression were a perfect square trinomial. Add those constant terms to each side of the equation. Write the resulting equation.

$$(x^2 - 4x + 4) + (y^2 - 6y + 9) = -9 + 4 + 9$$

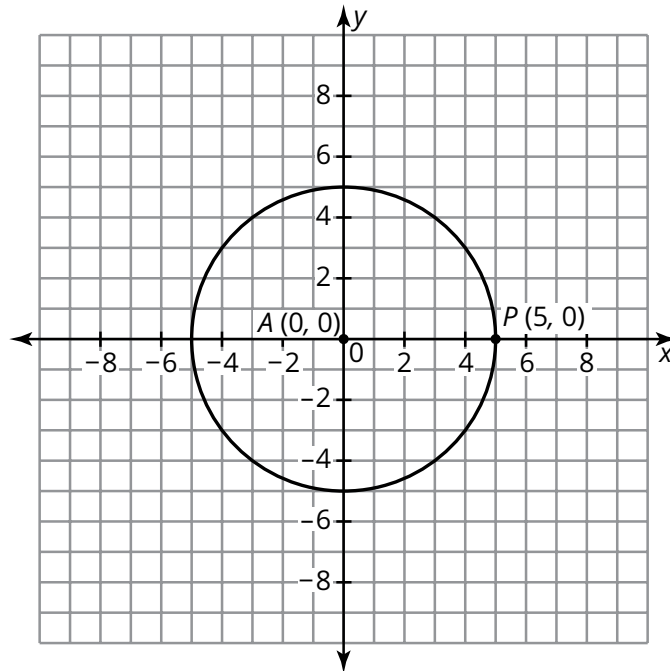
Finally, factor the left side of the equation, which should be a perfect square trinomial. Write the resulting equation.

$$(x - 2)^2 + (y - 3)^2 = 4$$

The center of this circle is $(2, 3)$, and the radius is 2.

Determining Points on a Circle

To determine the coordinates of points located on a circle, you can use the Pythagorean Theorem.



For example, consider circle A with its center point located at the origin and point P (5, 0) on the circle, as shown.

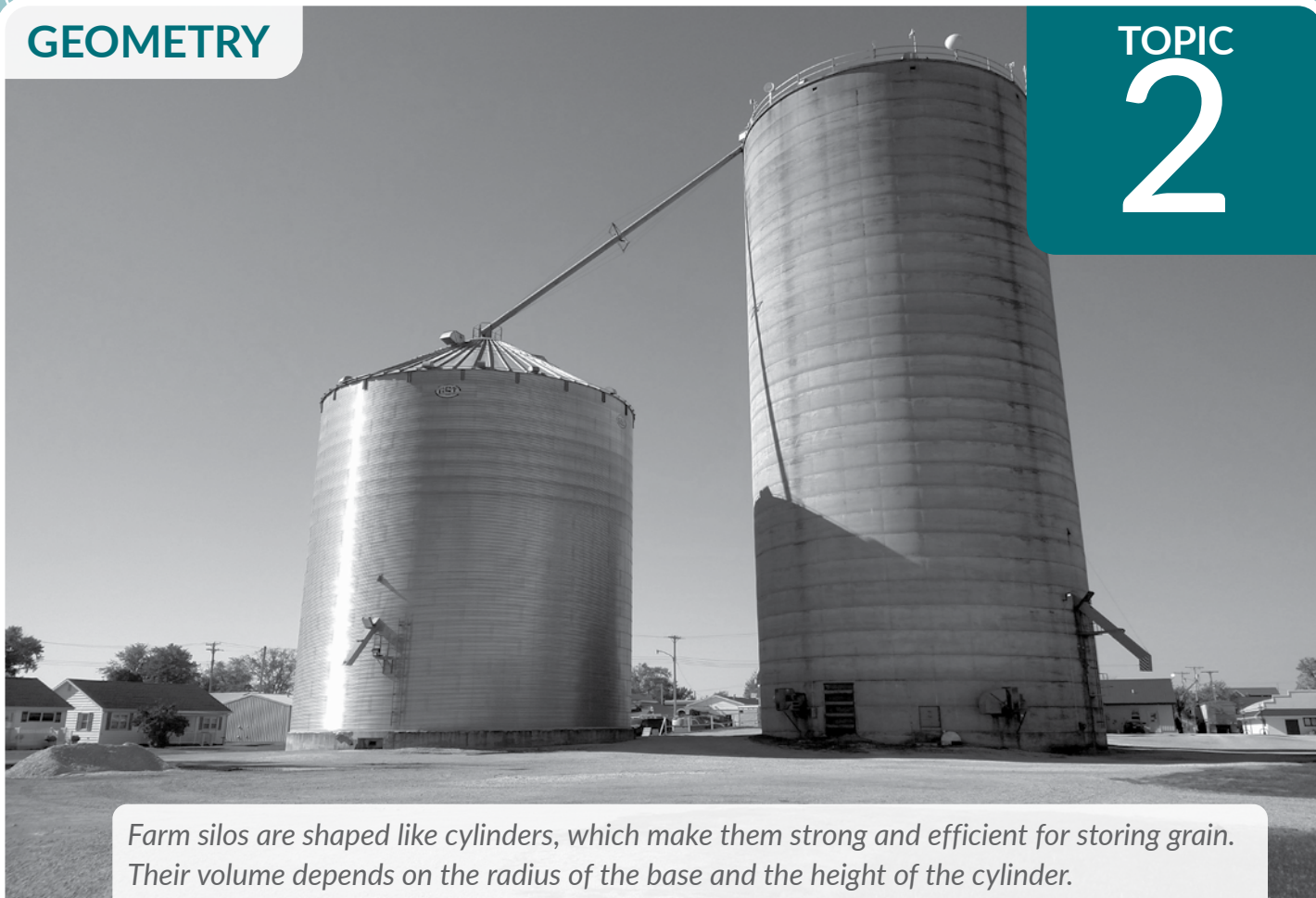
We can determine that (4, 3) is on the circle because we can enter it into our equation for a circle, knowing that the radius is 5 and the center is at (0, 0).

$$x^2 + y^2 = 25$$

$$4^2 + 3^2 = 25$$

$$25 = 25$$

This method also works even if the center of the circle is not at the origin. If you are given the center of a circle and a point that lies on the circle, you can determine if other points lie on the circle as well. First, determine the radius using the distance formula. Then, repeat the method that was shown in the example above and substitute the center and the point in the equation for a circle.



Farm silos are shaped like cylinders, which make them strong and efficient for storing grain. Their volume depends on the radius of the base and the height of the cylinder.

Building Three-Dimensional Shapes

LESSON 1	Building Three-Dimensional Figures	1089
LESSON 2	Cross-Sections	1107
LESSON 3	Building Volume Formulas for Pyramids, Cones, and Spheres	1125
LESSON 4	Building Lateral and Surface Area Formulas for Pyramids, Cones, and Spheres	1147



1

Building Three-Dimensional Figures

OBJECTIVES

- Apply rotations and translations to two-dimensional figures to create three-dimensional solids.
- Describe three-dimensional figures formed by rotations and translations of two-dimensional figures through space.
- Build three-dimensional figures by stacking congruent or similar two-dimensional figures.

KEY TERMS

- disc
- lateral face
- isometric paper
- right triangular prism
- oblique triangular prism
- right rectangular prism
- oblique rectangular prism
- right cylinder
- oblique cylinder

.....

You have used and described geometric transformations.

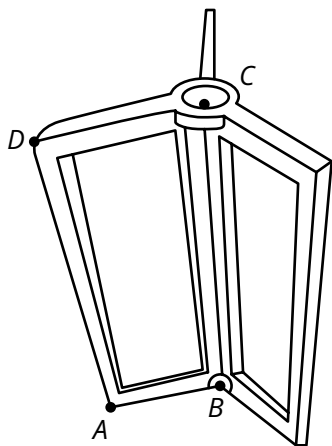
How can you use rigid motions to create and describe the volume of three-dimensional figures?

Getting Started

Revolutionary Ideas

An office building has a revolving door at the entrance. The door has congruent rectangular door panels that rotate around a center pole. Rectangle $ABCD$ represents one of the door panels.

1. Consider point A . Describe the shape the point traces as the door panel rotates around the center pole.



2. Consider side $\overline{AB}$. Describe the shape formed as the set of points included in the segment rotate around the center pole.

.....
The first revolving door was installed in 1899 at a restaurant in Times Square, New York City.
.....

While a circle is the set of all points that are equal distance from the center, a **disc** is the set of all points on the circle and in the interior of the circle.

3. Consider all the points that make up rectangle $ABCD$ and its area. What solid seems to be formed as the door panel rotates around the center pole? Explain your reasoning.

ACTIVITY
1.1

Rotating 2-D Shapes through Space

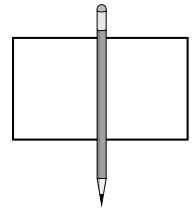
Previously, you learned about rotations on a coordinate plane. You can also perform rotations in three-dimensional space.

For each shape in Questions 1 through 3:

- Draw the shape on an index card.
- Cut out the shape and tape it along the center to a pencil below the eraser, as shown.
- Hold onto the eraser with your thumb and index finger such that the pencil is resting on its tip. Rotate the shape by holding onto the eraser and spinning the pencil. You can get the same effect by putting the lower portion of the pencil between both palms of your hands and rolling the pencil by moving your hands back and forth.

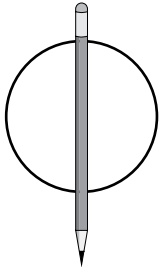
1. Rectangle

- a. Sketch the three-dimensional figure formed by spinning the two-dimensional shape.

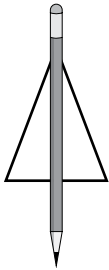


- b. Name the solid formed by rotating the rectangle about the pencil.
- c. Relate the dimensions of the rectangle to the dimensions of this solid.

2. Disc



- Sketch the three-dimensional figure formed by spinning the two-dimensional shape.
- Name the solid formed by rotating the disc about the pencil.
- Relate the dimensions of the disc to the dimensions of this solid.



3. Isosceles Triangle

- Sketch the three-dimensional figure formed by spinning the two-dimensional shape.
- Name the solid formed by rotating the triangle about the pencil.
- Relate the dimensions of the triangle to the dimensions of this solid.

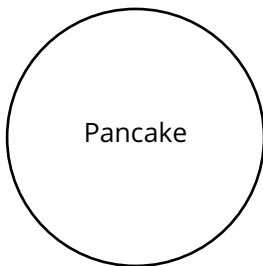
ACTIVITY
1.2

Transitioning to Three Dimensions

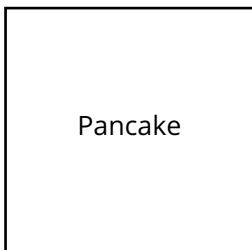
The math club at school is planning a pancake breakfast as a fundraiser. Because this is a fundraiser for the math club, the pancakes will use various geometric shapes!

1. Imagine you stack each of the geometric pancakes on top of each other. Name the solid formed by each stack of pancakes, and relate the dimensions of each pancake to the dimensions of the solid figure created.

a.



b.



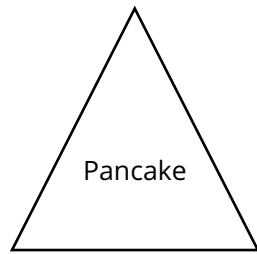
.....

Remember, similar figures have the same shape. Congruent figures have the same shape AND size.

.....

Ask Yourself . . .
What patterns do you notice?

c.



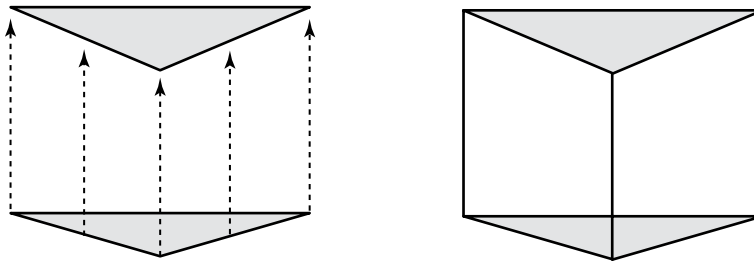
2. What do you notice about the three-dimensional figures created by stacking congruent figures?
3. What type of solid would be formed by stacking congruent rectangles? pentagons? hexagons?

ACTIVITY 1.3

Using Rigid Motions to Create Solid Figures

You can translate a two-dimensional figure through space to create a model of a three-dimensional figure.

- Suppose you and your classmate want to design a cake with triangular bases. You can imagine that the bottom triangular base is moved straight up to create the top triangular base.



- What is the shape of each lateral face of the triangular prism formed by this movement?
- What is the name of the rigid motion used to create the polyhedron?

Recall that a translation is a transformation that “slides” each point of a figure the same distance in the same direction.

A **lateral face** of a three-dimensional object is a face that is not a base.

You can use **isometric paper**, or dot paper, to create a two-dimensional representation of a three-dimensional figure. Engineers often use isometric drawings to show three-dimensional diagrams on “two-dimensional” paper.

- Translate each triangle to create a second triangle.
Use dashed line segments to connect the corresponding vertices.

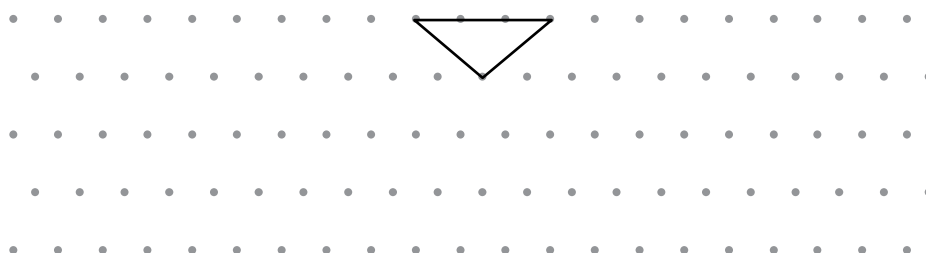
- Translate this triangle in a diagonal direction.



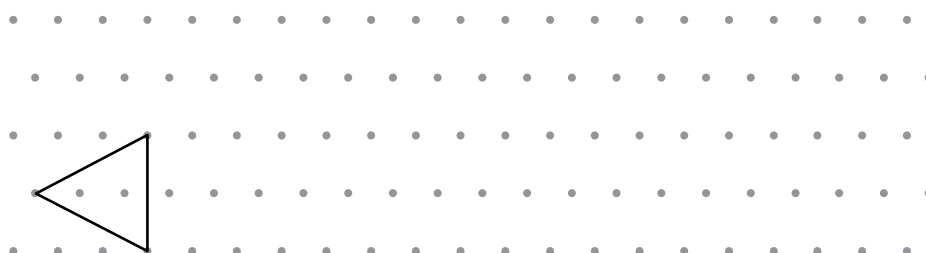
b. Translate this right triangle in a diagonal direction.



c. Translate this triangle vertically.



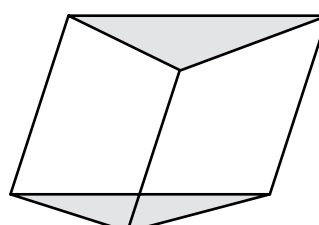
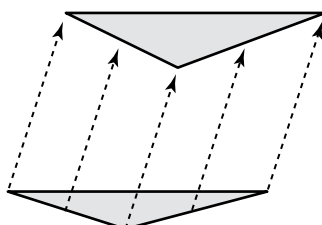
d. Translate this triangle horizontally.



3. What do you notice about the relationship among the line segments connecting the vertices in each of your drawings?

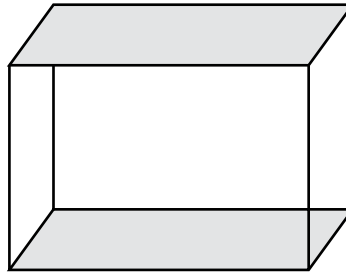
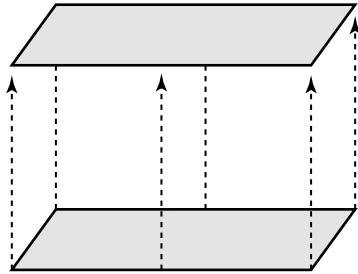
When you translate a triangle through space in a direction that is perpendicular to the plane containing the triangle, the solid formed is a **right triangular prism**. The triangular prism cake that you and your classmate created in Question 1 is an example of a right triangular prism. When you translate a triangle through space in a direction that is *not* perpendicular to the plane containing the triangle, the solid formed is an **oblique triangular prism**.

An example of an oblique triangular prism is shown.



4. What is the shape of each lateral face and base of an oblique triangular prism?

5. Suppose you and your classmate want to design a cake with rectangular bases. You can imagine that the bottom rectangular base is translated straight up to create the top rectangular base.



- a. What is the shape of each lateral face of the three-dimensional figure formed by this translation?
- b. What is the name of the solid formed by this translation?

A two-dimensional representation of a rectangular prism can be obtained by translating a rectangle in two dimensions and connecting corresponding vertices.

6. Draw a rectangle and translate it in a diagonal direction to create a second rectangle.



7. Analyze your drawing.
- a. What do you notice about the relationship among the line segments connecting the vertices in the drawing?
- b. What two-dimensional figure would you translate to create a cube? Sketch an example of a cube.

.....
Think About . . .

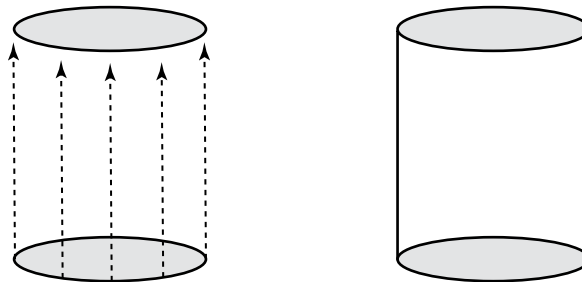
The set of all the translations along the parallel line segments creates the solid figure.
.....

When you translate a rectangle through space in a direction that is perpendicular to the plane containing the rectangle, the solid formed is a **right rectangular prism**. The rectangular prism cake that you and your classmate created in Question 5 is an example of a right rectangular prism. When you translate a rectangle through space in a direction that is *not* perpendicular to the plane containing the rectangle, the solid formed is an **oblique rectangular prism**.

8. Identify the shape of each lateral face of an oblique rectangular prism.

9. Sketch an oblique rectangular prism.

10. Suppose you and your classmate want to design a cake with circular bases. You can imagine that the bottom circular base, a disc, is translated straight up to create the top circular base.



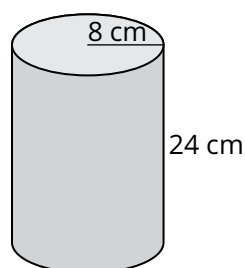
a. Identify the shape of the lateral face of the three-dimensional figure formed by this translation.

b. What is the name of the solid formed by this translation?

ACTIVITY
1.4**Building the Cylinder
Volume Formula**

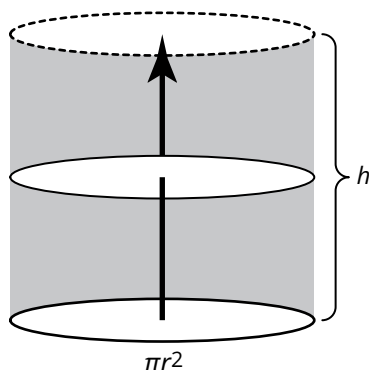
You have learned that when two-dimensional shapes are translated, rotated, or stacked, they can form three-dimensional solids. You can also use transformations and stacking to build formulas for three-dimensional figures.

1. Determine the volume for the cylinder shown. Show your work. Round your answer to the nearest hundredth.

**WORKED EXAMPLE**

To derive the formula for the volume of a cylinder, you can think of the cylinder as an infinite stack of discs, each with an area of πr^2 . These discs are stacked to a height of h , the height of the cylinder.

So, the volume of the cylinder is $(\pi r^2)(h)$, or $\pi r^2 h$.



2. Can you choose any disc in the cylinder and multiply the area by the height to calculate the volume of the cylinder? Explain your reasoning.

- A two-dimensional representation of a cylinder can be obtained by translating an oval in two dimensions and connecting the tops and bottoms of the ovals.

5. Translate the oval in a diagonal direction to create a second oval. Use dashed line segments to connect the tops and bottoms of the ovals.



- When you translate a disc through space in a direction that is perpendicular to the plane containing the disc, the solid formed is a **right cylinder**. The cylinder cake that you and your classmate created in the previous activity is an example of a right cylinder. When you translate a disc through space in a direction that is *not* perpendicular to the plane containing the disc, the solid formed is an **oblique cylinder**.



7. Samantha says the lateral face of an oblique cylinder is a parallelogram. Kaya says that it is neither a rectangle nor a parallelogram. Who's correct? Explain your reasoning.



Talk the Talk

Stacking and Rotating

In this lesson, you used transformations to create solid figures.

1. Explain the relationship between the volume formulas and the transformations you can use to create each solid.

- a. Prisms

$$V = Bh$$

$$V = \ell wh$$

- b. Cylinders

$$V = Bh$$

$$V = \pi r^2 h$$

2. A right triangle is rotated about one of its legs.

- a. Sketch the three-dimensional solid formed by spinning the two-dimensional shape.

- b. Name the solid formed by rotating the right triangle about its leg.
 - c. Relate the dimensions of the right triangle to the dimensions of this solid.
3. A cylindrical trash can has a diameter of 14 inches and a height of 16 inches. Determine the volume of the trash can. Round your answer to the nearest whole number.

Lesson 1 Assignment

Write

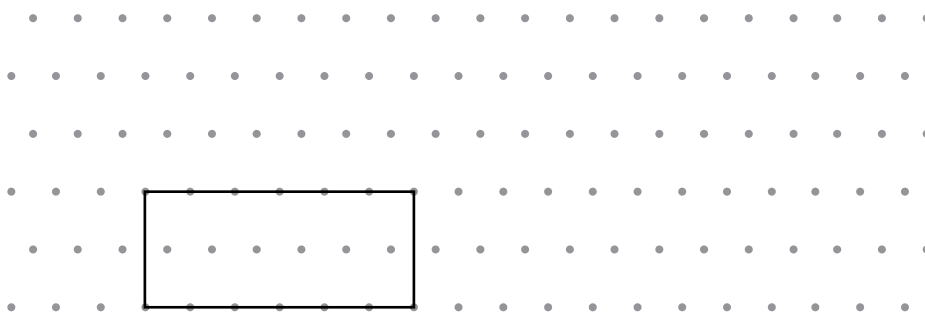
In your own words, describe the difference between a right prism and an oblique prism.

Remember

You can create three-dimensional figures by rotating or translating two-dimensional shapes. You can build the cylinder volume formula by stacking an infinite number of discs with an area of πr^2 . The formula for the volume of a cylinder is $V = \pi r^2 h$.

Practice

1. Identify the solid and its dimensions formed by a rotation using a vertical axis.
 - a. A 4 cm by 6 cm rectangle with its shorter side attached to the axis.
 - b. A 4 cm by 6 cm rectangle with its longer side attached to the axis.
2. Translate the rectangle in a diagonal direction to create a second rectangle. Identify the name of the solid formed by this translation when the corresponding vertices of the two rectangles are connected.

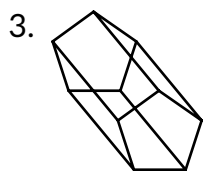
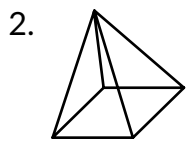
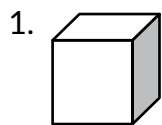


Lesson 1 Assignment

3. Congruent discs with an area of 30π square inches are stacked to a height of 8 inches.
- What is the name of the solid figure formed by the stacked discs?
 - What is the volume of the solid figure formed by the stacked discs? Round your answer to the nearest hundredth.

Prepare

Determine the number of faces, edges, and vertices of each solid.



2

Cross-Sections

OBJECTIVES

- Determine the shapes of cross-sections.
- Determine the shapes of intersections of solids and planes.
- Define the degenerate conics.

KEY TERM

- cross-section

.....

You have now explored how to form three-dimensional solids by stacking, translating, or rotating two-dimensional shapes.

How can cross-sections of a three-dimensional solid help you to analyze the characteristics of the inside of an object and how it is put together?

Getting Started

Put a Cork in It

A bottle company designs three bottles for olive oil with three different openings. The top view of the bottle openings is shown.



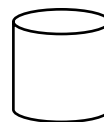
Bottle A

Bottle B

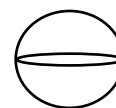
Bottle C

The company also designs a number of cork bottle stoppers in the shapes of different three-dimensional solids, as shown. A stopper is meant to fit into the bottle opening without leaving any gaps.

1. Which stopper could fit both Bottle A and Bottle B? Justify your choice.

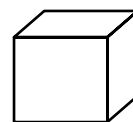


Stopper 1

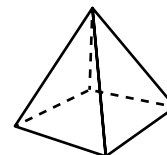


Stopper 2

2. Which stopper could fit both Bottle A and Bottle C? Justify your choice.

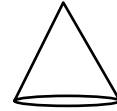


Stopper 3

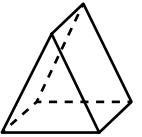


Stopper 4

3. Which stopper could fit both Bottle B and Bottle C?
Justify your choice.



Stopper 5



Stopper 6

4. Are there any stoppers that would fit in only one of the bottles?
Explain your reasoning.

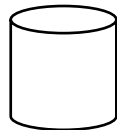
ACTIVITY
2.1

Point and Line Segment Cross-Sections

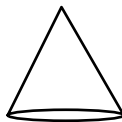
.....
A cube is also a
rectangular prism.
.....

A **cross-section** of a three-dimensional solid results from the intersection of a plane with the solid. When the plane passes through the solid, a two-dimensional cross-section is formed. Are there other geometric figures that result from the intersection of a plane and a solid?

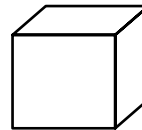
Consider the three-dimensional solids shown.



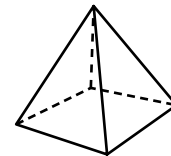
Cylinder



Cone



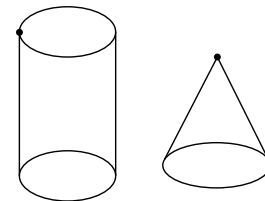
Cube



Pyramid



1. Trung says that when a solid has a circle as one of its bases, it can have a cross-section that is a point. Natalia says when a solid has a vertex, it can have a cross-section that is a point. Who is correct? Use sketches to justify your reasoning.



2. Describe how a plane can intersect each solid to form a cross-section that is a line segment.

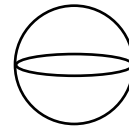
a. cylinder

b. cone

c. cube

d. pyramid

When you rotated the disc on the pencil in a previous lesson, a sphere appeared to be formed. Recall that a *sphere* is the set of all points in three dimensions that are equidistant from a given point, called the *center*.



Sphere

3. Can a sphere have a cross-section that is a point?
Explain your reasoning.

4. Can a sphere have a cross-section that is a line segment?
Explain your reasoning.

In the previous lesson, you translated shapes through space to create three-dimensional figures. When you translate the shape through space, it's like stacking copies of the shape on top of each other. The shape you translated is a slice or cross-section of the three-dimensional figure, cutting through in a way that is parallel to the bases.

1. Translate the quadrilateral in a diagonal direction. Use dashed line segments to connect the corresponding vertices.



2. Draw a plane through the prism that is parallel to a base.
3. Describe the cross-section.

A section of a tree trunk is roughly in the shape of a cylinder, as shown.

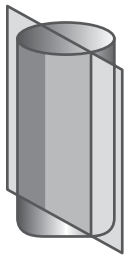


When a tree trunk is cut in order to see the tree rings, a cross-section of the trunk may be being studied.

4. Determine the shape of each cross-section. Then, sketch an example of the cross section.

a. Suppose a plane intersects a cylinder parallel to its bases.

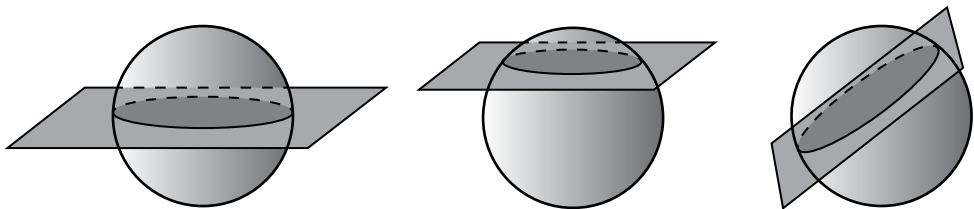
b. Suppose a plane intersects a cylinder perpendicular to its bases so that the plane passes through the centers of the bases.



c. Suppose a plane intersects a cylinder so that it is not parallel to its bases.

.....
Remember ...
A great circle of a sphere is a cross-section of a sphere when a plane passes through the center of the sphere.
.....

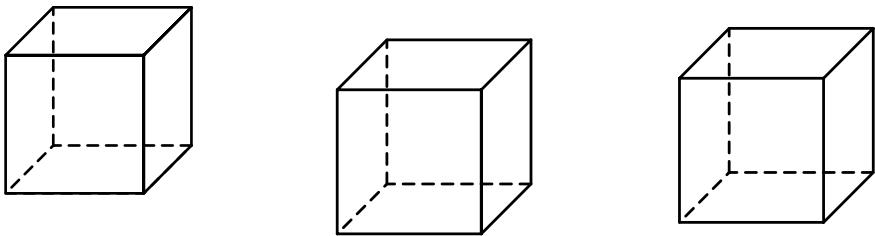
5. Consider the different cross-sections of a sphere shown.



- a. Identify the two-dimensional cross-sections of a sphere.
- b. What do you know about all the two-dimensional cross-sections of a sphere?

.....
Use models when you have them. They can help you “see” the cross-sections.
.....

6. Consider a cube. Sketch and describe five different two-dimensional cross-sections formed when a plane intersects a cube.



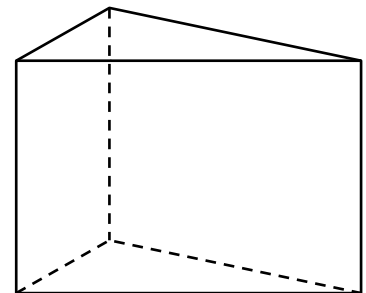
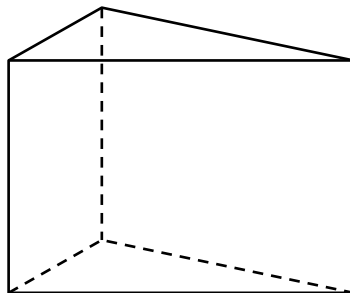
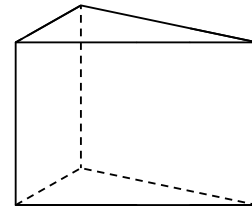
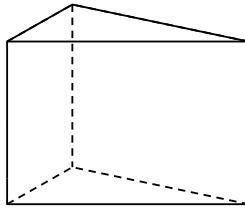
.....
Review the definition of **describe** in the Academic Glossary.
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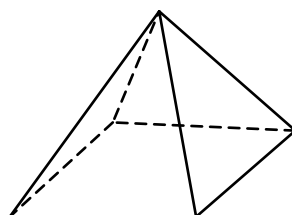
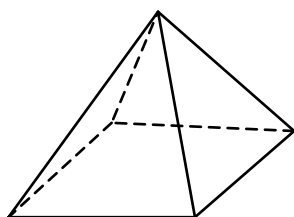
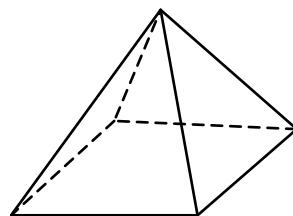
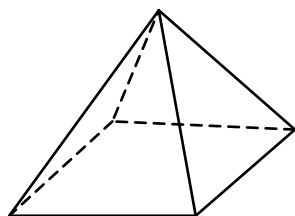


7. Alyssa says that it's not possible to form a circle as a cross-section of a cube. Is Alyssa correct? Explain your reasoning.

8. Consider a triangular prism. Sketch and describe four different two-dimensional cross-sections formed when a plane intersects the triangular prism.



9. Consider a square pyramid. Sketch and describe four different two-dimensional cross-sections formed when a plane intersects a square pyramid.



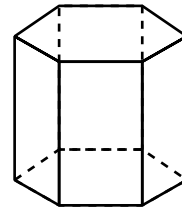
10. Josh says the maximum number of sides of a cross-section equals the number of faces of the solid, when it is a polyhedron. Is he correct? Explain your reasoning.

Talk the Talk

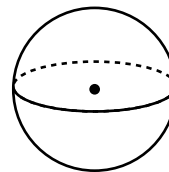
Cross Examination

Cross-sections can help you analyze the characteristics of the inside of a solid as well as how it is put together.

1. Use the solid to create two cross-sections. Create one cross-section parallel to a base and a second cross-section perpendicular to a base. Then, identify each of the cross sections.



2. Create two cross-sections for the sphere. Create one cross-section through the center of the sphere and one cross-section that is not through the center of the sphere. Then, identify each of the cross-sections.

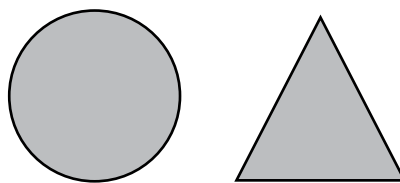


Ask Yourself . . .

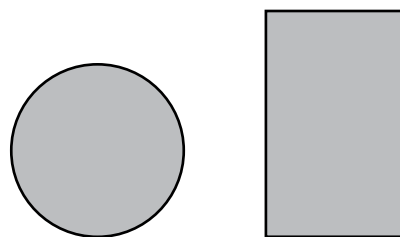
How does representing mathematics in multiple ways help to communicate reasoning?

3. Draw a solid that could have the given cross-sections.

- a. A cross-section parallel to a base and
a cross-section perpendicular to a base.



- b. A cross-section parallel to a base and
a cross-section perpendicular to a base.



4. Consider a prism placed on a table such that the base is horizontal. When a plane passes through the prism horizontally at any height, describe the cross-sections.

5. Define a cylinder in terms of its cross-sections.

6. Are the given figures possible cross-sections of a cube? When it is not possible, explain why.

a. a hexagon

b. an octagon

c. a rectangle

d. an equilateral triangle

e. a circle

f. a triangle that is not equilateral

g. a pentagon

h. a parallelogram that is not a rectangle

i. a rectangle that is not a square

j. a square

Lesson 2 Assignment

Write

Describe the three geometric figures that can result from the intersection of a plane with a solid, then sketch an example of each.

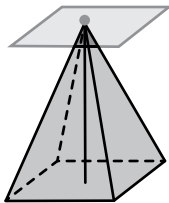
Remember

- A cross-section of a three-dimensional solid can be a point, a line segment, or a two-dimensional figure that is formed by the intersection of the solid and a plane.
- The maximum number of sides of a cross-section equals the number of sides of faces of the solid, if it is a polyhedron.

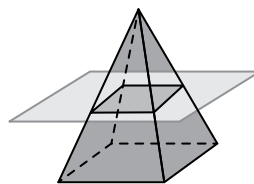
Practice

1. Describe the shape of each cross-section.

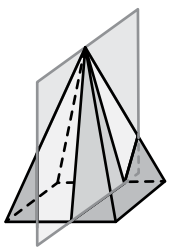
a.



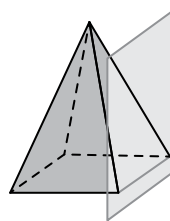
b.



c.



d.



Lesson 2 Assignment

2. Sketch two cross-sections of a pentagonal prism—one cross-section that is parallel to the base and another cross-section that is perpendicular to the base.
3. Sketch two cross-sections of a cone—one cross-section that is parallel to the base and another cross-section that is perpendicular to the base.
4. A solid's cross-section parallel to the base is an octagon.
A cross-section of the solid perpendicular to the base is a triangle.
Identify the solid.
5. A solid's cross-section parallel to the base is a triangle.
A cross-section of the solid perpendicular to the base is a rectangle. Identify the solid.

Lesson 2 Assignment

Prepare

1. Name the geometric solid associated with each situation.
 - a. a stack of 100 copies of the same book
 - b. a stack of 100 tires, all with the same radius
 - c. a stack of 100 stop signs
2. Describe a stack of objects that best resembles each geometric solid.
 - a. a hexagonal prism
 - b. a pentagonal prism
 - c. a triangular prism

3

Building Volume Formulas for Pyramids, Cones, and Spheres

OBJECTIVES

- Give informal arguments to explain the volume formulas of pyramids and cones.
- Verbally describe the relationship between the volume formulas of a prism and a pyramid.
- Verbally describe the relationship between the volume formulas of a cone and a cylinder.
- Use the formulas for the volume of a cylinder, cone, and sphere to solve problems.
- Determine the volume of composite figures.

KEY TERMS

- sphere
- radius of a sphere
- diameter of a sphere
- hemisphere

.....

You have used translated figures to build the formulas for the volumes of prisms.

How can you build the formulas for the volume of pyramids, cones, and spheres?

Getting Started

Short Stack

You have stacked congruent two-dimensional figures to create solid figures. Now, let's investigate how stacking similar figures can also create solid figures.

1. Imagine you stack similar circular pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.
 - a. Identify the name of the solid formed by this stack of pancakes.
 - b. Relate the dimensions of a single pancake to the dimensions of the solid.

2. Imagine you stack similar square pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.
 - a. Identify the name of the solid formed by this stack of pancakes.
 - b. Relate the dimensions of a single pancake to the dimensions of the solid.

3. Imagine you stack similar triangular pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.
 - a. Identify the name of the solid formed by this stack of pancakes.
 - b. Relate the dimensions of a single pancake to the dimensions of the solid.
4. What do you notice about the three-dimensional solids created by stacking similar figures?
5. Which type of solid is formed by stacking similar rectangles? Pentagons? Hexagons?

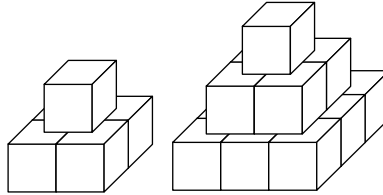
ACTIVITY
3.1

Building the Pyramid Volume Formula

Let's think about how to derive the formula for the volume of a pyramid. The diagrams show two pyramids—one with two layers and one with three layers.

.....
Review the definition
of **explain your
reasoning** in the
Academic Glossary.
.....

1. Use centimeter cubes to construct these pyramids. Then, answer the questions and explain your reasoning.



- a. Describe the two-dimensional shape of the top surface of each layer of cubes.
- b. Explain why the two-dimensional surfaces of the cube layers represent similar figures.
- c. What is the volume of each pyramid?
- d. For each of the two pyramids, determine the volume of a prism that has the same base.

2. Use centimeter cubes to build a pyramid with each number of layers. Complete the table.

Number of Layers	Volume of Pyramid (cubic cm)	Volume of Prism with Same Base (cubic cm)	$\frac{\text{Volume of Pyramid}}{\text{Volume of Prism}}$
1			
2			
3			
4			

3. How does the ratio volume of pyramid : volume of prism change as more layers are built onto each pyramid? Explain why you think this happens.

4. Build a spreadsheet to extend your exploration.

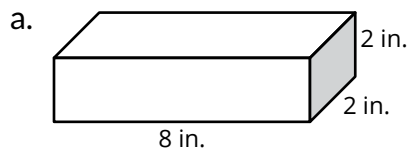
- In the first row of a spreadsheet, enter the column headings from the table in Question 2.
- Enter the data from the first 2 rows of the table, excluding the fourth column.
- In the fourth column, enter a formula for the ratio of the volume of the pyramid to the volume of the prism.

.....
In the spreadsheet, start each formula with “=”.

- d. Copy the formula into the fourth column of the second row.
Check that the calculated ratio has the same value as the ratio in your table from Question 2.
- e. Complete the next row by only using formulas. Be sure to use the patterns that you discovered when you were building the cube pyramids and prisms.
- f. Select the row of formulas that you just created and copy them into the next row. Continue to copy the row until you have 10 rows.
- g. How does your spreadsheet compare to your data table from Question 2?
- h. Extend your table to at least 100 rows—either by copying and pasting formulas or by using the “fill down” shortcut. Explain what you notice.

5. What happens to the ratio in the last column as the size of your pyramid increases?

6. Does your ratio start to approach a value? Explain your reasoning?
7. What generalization can you make about the relationship between the volume of a pyramid and the volume of a prism with the same base and height?
8. Write a formula for the volume of a pyramid with a square base.
9. Does this formula apply to pyramids with non-square bases? Explain your reasoning.
10. Determine the volume of each figure. Round your answer to the nearest hundredth.



- b. A rectangular pyramid with a base length of 2 inches and a height of 8 inches.
- c. Can you use the volume calculated in part a to determine the volume in part b? Explain your reasoning.

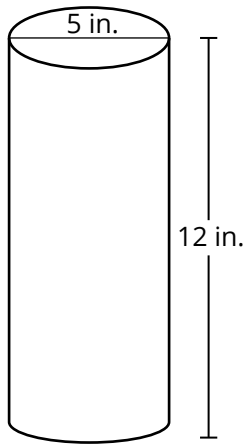
Comparing Shapes and Volumes

Recall, that the formula for the volume of a prism and a cylinder is $V = Bh$, where B represents the area of the base and h represents the height of the prism.

1. Explain why the same formula is used to determine the volume of both a prism and a cylinder.
2. Use the formula for the area of a circle to rewrite the formula for the volume of a cylinder with a radius of r and a height of h .
3. Describe the similarities between a pyramid and a cone.
4. Use the formula you wrote in Question 2 to write the formula for the volume of a cone with a radius of r and a height of h .

5. Determine the volume of each figure. Round your answer to the nearest hundredth.

a.



- b. The volume of a cone with a diameter of 5 inches and a height of 12 inches.

Let's investigate the volume of two different cylinders created with a standard sized sheet of paper.

6. Use two standard sized sheets of paper to create two cylinders. One cylinder should have a height of 11 inches, and the other cylinder should have a height of 8.5 inches.

.....
A standard sized
sheet of paper
measures 8.5 inches
by 11 inches.
.....



- a. Lizzie predicts that the cylinder with a height of 11 inches has a greater volume. Josh predicts that the cylinder with a height of 8.5 inches has a greater volume. Matthew predicts that the two cylinders have the same volume.

Predict which cylinder has the greatest volume.

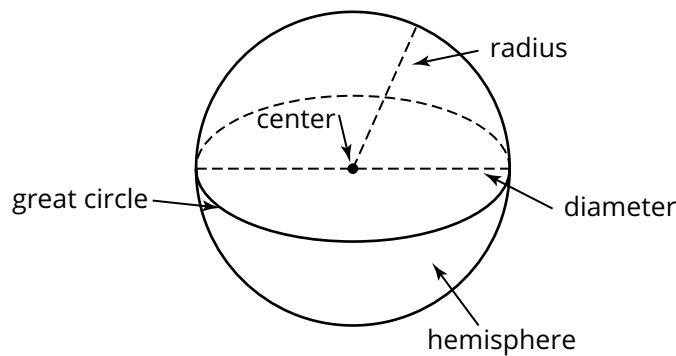
.....
The magnitude of
volume is the size of
the space that the
shape of the object
takes up.
.....

- b. Determine the radius and the height of each cylinder without using a measuring tool. Round each answer to the nearest hundredth.
 - c. Calculate the volume of each cylinder to prove or disprove your prediction and determine who was correct. Write each volume in terms of π . Then approximate each volume by rounding to the nearest hundredth.
 - d. Does the radius or the height have a greater impact on the magnitude of the volume? Explain your reasoning.
 - e. Consider the volume of the cylinder with a height of 8.5 inches. What radius would be required to create a cylinder with a height of 11 inches that has the same volume? Round your answer to the nearest hundredth.
7. Double the radius of each cylinder. How does doubling the radius affect the volume of a cylinder?

ACTIVITY 3.3

Volumes of Spheres

Recall that a circle is the set of all points in two dimensions that are equidistant from the center of the circle. A sphere can be thought of as a three-dimensional circle.



The volume of a sphere is given by the equation $V = \frac{4}{3}\pi r^3$, where r represents the radius of the sphere.

- Determine the volume of each sphere with the given radius length. Round each volume to the nearest hundredth.
 - 2.2 in.
 - 4 cm
 - $\frac{1}{2}$ yd
 - 4.4 in.
- Determine the volume of each hemisphere with the given radius length. Round each volume to the nearest hundredth.
 - 3.1 in.
 - $\frac{1}{4}$ ft
 - 6.2 in.

Remember ...

A great circle of a sphere is a cross-section of a sphere when a plane passes through the center of the sphere.

A **sphere** is the set of all points in three dimensions that are equidistant from a given point called the center.

The **radius of a sphere** is a line segment drawn from the center of the sphere to a point on the sphere.

The **diameter of a sphere** is a line segment drawn between two points on the sphere passing through the center.

A **hemisphere** is half of a sphere bounded by a great circle.

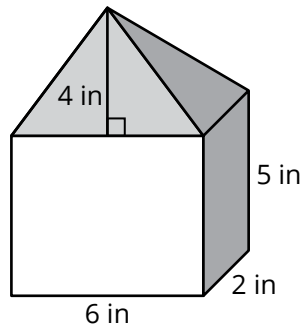
3. Compare the volumes of the spheres in Question 1, parts (a) and (d) How does doubling the radius affect the volume of a sphere?
4. Compare the volumes of the hemispheres in Question 2, parts (a) and (c). How does doubling the radius affect the volume of the hemisphere?
5. A cone and a sphere each have a radius of r units. The cone and the sphere also have equal volumes. Describe the height of the cone in terms of the radius. Show your work.

Volumes of Composite Figures

You know the formulas for the volume of prisms, cylinders, pyramids, cones, and spheres. You can use these formulas to determine the volume of composite figures.

WORKED EXAMPLE

Determine the volume of the composite figure.



Step 1: Determine which solids make up the figure.

This figure is composed of a rectangular prism and a pyramid.

Step 2: Determine the volume of each figure.

The formula for the volume of a prism is $V = Bh$.

$$B = (6)(2) = 12 \text{ square inches}$$

$$V = 12(5) = 60 \text{ cubic inches}$$

The volume of the prism is 60 cubic inches.

The formula for the volume of a pyramid is $V = \frac{1}{3}Bh$.

$$B = (6)(2) = 12 \text{ square inches}$$

$$V = \frac{1}{3}(12)(4) = 16 \text{ cubic inches}$$

Step 3: Add the volume of the figures that compose the figure to determine the volume of the composite figure.

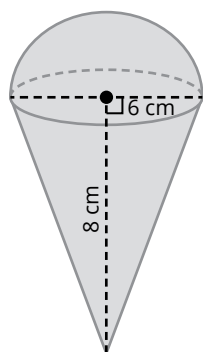
$$60 + 16 = 76 \text{ cubic inches}$$

PROBLEM SOLVING

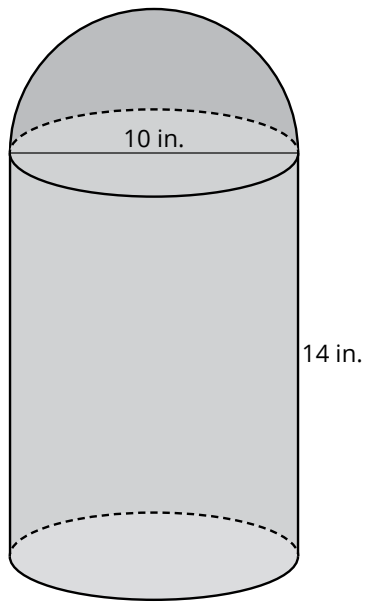


1. Determine the volume of each composite figure. Show your work. Round your answer to the nearest hundredth when necessary.

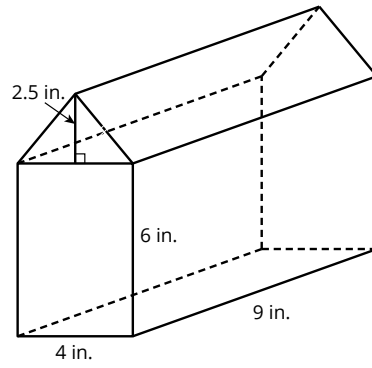
a.



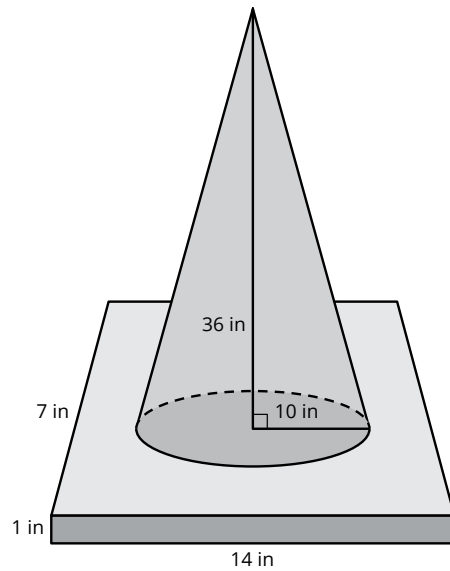
b.



c.



d.



Talk the Talk

Ice Cream Cone Piñata

Tiara asked her parents to make a piñata for her birthday party. Her parents decided to make the piñata in the shape of her favorite dessert, an ice cream cone. They stuffed only the cone portion of the piñata.

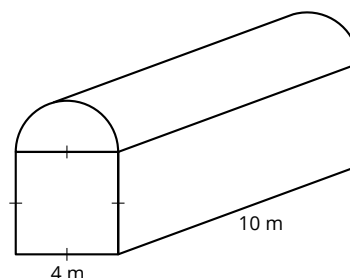


- The height of the cone is 34 in.
 - The length of the diameter of the base is 24 in.
1. Calculate the amount of space (cubic feet) in the cone that will be filled with goodies. Round your answer to the nearest whole number.

The Memphis Pyramid is a giant pyramid in downtown Memphis, Tennessee. It is a tourist attraction that draws many visitors. Visitors at the pyramid can view the Mississippi River and the city from a glass observation deck.

A side of the square base measures 591 feet long. The height of the pyramid is 321 feet.

2. Calculate the volume of the Memphis Pyramid. When necessary round your answer to the nearest hundredth.
3. Determine the volume of the composite figure. Round your answer to the nearest hundredth.



Remember ...

The volume of a pyramid is $\frac{1}{3}$ the volume of a prism with the same base and height.

4. Complete the table to record the formulas for the volume of the figures you studied in this lesson. Identify what the variables in your formulas represent.

Volume Formulas

Figure	Volume
Right Rectangular Prism	
Right Square Pyramid	
Right Cylinder	
Right Cone	
Sphere	

Lesson 3 Assignment

Write

1. Describe the relationship between the volume of a prism and the volume of a pyramid.
2. Describe the relationship between the volume of a cylinder and the volume of a cone.

Remember

The volume formula for a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone.

The volume formula for a pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height of the pyramid.

The volume formula for a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.

To determine the volume of a composite figure add the volume of the figures that make up the figure.

Practice

1. The Luxor Hotel in Las Vegas is a replica of the Great Pyramid at Giza. The Luxor's base is a square with a side length of 600 feet. Each lateral face has a slant height of 460 feet, and the hotel is 350 feet tall.
What is the volume of the Luxor Hotel?

2. Your municipality is replacing the storage tanks in the community. Which plan provides the greater total capacity?

Plan 1: Install one cylindrical tank that is 150 feet tall and has a radius of 50 feet.

Plan 2: Install two cylindrical tanks that are 75 feet tall. One cylindrical tank has a radius of 30 feet, and one tank has a radius of 25 feet.

Use 3.14 for π . Round your answers to the nearest tenth, if necessary.

Lesson 3 Assignment

3. A traffic cone has a radius of 9 inches and a height of 30 inches. What is the volume of this traffic cone? Round your answer to the nearest tenth.
4. Today's deal at the ice cream shop is a mini cone with one scoop of ice cream.
 - a. A mini ice cream cone has a diameter of 3.5 centimeters and a height of 6 centimeters. How much ice cream fits in the cone? Round your answer to the nearest tenth.
 - b. One scoop of ice cream has the same diameter as the cone, 3.5 centimeters. What's the volume of 1 scoop of ice cream? Round your answer to the nearest tenth.
5. A soccer ball has a circumference of 70 centimeters at its widest point. What is the volume of the soccer ball? Round your answer to the nearest whole number.
6. How does tripling the radius of a cylinder affect its volume?
7. How does tripling the height of a cylinder affect its volume?

Lesson 3 Assignment

8. How does halving the radius of a sphere affect its volume?

9. A spherical stone with a radius of 16 inches is placed on top of a cylindrical platform with the same radius. The height of the cylindrical platform is 48 inches. Determine the volume of the composite figure. Round your answer to the nearest tenth.

Prepare

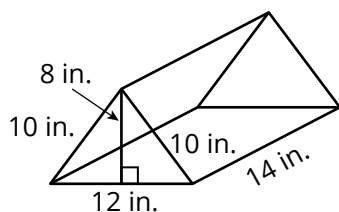
1. Determine the area of each figure. Round your answer to the nearest tenth when necessary.
 - a. a circle with a radius of 4 in.

 - b. a circle with a diameter of 6 ft

 - c. a right triangle with legs that measure 5 in. and 12 in.

Lesson 3 Assignment

2. Determine the area of each base and each lateral face of the triangular prism.



4

Building Lateral and Surface Area Formulas for Pyramids, Cones, and Spheres

OBJECTIVES

- Identify lateral faces and bases of three dimensional figures.
- Use the formulas for the lateral and surface areas of a cylinder, cone, sphere, and hemisphere to solve problems in written form.
- Determine the lateral and total surface areas of composite figures.

KEY TERMS

- lateral surface area
- total surface area

.....

You have determined the area of two-dimensional shapes including rectangles, triangles, and circles.

How can you use what you know about the area of two-dimensional figures to build the lateral and surface area formulas for cones, cylinders, and pyramids?

Getting Started

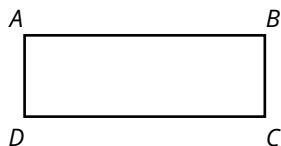
Remember...

A lateral face of a three-dimensional object is a face that is not a base.

Bases and Faces

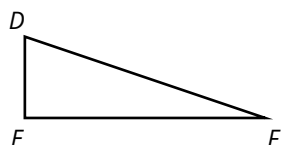
You have applied rotations to two-dimensional figures to create solids. Let's rotate two-dimensional figures and then identify lateral faces and bases of solids.

1. Consider rectangle $ABCD$.



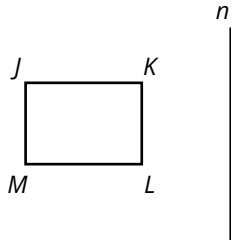
- a. Describe the shape formed as the rectangle $ABCD$ rotates around $\overline{DC}$.
- b. Identify and describe the shape and number of lateral faces and bases?

2. Consider $\triangle DEF$.



- a. Describe the shape formed as $\triangle DEF$ rotates around $\overline{EF}$.
- b. Identify and describe the shape and number of any lateral faces and bases.

3. Consider the rectangle $JKLM$.



- a. Describe the shape formed as rectangle $JKLM$ rotates around line n .

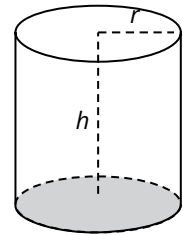
- b. Describe the shape and number of lateral faces and bases of the figure.

Surface Area

In the previous lesson, you explored the different volumes created by rolling a piece of paper in two different ways. The cylinders that you created had the same *lateral surface area*. The **lateral surface area** of a three-dimensional figure is the sum of the areas of its lateral faces. The **total surface area** of a three-dimensional figure is the sum of the areas of its bases and lateral faces.

1. Consider the right cylinder shown.

- a. Sketch the bases and lateral faces of the cylinder. Include the dimensions.



- b. Determine the area of each face.

.....

Recall that the width of the lateral face of a cylinder is equal to the circumference of the base.

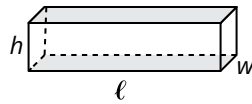
.....

- c. Use your sketch to write the formulas for the total surface area and lateral surface area of the cylinder. Explain your reasoning.

2. A cylindrical paint roller has a diameter of 2.5 inches and a length of 10 inches.

- a. Apply the formula to determine the lateral surface area of the paint roller.
- b. Apply the formula to determine the total surface area of the paint roller.

3. Consider the right rectangular prism shown. Its bases are shaded.



- Sketch the bases and lateral faces of the prism. Include the dimensions of each.
- Determine the area of each face.
- Use your sketch to write the formulas for the total surface area and lateral surface area of the right rectangular prism. Explain your reasoning.

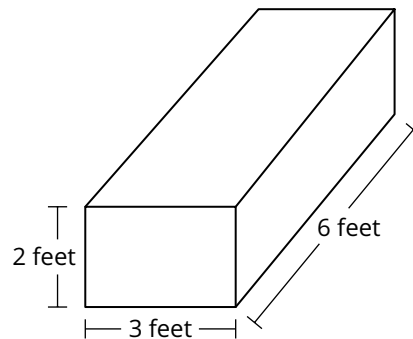
.....
For right rectangular prisms, you can call any pairs of opposite faces "bases."
.....



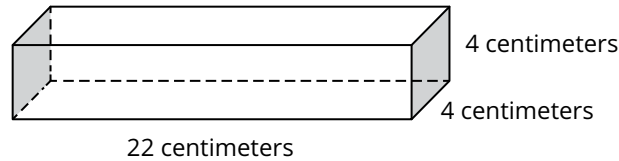
4. Jamal says that the lateral surface area of a right rectangular prism can change, depending on what the bases of the prism are. He calculates 3 different lateral surface areas, L , for the prism shown.

$$L_1 = 36 \text{ ft}^2 \quad L_2 = 60 \text{ ft}^2 \quad L_3 = 48 \text{ ft}^2$$

Is Jamal correct? Explain your reasoning.



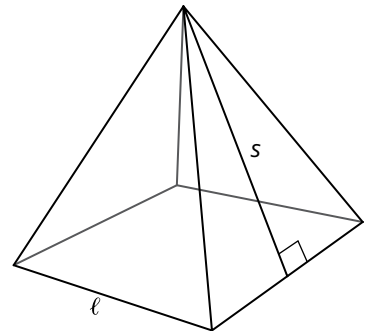
5. Determine the lateral surface area and total surface area of the right prism shown. The bases of the prism are shaded.



- Identify the length, width, and height of the prism.
- Apply the formula to determine the lateral surface area of the prism.
- Apply the formula to determine the total surface area of the prism.

6. Consider the right square pyramid shown.

- Sketch the bases and lateral faces of the pyramid. Include the dimensions.



- Determine the area of each face.

c. Use your sketch to write the formulas for the total surface area and lateral surface area of the pyramid. Explain your reasoning.

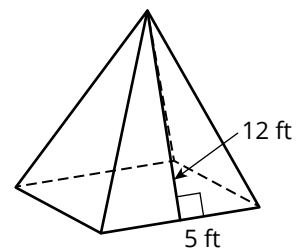
d. Use your sketch and the formula you determined to explain why Mario is correct.

Mario

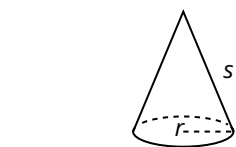


Lateral surface area of a right square pyramid = $\frac{1}{2}Ps$. Total surface area of a right square pyramid = $\frac{1}{2}Ps + B$.
 P = perimeter of base,
 s = slant height, and
 B = area of base.

7. Determine the lateral and total surface area of the square pyramid. Round to the nearest hundredth.



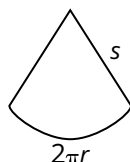
You can also apply the formulas for the lateral and total surface areas of cones to solve problems.



Base:



Lateral Face:



A right cone is made up of two faces—a circular base and a wedge-shaped lateral face.

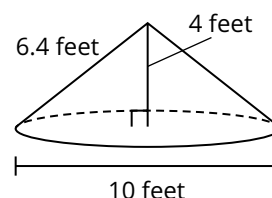
The area of the circular base is given by πr^2 .

The area of the lateral face is given by $\frac{1}{2} (2\pi r)(s)$, or πrs , where s is the slant height of the cone.

8. Write the formulas for the lateral surface area and total surface area of a right cone.

9. Determine the lateral and total surface area of the cone. Round to the nearest hundredth.

a. Determine the slant height, s , and the radius, r , of the cone.

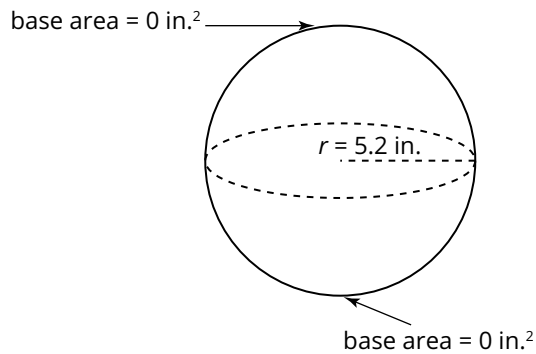


b. Apply the formula to determine the lateral surface area of the cone.

c. Apply the formula to determine the total surface area of the cone.

ACTIVITY
4.2**Surface Area of Spheres**

You can think of a sphere as a solid with one surface. The formula for the total surface area of a sphere is:



$$\begin{aligned}\text{total surface area} &= \text{lateral surface area} \\ &= 4\pi r^2\end{aligned}$$

1. Apply the formula to determine the total and lateral surface area of the sphere shown. Round your answer to the nearest tenth.
2. Determine the surface area of each sphere with the given radius length. Round each surface area to the nearest hundredth.
 - a. 2.2 in.
 - b. 4 cm
 - c. $\frac{1}{2}$ yd
 - d. 4.4 in.

Ask Yourself . . .

Why are the formulas for the lateral and total surface area of a sphere the same?

.....
Recall that a hemisphere is a sphere bounded by a great circle
.....

3. Compare the surface areas of the spheres in Question 2, parts (a) and (d). How does doubling the radius affect the surface area of the sphere?

The formula for the lateral surface area of a hemisphere is:

$$A = \frac{1}{2}4\pi r^2 = 2\pi r^2$$

The lateral surface area of a hemisphere only consists of one surface which is half of the total surface of the sphere.

4. Determine the lateral surface area of each hemisphere with the given radius length. Round your answer to the nearest hundredth.

a. 2 in.

b. 3 in.

Ask Yourself . . .

Does a hemisphere have a base?

The total surface area of a hemisphere is different than the lateral surface area of a hemisphere.

The formula for the total surface area of a hemisphere is:

$$A = 3\pi r^2.$$

5. Why is the formula for the lateral surface area of a hemisphere different than the formula for the total surface area of a hemisphere?
6. Determine the total surface area of each hemisphere with the given radius length. Round your answer to the nearest hundredth.
- a. 2 in.
- b. 3 in.

PROBLEM SOLVING



ACTIVITY 4.3

Lateral and Surface Areas of Composite Figures

You know the formulas for the lateral and total surface areas of prisms, pyramids, cones, and spheres. You can use these formulas to determine the lateral and total surface area of composite figures.

WORKED EXAMPLE

Determine the lateral surface area of the composite figure.

Step 1: Determine which solids make up the figure.

This figure is composed of a rectangular prism and a pyramid.

Step 2: Determine the lateral surface area of each figure.

Lateral surface area of the prism:

The formula for the lateral surface area of the prism is

$$A = 2(5) + 6(5) + 2(5) + 6(5) = 80.$$

The lateral surface area of the prism is 80 square inches.

Lateral surface area of the pyramid:

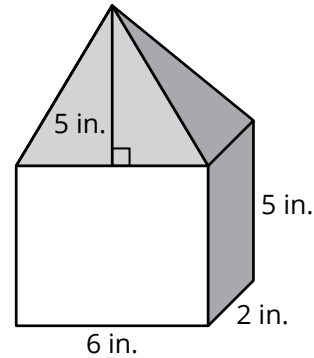
The formula for the lateral surface area of the pyramid is

$$A = \frac{1}{2} P_s = \frac{1}{2} (20)(5) = 50.$$

The lateral surface area of the pyramid is 50 square inches.

Step 3: Add the lateral areas of the figures that compose the figure to determine the lateral surface area of the composite figure.

$$80 + 50 = 130 \text{ square inches}$$



1. Consider the composite figure from the Worked Example.
 - a. Determine the total surface area of the prism in the figure.
 - b. Determine the total surface area of the pyramid in the figure.



Kaya

Kaya says the total surface area of the composite figure is

$$A = 104 + 62$$

$$A = 166 \text{ square inches}$$

Andrew

Andrew says the total surface area of the composite figure is

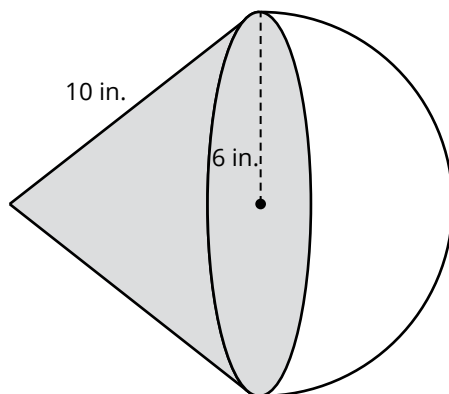
$$A = (104 - 12) + (62 - 12) = 142$$

$$A = 142 \text{ square inches}$$

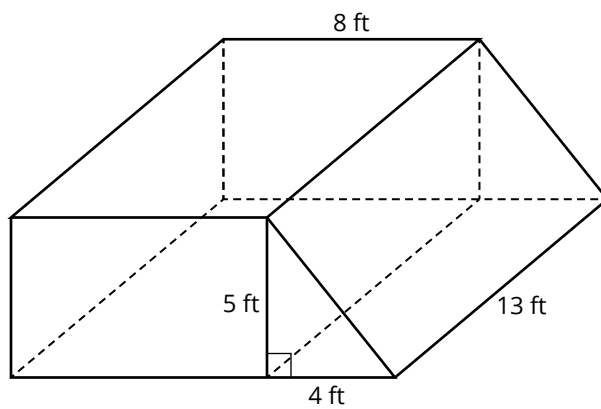
2. Who's correct? Explain your reasoning.

3. Determine the lateral and total surface area of each composite figure. Show your work. Round your answer to the nearest hundredth when necessary.

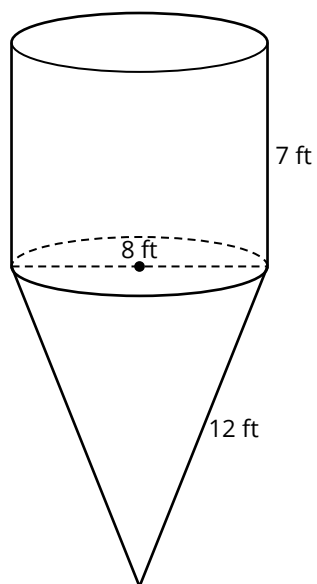
a.



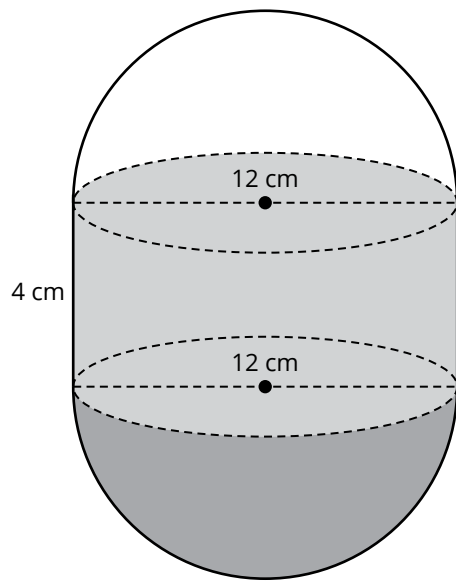
- b. The two bases of this composite figure are congruent composite two-dimensional shapes and are opposite each other.



C.



d.





Talk the Talk

Being Creative

Samantha is creating a hat for her friend's birthday party. She is decorating a cone-shaped party hat by covering the outside with construction paper.

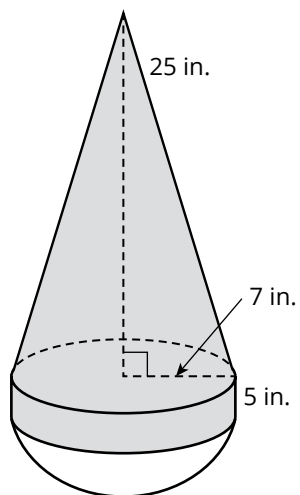
- The height of the cone is 17.8 cm.
 - The length of the diameter of the base is 19.5 cm.
1. Calculate the amount of construction paper (in square centimeters) that Samantha will need to cover the outside of the hat. Round your answer to the nearest tenth.



Mario is constructing a square pyramid for class out of cardboard. He wants to make a pyramid that has a base length of 20 cm and a slant height of 24 cm.

2. Calculate the amount of cardboard Mario needs for his project.

3. Determine the lateral, total surface area, and volume of the composite figure. Round your answer to the nearest hundredth.



4. Complete the table to record the formulas for the lateral surface area and total surface area of the figures you studied in this lesson. Identify what the variables in your formulas represent.

Surface Area Formulas

Figure	Lateral Surface Area	Total Surface Area
Right Rectangular Prism		
Right Square Pyramid		
Right Cylinder		
Right Cone		
Sphere		

Lesson 4 Assignment

Write

Describe a similarity and a difference between each term.

1. lateral surface area and total surface area
2. surface area and volume

Remember

The lateral surface area of a three-dimensional figure is the sum of the areas of its lateral faces. The total surface area of a three-dimensional figure is the sum of the areas of its bases and lateral faces.

The formula for the total surface area of a sphere is $SA = 4\pi r^2$, where r is the radius of the sphere.

Practice

1. The Luxor Hotel in Las Vegas is a replica of the Great Pyramid at Giza. The Luxor's base is a square with a side length of 600 feet. Each lateral face has a slant height of 460 feet, and the hotel is 350 feet tall.
 - a. What is the lateral surface area of the Luxor Hotel?
 - b. What is the total surface area of the Luxor Hotel?

Lesson 4 Assignment

2. Your municipality is replacing the storage tanks in the community. They need to determine the amount of metal needed to construct each tank. Which tank requires less metal to construct?

Plan 1: Install one cylindrical tank that is 150 feet tall and has a radius of 50 feet.

Plan 2: Install two cylindrical tanks that are 75 feet tall. One cylindrical tank has a radius of 30 feet, and one tank has a radius of 25 feet.

Use 3.14 for π . Round your answers to the nearest tenth, when necessary.

3. A traffic cone has a radius of 9 inches and a height of 30 inches. What is its lateral and total surface area when the slant height of the traffic cone is approximately 31 inches? Round your answers to the nearest whole number.

Lesson 4 Assignment

- A snow globe paperweight is in the shape of a hemisphere and has a diameter of 3.5 inches. What is the paperweight's lateral surface area, total surface area, and volume? Round your answers to the nearest tenth.
- A soccer ball has a circumference of 68 centimeters at its widest point. What is the lateral and total surface area of the soccer ball? Round your answer to the nearest whole number.
- How does tripling the radius of a cylinder affect its surface area?

Lesson 4 Assignment

7. How does tripling the radius and height of a cylinder affect its surface area?

8. How does halving the radius of a sphere affect its surface area?

9. A hemisphere with a radius of 5 centimeters is placed on top of a cylinder so its base is touching the cylinder's base with the same radius. The height of the cylinder is 12 centimeters. Determine the lateral area and total surface area of the composite figure. Round your answers to the nearest whole number.

Lesson 4 Assignment

Prepare

Consider the 7 days of a week. Suppose the name of each day is written on a separate piece of folded paper and placed into a bag. You reach into the bag and choose one piece of paper. Determine each probability.

1. Randomly choosing a day that begins with the letter S
2. Randomly choosing a weekday
3. Randomly choosing Wednesday

Building Three-Dimensional Shapes

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

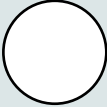
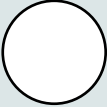
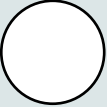
Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Building Three-Dimensional Shapes* topic by:

TOPIC 2 <i>Building Three-Dimensional Shapes</i>	Beginning of Topic	Middle of Topic	End of Topic
using translations and stacking of congruent and similar figures to compose prisms, cylinders, pyramids, and cones and to explain the volume formulas for prisms and cylinders.			
identifying two-dimensional cross-sections of prisms, pyramids, cylinders, cones, and spheres.			
explaining the formula for the volume of a pyramid.			
explaining the formula for the volume of a cone.			
using volume formulas to calculate the volume of geometric and real-world figures.			
using surface area formulas to calculate the surface area of geometric and real-world figures.			
describing how non-proportional changes in the linear dimensions of a figure and cylinder affect its volume or surface area.			

continued on the next page

TOPIC 2 SELF-REFLECTION
continued

TOPIC 2 <i>Building Three-Dimensional Shapes</i>	Beginning of Topic	Middle of Topic	End of Topic
determining the volume, lateral surface area, and total surface area of composite figures.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

- Describe a new strategy you learned in the *Building Three-Dimensional Shapes* topic.

- What mathematical understandings from the topic do you feel you are making the most progress with?

- Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 2 SUMMARY

Building Three-Dimensional Shapes

LESSON

1

Building Three-Dimensional Figures

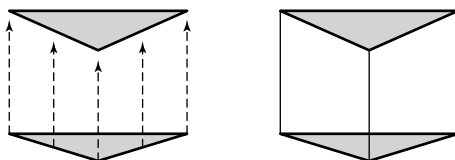
While a circle is the set of all points that are equal distance from the center, a **disc** is the set of all points on the circle and in the interior of the circle.

You can perform rotations in three-dimensional space to create a model of a three-dimensional figure.

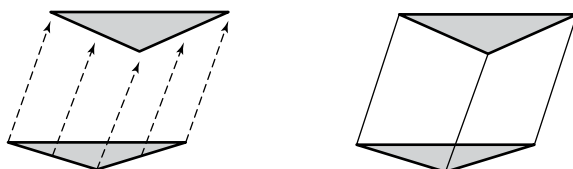
For example, a cylinder is formed by rotating a rectangle about one of its sides, or about an axis of symmetry. A sphere is formed by rotating a disc about an axis of symmetry. A cone is formed by rotating any triangle about an axis of symmetry, or by rotating a right triangle about one of its legs.

You can translate a two-dimensional figure through space to create a model of a three-dimensional figure. You can use **isometric paper**, or dot paper, to create a two-dimensional representation of a three-dimensional figure.

When you translate a triangle through space in a direction that is perpendicular to the plane containing the triangle, the solid formed is a **right triangular prism**, as shown. Each lateral face of a right triangular prism is a rectangle. A **lateral face** of a three-dimensional object is a face that is not a base.



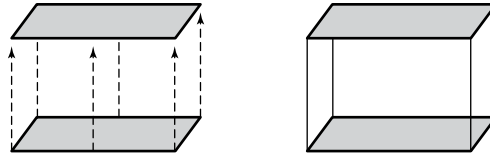
When you translate a triangle through space in a direction that is not perpendicular to the plane containing the triangle, the solid formed is an **oblique triangular prism**. An example of an oblique triangular prism is shown. Each lateral face is a parallelogram.



KEY TERMS

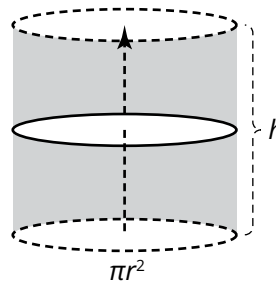
- disc [disco]
- lateral face
- isometric paper [papel isométrico]
- right triangular prism
- oblique triangular prism [prisma triangular oblicuo]
- right rectangular prism
- oblique rectangular prism [prisma rectangular oblicuo]
- right cylinder
- oblique cylinder [cilindro oblicuo]
- cross-section
- sphere [esfera]
- radius of a sphere [radio de una esfera]
- diameter of a sphere [diámetro de una esfera]
- hemisphere [hemisferio]
- lateral surface area [área de superficie lateral]
- total surface area [área de superficie total]

A two-dimensional representation of a rectangular prism can be obtained by translating a rectangle in two dimensions and connecting corresponding vertices. When you translate a rectangle through space in a direction that is perpendicular to the plane containing the rectangle, the solid formed is a **right rectangular prism**, as shown.

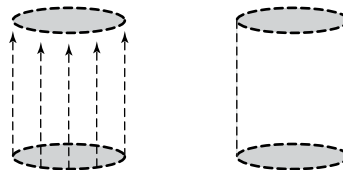


When you translate a rectangle through space in a direction that is not perpendicular to the plane containing the rectangle, the solid formed is an **oblique rectangular prism**.

To derive the formula for the volume of a cylinder, you can think of the cylinder as an infinite stack of discs, each with an area of πr^2 . These discs are stacked to a height of h , the height of the cylinder. So, the volume of the cylinder is $(\pi r^2)(h)$, or $\pi r^2 h$.



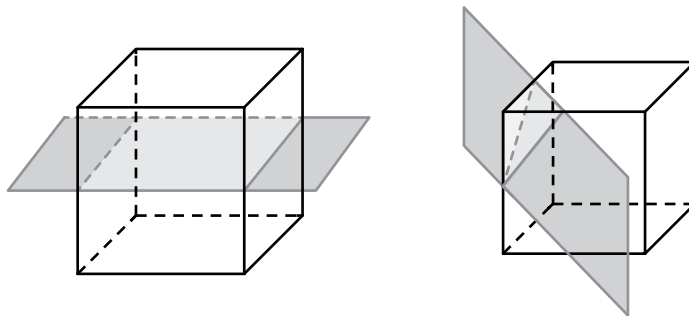
When you translate a disc through space in a direction that is perpendicular to the plane containing the disc, the solid formed is a **right cylinder**.



When you translate a disc through space in a direction that is not perpendicular to the plane containing the disc, the solid formed is an **oblique cylinder**.

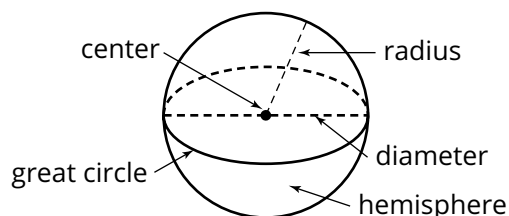
A **cross-section** of a three-dimensional solid results from the intersection of a plane with the solid. When the plane passes through the solid, a two-dimensional cross-section is formed. A point or a line segment may be the cross-section of a plane and a solid when the plane intersects just a vertex or an edge of the solid.

For example, consider some of the cross-sections formed by the intersection of a plane and a cube.



A pyramid has a volume that is one third the volume of a prism with the same base and height. Similarly, the volume of a cone is one third the volume of a cylinder with the same base area and height.

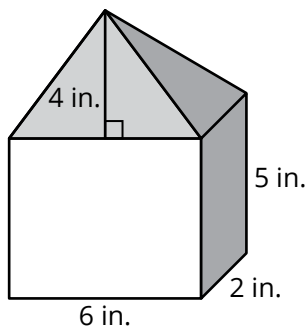
A **sphere** is the set of all points in three dimensions that are equidistant from a given point, called the *center*. The **radius of a sphere** is a line segment drawn from the center of the sphere to a point on the sphere. The **diameter of a sphere** is a line segment drawn between two points on the sphere passing through the center. A great circle of a sphere is a cross section of a sphere when a plane passes through the center of the sphere. A **hemisphere** is half of a sphere bounded by a great circle.



The volume formulas for the figures that we have explored are shown.

Prisms	Pyramids	Cylinders	Cones	Spheres	Hemisphere
$V = (\text{area of base}) \times \text{height}$ $V = Bh$	$V = \frac{1}{3}Bh$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$	$V = \frac{4}{3}\pi r^3$	$V = \frac{2}{3}\pi r^3$

The volume of a composite figure is the sum of the volumes of each solid in the figure.



This figure is composed of a rectangular prism and a pyramid.

The formula for the volume of a prism is $V = Bh$.

$$B = (6)(2) = 12 \text{ square inches}$$

$$V = 12(5) = 60 \text{ cubic inches}$$

The volume of the prism is 60 cubic inches.

The formula for the volume of a pyramid is $V = \frac{1}{3} Bh$.

$$B = (6)(2) = 12 \text{ square inches}$$

$$V = \frac{1}{3}(12)(4) = 16 \text{ cubic inches}$$

Add the volume of the figures that compose the figure to determine the volume of the composite figure.

$$60 + 16 = 76 \text{ cubic inches}$$

LESSON

4

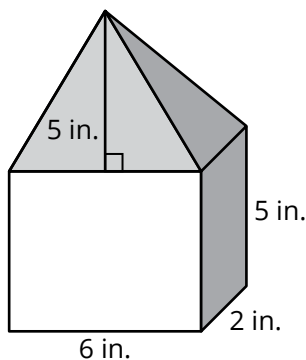
Building Lateral and Surface Area Formulas for Pyramids, Cones, and Spheres

The **lateral surface area** of a three-dimensional figure is the sum of the areas of its lateral faces. The **total surface area** of a three-dimensional figure is the sum of the areas of its bases and lateral faces. The surface area formulas for the figures you have explored are shown on the next page.

Surface Area Formulas

Figure	Lateral Surface Area	Total Surface Area
Right Rectangular Prism	$2\ell h + 2wh$ ℓ = length w = width h = height	$2\ell w + 2\ell h + 2wh$, or $2B + L$ ℓ = length w = width h = height B = area of base L = lateral surface area
Square Pyramid	$2\ell s$ ℓ = length of square s = slant height	$2\ell s + \ell^2$, or $B + L$ ℓ = length of square s = slant height B = area of base L = lateral surface area
Right Cylinder	$2\pi rh$ r = radius of cylinder h = height of cylinder	$2\pi r^2 + 2\pi rh$, or $2B + L$ r = radius of cylinder h = height of cylinder B = area of base L = lateral surface area
Right Cone	πrs r = radius of cone s = slant height	$\pi r^2 + \pi rs$, or $B + L$ r = radius of cone s = slant height B = area of base L = lateral surface area
Sphere	$4\pi r^2$ r = radius of sphere	$4\pi r^2$ r = radius of sphere
Hemisphere	$2\pi r^2$ r = radius of sphere	$3\pi r^2$ r = radius of sphere

The lateral surface area of a composite figure is the sum of the lateral areas of each solid in the figure. The total surface area of a composite figure is the sum of the lateral areas of each solid, plus any included bases that are visible on the outer surface of the composite figure.



This figure is composed of a rectangular prism and a pyramid.

Lateral Surface Area of the prism.

$$A = 2(5) + 6(5) + 2(5) + (65) = 80$$

The lateral surface area of the prism is 80 square inches.

The formula for the lateral surface area of the pyramid is

$$A = \frac{1}{2} Ps = \frac{1}{2} (16)(5) = 40$$

The lateral surface area of the pyramid is 40 square inches.

The lateral surface area of the composite figure is $80 + 40 = 120$ square inches.

To determine the total surface area, add the visible base on the outer surface of the composite figure.

The area of this visible base is $(6)(2) = 12$ square inches.

The total surface area of the composite figure is $120 + 12 = 132$ square inches.

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